Fracton superfluid hydrodynamics

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We examine the hydrodynamics of systems with spontaneously broken multipolar symmetries using a systematic effective field theory. We focus on the simplest nontrivial setting: a system with charge and dipole symmetry, but without momentum conservation. When no symmetries are broken, our formalism reproduces the quartic subdiffusion ($\omega \sim -ik^4$) characteristic of "fracton hydrodynamics" with conserved dipole moment. Our formalism also captures spontaneous breaking of charge and/or dipole symmetry. When charge symmetry is spontaneously broken, the hydrodynamic modes are quadratically propagating and quartically relaxing ($\omega \sim \pm k^2 - ik^4$). When the dipole symmetry is spontaneously broken but the charge symmetry is preserved, then we find quadratically relaxing (diffusive) transverse modes, plus another mode which, depending on parameters, may be either purely diffusive ($\omega \sim -ik^2$) or quadratically propagating and quadratically relaxing ($\omega \sim \pm k^2 - ik^2$). Our work provides concrete predictions that may be tested in near-term cold atom experiments, and also lays out a general framework that may be applied to study systems with spontaneously broken multipolar symmetries.

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I. INTRODUCTION

Multipolar symmetries are exciting widespread interest in modern condensed matter physics, quantum information, and quantum dynamics. They connect to exotic "fracton" phases of quantum matter [1,2] and can provide a new route to ergodicity breaking [3–5]. Multipolar symmetries can display partial or complete spontaneous symmetry breaking (SSB), through which they can stabilize new kinds of phases [6]. Of particular interest, the approach to equilibrium in systems with multipolar symmetries is described by hydrodynamics in an infinite family of nonstandard universality classes [7,8], collectively termed "fracton hydrodynamics." This "fracton hydrodynamics"—which has been realized in ultracold atoms [9]—provides an exciting new frontier, the exploration of which has become an important topic of research in its own right [10–21].

The thermodynamics of SSB of multipolar symmetries has been discussed in [22,23], where analogs of the Mermin-Wagner and Imry-Ma theorems were established. Different patterns of SSB, either of the entire multipole group or of its subgroups, correspond to condensing either monopole charges or higher multipole charges. We will call such SSB phases "fracton superfluids." The hydrodynamics of conventional superfluids is a well studied subject [24]—here, the Goldstone boson of the broken symmetry becomes a hydrodynamic mode. However, given the surprises attendant in the hydrodynamics of systems with unbroken multipolar symmetries, one might anticipate new features in hydrodynamics of fracton superfluids. By analogy, we may term the hydrodynamics of such generalized superfluids "fracton superfluid hydrodynamics." Previous literature has studied fracton superfluids at zero temperature [6,25]. Other work has shown that multipolar symmetries and translation symmetry together lead to exotic hydrodynamics in which one symmetry must be spontaneously broken [21], by certain definitions of SSB; see also [26].

In this work, we develop the theory of fracton superfluid hydrodynamics at nonzero temperatures. We do so in the simplest possible setting—a system with only charge and dipole symmetry, leaving generalization to arbitrary multipole groups and/or momentum conservation to future work. For such charge and dipole conserving systems, we develop a systematic effective field theory description which yields three phases, roughly organized as in Fig. 1. One phase corresponds to the "fracton hydrodynamics" of [7] with no symmetries broken. Another phase, with the symmetry fully broken, has been called the "fractonic superfluid" [25] or the "Bose-Einstein insulator" [6]. We will call it the "charge condensate." Finally, the "dipole condensate," with the dipole symmetry spontaneously broken and the monopole symmetry unbroken, exhibits new hydrodynamics. The two condensate phases are both fracton superfluids.

II. EFFECTIVE ACTION

We will use the recently developed hydrodynamic effective field theory [27–31] to explore our hydrodynamic phases. To build the effective action for a system with charge and dipole conservation we will use the phase fields ϕ and ψ_i. Recall that ordinary charge conservation symmetry corresponds to

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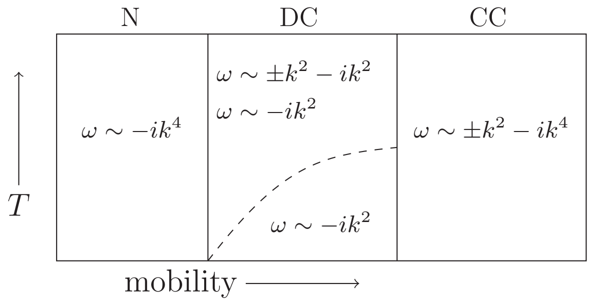

FIG. 1. A rough organization of the three phases we find. The axes are temperature T and mobility of the charges, which could be controlled by a hopping parameter in a microscopic theory. The normal phase (N) displays subdiffusion while the charge condensate (CC) displays quadratically propagating modes. The dipole condensate (DC) phase displays a transition between only diffusive modes at small T and coexisting diffusive and propagating modes at large T.

invariance under global shifts of the form $\phi \to \phi + b$. In order to consider conservation of charge and dipole moment, we additionally require the action to be invariant under shifts of the form

$$\phi \to \phi + b + x^k c_k, \quad \psi_i \to \psi_i + c_i. \tag{1}$$

The invariant objects are $\partial_t\phi$, $\partial_t\psi_i$, $\nabla_i\phi - \psi_i$, $\nabla_i\psi_j$, and $\nabla_i\nabla_j\phi$. Here ϕ is the "monopole" field and ψ_i is the "dipole" field, and the combination $\nabla_i\phi - \psi_i$ indicates that the motion of a monopole charge involves absorption (or emission) of a dipole. We notice that $\nabla_i\nabla_j\phi = \nabla_i(\nabla_j\phi - \psi_j) + \nabla_i\psi_j$, demonstrating that $\nabla_i\nabla_j\phi$ is a redundant degree of freedom.

