Symmetry-enriched topological order from partially gauging symmetry-protected topologically ordered states assisted by measurements

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Symmetry-protected topological (SPT) phases exhibit nontrivial short-ranged entanglement protected by symmetry and cannot be adiabatically connected to trivial product states while preserving the symmetry. In contrast, intrinsic topological phases do not need ordinary symmetry to stabilize them and their ground states exhibit long-range entanglement. It is known that for a given symmetry group G, the 2D SPT phase protected by G is dual to the 2D topological phase exemplified by the twisted quantum double model $D^{\omega}(G)$ via gauging the global symmetry G. Recently it was realized that such a general gauging map can be implemented by some local unitaries and local measurements when G is a finite, solvable group. Here, we review the general approach to gauging a G-SPT starting from a fixed-point ground-state wave function and applying a N-step gauging procedure. We provide an in-depth analysis of the intermediate states emerging during the N-step gauging and provide tools to measure and identify the emerging symmetry-enriched topological order (SET) of these states. We construct the generic lattice parent Hamiltonians for these intermediate states and show that they form an entangled superposition of a twisted quantum double (TQD) with an SPT-ordered state. Notably, we show that they can be connected to the TQD through a finite-depth, local quantum circuit which does not respect the global symmetry of the SET order. We introduce the so-called symmetry branch line operators and show that they can be used to extract the symmetry fractionalization classes (SFC) and symmetry defectification classes (SDC) of the SET phases with the input data G and $[\omega] \in H^3(G, U(1))$ of the pregauged SPT-ordered state. We illustrate the procedure of preparing and characterizing the emerging SET-ordered states for some Abelian and non-Abelian examples such as dihedral groups D_n and the quaternion group Q_8.

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I. INTRODUCTION

Topological order first originated from the study of the fractional quantum Hall effect [1,2]. It cannot be described by local order parameters and is beyond Landau's classification of matter. It exhibits ground-state degeneracy dependent on the topology of the underlying manifold and the excitations, displaying anyonic statistics [3,4]. More recently, it was recognized as possessing some kind of long-range quantum entanglement [5] and having nonzero topological entanglement [6–8]. In addition to fractional quantum Hall systems and certain spin liquids [9], there are models that manifest topological order, such as Kitaev's toric code and quantum double (QD) models [10], their twisted versions [11], Levin-Wen string-nets [12], and more recently fractons [13,14]. Topological features that characterize such a phase of matter are robust to local perturbations, which is a property highly desirable in quantum memories [15]. Some of the topological models also offer the capability of topological quantum computation (TQC) by exploiting the braiding of anyons [10,16–18], which has emerged as one of the schemes for fault-tolerant quantum computation due to its inherent robustness.

Interestingly, from the perspective of adiabatic connection and quantum circuits, ground states of different phases at zero temperature cannot be connected by either adiabatic evolution or a finite-depth quantum circuits [19]. Intrinsic topologically ordered states therefore cannot be created from a trivial ground state, such as product states, with a quantum circuit of finite depth. When all the local gates in the circuits are required to respect a certain global symmetry, trivial gapped phases can be further fine-grained into distinct classes: those that can be created from product states with symmetric, finite-depth circuits and those that cannot. The latter classes are referred to as nontrivial symmetry-protected topological (SPT) phases [20–22], and most of them can be classified by cohomology [5,23].

Quantum technology has been constantly improving and evolving. Several medium-scale quantum computers are available. Recently, certain topologically ordered states, such as those of the toric model, were created by quantum circuits [24], and furthermore, some braiding statistics has also been observed in experiments [25–28], Yet, preparation of high-fidelity ground states and precise manipulation of excitations in topological systems still remain challenging in the current era of noisy intermediate-scale quantum (NISQ) devices [29].

For the family of the QD models and their twisted versions, i.e., twisted quantum double (TQD) models [11], there is a well-known correspondence to models of SPT phases [5,30–32] via a procedure called gauging [33–36]. When two quantum states are topologically distinct with respect to a symmetry G, then they cannot be transformed to each other with a finite-depth, piecewise local unitary transformation that preserves the symmetry. The classification and characterization of SPT phases with global symmetry G in two dimensions is facilitated by a function of $g_i \in G$, namely a 3-cocycle $\omega_3(g_1, g_2, g_3)$, which is a representative element of the third cohomology group of G, denoted by $H^3(G, U(1))$ [5]. At the same time, the TQD model is also characterized by the 3-cocycle: inequivalent choices of the 3-cocycle give rise to distinct intrinsic topological phases [11]. Indeed, the wave function of the TQD model with the gauge group G is obtained by gauging the global symmetry G in the corresponding SPT wave function. An interesting question is whether such a gauging map is physically possible.

It has been known that the ground state of the toric code can be efficiently prepared by measuring half of qubits in a 2D cluster state [37]. The use of measurements thus can provide a route to creating long-range entangled states with finite-depth operations [38–40]. Reference [41] demonstrated that ground states of QD models with S_3 and D_4 groups can be prepared through finite-depth local unitary operations supplemented with onsite measurements. It was also argued in Ref. [38] that a similar procedure should work for QD models with any solvable group G, which was later elaborated in Ref. [42] through repeated rounds of finite-depth operations, where each round incorporates unitaries, measurements, feedforward, and corrections. This scheme was further generalized to the general TQD models with solvable groups in Ref. [43], where the number of measurement rounds was classified for various topological orders, leading to a conjecture of a new hierarchy of topological orders when one includes measurements as an ingredient. It is worth mentioning that further improvement is possible for the QD models with D_4 and Q_8 groups, which can be prepared with a single round of measurements, feedforward, and corrections [44]. Experimentally, measurement-based gauging is a promising method for realizing nontrivial topological orders in small-scale systems requiring only local unitary operations, midcircuit measurements, and feedforward corrections [45–47].

The present work re-examines the measurement-based gauging from the perspective of group representation theory and provides a characterization of the transformation and emergence of SPT, SET, and intrinsic topological order during gauging. In general, for a solvable group G, the corresponding TQD model can be prepared from a G-SPT through a multistep gauging procedure. In this work, we provide two approaches that realize such an N-step gauging which reduces to a one-step gauging when G is Abelian or to a two-step gauging when G is dihedral. For nonsolvable groups, it is argued that the measurement-assisted gauging procedure cannot be implemented by a finite-depth circuit [43].

Interestingly, we find that the intermediate states, that emerge during the multistep gauging, can be naturally described as symmetry-enriched topological (SET) orders [48–51]. We also show that, without respecting global symmetry, there is a finite-depth quantum circuit that takes the SET ground state to a ground state of a corresponding twisted quantum double model (TQD).

The essential data of an SET order, besides the intrinsic anyon theory $\mathcal{C}$, include the symmetry action as an automorphism on $\mathcal{C}$, the symmetry fractionalization class, and the defectification class [51]. A key result of our work is to characterize the resulting SET order given the 3-cocycle that describes the initial SPT wave function. If the emergent SET order has a global symmetry that *does not* change the anyon type, then we develop a general formalism based on symmetry branch line operators for the braiding phases between any Abelian anyon in the theory and the anyons obtained from fusing point defects, exactly characterizing the symmetry fractionalization patterns. If the SET order we enter has a global symmetry that *does* change anyon types, then we conjecture the form and algebra of non-Abelian symmetry branch line operators that can create the corresponding symmetry defects. Then, by calculating the tensor product of such operators, one can derive the fusion rules of these symmetry defects, which we believe is sufficient to characterize the symmetry fractionalization patterns. We consider the dihedral SPT states as an example to illustrate this case.

The remainder of this paper is organized as follows. In Sec. II, we review the duality between SPT states with global symmetry group G and ground states of a twisted quantum double model with a gauge group G in two dimensions. This duality is given by a formal gauging map, which turns the global symmetry G into a gauge symmetry. In Sec. III, we describe the general procedure of N-step gauging G-SPT-ordered states when G is a solvable group in terms of an algorithm (see Algorithm 1 below). In Secs. IV and V, we discuss one-step and two-step gauging, respectively, and consider Abelian and dihedral groups as illustrative examples. For the latter, we find that after the first gauging step, the system remains in a SET state where the remaining quotient group describes the global symmetry. Section VI contains the discussion on symmetry properties of the emergent SET phases from the perspective of symmetry defects. Using the framework of symmetry branch lines, we relate the transformation of symmetry defects under gauging to properties of the SET phase. We give several examples to illustrate our formalism. In Sec. VII we make some concluding remarks. The Appendixes provide materials that support the results in the main text. For example, we provide a constant-depth unitary circuit to map an SET state to a TQD state in Appendix E. In Appendix J, We also give an alternative gauging prescription based on a different presentation of solvable groups which is alternative but equivalent to the standard one, as proven in Appendix K.

II. FIXED-POINT SPTS, TWISTED QUANTUM DOUBLES, AND GAUGING

On an oriented triangulated lattice Λ, given a finite group G, we assign a Hilbert space $\mathcal{H}_v = \{\sum_{g\in G} c_g |g\rangle_v | c_g \in \mathbb{C}\}$ to each vertex v. Then we can write a fixed-point G-SPT wave function. To do this, we first assign a group cocycle to each simplex, where ω is a representative in $H^3(G, U(1))$ respecting the cocycle condition,

$$\frac{\omega(h,k,l)\omega(g,hk,l)\omega(g,h,k)}{\omega(gh,k,l)\omega(g,h,kl)} = 1, \tag{1}$$

for any $g, h, k, l \in G$.

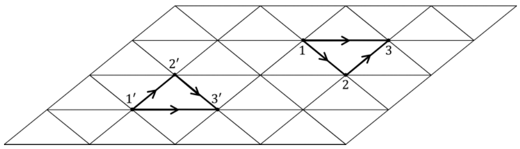


FIG. 1. On an oriented triangulated plane, two typical simplexes with opposite orientations are shown. Their corresponding cocycles are $\omega(g_3g_2^{-1}, g_2g_1^{-1}, g_1)$ and $\omega(g_{3'}g_{2'}^{-1}, g_{2'}g_{1'}^{-1}, g_{1'})^{-1}$.

The fixed-point SPT wave function is given by taking a product over all such cocycles,

$$|\Psi_{\rm SPT}\rangle = \sum_{\{g_v\}} \prod_{\text{simplex } \Delta_{123}} \omega\big(g_3g_2^{-1}, g_2g_1^{-1}, g_1\big)^{s(\Delta_{123})} \bigotimes_v |g_v\rangle_v, \tag{2}$$

where $s(\Delta) = \pm 1$ indicates the orientation of a simplex Δ_{123} (with a given branching structure, and $\{1, 2, 3\}$ labels the vertices on the simplex; see Fig. 1), the tensor product runs over all vertices v on the lattice, and all the configurations $\{g_v\}$ are summed over. Note that we use a convention from Ref. [32], which is slightly different from Ref. [5], for the sake of convenience in later discussions. This state can be obtained by the action of a unitary operator U_ω on the product state $\bigotimes_v \sum_g |g\rangle_v$,

$$U_\omega = \sum_{\{g_v\}} \prod_{\Delta_{123}} \omega\big(g_3g_2^{-1}, g_2g_1^{-1}, g_1\big)^{s(\Delta_{123})} \bigotimes_v |g_v\rangle_v \langle g_v|. \tag{3}$$

We define the left/right action of x on $\mathcal{H}_v$ as

$$L^x_{+v}|g\rangle_v = |xg\rangle_v, \quad L^x_{-v}|g\rangle_v = |gx^{-1}\rangle_v. \tag{4}$$

Then the global symmetry action (in our convention) $U^x \equiv \prod_v L^x_{-v}$ on SPT state yields

$$\begin{aligned} U^x|\Psi_{\rm SPT}\rangle &= \sum_{\{g_v\}} \prod_{\Delta} \omega^{s(\Delta)}(\{g_v\}) \bigotimes_v |g_v x^{-1}\rangle_v \\ &= \sum_{\{g_v\}} \prod_{\Delta} \omega^{s(\Delta)}(\{g_v x\}) \bigotimes_v |g_v\rangle_v \\ &\equiv \sum_{\{g_v\}} \prod_{\Delta} \text{Amp}(\{g_v\}, x) \omega^{s(\Delta)}(\{g_v\}) \bigotimes_v |g_v\rangle_v, \end{aligned} \tag{5}$$

where in the second line we used a change of variables and we have defined a phase factor Amp in the fourth line.

Suppose M is the two-dimensional spatial manifold on which the Hilbert space is defined, and $I = \{x_3 | 0 \leqslant x_3 \leqslant 1\}$ is an interval in the (Euclidean) time direction. The manifold $M \times I$ is now three-dimensional. We triangulate the $M \times I$ by 3-simplexes (tetrahedrons) with the constraint that each time slice at $x_3 = 0$ and $x_3 = 1$ matches the original two-dimensional lattice; see Fig. 2. The amplitude Amp is computed once the triangulation of $M \times I$ is specified. We now give more details.

We assign a 3-cocycle to each tetrahedron as in Fig. 3. The phase factor can be seen to be [5]

$$\text{Amp}(\{g_v\}, x) = \prod_{\text{tetrahedron}} \omega(\text{tetra})^{s(\text{tetra})}. \tag{6}$$

When the spatial manifold is closed, using cocycle conditions, one can show that Amp $\equiv 1$. Therefore, the SPT state is

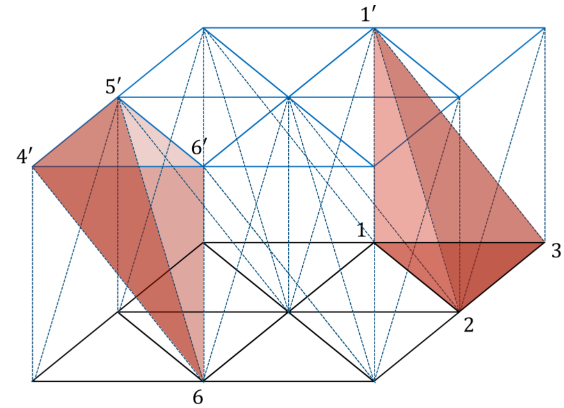


FIG. 2. The phase factor $\text{Amp}(\{g_v\}, x)$ can be given by the triangulation of such prisms, where $g_{v'} = g_v x$.

invariant under the global symmetry transformation,

$$U^x|\Psi_{\rm SPT}\rangle = |\Psi_{\rm SPT}\rangle. \tag{7}$$

Due to this symmetry, this state can be written schematically as (where we have suppressed the indices in Δ for simplicity),

$$\begin{aligned} |\Psi_{\rm SPT}\rangle &= \sum_{\{g_v\}} \prod_{\text{simplex}\Delta} \omega\big(g_3g_2^{-1}, g_2g_1^{-1}, g_1\big)^{s(\Delta)} \bigotimes_v |g_v\rangle_v \\ &= \sum_{\{g_v\}} \Omega\big(\big\{g_v g_{v'}^{-1}\big\}\big) \bigotimes_v |g_v\rangle_v, \end{aligned} \tag{8}$$

where Ω denotes the product of cocycles, and v and v' are vertices connected by an edge, $\langle v, v'\rangle \in E$.

A. Gauging global symmetry G

Under a gauging map as shown in Fig. 4, the vertex degrees of freedom (DOFs) are mapped to the edge DOFs,

$$\Gamma: \ \ |\{g_i\}\rangle_v \to \big|\big\{g_i g_j^{-1}\big\}\big\rangle_e. \tag{9}$$

This in turn maps the SPT state to an intrinsic topologically ordered state [52]

$$|\Psi_{\rm TQD}\rangle = \sum_{\{g_e\}} \Omega(\{g_e\}) \bigotimes_e |g_e\rangle, \tag{10}$$

which is a ground state of the twisted quantum double (i.e., described by the Dijkgraaf-Witten theory) $D^\omega(G)$ [53]. The twisted quantum double can be formulated on a triangulated lattice with a Kitaev's Quantum Double-like Hamiltonian

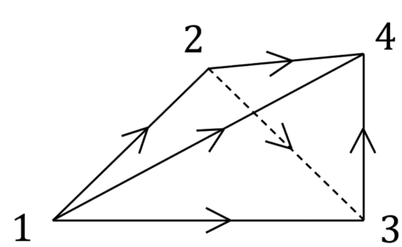


FIG. 3. A positively oriented tetrahedron. The 3-cocycle assigned to it is $\omega(g_4g_3^{-1}, g_3g_2^{-1}, g_2g_1^{-1})$.

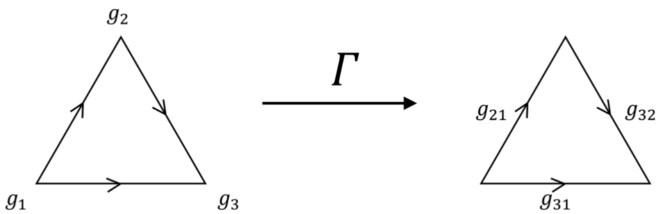


FIG. 4. The gauging map Γ maps from vertex DOFs to edge DOFs.

[11],

$$H = -\sum_v A_v - \sum_p B_p, \tag{11}$$

where v and p stand for the vertices and plaquettes, respectively, on the lattice. The vertex operator

$$A_v = \frac{1}{|G|}\sum_{g\in G}\left(\prod_{e\supset v} L^g_{\pm e}\right)\tilde{W}^g_v, \tag{12}$$

is Hermitian and is a projector (see Appendix A), where L^g_{+e} and L^g_{-e} are left and right action of the group element g on the edge e. When e emanates from the vertex v to another vertex, we apply L^g_{-e} in Eq. (12), When e flows to the vertex v, we apply L^g_{+e}.

The phase $\tilde{W}^g_v$ is a product of the cocycles corresponding to the tetrahedrons with appropriate orientations in the prism in Fig. 5, where the correspondence between a tetrahedron and a 3-cocycle is established in Fig. 6. Furthermore, this phase factor is the commutator between the right action of g on vertex v and the unitary operator introduced in Eq. (3),

$$\tilde{W}^g_v = \left(\prod_{e\supset v} L^g_{\pm e}\right)^{\dagger} U_\omega \left(\prod_{e\supset v} L^g_{\pm e}\right) U^{\dagger}_\omega, \tag{13}$$

where we use L^g_{+e} (L^g_{-e}) when the edge e ends at (emanates from) vertex v.

The plaquette operator is

$$B_p = \delta\left(\prod_{e\in p} g_e, 1\right), \tag{14}$$

where $\delta(x, y)$ is the Kronecker δ function. The resultant state from the gauging map is the ground state of this Hamiltonian,

$$H\,\Gamma(|\Psi_{\rm SPT}\rangle) = E_0\,\Gamma(|\Psi_{\rm SPT}\rangle). \tag{15}$$

The local excitations of TQD model are fractional charges called anyons, which can be classified by a unitary modular tensor category (UMTC); see, e.g., Ref. [54]. One thing to remark is that the convention here is slightly different from the one used in Refs. [11,52] for the sake of convenience in later discussions.

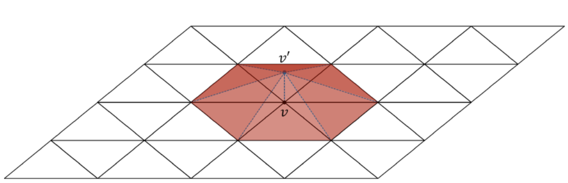


FIG. 5. The phase $\tilde{W}^g_v$ is defined as the multiplication of the phases corresponding to the tetrahedrons. Here, $g_{v'v} = g$.

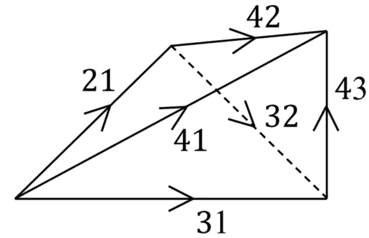


FIG. 6. A negatively oriented tetrahedron. The 3-cocycle assigned to it is $\omega^{-1}(g_{43}, g_{32}, g_{21})$.

B. Gauging a subgroup of G

One can introduce a gauging map Γ_N that corresponds to gauging only a normal subgroup N of G. We have the quotient group $Q = G/N$ with an embedding

$$s : Q \to G. \tag{16}$$

Any element $g \in G$ has a unique decomposition $g = qn$, where $q \in s(Q)$ and $n \in N$. Under the map, the normal part of vertex DOFs are mapped to edge DOFs, as illustrated in Fig. 7,

$$\Gamma_N :\ |\{g_i\}\rangle_v \to |\{q_i\}\rangle_v \otimes \left|\left\{n_i n_j^{-1}\right\}\right\rangle_e. \tag{17}$$

This maps the SPT state to

$$|\Psi_{\rm SET}\rangle = \sum_{\{q_v\}}\ \sum_{\{n_e\},\text{fluxless}} \Omega\left(\left\{q_i n_e q_j^{-1}\right\}\right) \bigotimes_v |q_v\rangle \bigotimes_e |n_e\rangle. \tag{18}$$

One point to notice is that the above state has a global symmetry Q under action $U^x_Q = U^x\phi^x$, where $U^x \equiv \prod_v L^x_{-v}$ is the right action of $x \in s(Q)$ on all vertices (e.g., $g \to gx^{-1}$) and $\phi^x \equiv \prod_e \phi^x_e$ is the conjugation by x on all edges defined as

$$\phi^x_e|n\rangle_e = |xnx^{-1}\rangle_e. \tag{19}$$

The second point is that the state $|\Psi_{\rm SET}\rangle$ is a ground state of a Kitaev's quantum double-like Hamiltonian,

$$H = -\sum_v A_v - \sum_p B_p - \sum_v K_v, \tag{20}$$

where v and p stand for the vertices and plaquettes on the lattice.

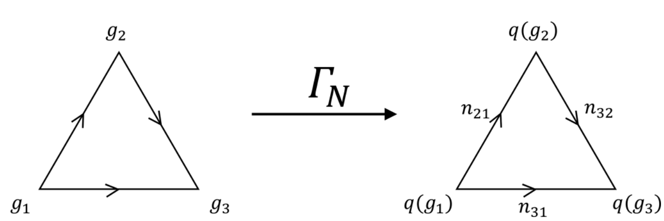


FIG. 7. The gauging map Γ_N maps the normal part from vertex DOFs to edge DOFs.

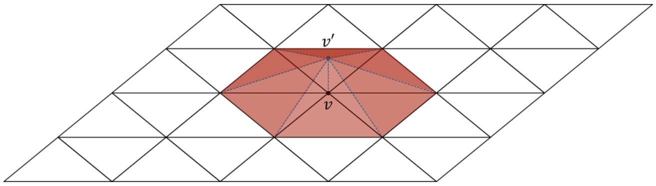


FIG. 8. The phase W_v^g is defined as the multiplication of the phases corresponding to the tetrahedrons. $h_{v'v} = x$, $q_{v'} = q_v$.

The vertex operator is

$$A_v = \frac{1}{|N|}\sum_{n\in N}\sum_{q\in s(Q)}\left(\prod_{e\supset v} L^n_{\pm e}\right) W_v^{qnq^{-1}} |q\rangle_v\langle q|. \tag{21}$$

The phase W_v^g is the product of the cocycles corresponding to the tetrahedrons with appropriate orientations of the prism in Fig. 8, where the correspondence between tetrahedron and 3-cocycle is established in Fig. 9. The plaquette operator is simply the following:

$$B_p = \delta\left(\prod_{e\in p} n_e, 1\right). \tag{22}$$

The additional vertex operator K_v is

$$K_v = \frac{1}{|Q|}\sum_{q,q'\in s(Q)} W_v^{qq'^{-1}} |q\rangle_v\langle q'|. \tag{23}$$

We can always apply a finite-depth local unitary to bring all the vertex DOFs to the identity element (see Appendix E) such that the state becomes

$$|\Psi_{\rm TQD}\rangle = \sum_{\{n_e\}} \Omega(\{n_e\})|\{n_e\}\rangle_e \otimes_v |1\rangle_v. \tag{24}$$

This is a TQD state with the 3-cocycle $\nu(n_1, n_2, n_3)$ being the restriction of $\omega(g_1, g_2, g_3)$ on subgroup N. Thus, we obtain the anyons and their braiding, which is the same as in $D^\nu(N)$. Therefore, the state $|\Psi_{\rm SET}\rangle$ is essentially an SET phase with the global Q symmetry, and it is in the same phase of a TQD $D^\nu(N)$ if ignoring the Q symmetry.

C. Classification of SETs

Here we briefly review some terminology relevant to SET phases for the convenience of later discussions. This section will be based on Refs. [51,55]. In general, an intrinsic topological phase in 2+1d is an anyon theory characterized by an UMTC, which is denoted by $\mathcal{C}$. Assuming the symmetry

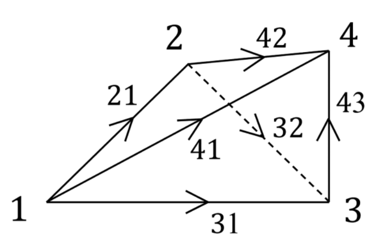


FIG. 9. A negatively oriented tetrahedron. The 3-cocycle assigned to it is $\omega^{-1}(g_4h_{43}g_3^{-1}, g_3h_{32}g_2^{-1}, g_2h_{21}g_1^{-1})$.

G preserves locality, an SET phase, which is described by a G-crossed braided fusion category $\mathcal{C}_G^\times$, is enriched from an anyon theory $\mathcal{C}$, by a G-symmetry action as an automorphism on $\mathcal{C}$, symmetry fractionalization class (SFC) and symmetry defectification class (SDC) [51]. We first assume the symmetry actions are unitary and always give the trivial automorphism on $\mathcal{C}$, i.e., the symmetry does not change anyon types. Consider a state $|\Psi_{a,b,c,\ldots}\rangle$ with anyons $\{a, b, c, \ldots\}$ present sufficiently far away from each other on a sphere. Therefore, the anyons $\{a, b, c, \ldots\}$ should be able to fuse into the vacuum charge. The symmetry operators respect the multiplication rules $U(g)U(h) = U(gh)$. Under our assumptions, the symmetry operator can be decomposed as some local unitaries $U_a(g), U_b(g), \ldots$, near the anyons,

$$U(g)|\Psi_{a,b,c,\ldots}\rangle = U_a(g)U_b(g)\cdots|\Psi_{a,b,c,\ldots}\rangle. \tag{25}$$

Each local symmetry action can be projective,

$$U_a(g)U_a(h) = \eta_a(g, h)U_a(gh), \tag{26}$$

where the phase $\eta_a(g, h)$ only depends on anyon type a and satisfy

$$\eta_a(g, h)\eta_b(g, h) = \eta_c(g, h), \tag{27}$$

whenever the multiplicity is $N_{ab}^c \neq 0$. We notice that the braiding phases B between an Abelian anyon e and any other anyons a, b, c satisfy the relation $B(e, a)B(e, b) = B(e, c)$ whenever the multiplicity is $N_{ab}^c \neq 0$. The similarity of this to Eq. (27) suggests that the latter may also arise from braiding anyons. It is indeed proved in Ref. [51] that a phase $\eta_a(g, h)$ with the above property is related to the braiding phase between anyon a and some other Abelian anyon $\mathfrak{w}(g, h)$ in the theory,

$$\eta_a(g, h) = B(\mathfrak{w}(g, h), a). \tag{28}$$

We note that the braiding phase $B(\mathfrak{w}, a)$ between anyons $\mathfrak{w}$ and a is defined in Ref. [51], and is related to the square of the R symbol. Therefore, applying two consecutive localized symmetry actions $U(g)$ and $U(h)$ in a region $\mathcal{R}$ will result in a symmetry action $U(gh)$ in $\mathcal{R}$ and an extra phase obtained by braiding anyon $\mathfrak{w}(g, h)$ around $\mathcal{R}$. This braiding phase indicates the symmetry fractionalization pattern of the SET.