Our most general Lagrangian is then

$$\mathcal{L} = \mathcal{L}(\partial_t\phi, \partial_t\psi_i, \nabla_i\phi - \psi_i, \nabla_i\psi_j). \tag{2}$$

From this, we can derive two Noether-like equations:

$$0 = \partial_t\rho + \nabla_i J_i, \tag{3}$$

$$0 = \partial_t\rho_i + J_i - \nabla_j J_{ij}, \tag{4}$$

where we have defined

$$\rho \equiv \frac{\partial\mathcal{L}}{\partial(\partial_t\phi)} \quad J_i \equiv \frac{\partial\mathcal{L}}{\partial(\nabla_i\phi)} = -\frac{\partial\mathcal{L}}{\partial\psi_i}$$
$$\rho_i \equiv \frac{\partial\mathcal{L}}{\partial(\partial_t\psi_i)} \quad J_{ij} \equiv -\frac{\partial\mathcal{L}}{\partial(\nabla_j\psi_i)} \tag{5}$$

as the densities and currents. We will call (3) the monopole continuity equation, and (4) the dipole continuity equation. To recognize (4) as dipole conservation, we can define the total dipole moment $d_i \equiv x_i\rho + \rho_i$ so that it obeys a continuity equation

$$0 = \partial_t d_i + \nabla_j J^{(d)}_{ij}, \tag{6}$$

where $J^{(d)}_{ij} = x_i J_j - J_{ij}$.

In order to turn this into a hydrodynamic effective field theory (EFT), following [31], we must put the action on a doubled Schwinger-Keldysh contour, and define forward-propagating fields ϕ_1 and ψ_{i1} and backward-propagating fields ϕ_2 and ψ_{i2} on the two contours. In the hydrodynamic limit, the forward and backward fields are close to equal so it is easier to work with the "classical fields" $\phi = (\phi_1 + \phi_2)/2$, $\psi_i = (\psi_{i1} + \psi_{i2})/2$ and the "noise fields" $\Phi = \phi_1 - \phi_2$, $\Psi_i = \psi_{i1} - \psi_{i2}$.

The full hydrodynamic Lagrangian will have the form

$$\mathcal{L}_{\text{eff}} = \rho\partial_t\Phi + \rho_i\partial_t\Psi_i + J_i(\nabla_i\Phi - \Psi_i) - J_{ij}\nabla_j\Psi_i, \tag{7}$$

where the densities and currents may depend on ϕ, ψ_i, Φ, and Ψ_i. We have no terms with $\nabla_i\nabla_j\Phi$ for the same reason we have no terms involving $\nabla_i\nabla_j\phi$ in (2): such terms can be converted into the terms already present in (7). For details, see the Appendix.

The action $I[\phi, \Phi, \psi_i, \Psi_i] = \int d^3x\, dt\, \mathcal{L}_{\text{eff}}$ must be symmetric under

$$\phi \to \phi + b + x^k c_k, \quad \psi_i \to \psi_i + c_i,$$
$$\Phi \to \Phi + b' + x^k c'_k, \quad \Psi_i \to \Psi_i + c'_i, \tag{8}$$

which are independent dipole transformations for the classical and noise fields. We can see that the hydroyanmic variables in (7) are the currents for the transformations of the noise fields.

Furthermore, the action must satisfy the EFT symmetries [31],

$$I^*[\phi, \Phi, \psi_i, \Psi_i] = -I[\phi, -\Phi, \psi_i, -\Psi_i],$$
$$I[\phi, \Phi = 0, \psi_i, \Psi_i = 0] = 0,$$
$$\operatorname{Im} I[\phi, \Phi, \psi_i, \Psi_i] \geqslant 0, \tag{9}$$

which can be derived from the full Schwinger-Keldysh formalism. It also must satisfy the Kubo-Martin-Schwinger (KMS) symmetry

$$\phi(x,t) \to -\phi(x,-t),$$
$$\Phi(x,t) \to -\Phi(x,-t) - i\beta\partial_t\phi(x,-t),$$
$$\psi_i(x,t) \to -\psi_i(x,-t),$$
$$\Psi_i(x,t) \to -\Psi_i(x,-t) - i\beta\partial_t\psi_i(x,-t), \tag{10}$$

which is a consequence of the fact that our hydrodynamic EFT describes relaxation toward an equilibrium thermal Gibbs state $e^{-\beta(H-\mu Q-\cdots)}$ [27]; similar ideas hold for more general steady state [20].