We comment that one can redefine the local unitary $U_a(g)$ by an arbitrary phase factor $v_a(g)$,

$$U'_a(g) = v_a(g)U_a(g), \tag{29}$$

where $v_a(g)$ satisfies

$$v_a(g)v_b(g) = v_c(g), \tag{30}$$

whenever the multiplicity is $N_{ab}^c \neq 0$. Again, the phase factor $v_a(g)$ can be written as a braiding phase between anyon a and an Abelian anyon $\mathfrak{v}(g)$, i.e., $v_a(g) = B(\mathfrak{v}(g), a)$. The Abelian anyon after the redefinition will be

$$\mathfrak{w}'(g, h) = \mathfrak{v}(g) \times \mathfrak{v}(h) \times \overline{\mathfrak{v}(gh)} \times \mathfrak{w}(g, h), \tag{31}$$

where $\overline{\mathfrak{v}(gh)}$ is the dual anyon for the Abelian anyon $\mathfrak{v}(gh)$. Further, according to the associativity condition $(U_a(g)U_a(h))U_a(k) = U_a(g)(U_a(h)U_a(k))$, we have

$$\mathfrak{w}(h, k) \times \overline{\mathfrak{w}(gh, k)} \times \mathfrak{w}(g, hk) \times \overline{\mathfrak{w}(g, h)} = 1. \tag{32}$$

To conclude, the distinctive patterns of symmetry fractionalization are characterized by the class $[\omega(g,h)]$ in cohomology group $H^2(Q,\mathcal{A})$, where $\mathcal{A}$ is the group formed by Abelian anyons via fusion algebra [55], and we have used Q instead of G as the global symmetry for the SET phase.

Another way to see the symmetry fractionalization classes is to construct a Q-crossed category $\mathcal{C}_Q^\times$ from the anyon theory including the point defects

$$\mathcal{C}_Q = \bigoplus_{q\in Q} \mathcal{C}_q, \tag{33}$$

where $\mathcal{C}_{\mathbb{1}} = \mathcal{C}$ and $\mathbb{1}$ denotes the identity element in Q. A distinctive Q-crossed category $\mathcal{C}_Q^\times$ is a candidate for an SET order. According to Ref. [51], when the symmetry *does not* change anyon types, one can always choose an Abelian defect from each sector $\mathcal{C}_q$ and label it as 0_q. In particular, $0_{\mathbb{1}}$ is the vacuum anyon. (When the symmetry action does not permute anyon types, one can find a bijective map f from $\mathcal{C}_{\mathbb{1}}$ to $\mathcal{C}_q$ that preserves quantum dimensions. This allows us to identify the inverse of the vacuum anyon, $f^{-1}(0_{\mathbb{1}})$, as 0_q.) The fusion of defects respects the group multiplication structure,

$$0_g \times 0_h = \omega(g,h)_{\mathbb{1}} \times 0_{gh}, \tag{34}$$

for some Abelian anyon ω. We denote the objects in $\mathcal{C}_{\mathbb{1}}$ by small letters a, b, c, etc. Their fusion is given by

$$a \times b = \sum_{c\in\mathcal{C}} N_{ab}^c\, c, \tag{35}$$

where N_{ab}^c is the multiplicity in the fusion. The objects in $\mathcal{C}_q$ is obtained by fusing 0_q with objects in $\mathcal{C}_{\mathbb{1}}$, i.e., $a_q \equiv a \times 0_q$. Their fusion is given by

$$a_g \times b_h = \sum_{c\in\mathcal{C}} N_{ab}^c\, c \times \omega(g,h)_{\mathbb{1}} \times 0_{gh}. \tag{36}$$

This Abelian anyon $\omega(g,h)$ is exactly what we have defined above for projective phase $\eta_a(g,h)$. The class $[\omega(g,h)]$ is in the cohomology group $H^2(Q,\mathcal{A})$, which classifies the SFC.

In generic cases, when the symmetry *does* change anyon types as an automorphism of $\mathcal{C}$,

$$\rho: Q \to Aut(\mathcal{C}), \tag{37}$$

it turns out that not every sector $\mathcal{C}_q$ has an Abelian object. Therefore, we cannot write the fusion rule as in Eq. (36).

The fusion rule of a Q-graded category $\mathcal{C}_Q^\times$ can be written as

$$a_g \times b_h = \sum_{c_{gh}} N_{a_g b_h}^{c_{gh}} c_{gh}. \tag{38}$$

Consequently, each element $[t] \in H_\rho^2(Q,\mathcal{A})$ specifies a potential way of modifying $\mathcal{C}_Q^\times$ (the SET order) via

$$a_g \times b_h = t(g,h) \times \sum_{c_{gh}} N_{a_g b_h}^{c_{gh}} c_{gh}. \tag{39}$$

Therefore, in generic cases, the potential symmetry fractionalization classes are elements of an $H_\rho^2(Q,\mathcal{A})$ torsor. In this work, we will not analyze the SDC in detail, and we simply note that one can enter a different SDC by stacking a Q-SPT state onto the SET state. In our framework, after gauging the normal subgroup, a global x-transformation will locally serve as an automorphism ρ_x of $\mathcal{C}$, mapping an anyon with flux n to an anyon with flux xnx^{-1}. Later in this work, we will analyze the phases of some SETs in which the automorphism ρ_x on $\mathcal{C}$ is either trivial or nontrivial.

III. *N*-STEP GAUGING OF 2D SPT VIA MEASUREMENT

In this section, we present the procedure to gauge a G-SPT state of a group G that can be factorized into N Abelian groups with N steps. (We note in this section, N refers to the number of steps rather than a normal subgroup. But it should be clear from the context.) A similar method was proposed by Refs. [42,43]. In Ref. [43], the authors considered the solvable group G and its derived series which consists of normal subgroups which are commutator subgroups of the previous group in the series. They proposed a gauging procedure for a particular sequence of normal subgroups. In Ref. [42], however, they proposed to implement the gauging procedure for a solvable group inductively, i.e., implement the gauging of a cyclic group, assuming the remaining quotient group is already gauged. In our procedure, we do not restrict ourselves to a particular derived series for the solvable group. This in turn helps us to prepare different types of SETs. We give the steps for gauging a G-SPT state explicitly.

Before presenting the gauging procedure in Algorithm 1, let us go through the most relevant definitions first. A group G is a solvable group if there are subgroups $1 = G_0 < G_1 < \cdots < G_N = G$ such that G_{k-1} is normal in G_k, and $G_k/G_{k-1} \equiv Q_k$ is Abelian for $k = 1, \cdots, N$. Given the embedding map s_k from each Q_k into $G_k \subset G$, every element $g \in G$ can be written as

$$g = q_N q_{N-1} \cdots q_2 q_1, \tag{40}$$

where $q_k \in s_k(Q_k)$. Similarly, for another group element h, we have the decomposition $h = \tilde{q}_N \cdots \tilde{q}_1$. Under this convention, we write down the multiplication between g and h^{-1} as

$$gh^{-1} = q_N \cdots q_3 q_2 \big(q_1 \tilde{q}_1^{-1}\big) \tilde{q}_2^{-1} \tilde{q}_3^{-1} \cdots \tilde{q}_N^{-1}. \tag{41}$$

Now, let us define the relevant unitaries and measured observables that will be used in the gauging procedure. We typically consider a state defined on a lattice (V,E) with vertices $i,j \in V$ and edges $\langle i,j\rangle \in E$ where the local Hilbert space $|g\rangle$ depends on the group G and is labeled by its group elements $g \in G$. Given an embedding s_k of Q_k into G_k as described above, we can define for all Q_k the following unitary controlled on vertices and targeting the shared edge:

$$U_{Q_k}^{(i,j)} := \sum_{g_1,g_2,g_3\in G} |g_1,g_2\rangle_{c_i,c_j}\langle g_1,g_2| \otimes |q_k(g_1) g_3 q_k(g_2)^{-1}\rangle_{\langle i,j\rangle}\langle g_3|, \tag{42}$$

where $q_k \in s_k(Q_k)$, $q_k(g)$ is the kth component of the normal decomposition of g and $c_i(c_j)$ denotes the control qudit on vertex $i(j)$. This unitary will be used to entangle vertex DOFs with edge DOFs.

Measurements of Abelian subgroups will play an important role in the gauging procedure which is why we will now introduce the generalized Pauli-observables for an Abelian group being product of cyclic groups $Q_k \equiv \prod_{j=1}^{l_k} Z_{d_k^j}$, where d_k^j are some integers indecating the group order. (We note that it should be clear from context whether the symbol Z

Algorithm 1. N-step gauging via measurements.

Require: (a) Solvable G with $1 = G_0 < G_1 < \cdots < G_N = G$ such that $Q_k \equiv G_k/G_{k-1}$ is Abelian $\forall k = 1, \dots, N$.
(b) G-SPT fixed point state $|\Psi_{\mathrm{SPT}}\rangle$ [defined in Eq. (8)] on a lattice (V, E) with vertices $i, j \in V$ and edges $\langle i, j\rangle \in E$.
(c) $U_{Q_k}^{(i,j)}$ as defined in Eq. (42).
(d) Generalized Pauli operators as defined in Eqs. (43) and (44) acting on the Abelian subspace defined by an embedding s_k of Q_k in G_k.
$k \leftarrow 1$
(1) Add ancillas: $|\Psi_{\mathrm{gauge}}\rangle \leftarrow |\Psi_{\mathrm{SPT}}\rangle \otimes \prod_{\langle i,j\rangle\in E} |e\rangle_{\langle i,j\rangle}$ where $e \in G$ is the identity element.
while $k \leqslant N$ **do**
(2) Entangle vertex and edge DOFs: $|\Psi_{\mathrm{gauge}}\rangle \leftarrow \prod_{\langle i,j\rangle\in E} U_{Q_k}^{(i,j)}|\Psi_{\mathrm{gauge}}\rangle$
(3) Measureg vertex DOFs in the basis given by Eq. (45) with outcomes $\{\tilde{i}_1^v, \dots, \tilde{i}_{l_k}^v\}_{v\in V}$ (neglecting normalization): $|\Psi_{\mathrm{gauge}}\rangle \leftarrow \prod_{v\in V} |\tilde{i}_1^v, \dots, \tilde{i}_{l_k}^v\rangle\langle \tilde{i}_1^v, \dots, \tilde{i}_{l_k}^v|\Psi_{\mathrm{gauge}}\rangle$
(4) Correct for random measurement outcomes by applying Z operators on a set of edges E_{Cor}: $|\Psi_{\mathrm{gauge}}\rangle \leftarrow \prod_{e\in E_{\mathrm{Cor}}} \mathbb{Z}_e|\Psi_{\mathrm{gauge}}\rangle$, where $\mathbb{Z}_e = \prod_{j=1}^{l_k} Z_j^{-p_{j;e}}$ (specifically, E_{Cor} and $p_{j;e}$) can be deduced from the measurement outcomes, given the symmetries [e.g., Eq. (55)] and so-called *transmutation rules* (e.g., Figs. 10, 11, or 12).
$k = k + 1$ $\triangleright |\Psi_{\mathrm{gauge}}\rangle$ is an SET state as analyzed in Sec. VI $\forall k < N$
end while

represents a group or a Pauli operator.) Any element $q \in Q_k$ can be written as $a_1^{i_1} a_2^{i_2} \cdots a_{l_k}^{i_{l_k}}$ where a_j are the generator of the subgroup $Z_{d_k^j} \subset Q_k$ and therefore, $a_j^{d_k^j} = e \,\forall j = 1, \dots, l_k$. Given this representation, we write the local Hilbert space basis as $|i_1, \dots, i_{l_k}\rangle \equiv |a_1^{i_1} a_2^{i_2} \cdots a_{l_k}^{i_{l_k}}\rangle$. This allows us to define the following generalized local Pauli operators by their action on this basis:

$$X_1^{t_1} \otimes \cdots \otimes X_{l_k}^{t_{l_k}} \big|i_1, \dots, i_{l_k}\big\rangle = \big|i_1 \oplus t_1, \dots, i_{l_k} \oplus t_{l_k}\big\rangle, \quad (43)$$

$$Z_1^{t_1} \otimes \cdots \otimes Z_{l_k}^{t_{l_k}} \big|i_1, \dots, i_{l_k}\big\rangle = \omega_1^{i_1 t_1} \cdots \omega_{l_k}^{i_{l_k} t_{l_k}} \big|i_1, \dots, i_{l_k}\big\rangle, \quad (44)$$

where $i_j \oplus x$ indicates addition modulo d_k^j and ω_j is the d_k^j-th root of unity $\forall j = 1, \dots, l_k$. Importantly, this allows us to define a Fourier-transformed basis as follows,

$$\big|\tilde{i}_1, \dots, \tilde{i}_{l_k}\big\rangle = Z_1^{i_1} \otimes \cdots Z_{l_k}^{i_{l_k}} |+\rangle, \quad (45)$$

where $|+\rangle = \sum_{a_1^{i_1} a_2^{i_2} \cdots a_{l_k}^{i_{l_k}} \in Q_k} |i_1, \dots, i_{l_k}\rangle$.

Note that in Algorithm 1, the local Hilbert space dimension is given by the non-Abelian group G, so we understand all the above unitaries and bases as defined on an embedded subspace given by s_k. See the discussion above Eq. (40).

We will now use the above equations to implement an N-step gauging procedure. We will gauge the G-symmetry of the state defined on the vertices of a lattice sequentially in N steps. We present the procedure in Algorithm 1 and consider the details below:

(1) *Include ancillas.* Add ancillas in the state $|e\rangle$, where $e \in G$ is the identity element, on the edges between the vertices.

(2) *Entangle gauge and matter DOFs.* Apply the following 2-controlled-shift operators with controls c_1, c_2 on neighboring vertices (oriented as $c_2 \to c_1$) and the target t on the in-between ancilla:

$$U_{Q_1} = \sum_{g_1, g_2, g_3 \in G} |g_1, g_2\rangle_{c_1,c_2}\langle g_1, g_2| \otimes |q_1(g_1) g_3 q_1(g_2)^{-1}\rangle_{\langle 1,2\rangle}\langle g_3|. \quad (46)$$

Here we have used $q_1(g)$ to denote the part of the decomposition g which lies in Q_1; that is, for $g = q_N \cdots q_1$ with $q_k \in s_k(Q_k)$, $q_1(g) = q_1$.

(3) *Measure* $\{X_1, X_2, \dots, X_{l_1}\}$ *on matter DOFs.* After measurement of the quotient part on each vertex (i.e., in the bases defined in $\{X_j\}$), with the outcome being $\{X_j = \omega_j^{-p_j}\}_{j=1}^{l_1}$ on a vertex (ω_j being d_jth root of unity), there is a corresponding (local) phase factor $\prod_{j=1}^{l_1} \omega_j^{-p_j i_j}$ from the wave function overlap in step (3) of Algorithm 1. These phase factors can be seen as some Abelian chargeons on vertices. See an example in Eq. (53).

(4) *Correct phase factors.* To obtain the ground state, the local phase factors arising from step (3) can be corrected as they can be expressed as a product of phase operators acting on the edge DOFs (see, e.g., Fig. 10). Therefore, we can apply counter Z operators on a set of edges E_{Cor}, i.e., $\prod_{e\in E_{\mathrm{Cor}}} \mathbb{Z}_e$, where the exact form of $\mathbb{Z}_e = \prod_{j=1}^{l_k} Z_j^{-p_{j;e}}$ can be deduced from the measurement outcomes. Physically, due to our N-step gauging procedure, the local phase factors always correspond to *Abelian* chargeons, therefore we can use ribbon operators to move them around. For example, we can move the phase factors on a vertex by performing Z operators on its neighboring edge and by doing this repeatedly we can move all the phase factors to one single vertex (via the transmutation rule, see, e.g., Fig. 10), therefore annihilating them altogether, due to the symmetry constraint [see Eq. (55) and the discussion below it]. The set of such edges is an example of E_{Cor}, but it is not necessarily optimized. (See also the following two sections for concrete examples.) After measurement and correction, the vertex DOF is mapped from $|g\rangle$ to $\frac{1}{|Q_1|}\sum_{q_1' \in s_1(Q_1)} |g^{(1)} q_1'\rangle$, where $g = q_N \cdots q_2 q_1$ and $g^{(1)} = q_N \cdots q_2$. The resultant state is a G_1-SET ground state.

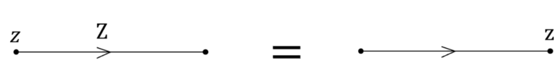


FIG. 10. The transmutation rule of z factor according to relation $z(g_{v'})z(g_v g_{v'}^{-1}) = z(g_v)$.

(5) *Repeat the procedure of entangling gauge DOFs on edges and matter DOFs on vertices.* Apply the following unitary similar to before:

$$U_{Q_2} = \sum_{g_1,g_2,g_3\in G} |g_1, g_2\rangle_{c_1,c_2}\langle g_1, g_2| \otimes |q_2(g_1)g_3q_2(g_2)^{-1}\rangle_{\langle 1,2\rangle}\langle g_3|. \tag{47}$$

(6) *Measure $\{X_1, \dots, X_{l_2}\}$ on the matter DOFs and correct the corresponding phase factors from the measurement.* This results in a G_2-SET ground state.

(7) *Repeat this process for all except the last quotient group Q_N.*

(8) *At the last step, apply the gauging and measurement procedure for Q_N.* Specifically, first apply

$$U_{Q_N} = \sum_{g_1,g_2,g_3\in G} |g_1, g_2\rangle_{c_1,c_2}\langle g_1, g_2| \otimes |q_N(g_1)g_3q_N(g_2)^{-1}\rangle_{\langle 1,2\rangle}\langle g_3|, \tag{48}$$

measure $\{X_1, \dots, X_{l_N}\}$, and then correct the corresponding phase factors. This gives us a G-TQD state.

It is worth remarking that in the expressions above, we always use the multiplication rules of the entire group G. For instance, if we take $q_2, q_2' \in s_2(Q_2)$, then their product q_2q_2' is not necessarily in $s_2(Q_2)$ (it is in $s_2(Q_2)$ only when the extension $G_2/G_1 = Q_2$ is central). This seemingly makes the remaining global symmetry algebra nonclosed, i.e., $g^{(1)}g'^{(1)}$ would produce components in the Q_1 subgroup [recall that $g^{(1)} = q_N \cdots q_2$ for some $q_k \in s_k(Q_k)$]. Nonetheless, instead of $|g^{(1)}\rangle$ in the above step (4), we have $\frac{1}{|Q_1|}\sum_{q_1'\in s_1(Q_1)}|g^{(1)}q_1'\rangle$, which can absorb the potential Q_1 components, making the multiplication closed. We can therefore define the global symmetry group of G_1 SET as such. Moreover, the state after applying U_{Q_2} is symmetric with X_{q_2}. It also follows that the phase operators resulting from the measurement of X_{q_2} can be corrected as the global symmetry gives a constraint on measurement outcomes, which will be discussed later.

In the following, we will consider the one-step gauging for Abelian groups in Sec. IV to illustrate correction processes. Then we will consider the two-step gauging for dihedral groups in Sec. V. The two-step and multistep gauging can be also applied to Abelian groups as well, and in the intermediate steps, SET states can emerge. We will discuss the phase of such SET states in Sec. V. Then in Sec. VI, we introduce the framework of the symmetry defect branch line and discuss the SET phases for several more examples.

We give an alternative procedure for the N-step gauging in Appendix J. This gauging procedure is implemented for a group G which admits sequential normal subgroups (see Appendix J for definition). This criterion is in fact equivalent to the group G being solvable (see Appendix K for a proof). The two procedures differ in the way in which the product of group elements are written down [compare Eqs. (41) and (J6)]. Apart from that, in the procedure given in this section, we gauge the quotient groups in every step, while in the procedure given in Appendix J we gauge normal subgroups in each step. As mentioned earlier, in Ref. [43], the commutator subgroups of the previous group in the series of a solvable group G are gauged successively.

IV. ONE-STEP GAUGING: ABELIAN GROUPS

In this section, we review the gauging procedure for Abelian groups [38,43]. We start with the SPT state given in Eq. (8). The gauging map can be implemented by first transferring the corresponding group elements to edges and then by measuring the vertices, where our state is projected to a quantum double state with (unwanted) charges whose configuration is given by the measurement outcomes. The excitation due to the randomness of measurement is then corrected by a certain finite-depth procedure. The steps are described in more detail as follows:

(0) *Prepare the SPT state on vertices.* We use local control-phase gates to prepare the SPT state from a direct product state.

(1) *Include ancillas.* Add ancillas in the state $|e\rangle$, where $e \in G$ is the identity element, on edges between adjacent vertices. The ancillas become the gauge DOFs.

(2) *Entangle gauge and matter DOFs.* Apply the following controlled-controlled-shift operators with controls c_1 & c_2 on the neighboring vertices of an edge e (oriented as $c_2 \to c_1$) and the target t being the ancilla on the edge between the two controls:

$$U_G = \sum_{g_1,g_2\in G} |g_1, g_2\rangle_{c_1,c_2}\langle g_1, g_2| \otimes L_{+e}^{g_1}L_{-e}^{g_2}. \tag{49}$$

At this point, the (premeasurement) state is

$$|\Psi_{\rm pre}\rangle = \sum_{\{g_v\}} \Omega(\{g_v\}) \bigotimes_{v\in V} |g_v\rangle_v \bigotimes_{\langle vv'\rangle\in E} \left|g_v g_{v'}^{-1}\right\rangle_{vv'}. \tag{50}$$

(3) *Choose a measurement basis in the G algebra, then project the matter DOFs onto the basis via measurement.* A natural basis can be chosen if we order elements in G as an ordered list $(g_0, g_1, \cdots, g_{n-1})$, where $g_0 = 1$, $n = |G|$. (Note that the subscript j in g_j here denotes the labeling of the group elements of G, not the vertex.) Then we simply use the Fourier basis $|k\rangle = \sum_{j=0}^{n-1} \exp 2\pi ijk/n|g_j\rangle/\sqrt{n} = Z^k|+\rangle$ to perform measurements on vertex DOFs, where we have defined $|+\rangle \equiv \sum_{j=0}^{n-1}|g_j\rangle/\sqrt{n}$. When $G = Z_n$, we project the matter DOFs onto this basis via measuring the generalized (qudit) X operator.

(4) *Correct Abelian chargeons in the G twisted quantum double.* The correction can be done with a finite-depth circuit, which consists of strings of Pauli-Z operators.

We give more explanation on the procedure for the case with $G = Z_n$ below. For the Z_n group, the wave function can be written using the qudit system. The basis vector $|a\rangle$ ($a \in \{0, \dots, n-1 \bmod n\}$) and the generalized Pauli operators satisfy

$$Z|a\rangle = \omega^a|a\rangle, \quad X|a\rangle = |a+1\rangle, \quad ZX = \omega XZ, \tag{51}$$

with $\omega = \exp(2\pi i/n)$, which is much simpler than the general Abelianized basis in Eqs. (43) and (44). Starting from a Z_n SPT on a triangulated lattice, we first add ancillas to all the edges in a product state with $|0\rangle$. Then we apply the controlled gate in Eq. (49), which is a set of controlled-X gates. Then the gauge DOFs are as given by the gauging map in Fig. 4. The next step is to disentangle matter DOFs by measuring the X operator on all vertices. After measurements, the matter DOF at a vertex v is projected onto $Z_v^{k_v}|+\rangle_v$, where

$k_v = 0, 1, \cdots, n-1$, and $|+\rangle = \sum_i |i\rangle/\sqrt{n}$. Suppose the measurement outcome on the vertex v is $X_v = \omega^{-k_v}$. We write the basis associated with measurement outcomes $\{k_v\}_{v\in V}$ as

$$|M\rangle := \bigotimes_v Z_v^{k_v}|+\rangle_v. \tag{52}$$

Then the total wave function after measurements (with the gauge part being projected to $\langle M|\Psi\rangle_{\rm pre}$) is written as

$$|\Psi\rangle_{\rm post} = \left(\sum_{\{g_v\}} \Omega(\{g_v\}) \prod_v z^{-k_v}(g_v) \bigotimes_e |g_v g_{v'}^{-1}\rangle_e\right) \bigotimes_v Z^{k_v}|+\rangle_v, \tag{53}$$

where $z^a(g_v) := \langle g_v|Z^a|g_v\rangle = (\omega^a)^{g_v}$ and Ω is the phase factor inherited from the SPT state (i.e., a product of 3-cocycles). Note that vv' is the edge e; many edges can share a vertex v but the factor z^{k_v} only appears once. Due to measurement, the vertex DOFs have been disentangled from the edge DOFs, and the edge DOFs form a state that is a ground state of the Z_n twisted quantum double in the flux-free sector up to a factor $z^{k_v}(g_v)$, which can be interpreted as an $\mathbf{e}^{k_v}$ chargeon on the vertex v.

In what follows, we describe how to remove the excitations in $|\Psi\rangle_{\rm post}$. First, the set of measurement outcomes is restricted to $[\sum_v k_v]_n = 0$, with $[x]_n$ being x mod n, due to the global symmetry of the SPT state. The global symmetry implies

$$\prod_{v\in V} X_v|\Psi_{\rm pre}\rangle = |\Psi_{\rm pre}\rangle, \tag{54}$$

so it should be satisfied that

$$\begin{aligned}\langle M|\Psi_{\rm pre}\rangle &= \langle M|\prod_{v\in V} X_v|\Psi_{\rm pre}\rangle,\\ &= \left(\prod_v \omega^{-k_v}\right)\langle M|\Psi_{\rm pre}\rangle,\end{aligned} \tag{55}$$

which gives the constraint $\prod_v \omega^{-k_v} = 1$, meaning $[\sum_v k_v]_n = 0$.

Next, the measurement with $|M\rangle$ gives us a phase $\prod_v \omega^{-k_v g_v}$ when contracted with the basis $\otimes_v|g_v\rangle$. Due to the constraint $[\sum_v k_v]_n = 0$, one can always find a set of paths such that we can rewrite the phase factor as $\prod_v \omega^{-k_v g_v}$, or equivalently $\prod_v z(g_v)^{-k_v}$, in terms of the phase operator Z_e supported on the paths. Concretely, we use a type of relations, which we call the *transmutation rules*, illustrated in Fig. 10. For $G = Z_n$, the relation is

$$z(g_v) = z(g_{v'})z\big(g_v g_{v'}^{-1}\big). \tag{56}$$

We apply the phase operator on the paths to remove the chargeons. Given that these operators commute, they can be applied all at once. Hence, our gauging procedure assisted by measurement requires only finite time steps or a finite-depth quantum circuit (with intermediate measurements).

Let us give two remarks here. The reason that we can correct the state by moving all factors to one vertex is because of the fact that all chargeons in $D^\omega(Z_n)$ are Abelian anyons. This procedure can be straightforwardly generalized to $Z_n \times Z_m \times \cdots$ group, where we measure $X \times 1 \times \cdots$, $1 \times X \times \cdots$, . . . , etc., on all vertices after we entangle gauge and matter DOFs. This occurred previously in the general N-step gauging in Sec. III. However, to explain the detailed correction there would incur cumbersome notations. The example of Z_n in this section should now make the procedure clearer. Different measurement outcomes will give rise to different chargeons in the flux-free sector, which are all Abelian anyons. (Note that we do not have fluxons, as we began with a flat-flux configuration followed by the controlled-controlled operation that does not create fluxons.) Therefore, the state after measurement is still correctable within finite steps.

V. TWO-STEP GAUGING: DIHEDRAL GROUP AND INTERMEDIATE SET STATES

When we attempt to gauge non-Abelian SPT states using measurement, although one can always choose a suitable basis such that the factors are *correctable*, one crucial problem is that the phase factors that arise from measurement do not necessarily correspond to Abelian anyons as in the $D^\omega(Z_n)$ case above; this makes correction with a finite depth circuit a nontrivial problem. In Refs. [41,42], there were two different ways proposed to prepare the ground state of the S_3 quantum double model. In Ref. [42], a Z_3 toric code ground state is prepared first, and it is coupled to the Z_2 product state using controlled gates. Then the Z_2 part is gauged via the measurement-assisted one-step gauging in Ref. [38].

We will show in this section that, for the symmetry group G being the extension of two Abelian groups, by choosing some Abelianized basis, we can still perform a two-step gauging procedure on G-SPT states via measurement. In the case with $G = S_3$, our procedure would be equivalent to first preparing the Z_3-TQD ground state, and then coupling to the Z_2-SPT state using entangling gates and controlled gates. The correction process for the two-step gauging is still fairly simple, i.e., via finite-depth quantum circuits. The complete procedure to gauge the Abelian N symmetry (i.e., the normal subgroup) of a G-SPT state and then to gauge the quotient Q symmetry of an SET is as follows:

(1) *Include ancillas.* Add ancillas in the state $|e\rangle$, where $e \in G$ is the identity element, on edges between adjacent vertices. The ancillas will become the gauge DOFs.

(2) *Entangle gauge and matter DOFs.* Apply the following controlled-controlled-shift operators with controls c_1 & c_2 on neighboring vertices (oriented as $c_2 \to c_1$) and the target t on the ancilla on the in-between edge e:

$$U_N = \sum_{g_1,g_2\in G} |g_1, g_2\rangle_{c_1,c_2}\langle g_1, g_2| \otimes L_{+e}^{n(g_1)} L_{-e}^{n(g_2)}. \tag{57}$$

The purpose of this step is to mimic the gauging map in Eq. (17).

(3&4) *Measure $X_{(n)}$ on matter DOFs and correct the z_n factors.* After measurement, with the outcome being $X_{(n)} = \omega^{-k}$, there is a corresponding phase factor z_n^k. Using the transmutation rule for z_n, one can correct all those factors by moving them to one single vertex, resulting in an SET ground state.

(5) *Further entangling the quotient part of the gauge and matter DOFs.* We apply a controlled-conjugate operator with the target e being the ancilla (oriented as $c_2 \to c_1$), and the

control being c_2:

$$U_Q = \sum_{g_1, g_2 \in G} |q(g_1), q(g_2)\rangle_{c_1,c_2} \langle q(g_1), q(g_2)| \otimes L_{+e}^{q(g_1)} L_{-e}^{q(g_2)}, \tag{58}$$

where $q(g)$ denotes the quotient part of g via an embedding in Eq. (16). Notice that the normal part of the matter DOF has been wiped out by measuring $X_{(n)}$, while the quotient part $Q = G/N$ still remains, which makes the above controlled-gates possible to implement. The edge DOFs are now $\{q(g_1)n(g_1)n(g_2)^{-1}q(g_2)^{-1}\} = \{g_1 g_2^{-1}\}$.

(6&7) *Measure $X_{(q)}$ on matter DOFs, and correct z_q factors.* Their correction is straightforward; we apply $Z_{(q)}$ operators on edges to move all z_q's to one vertex.

In the following, we will apply the above procedure to several cases.

A. Gauging S_3 SPT

The S_3 group is $G = \langle a, x | a^3 = e, x^2 = e, xax = a^{-1}\rangle$. Any element $g \in G$ can be written as $g = x^i a^j$, where $i = 0, 1$, $j = 0, 1, 2$. We define the decomposition of a group element, respectively, as

$$n(x^i a^j) = a^j, \tag{59}$$

$$q(x^i a^j) = x^i, \tag{60}$$

with the former being the normal part ($N = Z_3$), and the latter being the quotient part [$s(Q) = s(Z_2)$] of S_3. We then define the shift operator in each part as

$$X_{(n)} = \sum_{i,j} |x^i a^{j+1}\rangle\langle x^i a^j|,$$
$$X_{(q)} = \sum_{i,j} |x^{i+1} a^j\rangle\langle x^i a^j|. \tag{61}$$

The phase operators, which are known as the clock operators, in each respective part, are

$$Z_{(n)} = \sum_g z_n(g)|g\rangle\langle g|, \quad z_n(x^i a^j) = \omega^j,$$
$$Z_{(q)} = \sum_g z_q(g)|g\rangle\langle g|, \quad z_q(x^i a^j) = (-1)^i, \tag{62}$$

where $\omega = e^{i\frac{2\pi}{3}}$. The gauging step (2) transforms the ancilla DOF on edge $e = \langle v, v'\rangle$ from identity to $n(g_v)n(g_{v'})^{-1}$.

Then in step (3) we measure $X_{(n)}$ on all the vertex DOFs. Suppose the measurement outcome is $X_{(n)} = e^{-i\frac{2\pi k_v}{3}}$ on vertex v (where $k_v = 0, 1, 2$). The state after the measurement is projected into

$$|\Psi_3\rangle = \sum_{\{g_v\}} \left(\prod_v z_n^{-k_v}(g_v)\right) \Omega\big(\{g_v g_{v'}^{-1}\}\big) |\{n(g_v)n(g_{v'})^{-1}\}\rangle_e \bigotimes_v \left(Z_{(n)}^{k_v} \left(\sum_{r \in N} |q(g_v) r\rangle_v \right)\right). \tag{63}$$

The phase factor $\prod_v z_n^{-k_v}(g_v)$ depends on the measurement outcomes $\{k_v\}$. To correct them, we employ the transmutation

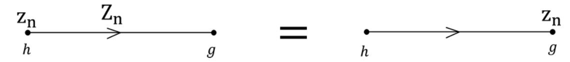


FIG. 11. The transmutation rule for z_n factors on step (4).

rule for the z factors (see Fig. 11)

$$z_n(h) z_n(n(g)n(h)^{-1}) = z_n(g). \tag{64}$$

As in the case with Z_n in the previous section, we have $[\sum_v k_v]_3 = 0$ due to the N symmetry. By inserting corresponding numbers of Z_n operators on the edges, we can move the factors z_n on vertices around and cancel them altogether. Equivalently, one can simultaneously apply Z_n operators supported on strings whose endpoints correspond to nontrivial measurement outcomes. This gives us the state

$$|\Psi_4\rangle = \sum_{\{g_v\}} \Omega\big(\{g_v g_{v'}^{-1}\}\big) |\{n(g_v)n(g_{v'})^{-1}\}\rangle_e \bigotimes_v \left(Z_{(n)}^{k_v} \left(\sum_{r \in N} |q(g_v) r\rangle_v \right)\right). \tag{65}$$

After the gauging step (5), the edge DOFs are conjugated and shifted by U_Q, giving rise to

$$|\Psi_5\rangle = \sum_{\{g_v\}} \Omega\big(\{g_v g_{v'}^{-1}\}\big) \big|\{g_v g_{v'}^{-1}\}\big\rangle_e \bigotimes_v \left(Z_{(n)}^{k_v} \sum_{r \in N} |q(g_v) r\rangle_v \right). \tag{66}$$

Step (6) is similar to step (3). By measurements, the state is projected to

$$|\Psi_6\rangle = \sum_{\{g_v\}} \left(\prod_v z_q^{-m_v}(g_v)\right) \Omega\big(\{g_v g_{v'}^{-1}\}\big) \big|\{g_v g_{v'}^{-1}\}\big\rangle_e \bigotimes_v \left(Z_{(q)}^{m_v} Z_{(n)}^{k_v} \left(\sum_{g \in G} |g\rangle_v \right)\right), \tag{67}$$

where we have assumed that $X_{(q)} = e^{-i\frac{2\pi m_v}{2}}$ from the measurement on vertex v (where $m_v = 0, 1$). To correct the phase factor $\prod_v z_q^{-m_v}(g_v)$, we employ the transmutation rules for z_q factors:

$$z_q(h) z_q(gh^{-1}) = z_q(g), \tag{68}$$

which is illustrated in Fig. 12. This rule, just as the rule for z_n in step (4), allows us to move all the z_q factors to a single vertex and annihilate them. This is guaranteed by the Q-global symmetry in $|\Psi_5\rangle$ (see Appendix C),

$$\left(\prod_v X_{(q)}\right) |\Psi_5\rangle = |\Psi_5\rangle 1, \tag{69}$$

which implies that the measurement outcomes satisfy $[\sum_v m_v]_2 = 0$ in this case, as the global symmetry is Z_2. After

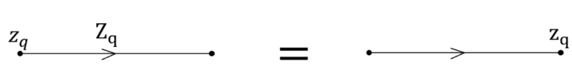


FIG. 12. The transmutation rule for z_q factors on step (7).

we apply the corresponding correcting phase factors to edges, we thus obtain a G-TQD ground state.

B. Z_3 SET with Z_2 Symmetry

Let us begin by recalling that the gauging map Γ_N in Eq. (17) gauges the normal subgroup N of G. In our procedure, after we correct the z_n factors in step (3), the state is essentially a ground state of the Z_3 SET phase. In what follows, we first look into the entanglement structure of the wave function after gauging the normal subgroup Z_3. Then we identify the class of this SET phase, namely, the unitary modular tensor category (UMTC) $\mathcal{C}$ that contains all anyonic excitations, the Z_2 symmetry action as an automorphism of $\mathcal{C}$, the symmetry fractionalization class and defectification class [51].

We write the element in S_3 as $\tilde{g} = (G, g) \equiv x^G a^g$ with some slight abuse of the notation, where $G = 0, 1$ and $g = 0, 1, 2$. It should be clear from the context when G is a number or a group. A representative of the cocycle in $H^3(S_3, U(1))$ is

$$\omega(\tilde{g}, \tilde{h}, \tilde{l}) = \exp \frac{2\pi i p_1}{9} g(-1)^{H+L}(h(-1)^L + l - [h(-1)^L + l]_3) \times \exp \pi i p_2 GHL, \tag{70}$$

where $p_1 = 0, 1, 2$, and $p_2 = 0, 1$. As pointed out in Sec. II, the anyon theory (UMTC) $\mathcal{C}$ is determined by the restriction of ω on subgroup Z_3 (i.e., setting $G = H = L = 0$),

$$\nu(g, h, l) = \exp \frac{2\pi i p_1}{9} g(h + l - [h + l]_3). \tag{71}$$

Different values of p_1 are in one-to-one correspondence with different Z_3 twisted quantum double phases $D^\nu(Z_3)$. An anyon in these phases is characterized by its flux $a \in Z_3$, and a projective representation of Z_3, satisfying

$$\mu_a(g)\mu_a(h) = \exp \frac{2\pi i p_1 a}{9}(g + h - [g + h]_3)\mu_a(gh), \tag{72}$$

which means $\mu_a(g) = e^{\frac{2\pi i a p_1 g}{9}} v(g)$, where $v(g)$ is an ordinary representation of Z_3.

Using Lyndon-Hochschild-Serre spectral sequence [22,56,57], we can decompose the cohomology class of the S_3 group as

$$H^3(S_3, U(1)) = H^3(Z_3, U(1)) \oplus H^3(Z_2, U(1)). \tag{73}$$

This suggests that this SET state is composed of a TQD $D^\nu(Z_3)$ and a Z_2-SPT state. A natural question is whether the wave function of the whole system is decomposed into a product of the two corresponding parts.

It turns out that we can write the 3-cocycle in Eq. (70) as

$$\omega(\tilde{g}, \tilde{h}, \tilde{l}) = \omega_n \cdot \omega_q \cdot \omega'(\tilde{g}, \tilde{h}, \tilde{l}), \tag{74}$$

where the phase ω_n is defined from the 3-cocycle of Z_3,

$$\begin{aligned}\omega_n\left(g_3 g_2^{-1}, g_2 g_1^{-1}, g_1\right) &\equiv \nu(n(g_3)n(g_2)^{-1}, n(g_2)n(g_1)^{-1}, n(g_1)) \\ &= \nu\left(\phi^{q(g_2)^{-1}}\left(n\left(g_3 g_2^{-1}\right)\right), \phi^{q(g_1)^{-1}}\left(n\left(g_2 g_1^{-1}\right)\right), n(g_1)\right),\end{aligned} \tag{75}$$

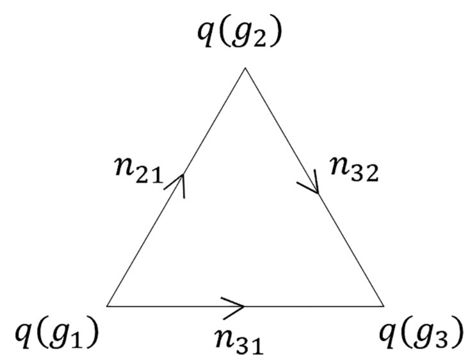


FIG. 13. One plaquette on a triangulated lattice.

and ν is a representative in $H^3(N, U(1))$. Every plaquette on the spatial manifold is associated with such a phase. The product of them over all plaquettes gives

$$\Omega_\nu(\{n(g_v)n(g_{v'})^{-1}\}) = \prod_\Delta \nu(\{n(g_v)\})^{s(\Delta)}. \tag{76}$$

The sum of the above phase over all possible $\{g_v\}$ configurations gives the wave function of a ground state of $D^\nu(Z_3)$.

The phase ω_q is defined from the 3-cocycle of Z_2, $\alpha \in H^3(Z_2, U(1))$,

$$\omega_q(\tilde{g}, \tilde{h}, \tilde{l}) = \exp \pi i p_2 GHL =: \alpha(G, H, L). \tag{77}$$

The product of this type of phases gives

$$\Omega_q(\{q(g_v)q(g_{v'})^{-1}\}) = \prod_\Delta \alpha(\{q(g_v)\})^{s(\Delta)}. \tag{78}$$

The sum of the above phases over all possible $\{g_v\}$ configurations results in the wave function of a Z_2-SPT state. The ω' part in Eq. (74) is

$$\omega'(\tilde{g}, \tilde{h}, \tilde{l}) = \exp\left\{\frac{2\pi i p_1}{3}(-1)^H g(1 - \delta_{h,0})\delta_{L,1}\right\}, \tag{79}$$

which is nontrivial when $p_1 \neq 0$. Similarly, we define the product of this type of phases over the spatial manifold as Ω',

$$\Omega'(\{q(g_v)\}, \{n(g_v)\}) = \prod_\Delta \omega'(\{q(g_v)\}, \{n(q_v)\})^{s(\Delta)}. \tag{80}$$

Thus, the resulting state after gauging Z_3 from an S_3-SPT state is (see Fig. 13 for an illustration of a plaquette)

$$|\Psi_{\text{SET}}\rangle = \sum_{\{g_v\}} \Omega'(\{q(g_v)\}, \{n(g_v)n(g_{v'})^{-1}\})(\Omega_n|\{n(g_v) \times n(g_{v'})^{-1}\}\rangle_e) \bigotimes (\Omega_q|\{q(g_v)\}\rangle_v), \tag{81}$$

which is an entangled state between a Z_3-TQD ground state and a Z_2-SPT state. When $p_1 = 0$, we have $\omega' = \Omega_n = 1$, hence the wave function of the system becomes

$$\begin{aligned}|\Psi\rangle &= \sum_{\{g_v\}} \Omega_q|\{n(g_v)n(g_{v'})^{-1}\}\rangle_e \bigotimes |\{q(g_v)\}\rangle_v \\ &= \sum_{\{g_v\}} (|\{n(g_v)n(g_{v'})^{-1}\}\rangle_e) \bigotimes (\Omega_q|\{q(g_v)\}\rangle_v) \\ &= |Z_3\text{TC}\rangle \otimes |Z_2\text{SPT}\rangle,\end{aligned} \tag{82}$$

which is a product state of a Z_3 Toric code ground state and a Z_2-SPT state.

Having obtained the SET wave functions, we now discuss the effect of the global symmetry action. The Z_2 symmetry action $U^x = \prod_v L^x_{-v}$ in the S_3-SPT state is mapped to $U^x_Q = U^x\phi^x$, under which an anyon with flux a^i will be mapped to one with flux $\phi^x(a) = a^{[-i]_3}$. And a chargeon will be mapped to its antiparticle under the symmetry. According to Sec. II C, the possible SFC will be given by elements in a $H^2_\rho(Z_2, \mathcal{A})$ torsor. With different values of p_1, the Abelian group $\mathcal{A}$ could be either $Z_3 \times Z_3$ or Z_9. In either cases, the cohomology group $H^2_\rho(Z_2, \mathcal{A})$ turns out to be trivial, and so is its torsor. Therefore, the TQD $D^\nu(Z_3)$ has only one possible Z_2 symmetry fractionalization pattern. Moreover, different values of p_2 in the 3-cocycle of S_3 result in different Z_2 symmetry defectification classes (SDC) in the SETs, which are obtained by gauging the normal Z_3 group. This is expected because as seen from Eq. (81), different p_2 values represent different Z_2-SPT states entangled with some Z_3-TQD state.

Let us remark that this construction for S_3 has a natural generalization on D_{2n+1} groups, where one first gauge Z_{2n+1}, resulting in an Z_{2n+1} SET on which the Z_2 symmetry acts to conjugate the gauge DOFs. Different parent SPT phases will result in different anyon theories and different SDCs, but always some unique symmetry fractionalization pattern. One can further gauge the quotient Z_2 symmetry to obtain D_{2n+1} TQD.

C. Gauging D_{2n} SPT

Now we discuss the process of two-step gauging a generic D_{2n} SPT state via measurement under a similar type of Abelianized basis. As in S_3, an element in group D_{2n} can be written as

$$g = x^i a^j, \quad i = 0, 1, \quad j = 0, 1, \cdots, 2n-1, \tag{83}$$

where $x^2 = 1$ and $a^{2n} = 1$. We will thus use a generalized definition of operators as for S_3 in Eqs. (59), (60), (61), and (62).

After applying the control gate to set the DOFs on edges, e.g., $\langle ij\rangle$, to $n(g_i)n(g_j)^{-1}$, measuring $X_{(n)}$ on vertices, and correcting all chargeon excitations, the resultant state is a ground state in a Z_{2n}-SET phase with a global Z_2 symmetry at this intermediate stage. According to the multiplication rule of D_{2n}, just as in S_3, the symmetry transformation U^x conjugates all the gauge DOFs.

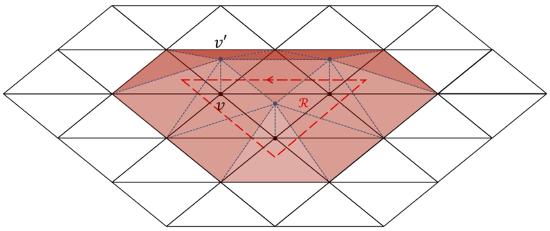
As an illustration, we will consider the D_4 group. Using the Lyndon-Hochschild-Serre sequence, the cohomology group can be decomposed as

$$H^3(D_4, U(1)) = H^3(Z_4, U(1)) \oplus H^3(Z_2, U(1)) \oplus H^2(Z_2, H^1(Z_4, U(1))). \tag{84}$$

Again, the cocycle factorizes as

$$\omega = \omega_n \cdot \omega_q \cdot \omega' \cdot \omega_{nq}, \tag{85}$$

where ω_n and ω_q are defined similarly as for the S_3 group in the last section, while ω' and ω_{nq} depend on both quotient and normal parts of the vertex DOFs. From this decomposition, it is clear that after gauging the normal Z_{2n}, we have an entangled state between the Z_{2n}-TQD state and the Z_2-SPT state.



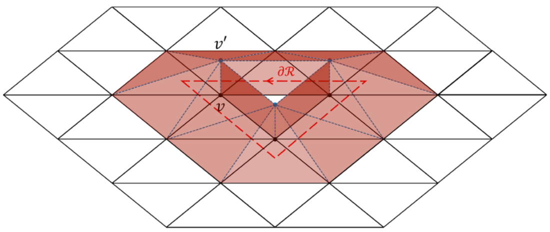


FIG. 14. (a) Symmetry action inside of region $\mathcal{R}$ "lifts" $\mathcal{R}$ such that all the simplex in the region correspond to $\tilde{\omega}$. (b) This symmetry action can be equivalently regarded as the insertion of symmetry branch line on $\partial\mathcal{R}$.

Because of the additional entanglement via ω_{nq}, we expect to have a nontrivial SFC from gauging the Z_4 symmetry of a D_4-SPT state.

Indeed, in the next section, we will show that the current two-step gauging setup could result in different SFCs. To do so, we first introduce symmetry branch line operators and other necessary tools to determine the SFC. We will also discuss several examples, including the above Z_4-SET phase.

VI. SYMMETRY DEFECT IN SPT AND SET

In this section, we will apply the notion of symmetry branch lines introduced in Ref. [51] and formulate corresponding symmetry branch line operators in SPT phases, as well as their relation to ribbon operators in TQD. We then show the gauging procedure transforms such operators in SPT phases into symmetry branch lines in SET phases and discuss how their fusions relate to the symmetry fractionalization classes (SFC) in a few examples.

A. Symmetry branch lines

To introduce symmetry branch lines, we start with the symmetry action in an SPT wave function. We recall in Eq. (2) the SPT wave function on a triangulated spatial manifold,

$$|\Psi_{\rm SPT}\rangle = \sum_{\{g_v\}} \prod_{\Delta} \omega\big(g_3 g_2^{-1}, g_2 g_1^{-1}, g_1\big)^{s(\Delta)} \bigotimes_v |g_v\rangle. \tag{86}$$

When the manifold is closed, the global symmetry action $U^x \equiv \prod_v L^x_{-v}$ leaves the entire SPT state invariant. We can also consider the symmetry action on a submanifold $\mathcal{R}$ [23], such as the one shown in Fig. 14(a),

$$U^x_{\mathcal{R}} \prod_{\Delta} \omega^{s(\Delta)} |\{g_v\}\rangle = \prod_{\Delta} {\rm Amp}^{\mathcal{R}}(\{g_v\}, x)\omega^{s(\Delta)} |\{g_v\}\rangle. \tag{87}$$

Triangulating the frustum created by lifting vertices in $\mathcal{R}$, we have multiple tetrahedrons. We associate each tetrahe-

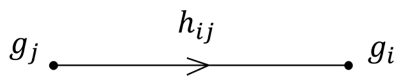


FIG. 15. $|h_{ij}\rangle_e$ is located on edge $e = \langle i, j\rangle$, $|g_i\rangle_v$ and $|g_j\rangle_v$ are located on the two endpoints i and j.

dron in the frustum as in Fig. 14(a) to a 3-cocycle, such as the one in Fig. 3, to which we assign a phase factor $\omega(g_4 g_3^{-1}, g_3 g_2^{-1}, g_2 g_1^{-1})$. The product of all such cocycles composes the factor $\text{Amp}^{\mathcal{R}}$, namely,

$$\text{Amp}^{\mathcal{R}} = \prod_{\text{tetra}\in\mathcal{R}} \omega(\text{tetra})^s. \tag{88}$$

Using the cocycle conditions, it turns out that $\text{Amp}^{\mathcal{R}}$ only depends the DOFs around $\partial\mathcal{R}$ and does not depend on those DOFs deep inside $\mathcal{R}$, see Fig. 14(b), and its expression is

$$\text{Amp}^{\mathcal{R}} = \tilde{\Theta}_{\partial\mathcal{R}}^{g_v x g_v^{-1}} \prod_{\text{tetra}\in\partial\mathcal{R}} \omega(\text{tetra})^s, \tag{89}$$

where the extra factor to the product on the right-hand side (r.h.s.) is

$$\tilde{\Theta}_{\partial\mathcal{R}}^{g_n x g_n^{-1}} = \theta_{g_n x g_n^{-1}}\left(g_n g_{n-1}^{-1}, g_{n-1} g_1^{-1}\right) \cdots \theta_{g_3 x g_3^{-1}}\left(g_3 g_2^{-1}, g_2 g_1^{-1}\right), \tag{90}$$

when $\partial\mathcal{R}$ contains vertices equipped with the branching structure, $1 \to 2 \to 3 \to \cdots \to n \leftarrow 1$; see Appendix D. In fact, one can introduce an operator $\mathcal{B}_{\partial\mathcal{R}}^x$ that is supported only on $\partial\mathcal{R}$ such that [23]

$$\text{“}\mathcal{B}_{\partial\mathcal{R}}^x\text{”}|\Psi_{\text{SPT}}\rangle = U_{\mathcal{R}}^x |\Psi_{\text{SPT}}\rangle. \tag{91}$$

To do this carefully, we need to first introduce a pregauge structure [58]. Namely, we introduce a G-DOF h_{ij} on every edge $\langle i, j\rangle$ (which we will be set to the identity group element $|1\rangle_e$ to begin with); see Fig. 15. One can think of the edge DOFs as the discrete gauge field. After introducing such a gauge field, one may write the local symmetry action on both vertex v and the surrounding edges $e \supset v$. This is also called the "gauge transformation" operator on a vertex v as

$$\mathcal{G}_v^x \equiv L_{-v}^x \prod_{e\supset v} L_{\pm e}^x, \tag{92}$$

where $e \supset v$ denotes those edges with one end being v and where $L_{\pm e}^{x_e}$ is the left (right) actions of x_e on $|h\rangle_e$, when the edge e flows to (emanates from) v. Furthermore, the "interactions" should also be written in a gauge invariant way. Namely, instead of the original SPT Hamiltonian, one can write the gauge invariant version as follows:

$$H_{\text{SPT-pre}} = -\sum_v \sum_{g\in G} \frac{1}{|G|} L_{+v}^g W_v^g, \tag{93}$$

where the W phase was previously introduced in Sec. II.