Lastly, we have the option of enforcing the "diagonal shift symmetries" [31]. These symmetries require that the action only depend on ϕ through $\partial_t\phi$, or only depend on ψ_i through $\partial_t\psi_i$. In ordinary fluids, the EFT in the presence of the diagonal shift symmetry describes the normal phase, while the EFT in the absence of the diagonal shift symmetry describes superfluidity. Thus, condensed degrees of freedom need not obey the diagonal shift symmetry, while normal degrees of freedom must.

III. HYDRODYNAMIC PHASES

We will approach the hydrodynamics by imposing the diagonal shift symmetries for each phase independently, and then finding the lowest-order action in that phase. To count scaling dimensions, we note that ψ_i must scale as $\nabla_i\phi$ in order to preserve the dipole symmetry. First, we will impose the diagonal shift symmetry on both ϕ and ψ_i. This should describe the normal phase, with no condensation. If we suppose that the dynamical scaling exponent is $z = 4$, the most general effective action consistent with the KMS and EFT symmetries is

$$\mathcal{L}_{\text{eff}} = \chi \partial_t \phi \partial_t \Phi + [-\sigma \partial_t (\nabla_i \phi - \psi_i)](\nabla_i \Phi - \Psi_i) - [B_1 \partial_t \nabla_i \nabla_j \phi + B_2 \partial_t \nabla_i \psi_j + B_3 \partial_t \nabla_j \psi_i] \nabla_j \Psi_i, \quad (11)$$

to leading order. All coefficients must be positive, by a combination of the KMS and EFT symmetries and thermodynamic stability. We have named χ and σ in reference to ordinary systems. Although χ is the susceptibility, σ does not play the role of a measurable electrical conductivity.

The density and currents are, at leading order,

$$\rho = \chi \partial_t \phi, \quad \rho_i = 0, \quad J_i = -\sigma \nabla_i \partial_t \phi + \sigma \partial_t \psi_i,$$
$$J_{ij} = B_1 \nabla_i \nabla_j \partial_t \phi + B_2 \nabla_i \partial_t \psi_j + B_3 \nabla_j \partial_t \psi_i. \quad (12)$$

Although ρ_i has nonzero contributions at higher order, we will not need to include them. The dipole continuity equation reads

$$0 = -\sigma \nabla_i \partial_t \phi + \sigma \partial_t \psi_i - B_1 \nabla^2 \nabla_i \partial_t \phi - B_2 \nabla_i \nabla_j \partial_t \psi_j - B_3 \nabla^2 \partial_t \psi_i, \quad (13)$$

which imposes that $\partial_t \psi_i = \nabla_i \partial_t \phi$, plus higher-order corrections. The monopole continuity equation then reads

$$0 = \partial_t \rho - \partial_t \nabla_i \rho_i + \nabla_i \nabla_j J_{ij} = \chi \partial_t^2 \phi + (B_1 + B_2 + B_3) \nabla^4 \partial_t \phi + \cdots, \quad (14)$$

so that the dispersion is

$$\omega = -i \frac{B_1 + B_2 + B_3}{\chi} k^4, \quad (15)$$

which describes subdiffusion. This is consistent with previous results [7], and also with experiments on cold atomic gases with approximate dipole symmetry [9].

For the remaining phases we will presciently suppose $z = 2$. Then, the most general effective action consistent with the KMS and EFT symmetries, but without any diagonal shift symmetries imposed, is

$$\mathcal{L}_{\text{eff}} = \chi \partial_t \phi \partial_t \Phi + \big[-\kappa_1^\phi (\nabla_i \phi - \psi_i) + \kappa_2^\phi \nabla^2 \nabla_i \phi + g_2 \nabla^2 \psi_i + g_3 \nabla_i \nabla_j \psi_j - \sigma \partial_t (\nabla_i \phi - \psi_i)\big](\nabla_i \Phi - \Psi_i) - \big[\kappa^{\phi\psi} \nabla_i \nabla_j \phi + \kappa_1^\psi \nabla_i \psi_j + \kappa_2^\psi \nabla_j \psi_i\big] \nabla_j \Psi_i, \quad (16)$$

to leading order. The κ coefficients act as generalized superfluid stiffnesses in the system. The symmetries require that all coefficients except g_2 and g_3 are positive. Furthermore, $\kappa^{\phi\psi} = \kappa_2^\phi + g_2 + g_3$ by KMS (see the Appendix). This action should describe the charge condensate. Under these conditions, all terms in the effective action are allowed and the density and currents are

$$\rho = \chi \partial_t \phi, \quad \rho_i = 0, \quad J_i = -\kappa_1^\phi \nabla_i \phi + \kappa_1^\phi \psi_i + \cdots,$$
$$J_{ij} = \kappa^{\phi\psi} \nabla_i \nabla_j \phi + \kappa_1^\psi \nabla_i \psi_j + \kappa_2^\psi \nabla_j \psi_i, \quad (17)$$

to leading order. The dipole continuity equation now imposes that $\psi_i = \nabla_i \phi$ plus higher-order corrections. The monopole continuity equation is