Taking any ground state of this Hamiltonian, we can impose the gauging map in the presence of the pregauge structure,

$$\Gamma: \quad |\{g_i\}\rangle_v |h_{ij}\rangle_e \to \left|\left\{g_i h_{ij} g_j^{-1}\right\}\right\rangle_e, \tag{94}$$

under which, the state would be mapped to a ground state of the TQD model. One special ground state of this Hamiltonian would be

$$|\Psi_{\text{SPT-pre}}\rangle = |\Psi_{\text{SPT}}\rangle \bigotimes_e |1\rangle_e, \tag{95}$$

where we have taken the original SPT state, which has $|h\rangle_e \equiv |1\rangle_e$ on all edges. One can easily verify that $\mathcal{G}_v^x|\Psi_{\text{SPT-pre}}\rangle$ is still a ground state, for any vertex v on lattice and $x \in G$. We define the gauge transformation over a region $\mathcal{R}$ as $\mathcal{G}_{\mathcal{R}}^x \equiv \prod_{v\in\mathcal{R}} \mathcal{G}_v^x$. If the spatial lattice Γ is closed, then we write $\mathcal{G}^x \equiv \prod_{v\in\Gamma} \mathcal{G}_v^x$ and one can check that

$$\mathcal{G}^x|\Psi_{\text{SPT-pre}}\rangle = |\Psi_{\text{SPT-pre}}\rangle. \tag{96}$$

Therefore, the operator $\mathcal{G}^x$ mimics the behavior of global symmetry operator U^x after introducing the pregauge structure.

Now we introduce the definition of symmetry branch line operators $\tilde{\mathcal{B}}_{\partial\mathcal{R}}^x$ on states with trivial edges $|h\rangle_e \equiv |1\rangle_e$,

$$\tilde{\mathcal{B}}_{\partial\mathcal{R}}^x|\Psi_{\text{SPT-pre}}\rangle = \mathcal{G}_{\mathcal{R}}^x|\Psi_{\text{SPT-pre}}\rangle, \tag{97}$$

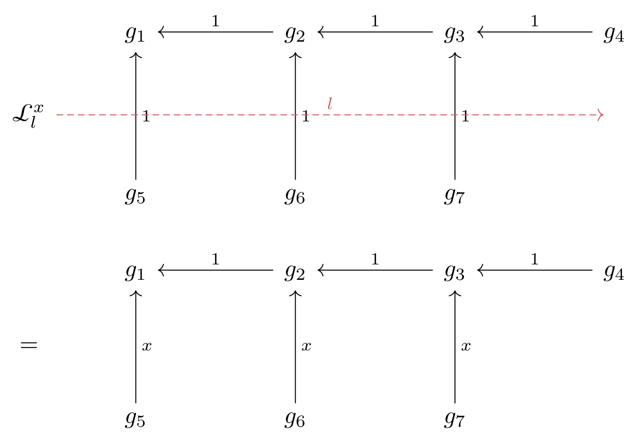
but unlike $\mathcal{G}_{\mathcal{R}}^x$, the operator $\tilde{\mathcal{B}}_{\partial\mathcal{R}}^x$ only takes effect on $\partial\mathcal{R}$. We find that the following expression of $\tilde{\mathcal{B}}_{\partial\mathcal{R}}^x$,

$$\tilde{\mathcal{B}}_{\partial\mathcal{R}}^x = \sum_{g_v} \mathcal{L}_{\partial\mathcal{R}}^x \tilde{W}_{\partial\mathcal{R}}^{g_v x g_v^{-1}} \tilde{\Theta}_{\partial\mathcal{R}}^{g_v x g_v^{-1}} |g_v\rangle_v\langle g_v|, \tag{98}$$

where v is a reference vertex on $\partial\mathcal{R}$ as shown in Fig. 14(b) and $\tilde{\Theta}_{\partial\mathcal{R}}^{g_v x g_v^{-1}}$ is defined in Eq. (90). The phase $\tilde{W}_{\partial\mathcal{R}}^{g_v x g_v^{-1}}$ is the product of cocycles associated to the tetrahedrons in Fig. 14(b), with $g_{v'} g_v^{-1} = g_v x g_v^{-1}$,

$$\tilde{W}_{\partial\mathcal{R}}^{g_v x g_v^{-1}} = \prod_{\text{tetra}\in\partial\mathcal{R}} \omega(\text{tetra})^s. \tag{99}$$

The operator $\mathcal{L}_{\partial\mathcal{R}}^x$ is a product of shift operators on the edges crossed by $\partial\mathcal{R}$. In general, $\mathcal{L}_l^x$ with $x \in G$ on a ribbon l is defined as follows:



(100)

We note that the branch line operator $\tilde{\mathcal{B}}_{\partial\mathcal{R}}^x$ cannot be written as a product of local terms on $\partial\mathcal{R}$. This operation $\tilde{\mathcal{B}}_{\partial\mathcal{R}}^x$ can be regarded as the nononsite symmetry action on the boundary $\partial\mathcal{R}$, and its definition can be extended to the case whenever there is no flux in the state (i.e., for every plaquette, $\prod_e h_e = 1$). From direct calculation using Eq. (97) and the fact that $\mathcal{G}_{\mathcal{R}}^x \mathcal{G}_{\mathcal{R}}^y = \mathcal{G}_{\mathcal{R}}^{xy}$, one can easily show that the multiplication rule of $\tilde{\mathcal{B}}_{\partial\mathcal{R}}^x$,

$$\tilde{\mathcal{B}}_{\partial\mathcal{R}}^x \tilde{\mathcal{B}}_{\partial\mathcal{R}}^y = \tilde{\mathcal{B}}_{\partial\mathcal{R}}^{xy}, \tag{101}$$

is exactly the multiplication rule of the group G, as expected for the symmetry branch lines. One can also show the multiplication rule of $\tilde{\mathcal{B}}^x_{\partial\mathcal{R}}$ directly using the form in Eq. (98); see Appendix D. Indeed, if we use the operator

$$\prod_v \left(\frac{1}{|G|} \sum_{x \in G} \mathcal{G}^x_v \right) \tag{102}$$

to project any ground state onto the gauge invariant sector, then we would also have a TQD ground state. The operator above in Eq. (102) can be seen as a superposition of different meshes of symmetry branch lines.

There are two important remarks here. The first is that inserting a symmetry branch line on an SPT state is to create a point symmetry defect 0_x, move it along $\partial\mathcal{R}$ and annihilate it with $0_{x^{-1}}$. The multiplication rule above indicates that the fusion between point defects is $0_x \cdot 0_y = 0_{xy}$. Second, we can consider the global symmetry action $\mathcal{G}^y$ on the state with a branch line on $\partial\mathcal{R}$, i.e., $\tilde{\mathcal{B}}^x_{\partial\mathcal{R}}|\Psi_{\text{SPT-pre}}\rangle$. It turns out that

$$\begin{aligned}
\mathcal{G}^y\tilde{\mathcal{B}}^x_{\partial\mathcal{R}}|\Psi_{\text{SPT-pre}}\rangle &= \mathcal{G}^y\mathcal{G}^x_{\mathcal{R}}|\Psi_{\text{SPT-pre}}\rangle \\
&= \mathcal{G}^y\mathcal{G}^x_{\mathcal{R}}\mathcal{G}^{y^{-1}}|\Psi_{\text{SPT-pre}}\rangle \\
&= \mathcal{G}^{yxy^{-1}}_{\mathcal{R}}|\Psi_{\text{SPT-pre}}\rangle \\
&= \tilde{\mathcal{B}}^{yxy^{-1}}_{\partial\mathcal{R}}|\Psi_{\text{SPT-pre}}\rangle,
\end{aligned} \tag{103}$$

where $\mathcal{G}^y_{\mathcal{R}} \equiv \prod_{v\in\mathcal{R}} \mathcal{G}^y_v$. In other words, the global symmetry transformation ρ^y on 0_x is $\rho^y(0_x) = 0_{yxy^{-1}}$.

B. More on symmetry branch lines

To discuss further the symmetry branch lines, we first remind the readers of some definitions in group cohomology. Given a 3-cocycle $\omega(g, h, l)$ as a representative of element in $H^3(G, U(1))$, the slant product is defined as [59]

$$\theta_x(g, h) \equiv \frac{\omega(x, g, h)\omega(g, h, (gh)^{-1}xgh)}{\omega(g, g^{-1}xg, h)}, \tag{104}$$

which is naturally a conjugated 2-cocycle, i.e., a representative in $H^2(G, U(1)[G])$. Namely, it satisfies the following condition:

$$\tilde{\delta}\theta_x(g, h, l) \equiv \frac{\theta_{g^{-1}xg}(h, l)\theta_x(g, hl)}{\theta_x(gh, l)\theta_x(g, h)} = 1. \tag{105}$$

When θ is a representative of the trivial element in $H^2(G, U(1)[G])$, there exists a conjugated 1-cochain ϵ, such that

$$\theta_x(g, h) = \tilde{\delta}\epsilon_x(g, h) \equiv \frac{\epsilon_{g^{-1}xg}(h)\epsilon_x(g)}{\epsilon_x(gh)}. \tag{106}$$

There is another product that will also become useful later:

$$\gamma_g(x, y) \equiv \frac{\omega(x, y, g)\omega(g, g^{-1}xg, g^{-1}yg)}{\omega(x, g, g^{-1}yg)}, \tag{107}$$

which, however, is not a 2-cocycle nor a conjugated one.

For a general state with a nontrivial pregauge structure, we give a conjecture for the expression of symmetry branch lines. We can follow the similar idea as in the previous section to introduce the branch line operator $\mathcal{B}^x_{\partial\mathcal{R}}$ from the symmetry action in the region $\mathcal{R}$, when all the plaquettes on $\partial\mathcal{R}$ are fluxless (i.e., $\prod_{e\in\partial p} h_e = 1$), and the flux on each plaquette $p \in \mathcal{R}$ ($\prod_{e\in\partial p} h_e$) is in the centralizer group $\mathcal{Z}_x$; see Appendix D for details. If we start from vertex v and go along $\partial\mathcal{R}$, then the holonomy, i.e., the product of group elements on all the edges along the path, is $g_v \prod_e h_e g_v^{-1}$, and the resulting symmetry branch line is

$$\mathcal{B}^x_{\partial\mathcal{R}} = \sum_g \mathcal{B}^{x,g}_{\partial\mathcal{R}} \epsilon_{g_v x g_v^{-1}}(g), \tag{108}$$

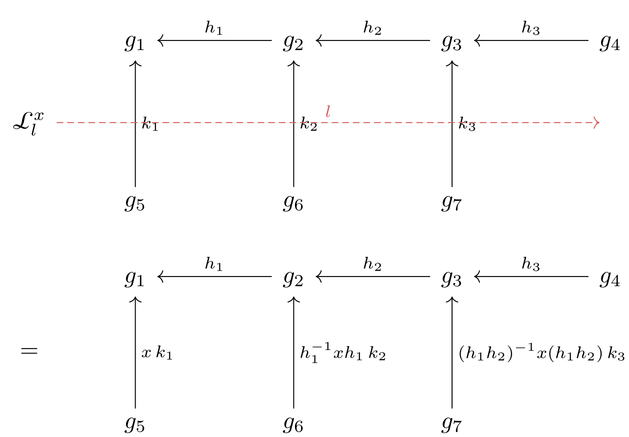
with the operator on the r.h.s. being

$$\mathcal{B}^{x,g}_{\partial\mathcal{R}} \equiv \sum_{g_v} \mathcal{L}^x_{\partial\mathcal{R}} W^{g_v x g_v^{-1}}_{\partial\mathcal{R}} \Theta^{g_v x g_v^{-1}}_{\partial\mathcal{R}} \delta_{g, g_v(\prod_e h_e) g_v^{-1}} |g_v\rangle_v\langle g_v|, \tag{109}$$

where $\epsilon_x(g)$ is a 1-cochain defined in Eq. (106), and v is a reference vertex. The phase $W^{g_v x g_v^{-1}}_{\partial\mathcal{R}}$ is the product of cocycles associated to the tetrahedrons in Fig. 14(b), with $h_{v'v} = x$ and $g_{v'} = g_v$, namely,

$$W^{g_v x g_v^{-1}}_{\partial\mathcal{R}} = \prod_{\text{tetra}\in\partial\mathcal{R}} \omega(\text{tetra})^s. \tag{110}$$

The operator $\Theta^{g_v x g_v^{-1}}_{\partial\mathcal{R}}$ is defined in Eq. (D15), and $\mathcal{L}^x_{\partial\mathcal{R}}$ is defined as follows:



(111)

To make $\mathcal{L}^x_{\partial\mathcal{R}}$ well-defined, we require that x and $\prod_e h_e$ commute. For an SPT state, we have that $\prod_e h_e = 1$, and this is trivially satisfied. With the above discussions, we can derive the multiplication of branch line operators (see Appendix D for the proof),

$$\mathcal{B}^{x,g}_{\partial\mathcal{R}} \mathcal{B}^{y,g'}_{\partial\mathcal{R}} = \mathcal{B}^{xy,g}_{\partial\mathcal{R}} \gamma_g\left(g_v x g_v^{-1}, g_v y g_v^{-1}\right) \delta_{g,g'}. \tag{112}$$

We also note that given a 3-cocycle ω, the phase factor ϵ is not unique. One can always replace ϵ_x by $\epsilon_x v_x$, where v_x satisfies $\tilde{\delta} v_x(g, h) \equiv 1$. This corresponds to a different choice of 0_x in $\mathcal{C}_x$, and we will illustrate this with concrete examples below.

We recall the gauging map in the presence of the pregauge structure,

$$\Gamma: \ |\{g_i\}\rangle_v |h_{ij}\rangle_e \to \left|\left\{g_i h_{ij} g_j^{-1}\right\}\right\rangle_e. \tag{113}$$

Under this, the operator $\mathcal{B}^x_{\partial\mathcal{R}}$ is mapped to

$$\Gamma\left(\mathcal{B}^x_{\partial\mathcal{R}}\right) = \frac{1}{|G|}\sum_{k\in G}\mathcal{L}^{kxk^{-1}}_{\partial\mathcal{R}} W^{kxk^{-1}}_{\partial\mathcal{R}} \Theta^{kxk^{-1}}_{\partial\mathcal{R}} \epsilon_{kxk^{-1}}(kgk^{-1}). \quad (114)$$

Notice that after gauging, the g_v dependence of the operator is summed over as in the sum of $k \in G$ above. Therefore, the resultant operator under the gauging map does not depend on any reference vertex.

In the case when $\omega \equiv 1$, we can choose $\epsilon \equiv 1$, then the operator $\Gamma(\mathcal{B}^x_{\partial\mathcal{R}})$ is reduced to the trace of the ribbon operator that creates an x-fluxed anyon in the quantum double (see, e.g., Ref. [60]),

$$F^{C_x,\mathbf{1}}_{\partial\mathcal{R}} = \frac{1}{|\mathcal{Z}_x|}\sum_{a\in\mathcal{Z}_x} F^{c_i,b_iab_j^{-1}}_{\partial\mathcal{R}}, \quad (115)$$

where on the left-hand side, C_x is the conjugacy class of x, and $\mathbf{1}$ is the trivial representation of the centralizer group $\mathcal{Z}_x$. To construct the operators $F^{C_x,\mathbf{1}}_{\partial\mathcal{R}}$, one enumerates the elements of the conjugacy class as $C_x = \{c_i\}$, together with a suitable subset $\{b_i\}_{i=1}^{|C_x|} \subset G$ such that $c_i = b_i x b_i^{-1}$. The operator $F^{C_x,\mathbf{1}}_{\partial\mathcal{R}}$ on the left-hand side is a ribbon operator labeled by topological charges, and $F^{c_i,b_inb_j^{-1}}_{\partial\mathcal{R}}$ on the right-hand side is a ribbon operator in a basis labeled by group elements.

For a generic 3-cocycle ω, when x is in the center of G, the x-fluxed anyon is Abelian. Then we have the following relation via the map Γ,

$$\Gamma\left(\mathcal{B}^{x,g}_{\partial\mathcal{R}}\right) = F^{x,g}_{\partial\mathcal{R}}, \quad (116)$$

where $F^{x,g}_{\partial\mathcal{R}}$ is a ribbon operator defined in TQD for Abelian groups in Ref. [49].

The algebra of ribbon operators can be inferred from the quasi-Hopf algebra [59,61]. To be more concrete, suppose we insert an x-flux on ribbon l. Operator $F^{x,g}_l$ thus satisfies the multiplication rule

$$F^{x,g}_l F^{y,g'}_l = F^{xy,g}_l \delta_{g,g'}\gamma_g(x,y), \quad (117)$$

which is consistent with our result of the branch line multiplication in Eq. (112), since we expect the gauging map to preserve the operator algebra, which is quasi-Hopf in this case.

One thing to notice is that, to write down $\mathcal{B}^x_{\partial\mathcal{R}}$, we have assumed the existence of ϵ. This is not always possible. When such ϵ does not exist, even when x is in the center of G, the x-fluxed anyon can still be non-Abelian [59]. For example, if ω is a type-3 cocycle of $Z_2 \times Z_2 \times Z_2$, then the anyons in the TQD are generally non-Abelian. Therefore, one could not expect to write down ribbon operators as we have defined above. However, when we gauge one of the Z_2 groups from the SPT, we would enter an SET order with global symmetry $Z_2 \times Z_2$. It turns out that we can try to write the branch line operators in the SET order, and from the algebra of which, one can infer the symmetry fractionalization patterns. We will leave this for further discussion later in this paper.

Throughout this paper, we have mostly used branch lines on closed curves. As we have seen earlier, for closed branch line operators, we need to specify a reference vertex v. Imagine if we could define branch line operators on an open ribbon (see Fig. 16) that starts from vertex v_1 and ends at vertex v_n, we have two natural reference vertices. In the case when G is Abelian, the gauging map will take such operators to ribbon operators defined in Ref. [62]. If we assume that the multiplication rule stays the same, then we have

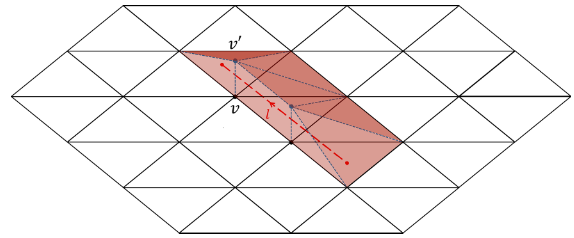


FIG. 16. An open defect on a triangulated lattice.

$$\mathcal{B}^x_l\mathcal{B}^{x^{-1}}_l = \sum_{g_1,g_n}\beta_{g_nxg_n^{-1}}\left(g_ng_1^{-1}\right)|g_1,g_n\rangle_{v_1,v_n}\langle g_1,g_n|, \quad (118)$$

where the factor

$$\beta_x(g) \equiv \epsilon_x(g)\epsilon_{x^{-1}}(g)\gamma_g(x,x^{-1}) \quad (119)$$

is the only object that is "pumped" out when we apply the anomalous nononsite transformation and its inverse accordingly. One can check from the cocycle condition (see also Ref. [61]) that

$$\frac{\beta_{g^{-1}xg}(h)\beta_x(g)}{\beta_x(gh)} = 1, \quad (120)$$

showing that $\beta_x(g)$ satisfies the "twisted" cocycle condition. One can thus use this to write

$$\beta_{g_nxg_n^{-1}}\left(g_ng_1^{-1}\right) = \beta_{g_nxg_n^{-1}}(g_n)\beta_{g_1xg_1^{-1}}(g_1)^{-1}. \quad (121)$$

If v is one of the endpoints of the defect operators, then we essentially pump a factor $\beta_{g_vxg_v^{-1}}(g_v)$ to the state. The phase factor $\beta_x(g)$ is a conjugated 1-cocycle, i.e., $\beta_x(g) \in H^1(G, U(1)[G]) \simeq \oplus_i H^1(\mathcal{Z}_i, U(1))$, where i labels conjugacy classes of G, and $\mathcal{Z}_i$ is the corresponding centralizer group. We will call this factor the SPT pumping factor, since, by fusing defect operators, we pump a lower dimensional SPT state (in this case a 0d SPT state) on the boundary of the line l after the application of $\mathcal{B}^x_l\mathcal{B}^{x^{-1}}_l$. This is a generalization of a previous Abelian case analyzed in Ref. [63]. If G is Abelian, then for every element $x \in G$, the factor $\beta_x(g)$ is a 1D representation of group G, i.e., one pumps a 0d G-SPT state on the endpoints of an open ribbon.

C. Symmetry branch lines in SET phase

As discussed in previous sections, we can gauge a normal subgroup N of G. Then we enter an N SET phase with global symmetry $Q = G/N$. Any group element g in G can be decomposed as $g = q(g)n(g)$. The SET wave function is

$$|\Psi_{\mathrm{SET}}\rangle = \sum_{\{q_v\},\{n_e\}}\Omega\left(\{q_in_eq_j^{-1}\}\right)\bigotimes_v|q_v\rangle\bigotimes_e|n_e\rangle. \quad (122)$$

When the group element $x \in G$ commutes with all the elements $n \in N$ in the normal subgroup, a branch line operator $\mathcal{B}^x_{\partial\mathcal{R}}$ we introduced for the G-SPT state will be mapped to another operator $H^x_{\partial\mathcal{R}}$ under the gauging map Γ_N. Further, the operators defined as such respect the same multiplication rules as $\mathcal{B}^x_{\partial\mathcal{R}}$ do. Denoting $q_v = q(g_v)$, the operator $H^x_{\partial\mathcal{R}}$ is written as

$$H^x_{\partial\mathcal{R}} = \sum_g H^{x,g}_{\partial\mathcal{R}} \epsilon_{q_v x q_v^{-1}}(g) \delta_{g, q_v(\prod_e n_e) q_v^{-1}}, \tag{123}$$

$$\text{with } H^{x,g}_{\partial\mathcal{R}} = \mathcal{L}^x_{\partial\mathcal{R}} W^{q_v x q_v^{-1}}_{\partial\mathcal{R}} \Theta^{q_v x q_v^{-1}}_{\partial\mathcal{R}} |q_v\rangle\langle q_v|. \tag{124}$$

When $x \in N$, the operator is a ribbon operator creating a gauge flux in the SET. When $n(x) = 1$, i.e., $x = s(q)$ for some element $q \in Q$, the operator creates a flux that corresponds to an element in the global symmetry group and thus is a branch line operator. The multiplication rule of such operators is

$$H^{x,g}_{\partial\mathcal{R}} H^{y,g'}_{\partial\mathcal{R}} = H^{xy,g}_{\partial\mathcal{R}} \gamma_g\left(q_v x q_v^{-1}, q_v y q_v^{-1}\right) \delta_{g,g'}, \tag{125}$$

which, in turn, leads to

$$H^x_{\partial\mathcal{R}} H^y_{\partial\mathcal{R}} = \sum_g H^{xy,g}_{\partial\mathcal{R}} \beta_{q_v x q_v^{-1}, q_v y q_v^{-1}}(g) \delta_{g, q_v(\prod_e n_e) q_v^{-1}}, \tag{126}$$

where the phase factor β on the r.h.s. is

$$\beta_{x,y}(g) = \epsilon_x(g)\epsilon_y(g)\gamma_g(x, y). \tag{127}$$

When we take two branch line operators $H^{x,g}_l$ and $H^{y,g}_l$, i.e., both $n(x) = n(y) = 1$, and multiply them together, then the resulting $H^{xy,g}_l$ is not necessarily a branch line operator, because $n(xy)$ is not always trivial. As we will see in more details later, this indicates a nontrivial symmetry fractionalization class (SFC) of the SET order.

We can always apply a finite-depth local unitary to take all the vertex DOFs to the identity element (see Appendix E), such that the state becomes

$$|\Psi\rangle = \sum_{\{n_e\}} \Omega(\{n_e\}) |\{n_e\}\rangle_e \otimes_v |1\rangle_v. \tag{128}$$

This is a TQD with the 3-cocycle $\nu(n_1, n_2, n_3)$ being the restriction of $\omega(g_1, g_2, g_3)$ on subgroup N. An Abelian anyon in this model is determined by its flux (i.e., conjugacy class C_a) and charge (i.e., a conjugated 1-cochain μ_a such that $\tilde{\delta}\mu_a = \theta_a|_N$, i.e., $\mu_a = \epsilon_a|_N$). The symmetry action on the anyons are given by

$$\rho^x : a\text{-flux} \rightarrow x^{-1}ax\text{-flux}, \tag{129}$$

$$\rho^x : \mu_a \rightarrow \mu'_{x^{-1}ax}. \tag{130}$$

From our previous analysis, we know that the automorphism ρ^x maps a-fluxed anyon to $x^{-1}ax$-fluxed anyon, thus we have Eq. (129). Furthermore, the automorphism will map a chargeon [i.e., a representation $\mu \in \text{Rep}(N)$] to μ', where $\mu'(x^{-1}ax) = \mu(a)$. Therefore, we can infer the general map of the anyon charge as

$$\mu'_{x^{-1}ax}(x^{-1}nx) = \frac{\theta_a(x, x^{-1}nx)}{\theta_a(n, x)} \mu_a(n). \tag{131}$$

However, the symmetry branch line operators we introduced above for the SET states assumed that x commutes with all the elements in normal subgroup N. Therefore, the results obtained from analyzing such operators are limited to the case when the symmetry does not change the anyon type. To determine the SFC in this case, without loss of generality, we will suppose that the global symmetry is Z_2 in most of the cases from now on. (Examples beyond Z_2 will be discussed later.) We first pick one Abelian anyon in $\mathcal{C}_x$, i.e., we pick one ϵ_x cochain such that the corresponding branch line operator creates 0_x. Then the SFC should be determined by the fusion rule $0_x \cdot 0_x = \omega_{\mathbb{1}}$.

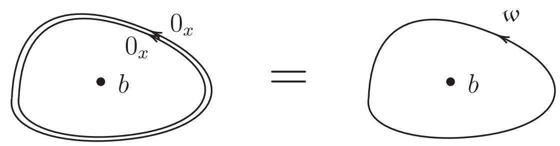


FIG. 17. The operator $(H^x_{\partial\mathcal{R}})^2$ applying on curve $\partial\mathcal{R}$ is equivalent to the braiding phase between the g-flux of an anyon b (g is the holonomy along $\partial\mathcal{R}$), and the charge of the anyon ω.

As shown in Fig. 17, the operator $(H^x_{\partial\mathcal{R}})^2$, when applied on the DOFs on curve $\partial\mathcal{R}$, is to create two point defects 0_x, move them along $\partial\mathcal{R}$, and finally make them cancel with each other. This process will introduce a phase factor that corresponds exactly to the braiding phase between the g-flux of an anyon b (g is the holonomy along $\partial\mathcal{R}$), and the charge of the anyon ω. According to Eq. (126), this phase factor is $\beta_{x,x}(g)$. Notice that, when the symmetry group is $Q = G/N = Z_2$, the q_v dependence in Eq. (126) disappears. In general, such as in more than two-step gauging, this factor still depends on q_v. Since the flux of ω is x^2 (which is not necessarily the identity element, as x is an embedding of the generator of $Q = Z_2$ into G), suppose the charge of anyon b is given by a 1-conjugated-cochain μ_g (such that $\tilde{\delta}\mu_g = \theta_g|_N$), then the braiding phase between the x^2-flux and μ_g charge is $\mu_g(x^2)$. To determine ω, we write down the braiding phase between anyon ω and b, which is the product of the above two factors,

$$B(\omega, b) = \beta_{x,x}(g)\mu_g(x^2), \tag{132}$$

and we illustrate this relation in Fig. 18.

From this result, one can determine the SFC of the SET state after gauging some normal subgroup. Since we can always attach a 1-conjugated-cocycle υ_x to ϵ_x such that $\epsilon'_x = \epsilon_x \upsilon_x$, the phase $B(\omega, b)$ we derived above also has this ambiguity. But it just corresponds to the freedom to choose any Abelian anyon in $\mathcal{C}_x$ that is labeled as 0_x. Furthermore, we can always attach a coboundary to the 3-cocycle ω such that $\omega' = \omega\,\delta\alpha$, where α is a 2-cochain. One can check that the

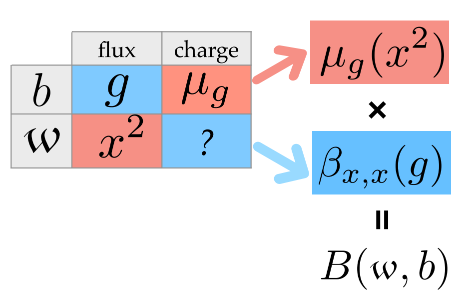


FIG. 18. The braiding phase between a g-fluxed μ_g-charged anyon b and the anyon $\omega(= 0_x \times 0_x)$ is written as a product of two braiding phases, as presented in Eq. (132). The second factor in blue is a consequence of Eq. (126). From the phase $B(\omega, b)$, we infer the unknown charge of the anyon ω (written as "?"), which determines the SFC.

phase $B(w, b)$ is always the same as long as we choose ω in the same cohomology class.

In the next few subsections, we will use the above result to determine the SFC in different SET phases, resulting from gauging a normal subgroup N of an SPT phase. In the later subsections, we will deal with the case when symmetry does change the anyon type and conjecture the form of branch line operators to use them to discuss the SFC of the SET orders that we obtain from gauging Dihedral SPT states.

D. SETs from partially gauging $Z_2 \times Z_2$ SPT

The third cohomology of $Z_2 \times Z_2$ has three generators, two of which are of type-1, and the other one is of type-2 [59]. Assume the two generators of the $Z_2 \times Z_2$ group are t and x, then we can denote any group elements as $g = (g^{(1)}, g^{(2)}) \equiv t^{g^{(1)}} x^{g^{(2)}}$, where $g^{(1)}, g^{(2)} \in \{0, 1\}$. The representative of the 3-cocycle is then

$$\omega_{g,h,l} = e^{\pi i(k_1 g^{(1)}h^{(1)}l^{(1)} + k_2 g^{(2)}h^{(2)}l^{(2)} + k_3 g^{(1)}h^{(2)}l^{(2)})}, \tag{133}$$

where $k_1, k_2, k_3 = 0, 1$.

In the SET phase obtained from gauging the first Z_2 group, the anyon theory is the same as that of a TQD $D^\nu(Z_2)$, where

$$\nu(g, h, l) = e^{\pi i k_1 g^{(1)} h^{(1)} l^{(1)}} \tag{134}$$

is a representative in $H^3(Z_2, U(1))$ obtained by the restriction of ω in the first Z_2 group (namely, one restricts the cocycles to those with $g^{(2)} = h^{(2)} = l^{(2)} = 0$) and is used as the "twisting" of Kitaev's Z_2 QD model.

Therefore, when $k_1 = 0$ it is a toric code model, and when $k_1 = 1$ it is a double-semion model. Since $Z_2 \times Z_2$ is the trivial central extension of Z_2 by Z_2, the symmetry action on anyons is trivial.

The second Z_2 group $\{1, x\}$ represents the global symmetry and therefore we can consider the multiplication of two branch line operators $H^x_{\partial\mathfrak{R}}$, which according to Eq. (126), gives

$$H^x_{\partial\mathfrak{R}} H^x_{\partial\mathfrak{R}} = \sum_g \beta_{x,x}(g)\delta_{g,\prod_e t_e}. \tag{135}$$

From Eqs. (127) and (132), we find the braiding phase between w and a t-fluxed anyon b,

$$B(w, b) = \epsilon_x(g)^2\gamma_g(x, x) = \theta_x(g, g)\gamma_g(x, x) = (-1)^{k_3 g^{(1)}}. \tag{136}$$

For the conjugated 1-cochain we have used $\mu_g(x^2) = \mu_g(\mathbb{1}) = 1$. Notice that since $g^2 = \mathbb{1}$ for $g \in G$, we have $\theta_x(g, g) = \epsilon_x(g)^2/\epsilon_x(g^2)$ by definition but $\epsilon_x(g^2) = \epsilon_x(\mathbb{1}) = 1$ so the second equality above follows. Therefore, different choices of t-fluxed b anyon and different choices of ϵ_x cochain (i.e., different choices of 0_x) give rise to the same braiding phase.

Now we discuss the consequence of the resultant braiding $B(w, b)$ in different cases. As mentioned above, when $k_1 = 0$, we have a toric code model. From our previous general analysis in Sec. VI C, we can infer that the anyon w braiding with b (m or em) gives rise to a phase $(-1)^{k_3}$. Therefore,

$$0_x \times 0_x = 0 \quad \text{or} \quad e, k_3 = 0 \quad \text{or} \quad 1. \tag{137}$$

As mentioned earlier, when $k_1 = 1$, we have a double-semion model. The fluxless anyon w braiding with b (s or $\bar{s}$) results in a phase $(-1)^{k_3}$. Therefore,

$$0_x \times 0_x = 0 \quad \text{or} \quad s\bar{s}, \quad k_3 = 0 \quad \text{or} \quad 1. \tag{138}$$

The discussion of k_1 and k_3 above completely specifies the SFC of the SET in this case. We have not discussed the consequence of k_2, but if we further gauge the second Z_2, different values of k_i will give rise to different topological orders, due to the 1-to-1 correspondence between the SPT and TQD phases [33,52]. Therefore, we know that the intermediate SETs with different values of k_2 must belong to different phases. Since all the topological order parameters of SET, except the SDC, are already fixed by k_1 and k_3, we can safely conclude that $k_2 = 0, 1$ corresponds to two defectification classes, respectively. Different defectifications intuitively can be regarded as stacking or gluing different SPT phases [51] to the SET. This particular case of SET phase was previously discussed in Ref. [49].

If we further gauge the global symmetry Z_2 in the SET, then it becomes a twisted quantum double $D^\omega(Z_2 \times Z_2)$. As we discussed earlier in Sec. VI B, for Abelian groups, the symmetry branch line operators will be mapped to ribbon operators creating certain Abelian anyons after gauging. Indeed, from the Slant product, the 1-conjugated cochain could be chosen as $\epsilon_{t_2}(g) = i^{k_2 g^{(2)}}$. The operator $H^x_{\partial\mathfrak{R}}$ is mapped to a ribbon operator creating a t_2-flux anyon η,

$$F^\eta_l = F^{t_2,1}_l + i^{k_2}F^{t_2,t_2}_l + F^{t_2,t_1}_l + i^{k_2}F^{t_2,t_1t_2}_l. \tag{139}$$

Since the gauge group is $Z_2 \times Z_2$, the fusing of two such ribbon operators becomes a ribbon operator exciting a flux-less anyon (chargeon) a, $(F^\eta_l)^2 = F^a_l$, similar to Eq. (135),

$$F^a_l = F^{1,1}_l + F^{1,t_2}_l + (-1)^{k_3}F^{1,t_1}_l + (-1)^{k_3}F^{1,t_1t_2}_l. \tag{140}$$

For example, when $k_1 = k_2 = k_3 = 0$, the anyon η is just a boson m in toric-code model, and the anyon a is the vacuum anyon. For any values of the parameters, we will see that the multiplication rules of branch line operators become the fusion rules of anyons under the gauging map.

E. SETs from partially gauging Z_4 SPT

We will use both multiplicative and additive representations of Abelian groups interchangeably, e.g., $g = 2$ means $g = x^2$ in multiplicative representation (for x being the generator of Z_4). We take representative cocycles in $H^3(Z_4, U(1)) = Z_4$ as

$$\omega_{g,h,l} = \exp\left\{\frac{2\pi i p}{16} g(h + l - [h + l]_4)\right\}, \tag{141}$$

where $p = 0, 1, 2, 3$. The slant product $\theta_{x^k}(g, h) = \exp\{\frac{2\pi i p}{16} k(g + h - [g + h]_4)\}$ corresponds to a projective representation given by $\epsilon_{x^k}(g) = \xi^{kg}$, where $\xi \equiv \exp\{\frac{2\pi i p}{16}\}$.

An SET phase can be obtained by gauging the normal $Z_2 = \{1, t\}$ group, where $t \equiv x^2$. By restricting ω in $H^3(Z_2, U(1))$, we have $\nu(g, h, l) = e^{\pi i p g h l}$, where now $g, h, l = 0, 1$ are Z_2-valued. Therefore, when $p = 0$ or 2, it is a toric code, and when $p = 1$ or 3, it is a double-semion model. Since Z_4 is a central extension of Z_2 by Z_2, the symmetry action on anyons is trivial.

Let us recall that the branch line operators in the SET ground state are

$$H^x_{\partial\mathcal{R}} \equiv \sum_g \epsilon_x(g) H^{x,g}_{\partial\mathcal{R}} = \sum_g \xi^g H^{x,g}_{\partial\mathcal{R}}. \quad (142)$$

The product of two such branch line operators $H^x_{\partial\mathcal{R}}$ gives rise to a factor [see Eq. (127)]

$$\beta_{x,x}(g) = \epsilon_x(g)^2 \gamma_g(x,x) = e^{\frac{2\pi i p g}{8}}, \quad (143)$$

for $g = 0, 2$. The charge of g-fluxed anyon b is given by

$$\mu_g(h) = e^{\frac{2\pi i p g h}{16} + \frac{2\pi i r_g h}{4}}, \quad (144)$$

for $g, h \in \{0, 2\} = s(Q)$. Furthermore, $r_g = 0, 1$ corresponds to different choices of charges of anyon b. Thus, the braiding phase between anyon w and b should be given as

$$B(w, b) = \beta_{x,x}(g)\mu_g(t) = e^{\frac{2\pi i p g}{4}}(-1)^{r_g}. \quad (145)$$

When $p = 0$, we have a toric code model. When $g = 0$, $B(w, b) = (-1)^{r_0}$ where charge of anyon b is given by $\mu_0(t) = (-1)^{r_0}$. There are two chargeons 0 and e, corresponding to $r_0 = 0$ or 1, respectively. Therefore, the braiding phase between w and 0 (e) is 1 (-1), according to Eq. (145). Moreover, when $g = 2$, the braiding $B(w, b) = (-1)^{r_2}$ where the charge of b is given by $\mu_2(h = t) = (-1)^{r_2}$ according to Eq. (144). Therefore, we could say that the braiding phase between w and m (em) is 1 (-1), which corresponds to $r_2 = 0, 1$, respectively. Therefore, we have a toric code with the following SFC:

$$0_x \times 0_x = m. \quad (146)$$

When $p = 1$, this is a double-semion model. The braiding phase is $B(w, b) = (-1)^{g/2 + r_g}$ where the charge of the g-fluxed anyon b is given by $\mu_g(h = t) = e^{\frac{\pi i}{2}\cdot\frac{g}{2} + \pi i r_g}$ according to Eq. (144). When $g = 0$, two chargeons 0 and $s\bar{s}$ correspond to $r_0 = 0$ and 1, respectively. Anyon w braiding with $s\bar{s}$ gives -1. When $g = 2$, the braiding phase between w and s ($\bar{s}$) is -1 (1), which corresponds to $r_2 = 0, 1$, respectively. Therefore, we have a double-semion model with SFC:

$$0_x \times 0_x = s. \quad (147)$$

When $p = 2$, this is a toric code model. The braiding phase is $B(w, b) = (-1)^{r_g}$ where the charge of the g-fluxed anyon b is given by $\mu_g(h = t) = e^{\pi i(\frac{g}{2} + r_g)}$ according to Eq. (144). When $g = 0$, two chargeons 0 and e correspond to $r_0 = 0$ and 1, respectively. Anyon w braiding with 0 (e) gives 1 (-1). When $g = 2$, the braiding phase between w and em (m) is 1 (-1), which corresponds to $r_2 = 0, 1$, respectively. Therefore, we have a toric code with SFC:

$$0_x \times 0_x = em. \quad (148)$$

When $p = 3$, this is a double-semion model. The braiding phase is $B(w, b) = (-1)^{g/2 + r_g}$, where the charge of the g-fluxed anyon b is given by $\mu_g(h = t) = e^{\frac{3\pi i}{2}\cdot\frac{g}{2} + \pi i r_g}$ according to Eq. (144). When $g = 0$, two chargeons 0 and $s\bar{s}$ correspond to $r_0 = 0$ and 1, respectively. Anyon w braiding with $s\bar{s}$ gives -1. When $g = 2$, the braiding phase between w and $\bar{s}$ (s) is -1 (1), which corresponds to $r_2 = 0, 1$, respectively. Therefore, we have a double-semion code with SFC:

$$0_x \times 0_x = \bar{s}. \quad (149)$$

One could check that, if we choose other ϵ_x instead of what we used above, then we would derive exactly the same fusion rule as above.

F. SETs from partially gauging Z_2^3 SPT

The third cohomology group of $Z_2^{(1)} \times Z_2^{(2)} \times Z_2^{(3)}$ has seven generators, three of which are of type-1, three of which are of type-2, and one of type-3 [59]. Assume the three generators of the Z_2^3 group are t, x_1, and x_2, then we can denote any group elements as $g = (g^{(1)}, g^{(2)}, g^{(3)}) \equiv t^{g^{(1)}} x_1^{g^{(2)}} x_2^{g^{(3)}}$, where $g^{(1)}, g^{(2)}, g^{(3)} \in \{0, 1\}$. For simplicity, in this section, we will demonstrate the analysis for representatives of some of the 3-cocycles and then derive the general result without further explanation. The representatives that we take are

$$\omega_{g,h,l} = e^{\pi i(k_1 g^{(1)} h^{(1)} l^{(1)} + k_2 g^{(1)} h^{(2)} l^{(3)})}, \quad (150)$$

where $k_1, k_2 = 0, 1$.

In the SET phase obtained from gauging the group $Z_2^{(1)}$, the anyon theory is the same as that of a TQD $D^\nu(Z_2)$, where

$$\nu(g, h, l) = e^{\pi i k_1 g^{(1)} h^{(1)} l^{(1)}} \quad (151)$$

is a representative in $H^3(Z_2, U(1))$ obtained by the restriction of ω in the first Z_2 group. Therefore, when $k_1 = 0$ it is a toric code model, and when $k_1 = 1$ it is a double-semion model. Since $Z_2^{(1)} \times Z_2^{(2)} \times Z_2^{(3)}$ is the trivial central extension of $Z_2^{(2)} \times Z_2^{(3)}$ by $Z_2^{(1)}$, the symmetry actions on anyons are trivial.

Since the slant product of the cocycle given above belongs to a class $[\theta]$ that is not the trivial element in $H^2(Z_2^3, U(1)[Z_2^3])$, it is impossible to find $\epsilon_h(g)$, such that

$$\theta_h(k, g) = \tilde{\delta}\epsilon_h(k, g) \equiv \frac{\epsilon_h(g)\epsilon_h(k)}{\epsilon_h(kg)}, \quad (152)$$

for any $g, h, k \in Z_2^3$. However, in defining the symmetry branch line operators, we only need phase factors $\epsilon_h(g)$ where the group element $h \in Z_2^{(2)} \times Z_2^{(3)}$ and $g \in N = Z_2^{(1)}$. Indeed, in this case there exists such a phase factor that satisfies Eq. (152) when restricting the group elements h, g in their corresponding subgroups.

The slant product of the cocycle is trivial,

$$\theta_h(k, g) = (-1)^{h^{(2)} k^{(1)} g^{(3)} + h^{(3)} k^{(1)} g^{(2)}} = 1, \quad (153)$$

when $h \in Z_2^{(2)} \times Z_2^{(3)}$ and $k, g \in Z_2^{(1)}$. Therefore, we can choose $\epsilon_h(g) \equiv 1$.

In general, when the symmetry group is $Q = Z_2 \times Z_2$, we take two elements $h_1, h_2 \in s(Q) \subset G$ that are the embedding of elements $\tilde{h}_1, \tilde{h}_2 \in Q$. The consistency condition of embedding is

$$q(h_1 h_2) = s(\tilde{h}_1 \tilde{h}_2). \quad (154)$$

The fusion rule of $\mathcal{C}_Q^\times$ is of the form

$$0_{\tilde{h}_1} \times 0_{\tilde{h}_2} = w(\tilde{h}_1, \tilde{h}_2) \times 0_{\tilde{h}_1\tilde{h}_2}. \quad (155)$$

From our previous analysis, the braiding phase between $0_{\tilde{h}_1} \times 0_{\tilde{h}_2}$ and anyon $b = (g, \mu_g)$ is $B_1 = \beta_{h_1,h_2}(g)\mu_g(n(h_1 h_2))$. According to Eq. (154), the braiding phase between $0_{\tilde{h}_1\tilde{h}_2}$ and anyon b is $B_2 = \epsilon_{q(h_1 h_2)}(g)$. As a result, the braiding phase between Abelian anyon $w(\tilde{h}_1,\tilde{h}_2)$ and b should be the ratio

$$B(w(\tilde{h}_1,\tilde{h}_2),b)=\frac{\epsilon_{h_1}(g)\epsilon_{h_2}(g)}{\epsilon_{q(h_1h_2)}(g)}\gamma_g(h_1,h_2)\mu_g(n(h_1h_2)). \quad (156)$$

Later on for simplicity, we will use $w(h_1,h_2)$ to denote $w(\tilde{h}_1,\tilde{h}_2)$. Since we choose $\epsilon_h(g)\equiv 1$ in this case and $n(h_1h_2)\equiv 1$, we have

$$B(w(h_1,h_2),b)=\gamma_g(h_1,h_2)=e^{\pi i k_2 g h_1^{(2)}h_2^{(3)}}, \quad (157)$$

for $g\in Z_2^{(1)}$. Since the group extension of $Z_2^{(2)}\times Z_2^{(3)}$ by $Z_2^{(1)}$ corresponds to the trivial element in $H^2(Z_2^{(2)}\times Z_2^{(3)},Z_2^{(1)})$, we know that the Abelian anyon $w(h_1,h_2)$ is always a chargeon for any $h_1,h_2\in Z_2^{(2)}\times Z_2^{(3)}$. When $k_1=0$, we have a Z_2 toric code model. From the above braiding phase we can conclude that when $k_2=0$,

$$w(h_1,h_2)\equiv 1; \quad (158)$$

when $k_2=1$,

$$w(h_1,h_2)=\begin{cases}e, & h_1^{(2)}=h_2^{(3)}=1,\\ 1, & \text{others}.\end{cases} \quad (159)$$

However, when $k_1=1$, we have a Z_2 double-semion model. From the above braiding phase we can conclude that when $k_2=0$,

$$w(h_1,h_2)\equiv 1; \quad (160)$$

when $k_2=1$,

$$w(h_1,h_2)=\begin{cases}s\bar{s}, & h_1^{(2)}=h_2^{(3)}=1,\\ 1, & \text{others}.\end{cases} \quad (161)$$

One can check that all the Abelian anyons $w(h_1,h_2)$'s above satisfy the cocycle condition,

$$\frac{w(h_2,h_3)w(h_1,h_2h_3)}{w(h_1h_2,h_3)w(h_1,h_2)}=1. \quad (162)$$

If one chooses different $\epsilon_h(g)$ other than what we used above, then the derived anyon $w(h_1,h_2)$ will be differed by a coboundary. Therefore, we conclude, different values of k_2 will give different symmetry fractionalization patterns that correspond to different elements in $H^2(Z_2^{(2)}\times Z_2^{(3)},\mathcal{A})$, where $\mathcal{A}=Z_2\times Z_2$ is the group of Abelian anyons.

One can generalize the above result to an arbitrary 3-cocycle. The cohomology group of Z_2^3 can be decomposed as such:

$$\begin{aligned} H^3(Z_2^3,U(1)) \;=\; & \underset{\text{anyon theory}}{\overset{\text{type-1 of } Z_2^{(1)}}{Z_2\uparrow}} \times \underset{\text{SDC}}{\overset{\text{type-1 and 2 of } Z_2^{(2)}\times Z_2^{(3)}}{Z_2^3\uparrow}} \times \underset{\text{SFC}_1}{\overset{\text{type-2 of } Z_2^{(1)}\times Z_2^{(2)}}{Z_2\uparrow}} \\ & \times \underset{\text{SFC}_1}{\overset{\text{type-2 of } Z_2^{(1)}\times Z_2^{(3)}}{Z_2\uparrow}} \times \underset{\text{SFC}_2}{\overset{\text{type-3 of } Z_2^{(1)}\times Z_2^{(2)}\times Z_2^{(3)}}{Z_2\uparrow}}\ . \end{aligned} \quad (163)$$

In this example, we have illustrated two of the seven generators in $H^3(Z_2^3,U(1))$ as in Eq. (150) and showed that k_1 gives the anyon theory and k_2 (which is associated with the type-3 cocycle) gives a symmetry fractionalization pattern named SFC_2 in the above diagram. To understand the rest of SET properties, we note that the two SFC_1's are the symmetry fractionalization pattern associated with type-2 cocycles of $Z_2^{(1)}\times Z_2^{(2)}$ and $Z_2^{(1)}\times Z_2^{(3)}$, respectively, which were already discussed in Sec. VI D. The SDC part is the symmetry defectification class associated with cocycles of $Z_2^{(2)}\times Z_2^{(3)}$, both of type-1 and type-2. In the cases when the Z_2^3-SPT phase corresponds to the cohomology class which is trivial in the first Z_2 subgroup in Eq. (163), one can choose a representative that is of some specific form. Then after gauging $Z_2^{(1)}$ subgroup, according to Ref. [62], one can determine the symmetry fractionalization patterns of the SET order, which agrees with our general results above.

G. SETs from partially gauging D_4 SPT

Now we consider the noncentral extension of Z_2 by Z_4. We write the element in D_4 as $\tilde{g}=(G,g)\equiv x^Ga^g$. We construct a representative of 3-cocycle in $H^3(D_4,U(1))$ as follows:

$$\begin{aligned}\omega(\tilde{g},\tilde{h},\tilde{l}) &= \exp\left\{\frac{2\pi i p_1}{16}g(-1)^{H+L}(h(-1)^L+l-[h(-1)^L+l]_4)\right\}\\ &\quad\times \exp \pi i p_2 GHL+\pi i p_3 gHL,\end{aligned} \quad (164)$$

where $p_1=0,1,2,3$, and $p_2,p_3=0$ or 1. There are four nontrivial Abelian normal subgroups in D_4, which leads to four options in the first step when gauging this group. We will consider three of them here.

1. Gauging Z_2

The first option is to gauge the normal subgroup $Z_2=\{1,a^2\}$, resulting in a state in an SET that has the same anyon theory as $D^\nu(Z_2)$, where

$$\nu(g,h,l)=\exp\left\{\frac{2\pi i p_1}{16}ghl\right\} \quad (165)$$

is the restriction of ω on Z_2, i.e., $g,h,l\in\{1,a^2\}$. When $[p_1]_2=0$ it is a toric code model, and when $[p_1]_2=1$ it is a double-semion model. Since the group extension of $Z_2 \times Z_2$ by Z_2 is central, the symmetry actions on the anyons are trivial. Therefore, according to Eq. (156), the braiding phase $B(\mathfrak{w}(h_1, h_2), b)$ is given by

$$B(\mathfrak{w}(h_1, h_2), b) = \frac{\epsilon_{h_1}(g)\epsilon_{h_2}(g)}{\epsilon_{q(h_1h_2)}(g)} \gamma_g(h_1, h_2)\mu_g(n(h_1h_2)). \quad (166)$$

We write the embedding of quotient group elements

$$g = (g^{(1)}, g^{(2)}) \in Z_2 \times Z_2 \equiv \{1, t_1\} \times \{1, t_2\} \quad (167)$$

in D_4 as

$$s(g) = x^{g^{(1)}} a^{g^{(2)}}. \quad (168)$$

From the group multiplication rule, one can infer that the Abelian anyons $\mathfrak{w}(t_2, t_1)$, $\mathfrak{w}(t_2, t_2)$, $\mathfrak{w}(t_1t_2, t_1)$ and $\mathfrak{w}(t_1t_2, t_2)$ have nontrivial flux, while $\mathfrak{w}(h_1, h_2)$ for other h_1, h_2 are chargeons. We list the detailed symmetry fractionalization patterns below.

When $[p_1]_2 = 0$, we have a Z_2 toric code model. From the above braiding phase we can conclude that the SFC is characterized by $[\mathfrak{w}(h_1, h_2)] \in H^3(Z_2 \times Z_2, Z_2 \times Z_2)$, where

$$\mathfrak{w}(h_1, h_2) = \begin{cases} m, & (h_1, h_2) = (t_2, t_1), (t_2, t_2), \quad (t_1t_2, t_1), (t_1t_2, t_2), \\ 1, & \text{others}. \end{cases} \quad (169)$$

When $[p_1]_2 = 1$, we have a Z_2 double-semion model. From the above braiding phase we can conclude that, the SFC is characterized by $[\mathfrak{w}(h_1, h_2)] \in H^3(Z_2 \times Z_2, Z_2 \times Z_2)$, where

$$\mathfrak{w}(h_1, h_2) = \begin{cases} s, & (h_1, h_2) = (t_2, t_1), (t_1t_2, t_2), (t_2, t_2), \\ \bar{s}, & (h_1, h_2) = (t_1t_2, t_1), \\ s\bar{s}, & (h_1, h_2) = (t_1, t_1), (t_1, t_1t_2), (t_1t_2, t_1t_2), \\ 1, & \text{others}. \end{cases} \quad (170)$$

Other parameters of the cohomology group $H^3(D_4, U(1))$, including $\frac{p_1 - [p_1]_2}{2}$, p_2 and p_3, will give rise to different SDCs that form an $H^3(Z_2 \times Z_2, U(1)) = Z_2^3$ torsor.