$$0 = \partial_t \rho - \partial_t \nabla_i \rho_i + \nabla_i \nabla_j J_{ij} = \chi \partial_t^2 \phi + (\kappa^{\phi\psi} + \kappa_1^\psi + \kappa_2^\psi) \nabla^4 \phi + \cdots, \quad (18)$$

so that the dispersion is

$$\omega^2 = \frac{\kappa^{\phi\psi} + \kappa_1^\psi + \kappa_2^\psi}{\chi} k^4, \quad (19)$$

which describes a propagating mode with $\omega \sim k^2$. Going beyond leading order, including generic dissipative terms, such as $\partial_t \nabla_i \psi_j \nabla_i \Psi_j$, in the action contributes a subleading $-ik^4$ to the dispersion.

The quadratic propagation matches previous expectations at $T = 0$ from a microscopic model [25], field theory [22], and a more generic model called the Dipolar Bose-Hubbard Model (DBHM) [6], so that the charge condensate behaves like a zero-temperature fluid. The effects of dissipation are subleading and do not modify the zero-temperature behavior at low wave vector. In Ref. [6], the authors show that the existence of only a single mode in the charge condensate phase of the DBHM is a result of a Higgs-like effect, where ψ_i plays the role of a gauge field. The same effect appears in the hydrodynamics as the requirement that $\psi_i = \nabla_i \phi$.

Finally, we can try imposing the diagonal shift symmetry on ϕ but not ψ_i. This corresponds to the dipole condensate, where dipole symmetry is spontaneously broken but monopole symmetry is not. The diagonal shift symmetry on ϕ requires that κ_1^ϕ, κ_2^ϕ, and $\kappa^{\phi\psi}$ vanish, which in turn requires that $g_3 = -g_2$. The density and currents are

$$\rho = \chi \partial_t \phi, \quad \rho_i = 0,$$
$$J_i = g_2 (\nabla^2 \psi_i - \nabla_i \nabla_j \psi_j) - \sigma \nabla_i \partial_t \phi + \sigma \partial_t \psi_i,$$
$$J_{ij} = \kappa_1^\psi \nabla_i \psi_j + \kappa_2^\psi \nabla_j \psi_i. \quad (20)$$

The dipole continuity equation will no longer result in a constraint because now J_i and $\nabla_j J_{ij}$ are of the same order. Instead, we will have to simultaneously solve both equations.

The two continuity equations are

$$0 = \chi \partial_t^2 \phi - \sigma \nabla^2 \partial_t \phi + \sigma \partial_t \nabla_i \psi_i,$$
$$0 = -\sigma \nabla_i \partial_t \phi + \sigma \partial_t \psi_i - \big(\kappa_2^\psi - g_2\big) \nabla^2 \psi_i - \big(\kappa_1^\psi + g_2\big) \nabla_i \nabla_j \psi_j. \quad (21)$$

We can simplify the analysis by splitting ψ_i into a transverse and longitudinal part $\psi_i = \psi_i^t + \psi_i^\ell$, where the longitudinal part is $\psi_i^\ell = k_i k_j / k^2 \psi_j$ and obeys $\nabla_i \psi_i^\ell = \nabla_i \psi_i$. The transverse part is $\psi_i^t = P_t \psi_j$, where $P_t = (\delta_{ij} - k_i k_j / k^2)$ is the transverse projector. Applying the transverse projector to the dipole continuity equation results in

$$0 = \sigma \partial_t \psi_i^t - (\kappa_2^\psi - g_2) \nabla^2 \psi_i^t, \quad (22)$$

with solution

$$\omega = -i \frac{\kappa_2^\psi - g_2}{\sigma} k^2, \quad (23)$$

which is an ordinary diffusive mode. Note that the value $\kappa_2^\psi - g_2$ is always positive (see the Appendix). Furthermore, this dispersion represents two hydrodynamic modes, corresponding to the two transverse polarizations of ψ_i.

To access the longitudinal part we may take the divergence of the dipole continuity equation. The monopole continuity equation and the divergence of the dipole continuity equation together read

$$0 = \begin{bmatrix} \chi\,\partial_t - \sigma\nabla^2 & \sigma\,\partial_t \\ -\sigma\nabla^2 & \sigma\,\partial_t - \kappa^\psi\nabla^2 \end{bmatrix} \begin{pmatrix} \partial_t\phi \\ \nabla_j\psi_j \end{pmatrix}, \tag{24}$$

where $\kappa^\psi = \kappa_1^\psi + \kappa_2^\psi$, showing that ϕ and $\nabla_i\psi_i$ are coupled. Their joint dispersion relation is

$$0 = \omega^2 + i\frac{\kappa^\psi}{\sigma}\omega k^2 - \frac{\kappa^\psi}{\chi}k^4 \tag{25}$$

or

$$\omega = -i\frac{\kappa^\psi}{2\sigma}k^2 \pm \sqrt{\frac{-(\kappa^\psi)^2}{4\sigma^2} + \frac{\kappa^\psi}{\chi}}k^2, \tag{26}$$