2. Gauging Z_4

The second option is to gauge the normal subgroup Z_4, resulting in a state in an SET that has the same anyon theory as $D^\nu(Z_4)$, where

$$\nu(g, h, l) = \exp\left\{\frac{2\pi i p_1}{16} g(h + l - [h + l]_4)\right\} \quad (171)$$

is the restriction of ω on Z_4. Different values of p_1 exactly correspond to different Z_4 TQD models. The symmetry action takes e to e^3, and takes m to $m^3 e^{2p_1}$ according to Eq. (131), which is not a trivial automorphism on $\mathcal{C}$. One can still manage to write a phase factor $B(\mathfrak{w}, b) = \beta_{x,x}(g)$ for $g \in N = Z_4$. (We remind readers that x is one of the generators of the group D_4 such that $x^2 = 1$.) However, two obvious problems will emerge from this factor. The first one is that unlike in the case when symmetry does not change the anyon type, when we change the representative 3-cocycle for the D_4-SPT state by a coboundary, $\omega' = \omega \cdot \delta\alpha$, the "braiding phase" is not invariant anymore, $B(\mathfrak{w}, b)' = B(\mathfrak{w}, b)\frac{\alpha(g^{-1}xg, g^{-1}xg)}{\alpha(x,x)}$. The second problem seems to be even worse. In a generic case, it might be impossible to find an Abelian object in sector $\mathcal{C}_x$ such as the 0_x from before. Therefore, there may not exist an Abelian anyon $\mathfrak{w}$ as the fusion between 0_x and itself. Indeed, in sector $\mathcal{C}_x$, there are 4 objects of quantum dimension 2. If we nonetheless pick one of them and still name it 0_x, by counting the dimension, then we can write a fusion rule of the form

$$0_x \times 0_x = a + b + c + d, \quad (172)$$

where $a, b, c, d \in \mathcal{C}$ are Abelian anyons.

Motivated by the ribbon operator in the quantum double model as in Eq. (115), we choose $b_1 = 1$ and $b_2 = a$ and we write a matrix-valued operator on an open ribbon as

$$\left(H_l^x\right)_{ii'} = \sum_{n \in \{1, a^2\}} H_l^{b_i x b_i^{-1}, b_i n b_{i'}^{-1}} \epsilon_{b_i x b_i^{-1}}\left(b_i n b_{i'}^{-1}\right), \quad (173)$$

where the matrix indices $i, i' = 1, 2$, and the operator $H_l^{x,g}$ satisfies the same multiplication rule as in Eq. (125),

$$H_l^{x,g} H_l^{y,g'} = H_l^{xy,g} \gamma_g(x, y)\delta_{g,g'}. \quad (174)$$

We conjecture that the operator as in Eq. (173) creates an object in sector $\mathcal{C}_x$ on the endpoint of l. We call this object 0_x even though it is of dimension 2. Then the object $0_x \times 0_x$ should be created on the endpoint of l by operator $(H_l^x)^{\otimes 2}$. It can be shown that, when we change the representative 3-cocycle for the D_4-SPT state by a coboundary, $\omega' = \omega \cdot \delta\alpha$, the matrix $(H_l^x)^{\otimes 2}$ differs by a similar transformation. Therefore, the fusion rule remains invariant under different representative choices. According to the detailed analysis in Appendix F, we see that different values of p_3 give different SFCs where the fusion rules are shifted by anyon $[e^2] \in H_\rho^2(Z_2, \mathcal{A})$.

3. Gauging $Z_2 \times Z_2$

One could also gauge the $Z_2 \times Z_2 = \{1, x, t, xt\}$ in D_4, resulting a state in the phase of $D^{\nu'}(Z_2 \times Z_2)$. We write $t \equiv a^2$ and $g = x^{g^{(1)}} t^{g^{(2)}} = x^{g^{(1)}} a^{2g^{(2)}}$. The 3-cocycle ω restricted in this group is obtained from Eq. (164) and is given as

$$\begin{aligned} \nu'(g, h, l) = \exp\Big\{&\frac{2\pi i p_1}{4} g^{(2)}(-1)^{h^{(1)}+l^{(1)}}\left(h^{(2)}(-1)^{l^{(1)}} + l^{(2)}\right. \\ &\left. - [h^{(2)}(-1)^{l^{(1)}} + l^{(2)}]_2\right) + \pi i p_2 g^{(1)} h^{(1)} l^{(1)}\Big\}, \\ = &(-1)^{p_1(g^{(2)}h^{(2)}l^{(2)} + g^{(2)}h^{(2)}l^{(1)}) + p_2 g^{(1)}h^{(1)}l^{(1)}}. \end{aligned} \quad (175)$$

Notice that there is no contribution from the third part in Eq. (164) as $e^{\pi i p_3 (2g^{(2)}) h^{(1)} l^{(1)}} \equiv 1$. In Appendix H, we analyze the fluxes and charges of all the anyons in the theory from 3-cocycle $[\nu']$. Let $b_1 = 1$ and $b_2 = x$, one can write the matrix-valued operator on an open ribbon l as

$$\left(H_l^a\right)_{ii'} = \sum_{n\in\{1,t\}} H_l^{b_iab_i^{-1},b_inb_{i'}^{-1}} \epsilon_{b_iab_i^{-1}}\left(b_inb_{i'}^{-1}\right), \tag{176}$$

where the matrix indices have the range $i, i' = 1, 2$. As we conjectured, the object $0_a \times 0_a$ should be created on the endpoint of l by operator $(H_l^a)^{\otimes 2}$. According to the detailed analysis in Appendix G, we see that different values of p_3 give different SFCs where the fusion rules are shifted by anyon $[e^{(1)}] \in H_\rho^2(Z_2, \mathcal{A})$.

H. SETs from partially gauging S_3 SPT

Now with the conjecture made in Sec. VI G, we can revisit our first example in Sec. V B. Recall that we write the element in S_3 as $\tilde{g} = (G, g) \equiv x^G a^g$. We construct a representative of 3-cocycle in $H^3(S_3, U(1))$ as in Eq. (70). Gauging the normal subgroup Z_3 of a S_3-SPT state, results in a state in an SET phase that has the same anyon theory as $D^\nu(Z_3)$, where

$$\nu(g, h, l) = \exp\left\{\frac{2\pi i p_1}{9} g(h + l - [h + l]_3)\right\} \tag{177}$$

is the restriction of ω on Z_3. Different values of p_1 exactly correspond to different Z_3 TQD models. The symmetry action takes e to e^2, and takes m to $m^2e^{2p_1}$ according to Eq. (131), which is also not a trivial automorphism on $\mathcal{C}$, as we have seen something similar in the previous D_4 case.

In sector $\mathcal{C}_x$, there is only one object of quantum dimension 3. We name it 0_x. By the dimension counting, we can write a fusion rule of the form

$$0_x \times 0_x = \sum_{i=1}^{9} a_i, \tag{178}$$

where $a_i \in \mathcal{C}$ are Abelian anyons. Let $b_1 = 1$, $b_2 = a$, and $b_3 = a^2$, one can write a matrix-valued operator on an open ribbon as

$$\left(H_l^x\right)_{ii'} = H_l^{b_ixb_i^{-1},b_ib_{i'}^{-1}} \epsilon_{b_ixb_i^{-1}}\left(b_ib_{i'}^{-1}\right), \tag{179}$$

where the matrix indices are in the range $i, i' = 1, 2, 3$. As we conjectured in the last example, the object $0_x \times 0_x$ should be created on the endpoint of l by operator $(H_l^x)^{\otimes 2}$. According to the detailed analysis in Appendix I, we can obtain the fusion rule of the Z_3 SET from the conjectured branch line operator,

$$0_x \times 0_x = 1 + e + e^2 + m + em + e^2m + m^2 + em^2 + e^2m^2. \tag{180}$$

From the analysis in Sec. VI G, we know that there is only one symmetry fractionalization pattern for every value of p_1. This unique SFC result is consistent with the fact that the fusion in the above equation is the same for all values of p_2 (with a fixed p_1, i.e., fixing a distinct anyon theory), and thus p_2 gives different SDCs, unrelated to the SFC.

I. SETs from D_n SPT

Here we comment on the SET phase obtained from gauging the $N = Z_n$ subgroup in the D_n SPT state. When $n = 2m + 1$ is odd, from the similar argument we used for the S_3-SPT state, there is only one symmetry fractionalization pattern of such SET. The cohomology group can be decomposed as

$$\begin{aligned} H^3(D_{2m+1}, U(1)) &= H^3(Z_{2m+1}, U(1)) \oplus H^3(Z_2, U(1)) \\ &= Z_{2m+1} \oplus Z_2. \end{aligned} \tag{181}$$

Therefore, just as in $S_3 = D_3$ case, a representative 3-cocycle $[\omega] \in H^3(D_{2m+1}, U(1))$ will have two parameters $p_1 = 0, \dots, 2m$ and $p_2 = 0, 1$. Different values of p_1 give different anyon theory of the SET order, while different values of p_2 give different SDCs. Furthermore, there is only one object 0_x in sector $\mathcal{C}_x$ of dimension $2m + 1$, and from a similar calculation, one expects the fusion rule to be

$$0_x \times 0_x = \sum_{a\in\mathcal{C}} a. \tag{182}$$

When $n = 2m$ is even, the cohomology group can be decomposed as

$$\begin{aligned} H^3(D_{2m}, U(1)) =& H^3(Z_{2m}, U(1)) \oplus H^3(Z_2, U(1)) \\ &\oplus H^2(Z_2, H^1(Z_{2m}, U(1))) \\ =& Z_{2m} \oplus Z_2 \oplus Z_2. \end{aligned} \tag{183}$$

Therefore, just as in D_4 case, a representative 3-cocycle $[\omega] \in H^3(D_{2m}, U(1))$ will have three parameters $p_1 = 0, \dots, 2m - 1$, $p_2 = 0, 1$, and $p_3 = 0, 1$. Different values of p_1 give different anyon theory of the SET order and different values of p_2 give different SDCs. Furthermore, different values of p_3 will differ in the fusion of $0_x \times 0_x$ by anyon e^m, and, therefore, correspond to different symmetry fractionalization patterns.

J. SETs from partially gauging Q_8 SPT

Another group extension of Z_2 by Z_4 is the Q_8 group. We write the element in Q_8 as $\tilde{g} = (G, g) \equiv x^G a^g$ just as for D_4. The only difference is that $x^2 = a^2$ instead of identity now. A representative of 3-cocycle in $H^3(Q_8, U(1))$ is [59]

$$\begin{aligned} \omega(\tilde{g}, \tilde{h}, \tilde{l}) = \exp\Big[&\frac{2\pi i p}{8}(-2GHL + g(-1)^{H+L} \\ &\times \{h(-1)^L + l - [h(-1)^L + l + 2HL]_4\})\Big], \end{aligned} \tag{184}$$

where $p = 0, 1, 2, 3$. We note that despite the fact that $H^3(Q_8, U(1)) = Z_8$, we only present half of the cocycles here, as we are not aware of the other half. After gauging the normal subgroup $Z_2 = \{1, a^2\}$, we obtain an SET states in which the anyon theory is the same as in $D^\nu(Z_2)$ where $\nu(g, h, l) = \exp\{\frac{2\pi i p}{8} ghl\}$, with $g, h, l = 0, 2$ representing elements from the set $\{1, a^2\}$. Therefore, $p = 0, 1, 2, 3$ all correspond to the Z_2 toric code model after gauging. The symmetry action on anyons are trivial since the group extension is central. According to Eq. (156), the braiding phase $B(w(h_1, h_2), b)$ is given by

$$B(w(h_1, h_2), b) = \frac{\epsilon_{h_1}(g)\epsilon_{h_2}(g)}{\epsilon_{q(h_1h_2)}(g)} \gamma_g(h_1, h_2)\mu_g(n(h_1h_2)). \tag{185}$$

Let us write the quotient group elements as $Z_2 \times Z_2 \equiv \{1, t_1\} \times \{1, t_2\}$, where the embedding of $t_1(t_2)$ is $x(a)$. After carrying out the detailed calculations from the cocycles above, the SFC corresponds to $[\omega(h_1, h_2)] \in H^2(Z_2 \times Z_2, Z_2 \times Z_2)$, where

$$\omega(h_1, h_2) = \begin{cases} m, & (h_1, h_2) = (t_1, t_1), (t_2, t_1), (t_2, t_2), \quad (t_1, t_1 t_2), (t_1 t_2, t_2), \\ 1, & \text{others}. \end{cases} \tag{186}$$

Therefore, the parameter p characterizes different SDCs of the SET order. One expects that for the other 4 classes of cocycles in $H^3(Q_8, U(1))$, the anyon theory after gauging the normal Z_2 subgroup would be Z_2 double-semion model, and from similar calculations, one can determine the SFC accordingly.

VII. CONCLUSION

Recently, it has been realized that a wide class of topologically ordered states described by the (twisted) quantum double models with solvable gauge groups can be prepared with finite depth local operations as long as local measurements are included [41–44,46,47]. We have re-examined such a measurement-based gauging approach which transforms a nontrivial SPT state into a corresponding TQD state. We provided two alternative gauging procedures: one using a particular decomposition in terms of successive quotient groups and another one exploiting a new and equivalent definition of solvable groups. This flexibility in our method may allow us different options in preparing midgauging SET states.

In the case of non-Abelian groups, the gauging procedure involves multiple steps where intermediate steps only partially gauge the system so that some symmetry remains. Starting from an initial G-SPT state, we have presented an in-depth analysis of the intermediate states and have found them to be topologically ordered states enriched by the remaining ungauged symmetry. We have constructed the generic lattice (parent) Hamiltonian for these states, and showed that they are connected to twisted quantum double (TQD) ground states via a finite-depth local unitary circuit (without measurements) which does not respect the global symmetry.

Furthermore, we have shown that the algebra of the symmetry branch line operators can be used to extract the symmetry fractionalization classes and infer symmetry defectification classes of the SET phases given the input data G and $[\omega] \in H^3(G, U(1))$. When the SET order in the intermediate step of the N-step gauging has a global symmetry that *does not* change the anyon type, using the algebra of symmetry branch line operators, we have developed a general formula for the braiding phases between any Abelian anyon in the theory and the anyons obtained from fusing point defects, which exactly characterize the symmetry fractionalization patterns. We have given various examples for this case. When the SET order we enter has a global symmetry that *does* change anyon types, we conjectured the form and algebra of non-Abelian symmetry branch line operators that can create the corresponding symmetry defects. Then by calculating the tensor product of such operators, we showed that fusion rules of these symmetry defects can be derived, which is sufficient to characterize the symmetry fractionalization patterns. We have used the dihedral SPT states and the associated SET states as examples to illustrate this latter case.

In this work, we mainly focused on the SFC, and a framework to characterize the SDC is left for future study. We note that, according to Ref. [51], the SDC forms a $H^3(Q, U(1))$ torsor, and two defectification classes are differed by an element in $H^3(Q, U(1))$. One can always enter another SDC by applying a unitary $U_{\omega'}$ to vertex DOFs (where $[\omega'] \in H^3(Q, U(1))$). This is equivalent to stacking a Q-SPT state onto the current SET state.

Our method to probe SET phases using fusion was inspired by Ref. [43] and it turns out to be specifically useful when the gauge group (or the normal subgroup in the case of multistep gauging) is Abelian. We expect our formalism holds for non-Abelian TQD models with some global symmetry. We leave as open questions how to consistently define ribbon operators for probing more complex SET phases with non-Abelian gauge groups. For this purpose, it may be useful to re-examine some literature regarding the quasi-Hopf algebra [59,61,64].

As a technical issue, in writing down the branch line operators, we have assumed the existence of $\epsilon_x(g)$ [see Eq. (106)] for the 3-cocycle $[\omega] \in H^3(G, U(1))$. However, this factor may not exist in general. We did encounter this situation in the example of gauging the $Z_2^{(1)}$ symmetry in the $Z_2^{(1)} \times Z_2^{(2)} \times Z_2^{(3)}$-SPT phase when the cocycle is of type-3. Nonetheless, when restricting x and g to some specific subgroups, we can still define $\epsilon_x(g)$, and we used them for the branch line operators to characterize the symmetry fractionalization patterns. It is not clear how to overcome the nonexistence of $\epsilon_x(g)$ in general, and this is also left for future exploration.

As conjectured by the hierarchy of topological orders conceptualized in Ref. [43], topologically ordered states with nonsolvable groups or even more general anyon models without any group structure cannot be prepared using finite-depth measurement-assisted circuits. Nonetheless, it was recently shown in Ref. [65] that the Fibonacci anyon state can be prepared using $\log L$-depth circuits with midcircuit measurements (where L is the linear size of the system). It would be worth exploring midgauging topological phases beyond solvable groups and group-based anyon theories.

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APPENDIX A: SOME PROPERTIES OF TWISTED QUANTUM DOUBLE (TQD) HAMILTONIAN

In this Appendix, we check some properties of twisted quantum double (TQD) Hamiltonian. Although some proofs of the claims here are given in the original paper [11], we demonstrate those in our notation. The TQD Hamiltonian is given by

$$H = -\sum_v A_v - \sum_p B_p, \tag{A1}$$

where A_v is the vertex operator and B_p is the plaquette operator explicitly given by

$$A_v = \frac{1}{|G|}\sum_{g\in G}\left(\prod_{e\supset v} L^g_{\pm e}\right)\tilde{W}^g_v = \frac{1}{|G|}\sum_{g\in G} A_{v,g}, \tag{A2a}$$

$$B_p = \delta\left(\prod_{e\in p} g_e, 1\right). \tag{A2b}$$

The operator $(\prod_{e\supset v} L^g_{\pm e})$ denotes the operator which implements left action (L^g_{+e}) or right action (L^g_{-e}) on the edges adjacent to the vertex v when the edge flows to vertex v or emanates from vertex v, respectively. $\tilde{W}^g_v$ is the phase operator defined as follows:

$$\tilde{W}^g_v = \left(\prod_{e\supset v} L^g_{\pm e}\right)^\dagger U_\omega \left(\prod_{e\supset v} L^g_{\pm e}\right) U^\dagger_\omega, \tag{A3}$$

where U_ω is the phase operator which assigns a phase to a given configuration of edges in the TQD ground state [see Eq. (10)]. The purpose of $\tilde{W}^g_v$ is to change the phase factor in TQD wave-function after the operation $\prod_{e\supset v} L^g_{\pm e}$ is applied on the vertex v.

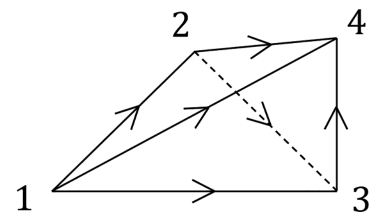

FIG. 19. Tetrahedron with orientation.

Claim A.1. The action of $\tilde{W}^g_v$ on vertex v can be interpreted geometrically as a product of cocylces of tetrahedrons in Fig. 5 with appropriate signs in the exponent.

Proof. To prove this equivalence, we mention the following fact: given a tetrahedron with a branching structure, equating one with the product of (a) cocycles (with appropriate signs in the exponent) on the faces and (b) the cocylce on the tetrahedron (also with appropriate sign in the exponent) gives the cocycle condition. The signs in the exponent can be found using the following rule:

(1) First we define the orientation of a face of a tetrahedron. Consider the branching structure of a face. Curl your fingers on the right hand along the direction of two arrows which point one after the other. The direction your thumb points at gives you the orientation of the face.

(2) For the tetrahedron, consider the vertex where all the arrows end and the face opposite to it. If the orientation of the face points inward to the tetrahedron, then the sign is $+1$; otherwise, it is -1.

(3) For a face, simply consider its orientation. If the orientation points inward to the tetrahedron, then assign the sign to be $+1$; otherwise, assign -1.

As an example, consider the following tetrahedron given in Fig. 19.

The product of cocycles with appropriate signs in the exponent gives the cocycle condition,

$$\frac{\omega(g_4g_2^{-1}, g_2g_1^{-1}, g_1)\,\omega(g_4g_3^{-1}, g_3g_2^{-1}, g_2)}{\omega(g_4g_3^{-1}, g_3g_2^{-1}, g_2g_1^{-1})\,\omega(g_3g_2^{-1}, g_2g_1^{-1}, g_1)\,\omega(g_4g_3^{-1}, g_3g_1^{-1}, g_1)} = 1. \tag{A4}$$

Now, multiplying all the cocycle conditions coming from all the tetrahedron adjacent to vertex v as in Fig. 5, one can clearly see that the cocycles coming from the faces shared by two tetrahedron cancel in pairs since the signs in the exponent coming from the two adjacent tetrahedrons of a face are opposite. Finally, the remaining product of cocycles can be rewritten as

$$\prod_{\text{tetra}} \omega(\text{tetra})^{s(\text{tetra})} = \prod_\Delta \frac{1}{\omega(\Delta')^{s(\Delta')}\omega(\Delta)^{s(\Delta)}} = \prod_\Delta \left(\frac{\omega(\Delta')}{\omega(\Delta)}\right)^{s(\Delta)}, \tag{A5}$$

where Δ and Δ' denote the triangles in the original and the lifted plane (after action by A_v), $s(\Delta)$ and $s(\Delta')$ denote the signs in the exponent for the cocycle coming from Δ and Δ'. The last equality in Eq. (A5) follows from the fact that $s(\Delta') = -s(\Delta)$. The expression which follows the last equality is exactly what Eq. (A3) achieves. ■

Claim A.2. A_v is Hermitian, i.e., $A_v^\dagger = A_v$.

Proof. By definition, we have

$$A^\dagger_{v,g} = \left(\left(\prod_{e\supset v} L^g_{\pm e}\right)\tilde{W}^g_v\right)^\dagger = (\tilde{W}^g_v)^*\left(\prod_{e\supset v} L^g_{\pm e}\right)^\dagger, \tag{A6}$$

where $(\tilde{W}^g_v)^*$ denotes the complex conjugate of $\tilde{W}^g_v$. We note that $(\prod_{e\supset v} L^g_{\pm e})^\dagger = (\prod_{e\supset v} L^{g^{-1}}_{\pm e})$. From the definition

Eq. (A3), we have

$$\left(\tilde{W}_v^g\right)^* = U_\omega\left(\prod_{e\supset v} L_{\pm e}^g\right)^\dagger U_\omega^\dagger\left(\prod_{e\supset v} L_{\pm e}^g\right). \quad \text{(A7)}$$

Now we compute $A_{v,g}^\dagger$:

$$\left(\tilde{W}_v^g\right)^*\left(\prod_{e\supset v} L_{\pm e}^g\right)^\dagger = U_\omega\left(\prod_{e\supset v} L_{\pm e}^g\right)^\dagger U_\omega^\dagger \quad \text{(A8a)}$$

$$= \left(\prod_{e\supset v} L_{\pm e}^g\right)^\dagger\left(\prod_{e\supset v} L_{\pm e}^g\right)U_\omega\left(\prod_{e\supset v} L_{\pm e}^g\right)^\dagger U_\omega^\dagger \quad \text{(A8b)}$$

$$= \left(\prod_{e\supset v} L_{\pm e}^{g^{-1}}\right)\left(\prod_{e\supset v} L_{\pm e}^{g^{-1}}\right)^\dagger U_\omega\left(\prod_{e\supset v} L_{\pm e}^{g^{-1}}\right)U_\omega^\dagger \quad \text{(A8c)}$$

$$= \left(\prod_{e\supset v} L_{\pm e}^{g^{-1}}\right)\tilde{W}_v^{g^{-1}} = A_{v,g^{-1}}. \quad \text{(A8d)}$$

Hence, it holds that

$$A_v^\dagger = \frac{1}{|G|}\sum_{g\in G} A_{v,g}^\dagger = \frac{1}{|G|}\sum_{g\in G} A_{v,g^{-1}} = A_v. \quad \text{(A9)}$$

■

Claim A.3. A_v is a projector. $A_v^2 = A_v$.

Proof. Note the following observation:

$$A_{v,g}A_{v,h} = U_\omega\left(\prod_{e\supset v} L_{\pm e}^g\right)U_\omega^\dagger U_\omega\left(\prod_{e\supset v} L_{\pm e}^h\right)U_\omega^\dagger \quad \text{(A10a)}$$

$$= U_\omega\left(\prod_{e\supset v} L_{\pm e}^{gh}\right)U_\omega^\dagger = A_{v,gh}. \quad \text{(A10b)}$$

Hence, we can square A_v and arrive at

$$A_v^2 = \frac{1}{|G|^2}\sum_{g,h\in G} A_{v,g}A_{v,h} = \frac{1}{|G|^2}\sum_{g,h\in G} A_{v,gh}$$

$$= \frac{1}{|G|}\sum_{g\in G} A_{v,g} = A_v. \quad \text{(A11)}$$

■

Claim A.4. B_p is Hermitian as well as a projector.

Proof. This follows trivially from the definition Eq. (14). ■

Claim A.5. $[A_v, A_{v'}] = 0$ for any vertices v and v'.

Proof. First we prove $[A_{v,g}, A_{v',h}] = 0$ when $v \neq v'$:

$$[A_{v,g}, A_{v',h}] = \left[U_\omega\left(\prod_{e\supset v} L_{\pm e}^g\right)U_\omega^\dagger, U_\omega\left(\prod_{e\supset v'} L_{\pm e}^h\right)U_\omega^\dagger\right]$$

$$= U_\omega\left[\prod_{e\supset v} L_{\pm e}^g, \prod_{e\supset v'} L_{\pm e}^h\right]U_\omega^\dagger = 0. \quad \text{(A12)}$$

The last line is trivial when v and v' are not adjacent. When they are adjacent, the two vertices have opposite (right/left) action on the edge DOF. So their commutator is again zero. Hence, $[A_{v,g}, A_{v',h}] = 0$ when $v \neq v'$. This imply $[A_v, A_{v'}] = 0$ when $v \neq v'$. When $v = v'$, the commutator is trivially zero. So $[A_v, A_{v'}] = 0$, $\forall$ v and v'. ■

Claim A.6. $[A_v, B_p] = 0$ $\forall$ v and p.

Proof. When the vertex v is not on the boundary of the plaquette p, the two terms commute trivially. When the vertex is on the boundary of p, we consider the two edges which are adjacent to the vertex v as well as lie on the boundary of p. Now consider three cases.

Case 1: Both edges point toward the vertex v. Action of $A_{v,g}$ on this configuration is given by left multiplying g on the corresponding edges. This preserves the fluxless condition imposed by the B_p operator.

Case 2: Both edges point away from the vertex v. Action of $A_{v,g}$ on this configuration is given by right multiplying g^{-1} on the corresponding edges. This preserves the fluxless condition imposed by the B_p operator.