which displays a transition from pure diffusion to quadratic propagation, controlled by the dimensionless parameter $\kappa^\psi\chi/\sigma^2$. For $\kappa^\psi\chi \gg 4\sigma^2$, the dispersion approaches

$$\omega = -i\frac{\kappa^\psi}{\sigma}k^2, \qquad \omega = -i\frac{\sigma}{\chi}k^2, \tag{27}$$

with two quadratically diffusing modes. In the opposite limit the dispersion approaches

$$\omega = -i\frac{\kappa^\psi}{2\sigma}k^2 \pm \sqrt{\frac{\kappa^\psi}{\chi}}k^2, \tag{28}$$

which is simultaneously quadratically propagating and quadratically diffusive. While we might have expected the dissipative coefficient σ to play a damping role, the large-σ regime is underdamped and the small-σ regime is overdamped. This effect, opposite to expectations, would be an interesting avenue for further research.

IV. EXPLORING THE DIPOLE CONDENSATE

Since the subdiffusion of the normal phase and quadratic propagation of the charge condensate have been studied in the literature [6,7], we here focus on better understanding different regimes of the dipole condensate. We can tune various parameters to be small, bringing us to limiting points of the phase diagram. The small parameters define a quasihydrodynamic timescale τ [32], which is parametrically long.

To begin, let us study the hydrodynamics in the charge condensate but near the phase transition to the dipole condensate. We allow terms that break the diagonal shift symmetry for ϕ, but require them to be small. The important coefficients are κ_1^ϕ and $\kappa^{\phi\psi}$ (κ_2^ϕ, g_2, and g_3 will all be subleading). This gives a longitudinal continuity equation,

$$0 = \omega^2 + \frac{\kappa^\psi}{\kappa_1^\phi - i\omega\sigma}\omega^2 k^2 - \frac{\kappa^{\phi\psi} + \kappa^\psi}{\chi}k^4, \tag{29}$$

with $\kappa^\psi = \kappa_1^\psi + \kappa_2^\psi$ as before. This dispersion defines a quasihydrodynamic time scale $\tau = \sigma/\kappa_1^\phi$. At times much longer than τ, the dispersion is $\omega^2 = (\kappa^{\phi\psi} + \kappa^\psi)k^4/\chi$, recovering the subdiffusive dispersion of the charge condensate.

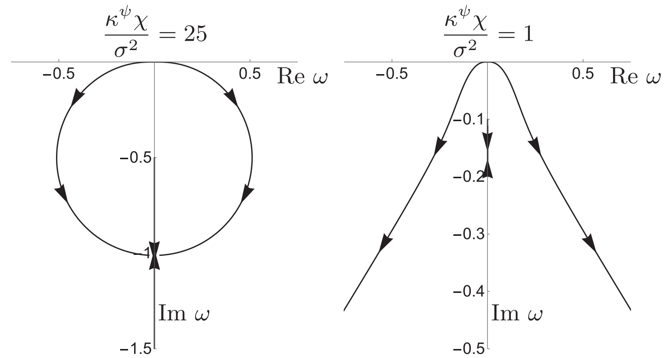

FIG. 2. Parametric plot of the dispersion in the charge condensate but close to the dipole condensate. Arrows point from small to large k. The left figure is plotted near the diffusive regime of the dipole condensate ($\kappa^\psi\chi/\sigma^2 = 25$), while the right figure is plotted near the propagating regime of the dipole condensate ($\kappa^\psi\chi/\sigma^2 = 1$). At small k both dispersions look like the charge condensate, while at large k they look like their respective DC dispersions.

This shows that we are truly in the charge condensed phase. At times below the scale τ, the dispersion looks like the dipole condensate [compare to (25)]. There is no transverse mode in the charge condensate, even near the dipole condensate phase transition.

The quasihydrodynamic crossover between the charge condensate at the low k and the dipole condensate at large k can be visualized by examining parametric plots of the dispersion (29). The dimensionless parameter $\kappa^\psi\chi/\sigma^2$ places us on either side of the transition in the dipole condensate. In Fig. 2 we can see the dispersion with $\kappa^\psi\chi/\sigma^2 = 25$ chosen to place us firmly within the diffusive regime. At small k (the true hydrodynamic regime), the dispersion looks like that of the charge condensate, $\omega^2 \sim k^4$, together with an additional gapped mode. At large k (the quasihydrodynamic regime) the dispersion becomes completely imaginary and looks like the diffusive regime of the dipole condensate ($\omega \sim -ik^2$). One of the diffusive modes becomes gapped at small k, while the other diffusive mode collides with the large-k gapped mode to give the propagating modes.

Increasing κ_1^ϕ moves the system deeper into the charge condensate and results in a smaller τ, which pushes the quasihydrodynamic regime to larger k. Accordingly, this causes the circle in Fig. 2 to grow, so that the $\omega^2 \sim k^4$ behavior of the charge condensate persists for larger k. On the other hand, a larger τ results in a smaller circle in Fig. 2, and the crossover to quasihydrodynamics occurs for smaller and smaller values of k. The phase transition occurs when $\tau \to \infty$, the circle shrinks entirely, and the small k behavior is purely diffusive.

The right side of Fig. 2 shows the dispersions at a value of $\kappa^\psi\chi/\sigma^2 = 1$, near a point deep in the propagating regime of the dipole condensate (recall the critical value is four). The propagating modes of the charge condensate at small k cross over to the propagating and diffusing modes of the dipole condensate at large k with no collision. Additionally, there is an extra mode that is gapped at large and small k. This mode is not a hydrodynamic or quasihydrodynamic mode, but it cannot be removed from the analysis because it is the same mode that goes from diffusive to gapped in the other regime.

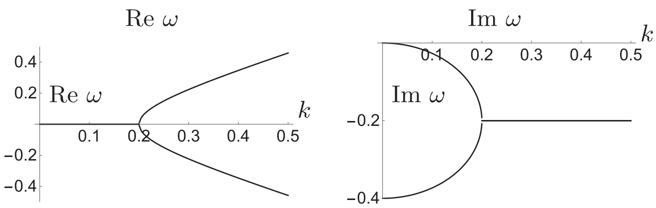

FIG. 3. Real and imaginary parts of the dispersion in the dipole condensate phase close to the dissipationless limit. The transverse mode and one of the modes of the longitudinal/monopole composite display the same behavior, shown here. At small k the dispersion looks like the diffusive regime of the dipole condensate ($\omega \sim -ik^2$), while at large k it looks like the $T = 0$ limit of the dipole condensate ($\omega \sim \pm k$).

As before, a larger κ_1^ϕ results in a smaller τ and pushes the quasihydrodynamic regime to larger k. This "opens" the initially rounded region near $k = 0$ so that the $\omega^2 \sim k^4$ hydrodynamics persists for larger k. Conversely, a larger τ leads to a smaller rounded region so that the crossover to the quasihydrodynamic $\omega \sim \pm k^2 - ik^2$ occurs for smaller and smaller k.

Another surprising facet of the dipole condensate phase is its quadratic propagation. At $T = 0$ in the DBHM, the dipole condensate consists of d modes (one for each space dimension), all propagating linearly [6,22]. We can view the hydrodynamic phase explored here as consisting of both the dipole condensate and a background normal (subdiffusive) fluid. Although the hydrodynamics EFT does not provide a mechanism for studying the behavior of the fluid as $T \to 0$, we can instead reproduce the $T = 0$ behavior in the nondissipative (small-σ) limit.

Specifically, let us take the small-σ limit of (20). This forces us to retain ρ_i in the continuity equations, with the important contribution being $\rho_i = \chi^\psi \partial_t \psi_i$. The transverse part of the dipole continuity equation reads

$$0 = \chi^\psi \partial_t^2 \psi_i^t + \sigma \partial_t \psi_i^t - \left(\kappa_2^\psi - g_2\right)\nabla^2 \psi_i^t, \qquad (30)$$

with solution

$$\omega = \frac{-i\sigma}{2\chi^\psi} \pm \sqrt{\frac{-\sigma^2}{2(\chi^\psi)^2} + \frac{\left(\kappa_2^\psi - g_2\right)k^2}{\chi^\psi}}, \qquad (31)$$

introducing a timescale $\tau = \chi^\psi/\sigma$. The new timescale τ is large in the small-σ limit. On timescales smaller than τ the quasihydrodynamics consists of a linear propagating mode,

$$\omega = \pm\sqrt{\frac{\left(\kappa_2^\psi - g_2\right)k^2}{\chi^\psi}}, \qquad (32)$$

matching the $T = 0$ expectation. At timescales larger than τ the propagating mode splits into a gapped mode and a diffusive mode with diffusion constant $(\kappa_2^\psi - g_2)/\sigma$, as in (23). This behavior is shown in Fig. 3.

With the introduction of χ^ψ, the analog of (24) for the transverse mode is

$$0 = \begin{bmatrix} \chi\partial_t - \sigma\nabla^2 & \sigma\partial_t \\ -\sigma\nabla^2 & \chi^\psi\partial_t + \sigma\partial_t - \kappa^\psi\nabla^2 \end{bmatrix}\begin{pmatrix} \partial_t\phi \\ \nabla_j\psi_j \end{pmatrix}, \qquad (33)$$

with the same timescale $\tau = \chi^\psi/\sigma$. In the small-σ limit, the solutions are

$$\omega = -i\frac{\sigma}{\chi}k^2, \quad \omega = \frac{-i\sigma}{2\chi^\psi} \pm \sqrt{\frac{-\sigma^2}{2(\chi^\psi)^2} + \frac{\kappa^\psi k^2}{\chi^\psi}}. \qquad (34)$$

The first solution matches one of the diffusion modes from the dipole condensate phase, with a diffusion constant that vanishes in the small-σ limit. The other mode behaves similar to the transverse mode, crossing over from linear propagation in quasihydrodynamics to a gapped mode and a diffusive mode in the late-time hydrodynamics, as shown in Fig. 3.