Case 3: One edge point toward the vertex v and the other edges point away from it. Action of $A_{v,g}$ on this configuration is given by left multiplying by g and right multiplying by g^{-1}, respectively. This also preserves the fluxless condition.

From this observation it follows that $[A_v, B_p] = 0$ $\forall$ v and p. ■

Claim A.7. $[B_p, B_{p'}] = 0$ $\forall$ plaquettes p and p'.

Proof. The proof follows straightforwardly from the fact $B_p = 1$ on the configurations for which there is no flux around plaquette p, and zero otherwise. ■

Now we consider the quantum double like Hamiltonian in the presence of a global symmetry given in Eq. (20)

$$H = -\sum_v A_v - \sum_p B_p - \sum_v K_v, \quad \text{(A13)}$$

where A_v, B_p, and K_v are defined in Eqs. (21), (22), and (23), respectively. Again A_v is Hermitian as well as a projector. Similarly B_p is also Hermitian as well as a projector. A_v and B_p commute among themselves and with each other. The proofs follow by repeating the steps in the twisted quantum-double case. Now we consider the last term K_v.

Claim A.8. K_v is Hermitian as well as a projector.

Proof. Let us write

$$K_v = \frac{1}{|Q|}\sum_{k,l=0}^{|Q|-1} K_{v,kl}, \quad \text{(A14)}$$

where $K_{v,kl} = W_v^{q_k q_l^{-1}}|q_k\rangle_v\langle q_l|$. We can write the phase operator $W_v^{q_k q_l^{-1}}$ as

$$W_v^{q_k q_l^{-1}}|q_k\rangle_v\langle q_l| = U_\omega|q_k\rangle_v\langle q_l|U_\omega^\dagger. \quad \text{(A15)}$$

Using the Hermitian conjugation of the above equation, we have

$$K_{v,kl}^\dagger = U_\omega|q_l\rangle\langle q_k|U_\omega^\dagger = K_{v,lk}. \quad \text{(A16)}$$

Hence, we have $K_v^\dagger = K_v$. Next, we prove $K_v^2 = K_v$ using the following steps,

$$K_v^2 = \frac{1}{|Q|^2}\sum_{k,l,m,n=0}^{|Q|-1} U_\omega|q_k\rangle_v\langle q_l|U_\omega^\dagger U_\omega|q_m\rangle_v\langle q_n|U_\omega^\dagger \quad \text{(A17a)}$$

$$= \frac{1}{|Q|^2}\sum_{k,l,m,n=0}^{|Q|-1} U_\omega |q_k\rangle_v\langle q_n| U_\omega^\dagger \delta_{l,m} \quad \text{(A17b)}$$

$$= \frac{1}{|Q|}\sum_{k,l=0}^{|Q|-1} K_{v,kn} = K_v. \quad \text{(A17c)}$$

■

Claim A.9. K_v commute with $A_{v'}$, B_p, and $K_{v'}$ $\forall$ v and v'.

Proof. First we prove $[K_v, K_{v'}] = 0$. For this, we show $[K_{v,kl}, K_{v',mn}] = 0$ when $v \neq v'$, as follows:

$$[K_{v,kl}, K_{v',mn}] = [U_\omega|q_k\rangle_v\langle q_l|U_\omega^\dagger, U_\omega|q_m\rangle_{v'}\langle q_n|U_\omega^\dagger] \quad \text{(A18a)}$$

$$= U_\omega[|q_k\rangle_v\langle q_l|, |q_m\rangle_{v'}\langle q_n|]U_\omega^\dagger = 0. \quad \text{(A18b)}$$

Hence, $[K_v, K_{v'}] = 0$ $\forall$ v and v'. Note that K_v operator only changes the vertex DOF on the lattice, so K_v does not change the fluxless condition around any of the plaquettes. Hence, $[K_v, B_p] = 0$.

Now let us prove $[K_v, A_{v'}] = 0$ by the following steps:

$$[K_v, A_{v'}] = \left[\frac{1}{|Q|}\sum_{k,l=0}^{|Q|-1} U_\omega|q_k\rangle_v\langle q_l|U_\omega^\dagger, \frac{1}{|G|}\sum_{g\in G} U_\omega \prod_{e\supset v'} L^g_{\pm e} U_\omega^\dagger\right] \quad \text{(A19a)}$$

$$= \frac{1}{|Q|}\sum_{k,l=0}^{|Q|-1}\frac{1}{|G|}\sum_{g\in G}\left[U_\omega|q_k\rangle_v\langle q_l|U_\omega^\dagger, U_\omega\prod_{e\supset v'} L^g_{\pm e}U_\omega^\dagger\right] \quad \text{(A19b)}$$

$$= \frac{1}{|Q|}\sum_{k,l=0}^{|Q|-1}\frac{1}{|G|}\sum_{g\in G} U_\omega\left[|q_k\rangle_v\langle q_l|, \prod_{e\supset v'} L^g_{\pm e}\right]U_\omega^\dagger = 0. \quad \text{(A19c)}$$

From the above equations, we thus conclude that K_v commutes with $A_{v'}$, B_p, and $K_{v'}$. ■

APPENDIX B: GROUP EXTENSION

Suppose we are given two groups Q and N, then one can construct an extension of Q by N which we denote by G if one has the following short exact sequence:

$$1 \to N \xrightarrow{i} G \xrightarrow{\pi} Q \to 1, \quad \text{(B1)}$$

where i denotes the inclusion map and π denotes the projection map. Given this short exact sequence, one can define a choice of embedding of Q in G,

$$Q \xrightarrow{s} G, \quad \text{(B2)}$$

such that $\pi \circ s = id_Q$. Although the inclusion i and projection π are homomorphisms, the section s is not a homomorphism (however, we have $s(1_Q) = 1_G$). The failure to become a homomorphism is captured by a cocycle in the group cohomology $H^2(Q, N)$. The failure of the section to be a homomorphism is given by

$$s(q_1q_2)^{-1}s(q_1)s(q_2) = i(\omega(q_1, q_2)), \quad \text{(B3)}$$

where $\omega \in H^2(Q, N)$. We consider conjugation operation $\phi : Q \to Aut(N)$ which satisfies

$$i\big(\phi^{q_2^{-1}}(n_1)\big) = s(q_2)^{-1}i(n_1)s(q_2). \quad \text{(B4)}$$

Note that the conjugation operation is dependent on the choice of s. We denote an element $g \in G$ as $g = (q, n)$, where $q \in Q$ and $n \in N$. With the given choice of section s, one can equivalently write

$$g = s(q)i(n). \quad \text{(B5)}$$

Suppose $g_1 = (q_1, n_1)$ and $g_2 = (q_2, n_2)$, then

$$g_1.g_2 = (q_1, n_1).(q_2, n_2) = \big(q_1q_2, \omega(q_1, q_2)\phi^{q_2^{-1}}(n_1)n_2\big). \quad \text{(B6)}$$

The associativity condition on the group multiplication gives the cocycle condition for ω,

$$\omega(q_1, q_2q_3)\omega(q_2, q_3) = \omega(q_1q_2, q_3)\phi^{q_3^{-1}}[\omega(q_1, q_2)], \quad \text{(B7)}$$

when N is Abelian. Note that this cocycle condition is different from the one where the conjugation acts on $\omega(q_2, q_3)$ considered in Ref. [43].

As an example, consider the central extension of $\mathbb{Z}_2$ by $\mathbb{Z}_2$. Since $H^2(\mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{Z}_2$, there are two possible extensions.

Case 1: ω is trivial. In this case, we have $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ since $Aut(\mathbb{Z}_2)$ consists only of the identity map.

Case 2: ω is the nontrivial class. Then the group G is $\mathbb{Z}_4$. Suppose we denote $N = \mathbb{Z}_2 = \{1, t_1\}$ and $Q = \mathbb{Z}_2 = \{1, t_2\}$, then $G = \mathbb{Z}_4$ is generated by $(t_2, 1)$.

APPENDIX C: *Q*-GLOBAL SYMMETRY IN TWO-STEP GAUGING

Here we show the Q-global symmetry of $|\Psi_5\rangle$ in Eq. (66):

$$U_{(q)}|\Psi_5\rangle = |\Psi_5\rangle, \quad \text{(C1)}$$

with

$$|\Psi_5\rangle := \sum_{\{g_v\}} \Omega\big(\{g_v g_{v'}^{-1}\}\big)\big|\{g_v g_{v'}^{-1}\}\big\rangle_e \bigotimes_v \left(Z^{k_v}_{(n)}\sum_{r\in N}|q(g_v)r\rangle_v\right),$$

$$U_{(q)} := \prod_v X_{(q)}. \quad \text{(C2)}$$

We have that $\sum_{r\in N} X_{(q_1)}|q(g_v)r\rangle := \sum_{r\in N}|s(q_1)q(g_v)r\rangle = \sum_{r\in N}|q(s(q_1)g_v))n(s(q_1)q(g_v))r\rangle = \sum_{r'\in N}|q(g_v s(q_1))r'\rangle$, where the first equality is by definition [see Eq. (61)], the second equality is by the definitions of $q(\dots)$ and $n(\dots)$ in Eq. (59), and in the last equality we have relabeled the group element $r \in N$ by $r' = n(s(q_1)q(g_v))r$ and used $q(s(q_1)g_v) = q(g_v s(q_1))$. (Note that in the second equality, for $G = S_3$ the element $n(s(q_1)q(g_v))$ is trivial, but for $G = Q_8$, for example, it is nontrivial, and thus we kept it present in the equation.) We find that the change of variable $g_v = \widetilde{g}_v s(q_1)^{-1}$ gives us

$$U_{(q_1)}|\Psi_5\rangle = \sum_{\{\widetilde{g}_v\}} \Omega\big(\{\widetilde{g}_v\widetilde{g}_{v'}^{-1}\}\big)\big|\{\widetilde{g}_v\widetilde{g}_{v'}^{-1}\}\big\rangle_e \bigotimes_v \left(Z^{k_v}_{(n)}\sum_{r'\in N}|q(\widetilde{g}_v)r'\rangle_v,\right), \quad \text{(C3)}$$

and thus the state is invariant.

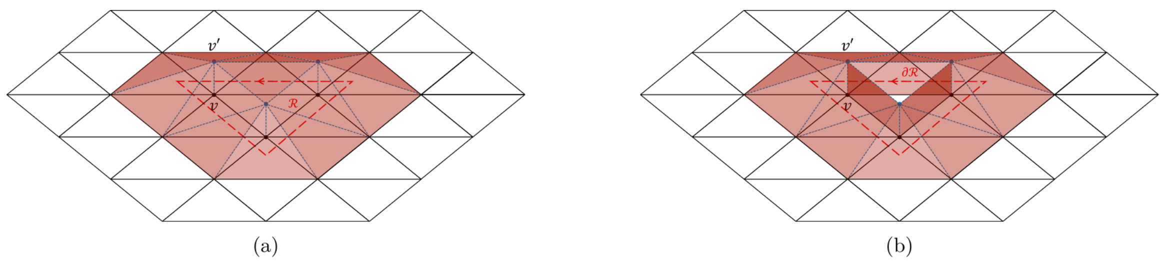

FIG. 20. (a) Symmetry action inside of region $\mathcal{R}$ "lifts" $\mathcal{R}$ such that all the simplex in the region correspond to $\tilde{\omega}$. (b) This symmetry action can be equivalently regarded as the insertion of symmetry branch line on $\partial\mathcal{R}$.

APPENDIX D: BRANCH LINE OPERATOR $\mathcal{B}^x_{\partial\mathcal{R}}$

With a pregauge structure we have introduced the gauge transformation as

$$\mathcal{G}^x_v \equiv L^x_{-v}\prod_{e\supset v} L^x_{\pm e}. \tag{D1}$$

To write down the branch line operator on $\partial\mathcal{R}$, we impose some gauge transformation in region $\mathcal{R}$, the effect of which can be contained only on its boundary. In Sec. VI A, we claim that when the edge configuration is trivial (i.e., $h_e = 1$ for all edges), the operator $\mathcal{G}^x_{\mathcal{R}} = \prod_{v\in\mathcal{R}} \mathcal{G}^x_v$ has this property. Now we make some more detailed analysis. Recall that

$$|\Psi_{\text{SPT-pre}}\rangle = |\Psi_{\text{SPT}}\rangle \bigotimes_e |1\rangle_e. \tag{D2}$$

Now we insert a $\mathcal{G}^x_{\mathcal{R}}$ on the state and have the following:

$$\begin{aligned}
\mathcal{G}^x_{\mathcal{R}}|\Psi_{\text{SPT-pre}}\rangle &= \prod_{v\in\mathcal{R}} L^x_{-v}\left(\prod_{e\supset v} L^x_{\pm e}\right)|\Psi_{\text{SPT}}\rangle\bigotimes_e |1\rangle_e\\
&= \sum_{\{g_v\}}\prod_{\Delta}\omega(\{g_v\})^{s(\Delta)}\bigotimes_{v_i\in\mathcal{R}}|g_{v_i}x^{-1}\rangle\bigotimes_{v_o\notin\mathcal{R}}|g_{v_o}\rangle\bigotimes_e\left(\prod_{e'\cap\partial\mathcal{R}} L^x_{\pm e'}\right)|1\rangle\\
&= \sum_{\{g_v\}}\prod_{\Delta}\omega(\{g_{v_i}x\},\{g_{v_o}\})^{s(\Delta)}\bigotimes_v|g_v\rangle\bigotimes_e\left(\prod_{e'\cap\partial\mathcal{R}} L^x_{\pm e'}\right)|1\rangle\\
&= \sum_{\{g_v\}}\text{Amp}^{\mathcal{R}}(\{g_v\},x)\prod_{\Delta}\omega(\{g_v\})^{s(\Delta)}\bigotimes_v|g_v\rangle\bigotimes_e\left(\prod_{e'\cap\partial\mathcal{R}} L^x_{\pm e'}\right)|1\rangle.
\end{aligned}\tag{D3}$$

As shown above in Fig. 20(a), given the configuration $\{g_v\}$, the product of cocycles $\prod_\Delta \omega(\{g_{v_i}x\},\{g_{v_o}\})^{s(\Delta)}$ corresponds to the upper surface, while $\prod_\Delta \omega(\{g_{v_i}\},\{g_{v_o}\})^{s(\Delta)}$ corresponds to the lower surface. According to the cocycle conditions, their ratio corresponds to the tetrahedrons that they enclose,

$$\text{Amp}^{\mathcal{R}} = \prod_{\text{tetra}\in\mathcal{R}}\omega(\text{tetra})^s. \tag{D4}$$

In Sec. VI A, we claim that according to cocycle conditions, this phase factor only depends on the configurations on $\partial\mathcal{R}$. Now we take the simplest case when $\mathcal{R}$ only encloses one plaquette to illustrate. As shown in Fig. 21, when the plaquette (123) is lifted, there are three tetrahedrons, where the vertices on the upper surface are associated with $g_{i'} = g_i x$. According to the rule introduced in Sec. II, the three tetrahedrons correspond to the expression

$$\begin{aligned}
&\frac{\omega\big(g_3xg_3^{-1}, g_3g_2^{-1}, g_2g_1^{-1}\big)\omega\big(g_3g_2^{-1}, g_2g_1^{-1}, g_1xg_1^{-1}\big)}{\omega\big(g_3g_2^{-1}, g_2xg_2^{-1}, g_2g_1^{-1}\big)}\\
&\quad= \theta_{g_3xg_3^{-1}}\big(g_3g_2^{-1}, g_2g_1^{-1}\big),
\end{aligned}\tag{D5}$$

where $\theta_x(g,h)$ is the slant product introduced in Eq. (104). Therefore, in this simplest case, we have

$$\text{Amp}^{\mathcal{R}} = \theta_{g_3xg_3^{-1}}\big(g_3g_2^{-1}, g_2g_1^{-1}\big)\prod_{\text{tetra}\in\partial\mathcal{R}}\omega(\text{tetra})^s. \tag{D6}$$

For simplicity and without loss of generality, we assume $\partial\mathcal{R}$ with branching structure $1\to 2\to 3\to\cdots\to n\leftarrow 1$, then by using cocycle conditions, we can show that the tetrahedrons inside a union of prisms [see Fig. 20(b)], which is formed by lifted plaquettes, will give rise to

$$\tilde{\Theta}^{g_nxg_n^{-1}}_{\partial\mathcal{R}} = \theta_{g_nxg_n^{-1}}\big(g_ng_{n-1}^{-1}, g_{n-1}g_1^{-1}\big)\cdots\theta_{g_3xg_3^{-1}}\big(g_3g_2^{-1}, g_2g_1^{-1}\big). \tag{D7}$$

Therefore,

$$\text{Amp}^{\mathcal{R}} = \tilde{\Theta}^{g_nxg_n^{-1}}_{\partial\mathcal{R}}\prod_{\text{tetra}\in\partial\mathcal{R}}\omega(\text{tetra})^s. \tag{D8}$$

The shift operator $\prod_{e'\cap\partial\mathcal{R}} L^x_{\pm e'}$ in Eq. (D3) is exactly the operator $L^x_{\partial\mathcal{R}}$ defined in Eq. (100). Therefore, we arrive at

$$\tilde{\mathcal{B}}^x_{\partial\mathcal{R}}|\Psi_{\text{SPT-pre}}\rangle = \mathcal{G}^x_{\mathcal{R}}|\Psi_{\text{SPT-pre}}\rangle, \tag{D9}$$

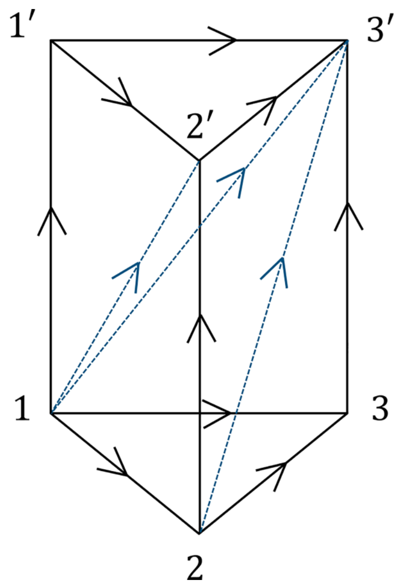

FIG. 21. There are three tetrahedrons when one plaquette is lifted.

when

$$\tilde{\mathcal{B}}^x_{\partial\mathcal{R}} = \sum_{g_v} \mathcal{L}^x_{\partial\mathcal{R}} \tilde{W}^{g_v x g_v^{-1}}_{\partial\mathcal{R}} \tilde{\Theta}^{g_v x g_v^{-1}}_{\partial\mathcal{R}} |g_v\rangle_v \langle g_v|. \tag{D10}$$

For a state with nontrivial configuration $\{h_e\}$, there could be fluxes on some plaquette, i.e., for some plaquette p, $\prod_{e\in\partial p} h_e \neq 1$. For the purpose of illustration, we now focus on states that have fluxes only on 1 plaquette, and only violates terms that are on this plaquette in $H_{\text{SPT-pre}}$ in Eq. (93). To write down the branch line operators for these states, we repeat a similar procedure: we impose some gauge transformation on region $\mathcal{R}$, the effect of which would be only on its boundary. It turns out that when all the plaquettes on $\partial\mathcal{R}$ are fluxless (i.e., $\prod_{e\in\partial p} h_e = 1$), and the flux on each plaquette $p \in \mathcal{R}$ ($\prod_{e\in\partial p} h_e$) is in the centralizer group $\mathcal{Z}_x$ (since the flux on a plaquette is ambiguous up to a conjugation when choosing a different starting point, we assume that the whole conjugacy class of flux $[\prod_{e\in\partial p} h_e] \subset \mathcal{Z}_x$), we can indeed find such a gauge transformation as we now explain. We first choose a reference vertex v in $\mathcal{R}$, then for any vertex v', we can find a path l that flows from v to v' and define $h_{v'v} = \prod_{e\in l} h_e$. Given that all the fluxes are assumed to be in $\mathcal{Z}_x$, i.e., they commute with x, therefore, different choices of path l will give rise to the same gauge transformation (taking $h_{vv} = 1$),

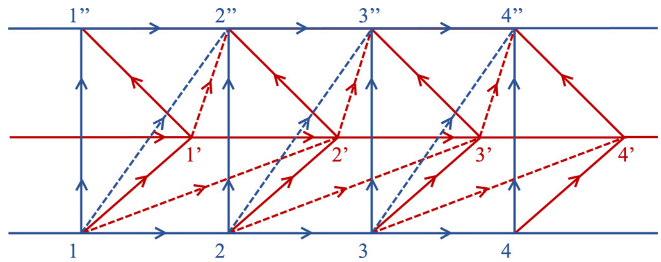

FIG. 22. Geometric diagram illustrating the phase combination in Eq. (D16). Here $g_{v'} g_v^{-1} = y$ and $g_{v''} g_{v'}^{-1} = x$ for $v \in V$ (which are enumerated by 1,2,3,4).

$$\mathcal{G}^x_{v,\mathcal{R}} \equiv \prod_{v'\in\mathcal{R}} \mathcal{G}^{h_{v'v} x h_{v'v}^{-1}}_{v'}, \tag{D11}$$

which will leave all edge DOFs invariant except for the ones on $\partial\mathcal{R}$, and the shift on those edges is exactly the operator $L^x_{\partial\mathcal{R}}$ defined in Eq. (111). Now since there are plaquettes in $\mathcal{R}$ that have nontrivial fluxes, we cannot write the phase part of the gauge transformation as

$$\text{Amp}^{\mathcal{R}} = \prod_{\text{tetra}\in\mathcal{R}} \omega(\text{tetra})^s. \tag{D12}$$

However, since we assume the plaquettes on $\partial\mathcal{R}$ are all fluxless, we still have a well-defined factor $\prod_{\text{tetra}\in\partial\mathcal{R}} \omega(\text{tetra})^s$, the holonomy along $\partial\mathcal{R}$ is $h \equiv \prod_{e\in\partial\mathcal{R}} h_e \neq 1$. We conjecture the branch line operator to be

$$\mathcal{B}^x_{\partial\mathcal{R}} = \sum_g \mathcal{B}^{x,g}_{\partial\mathcal{R}} \epsilon_{g_v x g_v^{-1}}(g), \tag{D13}$$

with

$$\mathcal{B}^{x,g}_{\partial\mathcal{R}} \equiv \sum_{g_v} \mathcal{L}^x_{\partial\mathcal{R}} W^{g_v x g_v^{-1}}_{\partial\mathcal{R}} \Theta^{g_v x g_v^{-1}}_{\partial\mathcal{R}} \delta_{g, g_v(\prod_e h_e) g_v^{-1}} |g_v\rangle_v \langle g_v|. \tag{D14}$$

Again we suppose that $\partial\mathcal{R}$ has the branching structure $1 \to 2 \to 3 \to \cdots \to n \leftarrow 1$. Then, we have

$$\Theta^{g_n x g_n^{-1}}_{\partial\mathcal{R}} = \theta_{g_n x g_n^{-1}}\left(g_n h_{n,n-1} g_{n-1}^{-1}, g_{n-1} h_{n,n-1}^{-1} h g_1^{-1}\right) \cdots \theta_{g_3 x g_3^{-1}}\left(g_3 h_{3,2} g_2^{-1}, g_2 h_{2,1} g_1^{-1}\right) \theta^{-1}_{g_n x g_n^{-1}}\left(g_n h g_n^{-1}, g_n h_{n,1} g_1^{-1}\right), \tag{D15}$$

where $h \equiv h_{n,n-1} \cdots h_{2,1} h_{n,1}^{-1}$ is the holonomy along $\partial\mathcal{R}$. We can calculate the multiplication rule for $\mathcal{B}^x_{\partial\mathcal{R}}$. First off, notice that operator $\mathcal{L}^x_{\partial\mathcal{R}} W^{g_n x g_n^{-1}}_{\partial\mathcal{R}}$ and $\Theta^{g_n y g_n^{-1}}_{\partial\mathcal{R}}$ commute for any $x, y \in G$. Using the cocycle condition, one can show that

$$\mathcal{L}^x_{\partial\mathcal{R}} W^{g_n x g_n^{-1}}_{\partial\mathcal{R}} \mathcal{L}^y_{\partial\mathcal{R}} W^{g_n y g_n^{-1}}_{\partial\mathcal{R}} = \mathcal{L}^x_{\partial\mathcal{R}} W^{g_n x y g_n^{-1}}_{\partial\mathcal{R}} \gamma_{g_n h_{n,n-1} g_{n-1}^{-1}}\left(g_n x g_n^{-1}, g_n y g_n^{-1}\right) \cdot \text{phase}, \tag{D16}$$

where the "phase" in the above equation corresponds to Fig. 22. Using the correspondence between tetrahedrons and cocycles, one can write

$$\mathcal{L}^x_{\partial\mathcal{R}} W^{g_n x g_n^{-1}}_{\partial\mathcal{R}} \mathcal{L}^y_{\partial\mathcal{R}} W^{g_n y g_n^{-1}}_{\partial\mathcal{R}} = \mathcal{L}^{xy}_{\partial\mathcal{R}} W^{g_n x y g_n^{-1}}_{\partial\mathcal{R}} \gamma_{g_n h_{n,n-1} g_{n-1}^{-1}}\left(g_n x g_n^{-1}, g_n y g_n^{-1}\right) \cdots \gamma_{g_2 h_{2,1} g_1^{-1}}\left(g_2 x g_2^{-1}, g_2 y g_2^{-1}\right) \gamma^{-1}_{g_n h_{n,1} g_1^{-1}}\left(g_n x g_n^{-1}, g_n y g_n^{-1}\right). \tag{D17}$$

By using the definition, we can compute the following product of the two phases,

$$\Theta_{\partial\mathcal{R}}^{g_nxg_n^{-1}}\Theta_{\partial\mathcal{R}}^{g_nyg_n^{-1}} = \theta_{g_nxg_n^{-1}}\big(g_nh_{n,n-1}g_{n-1}^{-1}, g_{n-1}h_{n,n-1}^{-1}hg_1^{-1}\big)\theta_{g_nyg_n^{-1}}\big(g_nh_{n,n-1}g_{n-1}^{-1}, g_{n-1}h_{n,n-1}^{-1}hg_1^{-1}\big)\cdots \theta_{g_3xg_3^{-1}}\big(g_3h_{3,2}g_2^{-1}, g_2h_{2,1}g_1^{-1}\big)\theta_{g_3yg_3^{-1}}\big(g_3h_{3,2}g_2^{-1}, g_2h_{2,1}g_1^{-1}\big) \times \theta_{g_nxg_n^{-1}}^{-1}\big(g_nhg_n^{-1}, g_nh_{n,1}g_1^{-1}\big)\theta_{g_nyg_n^{-1}}^{-1}\big(g_nhg_n^{-1}, g_nh_{n,1}g_1^{-1}\big). \tag{D18}$$