In contrast to the charge condensate, where dissipation had little effect on the physics, in the dipole condensate the mode propagation at low wave number is immediately modified in the presence of dissipation. The dissipative coefficient σ is crucial in that it completely changes the nature of the dispersion relation from $T = 0$ to finite T, going from a ballistic to a quadratic scaling. While we determined the presence of this transport coefficient in terms of simple symmetry arguments, we note that this term can be argued to be finite based on microscopic reasoning. Consider a lattice model described by a complex boson $b_{\boldsymbol{x}}$ and with dipole symmetry $b_{\boldsymbol{x}} \to b_{\boldsymbol{x}} e^{i\boldsymbol{\alpha}\cdot\boldsymbol{x}}$. In the condensed dipole phase, hopping of a single boson is allowed through the term $b_{\boldsymbol{x}} b_{\boldsymbol{x}+\boldsymbol{e}_j} e^{i\psi_j}$ + H.c., where $\boldsymbol{e}_j$ denotes a unit vector in the j direction [6]. Note that ψ_i can exactly be viewed as a spatial gauge field $A_i = \psi_i$. Treating ψ_i as a background nondynamical field, at finite temperature, this coupling will generically lead to a finite conductivity term in the current $J_i = \sigma E_i = \sigma\partial_t\psi_i$, which is precisely the last term in the third line of (20). This argument not only confirms that σ must generically be finite, it also shows that, given a $U(1)$-invariant system without dipole symmetry, this can be straightforwardly extended to a dipole symmetric system in the dipole condensed phase.

V. DISCUSSION

We have developed a systematic, effective field theory based treatment of hydrodynamics in systems with charge and dipole symmetry, allowing for the possibility of spontaneous symmetry breaking. In the absence of any SSB, we find quartic subdiffusion, consistent with [7]. With both charge and dipole symmetries broken, we find a quadratically propagating (and quartically subdiffusing) mode, consistent with [6]. We also introduced the phase where dipole symmetry is spontaneously broken but monopole symmetry is preserved, corresponding to a "dipole condensate." In this phase we find that there exist diffusive transverse modes, as well as longitudinal modes which, depending on parameters, can be either purely diffusive or quadratically propagating and relaxing. This phase does not match any in the literature, and reflects intrinsically nonzero-temperature effects.

Our results can be tested in ultracold atom experiments analogous to [9]. Further afield, they could be generalized to systems with momentum conservation and/or systems with more complex multipolar symmetries [33]. We leave such generalizations to future work.

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APPENDIX: DERIVATION OF THE EFFECTIVE ACTION

The effective actions we consider must obey the KMS symmetry in (10). Reference [34] shows that we can construct KMS-invariant terms in two distinct ways, which correspond to dissipative and nondissipative terms in the effective action. The nondissipative terms are

$$\mathcal{L}_{\mathrm{nd}} = \left(\Phi\frac{\delta}{\delta\phi} + \Psi_i\frac{\delta}{\delta\psi_i}\right)\int d^3x\, dt\, \Omega, \tag{A1}$$

where Ω is a Lagrangian that depends on ϕ and ψ_i but not on Φ or Ψ_i. Thermodynamic stability of the effective action requires that Ω is negative when Wick-rotated. The dissipative terms are

$$\mathcal{L}_{\mathrm{d}} = \tfrac{1}{2}(X(\phi,\psi_i,\Phi,\Psi_i) + X_{\mathrm{KMS}}(\phi,\psi_i,\Phi,\Psi_i) - X(\phi,\psi_i,0,0) - X_{\mathrm{KMS}}(\phi,\psi_i,0,0)), \tag{A2}$$

where X is quadratic in Φ and Ψ_i, and is even under time reversal. The function X_{KMS} is the result of the transformation in (10) applied to X.

For the nondissipative part, we will consider terms of order ω^2, ω^2k^2, k^2, and k^4. This is not a strictly valid gradient expansion at any value of z, but will give us all the terms we need for our analysis. Then, we have

$$\begin{aligned}2\Omega &= \chi(\partial_t\phi)^2 + \chi_2^\phi(\partial_t\nabla_i\phi)^2 + 2g_1\partial_t\nabla_i\phi\partial_t\psi_i + \chi^\psi(\partial_t\psi_i)^2 - \kappa_1^\phi(\nabla_i\phi-\psi_i)^2\\ &\quad - \kappa_2^\phi(\nabla_i\nabla_j\phi)^2 - 2g_2\nabla_i\nabla_j\phi\nabla_i\psi_j - 2g_3\nabla^2\phi\nabla_i\psi_i - \tilde\kappa_2^\psi(\nabla_i\psi_j)^2 - \tilde\kappa_1^\psi(\nabla_i\psi_i)^2,\end{aligned} \tag{A3}$$

where we have included various factors of two for convenience. All χ and κ coefficients must be nonnegative. The g coefficients may be positive or negative, but must obey the stability conditions $|g_1| \leqslant \min(\chi_2^\phi, \chi^\psi)$, $|g_2| \leqslant \min(\kappa_2^\phi, \tilde\kappa_2^\psi)$, $|g_3| \leqslant \min(\kappa_2^\phi, \tilde\kappa_1^\psi)$, and $|g_2+g_3| \leqslant \kappa_2^\phi$. The Lagrangian becomes

$$\begin{aligned}\mathcal{L}_{\mathrm{nd}} &= \big[\chi\partial_t\phi - \chi_2^\phi\partial_t\nabla^2\phi - g_1\partial_t\nabla_i\psi_i\big]\partial_t\Phi + [g_1\partial_t\nabla_i\phi + \chi^\psi\partial_t\psi_i]\partial_t\Psi_i + \big[-\kappa_1^\phi(\nabla_i\phi-\psi_i)\big](\nabla_i\Phi-\Psi_i)\\ &\quad + \big[-\kappa_2^\phi\nabla_i\nabla_j\phi - g_2\nabla_i\psi_j - g_3\delta_{ij}\nabla_k\psi_k\big]\nabla_i\nabla_j\Phi + \big[-g_2\nabla_i\nabla_j\phi - g_3\delta_{ij}\nabla^2\phi - \tilde\kappa_2^\psi\nabla_i\psi_j - \tilde\kappa_1^\psi\delta_{ij}\nabla_k\psi_k\big]\nabla_i\Psi_j\\ &= \big[\chi\partial_t\phi - \chi_2^\phi\partial_t\nabla^2\phi - g_1\partial_t\nabla_i\psi_i\big]\partial_t\Phi + \big[g_1\partial_t\nabla_i\phi + \chi^\psi\partial_t\psi_i\big]\partial_t\Psi_i + \big[-\kappa_1^\phi(\nabla_i\phi-\psi_i) + \kappa_2^\phi\nabla^2\nabla_i\phi + g_2\nabla^2\psi_i\\ &\quad + g_3\nabla_i\nabla_j\psi_j\big](\nabla_i\Phi-\Psi_i) - \big[\big(\kappa_2^\phi+g_2+g_3\big)\nabla_i\nabla_j\phi + \big(\tilde\kappa_1^\psi+g_3\big)\nabla_i\psi_j + \big(\tilde\kappa_2^\psi+g_2\big)\nabla_j\psi_i\big]\nabla_j\Psi_i,\end{aligned} \tag{A4}$$

where we used $\nabla_i\nabla_j\Phi = \nabla_i(\nabla_j\Phi-\Psi_j) + \nabla_i\Psi_j$ and integration by parts. Note the sign and order of indices in the last line, chosen to match the convention in (7). We can identify the new coefficients $\kappa^{\phi\psi} = \kappa_2^\phi + g_2 + g_3$, $\kappa_1^\psi = \tilde\kappa_1^\psi + g_3$, and $\kappa_2^\psi = \tilde\kappa_2^\psi + g_2$, all of which are nonnegative.

The dissipative terms we need for our analysis descend from the expression

$$2\beta X = ib_0(\nabla_i\Phi-\Psi)^2 + ib_1(\nabla_i\nabla_j\Phi)^2 + 2i\xi\nabla_i\nabla_j\Phi\nabla_i\Psi_j + 2i\xi_2\nabla^2\Phi\nabla_i\Psi_i + ib_2(\nabla_i\Psi_j)^2 + ib_3(\nabla_i\Psi_i)^2, \tag{A5}$$

where the b coefficients must be positive and $|\xi_1| \leqslant \min(b_1,b_2)$, $|\xi_2| \leqslant \min(b_1,b_3)$, and $|\xi_1+\xi_2| \leqslant b_1$ by (9). Then,

$$\begin{aligned}\mathcal{L}_{\mathrm{d}} &= X - b_0\partial_t(\nabla_i\phi-\psi_i)(\nabla_i\Phi-\Psi_i)\\ &\quad - [b_1\partial_t\nabla_i\nabla_j\phi + \xi_1\partial_t\nabla_i\psi_j + \xi_2\delta_{ij}\partial_t\nabla_k\psi_k]\nabla_i\nabla_j\Phi\\ &\quad - [\xi_1\partial_t\nabla_i\nabla_j\phi + \xi_2\delta_{ij}\partial_t\nabla^2\phi + b_2\partial_t\nabla_i\psi_j + b_3\delta_{ij}\partial_t\nabla_k\psi_k]\nabla_i\Psi_j\\ &= X + [-b_0\partial_t(\nabla_i\phi-\psi_i) + b_1\partial_t\nabla^2\nabla_i\phi + \xi_1\partial_t\nabla^2\psi_i + \xi_2\partial_t\nabla_i\nabla_j\psi_j](\nabla_i\Phi-\Psi_i)\\ &\quad - [(b_1+\xi_1+\xi_2)\partial_t\nabla_i\nabla_j\phi + (b_2+\xi_1)\partial_t\nabla_i\psi_j + (b_3+\xi_2)\delta_{ij}\partial_t\nabla_k\psi_k]\nabla_i\Psi_j,\end{aligned} \tag{A6}$$

from which we can identify $\sigma = b_0$, $B_1 = b_1+\xi_1+\xi_2$, $B_2 = b_2+\xi_1$, and $B_3 = b_3+\xi_2$. The other terms end up being subleading so we may drop them. The terms in X itself are quadratic in Φ and Ψ_i, so they contribute to the fluctuating hydrodynamics but can be ignored for the purpose of computing the dispersion relations.

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