Using the identity,

$$\theta_g(x,y)\theta_h(x,y)\gamma_x(g,h)\gamma_y(x^{-1}gx, x^{-1}hx) = \theta_{gh}(x,y)\gamma_{xy}(g,h), \tag{D19}$$

one can derive that

$$\mathcal{L}_{\partial\mathcal{R}}^{x}W_{\partial\mathcal{R}}^{g_nxg_n^{-1}}\Theta_{\partial\mathcal{R}}^{g_nxg_n^{-1}}\mathcal{L}_{\partial\mathcal{R}}^{y}W_{\partial\mathcal{R}}^{g_nyg_n^{-1}}\Theta_{\partial\mathcal{R}}^{g_nyg_n^{-1}} = \mathcal{L}_{\partial\mathcal{R}}^{xy}W_{\partial\mathcal{R}}^{g_nxyg_n^{-1}}\Theta_{\partial\mathcal{R}}^{g_nxyg_n^{-1}}\gamma_{g_nhg_n^{-1}}\big(g_nxg_n^{-1}, g_nyg_n^{-1}\big). \tag{D20}$$

When the pregauge structure is trivial (i.e., $h_e \equiv 1$), the phases $W_{\partial\mathcal{R}}^{g_nxg_n^{-1}}$ and $\Theta_{\partial\mathcal{R}}^{g_vxg_v^{-1}}$ are reduced to $\tilde{W}_{\partial\mathcal{R}}^{g_nxg_n^{-1}}$ and $\tilde{\Theta}_{\partial\mathcal{R}}^{g_vxg_v^{-1}}$, respectively, and also the holonomy is trivial, $h \equiv 1$. Therefore, we arrive at the multiplication rule,

$$\tilde{\mathcal{B}}_{\partial\mathcal{R}}^{x}\tilde{\mathcal{B}}_{\partial\mathcal{R}}^{y} = \tilde{\mathcal{B}}_{\partial\mathcal{R}}^{xy}. \tag{D21}$$

For a more general pregauge structure, where the fluxes are in $\mathcal{Z}_x$, one has

$$\mathcal{B}_{\partial\mathcal{R}}^{x,g}\mathcal{B}_{\partial\mathcal{R}}^{y,g'} = \mathcal{B}_{\partial\mathcal{R}}^{xy,g}\gamma_g\big(g_vxg_v^{-1}, g_vyg_v^{-1}\big)\delta_{g,g'}. \tag{D22}$$

APPENDIX E: FINITE-DEPTH LOCAL UNITARY TO MAP AN SET STATE TO A TQD STATE

For an SPT state, we define an operator

$$\hat{O} = U_\omega\left(\prod_v \mathcal{V}_v^q\right)U_\omega^\dagger, \tag{E1}$$

where U_ω is the operator that brings a direct product state to a G-SPT state,

$$U_\omega = \sum_{\{g_v\}}\prod_\Delta \omega\big(g_3g_2^{-1}, g_2g_1^{-1}, g_1\big)^{s(\Delta)}\bigotimes_v |g_v\rangle\langle g_v|, \tag{E2}$$

and $\mathcal{V}_v^q$ is the Fourier transform of the quotient part. Namely, suppose $|Q| = m$, we label the embedding of Q in G as $s(Q) = \{q_0, \cdots, q_{m-1}\}$, where $q_0 = 1$. Then we write

$$\mathcal{V}_v^q = \frac{1}{\sqrt{m}}\sum_{k,l=0}^{m-1} e^{\frac{2\pi ikl}{m}} L_{+v}^{q_kq_l^{-1}}\delta(q(g_v), q_l), \tag{E3}$$

whose Hermitian conjugate is calculated to be

$$\big(\mathcal{V}_v^q\big)^\dagger = \frac{1}{\sqrt{m}}\sum_{k,l=0}^{m-1} e^{-\frac{2\pi ikl}{m}} L_{+v}^{q_kq_l^{-1}}\delta(q(g_v), q_l). \tag{E4}$$

We note that this Fourier transform is indeed unitary: $(\mathcal{V}_v^q)^\dagger\mathcal{V}_v^q = 1$. Applying $\hat{O}$ on the G-SPT state brings a G-SPT state to an N-SPT state:

$$\begin{aligned}\hat{O}|\Psi_{G\text{-SPT}}\rangle &= U_\omega\left(\prod_v \mathcal{V}_v^q\right)U_\omega^\dagger|\Psi_{G\text{-SPT}}\rangle \\ &= U_\omega\left(\prod_v \mathcal{V}_v^q\right)\bigotimes_v\left(\sum_{g\in G}|g\rangle_v\right) \\ &= U_\omega\bigotimes_v\left(\sum_{n\in N}|n\rangle_v\right) \\ &= \sum_{\{n_v\}}\prod_\Delta \omega\big(n_3n_2^{-1}, n_2n_1^{-1}, n_1\big)^{s(\Delta)}\bigotimes_v|n_v\rangle \\ &= \sum_{\{n_v\}}\prod_\Delta \nu\big(n_3n_2^{-1}, n_2n_1^{-1}, n_1\big)^{s(\Delta)}\bigotimes_v|n_v\rangle \\ &= |\Psi_{N\text{-SPT}}\rangle,\end{aligned} \tag{E5}$$

where ν is the restriction of ω on N and naturally a cocycle in $H^3(N, U(1))$. The operator $\mathcal{V}_v^q$ on different vertices commute,

$$\mathcal{V}_v^q\mathcal{V}_{v'}^q - \mathcal{V}_{v'}^q\mathcal{V}_v^q = 0. \tag{E6}$$

Therefore, we can write an operator on vertex v,

$$\hat{O}_v \equiv U_\omega\mathcal{V}_v^qU_\omega^\dagger = \frac{1}{\sqrt{m}}\sum_{k,l=0}^{m-1} e^{\frac{2\pi ikl}{m}} L_{+v}^{q_kq_l^{-1}}W_v^{q_kq_l^{-1}}\delta(q(g_v), q_l), \tag{E7}$$

such that $\hat{O}_v$ on different vertices commute, and their product over all the vertices is exactly $\hat{O}$,

$$\hat{O} \equiv \prod_v \hat{O}_v. \tag{E8}$$

After gauging the normal subgroup N, the operator $\hat{O}$ is mapped to

$$\hat{O}_{\mathrm{SET}} = \prod_v \hat{O}_{v,\mathrm{SET}}, \tag{E9}$$

where

$$\hat{O}_{v,\mathrm{SET}} = \frac{1}{\sqrt{m}}\sum_{k,l=0}^{m-1} e^{\frac{2\pi ikl}{m}} W_v^{q_kq_l^{-1}}|q_k\rangle_v\langle q_l|. \tag{E10}$$

The operators $\hat{O}_{v,\mathrm{SET}}$ on different vertices commute. Also note that since $\hat{O}_{v,\mathrm{SET}}$ is supported on v, adjacent vertices, and edges, and thus it can be implemented locally in the state after gauging.

Note that although all $\hat{O}_{v,\mathrm{SET}}$ commute, the operators $W_v^{q_kq_l^{-1}}$ generally depends on $q_{v'}$ and n_e configuration on adjacent vertices and edges. To implement the transformation, we specify an ordering of $\hat{O}_{v,\mathrm{SET}}$ in $\hat{O}_{\mathrm{SET}}$ which can be implemented in finite depth as follows. We divide the spatial lattice into sublattices such that, the vertices in each sublattice are not adjacent in the spatial lattice. Then the operators $L_{+v}^{q_k q_l^{-1}}$ and $W_{v'}^{q_k q_l^{-1}}$ always commute when v and v' are different vertices in one sublattice. As a result, we can implement $\hat{O}_{v,\mathrm{SET}}$ within one sublattice simultaneously, and implement one sublattice in each step. As long as there is only a finite number of such sublattices in the spatial lattice, the circuit we described above is a finite-depth local unitary,

$$\hat{O}_{\mathrm{SET}} = \prod_{\text{sublattices}} \prod_{v \in \text{sublattice}} \hat{O}_{v,\mathrm{SET}}. \tag{E11}$$

Applying $\hat{O}_{\mathrm{SET}}$ on an SET ground state will take all the vertex DOFs to $|q_v\rangle = |1\rangle$, i.e., disentangle all the vertex DOFs. Therefore, under the action of this operator, we get a TQD ground state with gauge group N. We emphasize that here we aim to probe the underlying anyons, and the circuit does not respect the global Q symmetry, which is why we can take an SET state to a pure TQD state.

APPENDIX F: FUSION RULE IN SET VIA GAUGING Z_4 GROUP FROM D_4 SPT

We write the element in D_4 as $\tilde{g} = (G, g) \equiv x^G a^g$. We construct a representative of 3-cocycle in $H^3(D_4, U(1))$ as follows:

$$\begin{aligned}\omega(\tilde{g}, \tilde{h}, \tilde{l}) &= \exp\bigg\{\frac{2\pi i p_1}{16} g(-1)^{H+L}(h(-1)^L + l - [h(-1)^L + l]_4) \\ &\quad + \pi i p_2 GHL + \pi i p_3 gHL\bigg\},\end{aligned} \tag{F1}$$

where $p_1 = 0, 1, 2, 3$, and $p_2, p_3 = 0$ or 1. Gauging the normal subgroup Z_4 of a D_4-SPT results in a state in an SET that has the same anyon theory as $D^{\nu}(Z_4)$, where

$$\nu(g, h, l) = \exp\left\{\frac{2\pi i p_1}{16} g(h + l - [h + l]_4)\right\} \tag{F2}$$

is the restriction of ω on Z_4. Different values of p_1 correspond to different Z_4-TQD models. The symmetry action is nontrivial, and it takes an anyon to its inverse. In the sector $\mathcal{C}_x$, there are four objects of quantum dimension 2. If we pick one of them and name it as 0_x, by dimension counting, then we can write a fusion rule of the form,

$$0_x \times 0_x = a + b + c + d, \tag{F3}$$

where $a, b, c, d \in \mathcal{C}$ are Abelian anyons. Let $b_1 = 1$ and $b_2 = a$. Then one can write a matrix-valued operator on an open ribbon as

$$\left(H_l^x\right)_{ii'} = \sum_{n \in \{1, a^2\}} H_l^{b_i x b_i^{-1}, b_i n b_{i'}^{-1}} \epsilon_{b_i x b_i^{-1}}\left(b_i n b_{i'}^{-1}\right), \tag{F4}$$

where the matrix indices $i, i' = 1, 2$, and the operator $H_l^{x,g}$ satisfies the same multiplication rule as in Eq. (125),

$$H_l^{x,g} H_l^{y,g'} = H_l^{xy,g} \gamma_g(x, y) \delta_{g,g'}. \tag{F5}$$

We conjecture that the operator H_l^x creates an object in the sector $\mathcal{C}_x$ on the end point of l, and we name it 0_x. Then the object $0_x \times 0_x$ should be created on the endpoint of l by operator $(H_l^x)^{\otimes 2}$. Let $\xi \equiv \exp\{\frac{2\pi i}{16}\}$. We calculate the tensor product of the two open ribbon operators, and by diagonalizing it, we find an expression as follows:

$$\begin{aligned}
\left(H_l^x\right)^{\otimes 2} &= \begin{pmatrix} H_l^{x,1} + \epsilon_x(a^2) H_l^{x,a^2} & \epsilon_x(a) H_l^{x,a} + \epsilon_x(a^3) H_l^{x,a^3} \\ \epsilon_{xa^2}(a) H_l^{xa^2,a} + \epsilon_{xa^2}(a^3) H_l^{xa^2,a^3} & H_l^{xa^2,1} + \epsilon_{xa^2}(a^2) H_l^{xa^2,a^2} \end{pmatrix}^{\otimes 2} \\
&= \begin{pmatrix} H_l^{x,1} + \xi^{-4p_1} H_l^{x,a^2} & \xi^{p_1} H_l^{x,a} + \xi^{-3p_1} H_l^{x,a^3} \\ \xi^{-p_1} H_l^{xa^2,a} + \xi^{-9p_1} H_l^{xa^2,a^3} & H_l^{xa^2,1} + H_l^{xa^2,a^2} \end{pmatrix}^{\otimes 2} \\
&= \begin{pmatrix} H_l^{1,1} + \xi^{-8p_1} H_l^{1,a^2} & \xi^{2p_1}(-1)^{p_3} H^{1,a} + \xi^{-6p_1}(-1)^{p_3} H^{1,a^3} \\ \xi^{6p_1}(-1)^{p_3} H^{1,a} + \xi^{-2p_1}(-1)^{p_3} H^{1,a^3} & H_l^{1,1} + \xi^{-8p_1} H_l^{1,a^2} \end{pmatrix} \\
&\quad \oplus \begin{pmatrix} H_l^{a^2,1} + \xi^{-4p_1} H_l^{a^2,a^2} & \xi^{-4p_1}(-1)^{p_3} H_l^{a^2,a} + \xi^{8p_1}(-1)^{p_3} H_l^{a^2,a^3} \\ (-1)^{p_3} H_l^{a^2,a} + \xi^{-4p_1}(-1)^{p_3} H_l^{a^2,a^3} & H_l^{a^2,1} + \xi^{-4p_1} H_l^{a^2,a^2} \end{pmatrix} \\
&= \begin{pmatrix} H_l^{1,1} + \xi^{-8p_1} H_l^{1,a^2} & \xi^{2p_1}(-1)^{p_3} H^{1,a} + \xi^{-6p_1}(-1)^{p_3} H^{1,a^3} \\ \xi^{6p_1}(-1)^{p_3} H^{1,a} + \xi^{-2p_1}(-1)^{p_3} H^{1,a^3} & H_l^{1,1} + \xi^{-8p_1} H_l^{1,a^2} \end{pmatrix} \\
&\quad \oplus \begin{pmatrix} H_l^{a^2,1} + \xi^{-8p_1} \epsilon_{a^2}(a^2) H_l^{a^2,a^2} & \xi^{-6p_1}(-1)^{p_3} \epsilon_{a^2}(a) H_l^{a^2,a} + \xi^{2p_1}(-1)^{p_3} \epsilon_{a^2}(a^3) H_l^{a^2,a^3} \\ \xi^{-2p_1}(-1)^{p_3} \epsilon_{a^2}(a) H_l^{a^2,a} + \xi^{6p_1}(-1)^{p_3} \epsilon_{a^2}(a^3) H_l^{a^2,a^3} & H_l^{a^2,1} + \xi^{-8p_1} \epsilon_{a^2}(a^2) H_l^{a^2,a^2} \end{pmatrix} \\
&\simeq \left(H_l^{1,1} + \xi^{4p_1}(-1)^{p_3} H_l^{1,a} + \xi^{8p_1} H_l^{1,a^2} + \xi^{12p_1}(-1)^{p_3} H_l^{1,a^3}\right) \\
&\quad \oplus \left(H_l^{1,1} - \xi^{4p_1}(-1)^{p_3} H_l^{1,a} + \xi^{8p_1} H_l^{1,a^2} - \xi^{12p_1}(-1)^{p_3} H_l^{1,a^3}\right) \\
&\quad \oplus \left(H_l^{a^2,1} + \xi^{4p_1}(-1)^{p_3} \epsilon_{a^2}(a) H_l^{a^2,a} + \xi^{8p_1} \epsilon_{a^2}(a^2) H_l^{a^2,a^2} + \xi^{12p_1}(-1)^{p_3} \epsilon_{a^2}(a^3) H_l^{a^2,a^3}\right) \\
&\quad \oplus \left(H_l^{a^2,1} - \xi^{4p_1}(-1)^{p_3} \epsilon_{a^2}(a) H_l^{a^2,a} + \xi^{8p_1} \epsilon_{a^2}(a^2) H_l^{a^2,a^2} - \xi^{12p_1}(-1)^{p_3} \epsilon_{a^2}(a^3) H_l^{a^2,a^3}\right).
\end{aligned} \tag{F6}$$

In the third equality, we used the multiplication rule Eq. (F5), and the p_3 dependence arose from $\gamma_g(x, y)$.

Let us first consider the case with $p_3 = 0$. When $p_1 = 0, 2$, the four parts in the last line—each part is a sum of four $H_l^{\bullet,\bullet}$ operators—are exactly the four ribbon operators in the TQD $D^\nu(Z_4)$ that create anyons, $1, e^2, m^2$, and e^2m^2, respectively. Therefore, we obtain the fusion rule,

$$0_x \times 0_x = 1 + e^2 + m^2 + e^2m^2. \tag{F7}$$

When $p_1 = 1, 3$, the four parts in the last line after the decomposition are four ribbon operators in the TQD $D^\nu(Z_4)$ that create anyons, e, e^3, em^2, and e^3m^2. Therefore, we obtain the fusion rule,

$$0_x \times 0_x = e + e^3 + em^2 + e^3m^2. \tag{F8}$$

According to Eq. (F6), when p_3 takes value 1 instead of 0, the ribbon operators creating anyon $e/e^3/em^2/e^3m^2$ after the decomposition become ribbon operators creating, respectively, $e^3/e/e^3m^2/em^2$ instead. According to Eq. (39), we conclude that different values of p_3 indeed give rise to different SFCs such that the fusion rules are shifted by the anyon $[e^2] \in H^2_\rho(Z_2, \mathcal{A})$. We note that after the shift by e^2, the fusion rules in Eqs. (F7) and (F8) are actually invariant. However, it does not mean that we are in the same SET order: further analysis, such as the F-symbol, the S-matrix of the category $\mathcal{C}_{Z_2}^\times$ [51], is necessary. Indeed, in this case it is ensured that SET orders corresponding to different p_3 values are distinct, since further gauging the Z_2 symmetry it should result in different TQD orders $D^\omega(D_4)$.

APPENDIX G: FUSION RULE IN SET VIA GAUGING $Z_2 \times Z_2$ GROUP FROM D_4 SPT

Again, we start from the representative 3-cocycle in $H^3(D_4, U(1))$ as follows:

$$\omega(\tilde{g}, \tilde{h}, \tilde{l}) = \exp\left\{\frac{2\pi i p_1}{16} g(-1)^{H+L}(h(-1)^L + l - [h(-1)^L + l]_4) + \pi i p_2 GHL + \pi i p_3 gHL\right\}, \tag{G1}$$

where $p_1 = 0, 1, 2, 3$, and $p_2, p_3 = 0$ or 1. Gauging the subgroup $Z_2 \times Z_2 = \{1, x, t, xt\}$ in D_4 results in a state within $D^{\nu'}(Z_2 \times Z_2)$. Let us write $t \equiv a^2$ and $g = x^{g^{(1)}} t^{g^{(2)}} = x^{g^{(1)}} a^{2g^{(2)}}$. The 3-cocycle for this group is

$$\begin{aligned}\nu'(g, h, l) &= \exp\left\{\frac{2\pi i p_1}{4} g^{(2)}(-1)^{h^{(1)}+l^{(1)}}\big(h^{(2)}(-1)^{l^{(1)}} + l^{(2)} - [h^{(2)}(-1)^{l^{(1)}} + l^{(2)}]_2\big) + \pi i p_2 g^{(1)}h^{(1)}l^{(1)}\right\},\\ &= (-1)^{p_1(g^{(2)}h^{(2)}l^{(2)} + g^{(2)}h^{(2)}l^{(1)}) + p_2 g^{(1)}h^{(1)}l^{(1)}}.\end{aligned} \tag{G2}$$

We conjecture that the operator H_l^a creates an object in sector $\mathcal{C}_a$ on the end point of l and we name it 0_a. Then the object $0_a \times 0_a$ should be created on the endpoint of l by operator $(H_l^a)^{\otimes 2}$. Let $\xi \equiv \exp\{\frac{2\pi i}{16}\}$, then a calculation similar to that in Appendix F gives us

$$\begin{aligned}
(H_l^a)^{\otimes 2} &= \begin{pmatrix} H_l^{a,1} + \epsilon_a(a^2)H_l^{a,a^2} & \epsilon_a(x)H_l^{a,x} + \epsilon_a(xa^2)H_l^{a,xa^2} \\ \epsilon_{a^3}(x)H_l^{a^3,x} + \epsilon_{a^3}(xa^2)H_l^{a^3,xa^2} & H_l^{a^3,1} + \epsilon_{a^3}(a^2)H_l^{a^3,a^2} \end{pmatrix}^{\otimes 2}\\
&= \begin{pmatrix} H_l^{a,1} + \xi^{2p_1}H_l^{a,a^2} & \xi^{4p_3}H_l^{a,x} + \xi^{6p_1+4p_3}H_l^{a,xa^2} \\ \xi^{4p_3}H_l^{a^3,x} + \xi^{2p_1+4p_3}H_l^{a^3,xa^2} & H_l^{a^3,1} + \xi^{6p_1}H_l^{a^3,a^2} \end{pmatrix}^{\otimes 2}\\
&= \begin{pmatrix} H_l^{a^2,1} + \xi^{4p_1}H_l^{a^2,a^2} & \xi^{4p_1+8p_3}H^{a^2,x} + \xi^{8p_1+8p_3}H^{a^2,xa^2} \\ \xi^{-4p_1+8p_3}H^{a^2,x} + \xi^{8p_3}H^{a^2,xa^2} & H_l^{a^2,1} + \xi^{4p_1}H_l^{a^2,a^2} \end{pmatrix}\\
&\quad \oplus \begin{pmatrix} H_l^{1,1} + H_l^{1,a^2} & \xi^{4p_1+8p_3}H_l^{1,x} + \xi^{4p_1+8p_3}H_l^{1,xa^2} \\ \xi^{-4p_1+8p_3}H_l^{1,x} + \xi^{-4p_1+8p_3}H_l^{1,xa^2} & H_l^{1,1} + H_l^{1,a^2} \end{pmatrix}\\
&= \begin{pmatrix} H_l^{a^2,1} + \epsilon_{a^2}(a^2)H_l^{a^2,a^2} & \xi^{4p_1+8p_3}\epsilon_{a^2}(x)H^{a^2,x} + \xi^{4p_1+8p_3}\epsilon_{a^2}(xa^2)H^{a^2,xa^2} \\ \xi^{-4p_1+8p_3}\epsilon_{a^2}(x)H^{a^2,x} + \xi^{-4p_1+8p_3}\epsilon_{a^2}(xa^2)H^{a^2,xa^2} & H_l^{a^2,1} + \epsilon_{a^2}(a^2)H_l^{a^2,a^2} \end{pmatrix}\\
&\quad \oplus \begin{pmatrix} H_l^{1,1} + H_l^{1,a^2} & \xi^{4p_1+8p_3}H_l^{1,x} + \xi^{4p_1+8p_3}H_l^{1,xa^2} \\ \xi^{-4p_1+8p_3}H_l^{1,x} + \xi^{-4p_1+8p_3}H_l^{1,xa^2} & H_l^{1,1} + H_l^{1,a^2} \end{pmatrix}\\
&\simeq \big(H_l^{a^2,1} + (-1)^{p_3}\epsilon_{a^2}(x)H_l^{a^2,x} + \epsilon_{a^2}(a^2)H_l^{a^2,a^2} + (-1)^{p_3}\epsilon_{a^2}(xa^2)H_l^{a^2,xa^2}\big)\\
&\quad \oplus\big(H_l^{a^2,1} - (-1)^{p_3}\epsilon_{a^2}(x)H_l^{a^2,x} + \epsilon_{a^2}(a^2)H_l^{a^2,a^2} - (-1)^{p_3}\epsilon_{a^2}(xa^2)H_l^{a^2,xa^2}\big)\\
&\quad \oplus\big(H_l^{1,1} + (-1)^{p_3}H_l^{1,x} + H_l^{1,a^2} + (-1)^{p_3}H_l^{1,xa^2}\big) \oplus \big(H_l^{1,1} - (-1)^{p_3}H_l^{1,x} + H_l^{1,a^2} - (-1)^{p_3}H_l^{1,xa^2}\big).
\end{aligned} \tag{G3}$$

For different values of p_1 and p_2, the anyon theory would be different after gauging. In Appendix H, we give a complete classification of the anyons in all cases. When $p_3 = 0$ here, one can then find the fusion rule from the above calculation as

$$0_a \times 0_a = 1 + e^{(1)} + m^{(2)} + e^{(1)}m^{(2)}. \tag{G4}$$

When p_3 takes value 1 instead of 0, the ribbon operators that create anyon $1/e^{(1)}/m^{(2)}/e^{(1)}m^{(2)}$ after the decomposition become ribbon operators that create $e^{(1)}/1/e^{(1)}m^{(2)}/m^{(2)}$, respectively. According to Eq. (39), we can conclude that different values of p_3 indeed give rise to different SFCs such that the fusion rules are shifted by the anyon $[e^{(1)}] \in H^2_\rho(Z_2, \mathcal{A})$.

APPENDIX H: COMPUTATION OF BRAIDING PHASES OF ANYONS IN TQD

In this Appendix, we compute the braiding phases of anyons in $Z_2 \times Z_2$ TQD obtained from gauging $Z_2 \times Z_2 \subset D_4$. First we define the slant product

$$H^n(G, U(1)) \xrightarrow{i_g^n} H^{n-1}(G, U(1)) \tag{H1}$$

as follows:

$$\begin{aligned} i_g^n\omega(g_1, \dots, g_{n-1}) &= \omega(g, g_1, \dots, g_{n-1})^{(-1)^{n-1}} \\ &\times \prod_{i=1}^{n-1} \omega(g_1, \dots, g_i, g, g_{i+1}, \dots, g_{n-1})^{(-1)^{n-1+i}}. \end{aligned} \tag{H2}$$

Now let us consider the cocycle given in Eq. (175):

$$\nu'(g, h, l) = (-1)^{p_1(g^{(2)}h^{(2)}l^{(2)} + g^{(2)}h^{(2)}l^{(1)}) + p_2 g^{(1)}h^{(1)}l^{(1)}}. \tag{H3}$$

One can calculate the slant product with $g = x, t$, and xt:

$$i_x\nu'(h, l) = (-1)^{p_1 h^{(2)}l^{(2)} + p_2 h^{(1)}l^{(1)}}, \tag{H4}$$

$$i_t\nu'(h, l) = (-1)^{p_1 h^{(2)}l^{(2)}}, \tag{H5}$$

$$i_{xt}\nu'(h, l) = (-1)^{p_2 h^{(1)}l^{(1)}}. \tag{H6}$$

These slant products give the projective phases in the projective representations μ_x, μ_t, and μ_{xt}, respectively, as follows:

$$\mu_x(h)\mu_x(l) = i_x\nu'(h, l)\mu_x(hl), \tag{H7a}$$

$$\mu_t(h)\mu_t(l) = i_t\nu'(h, l)\mu_t(hl), \tag{H7b}$$

$$\mu_{xt}(h)\mu_{xt}(l) = i_{xt}\nu'(h, l)\mu_{xt}(hl). \tag{H7c}$$

From Eq. (H6), we see that the projective representations are given, respectively, by

$$\mu_x(h) = i^{p_1 h^{(2)} + p_2 h^{(1)}}, \quad \mu_t(h) = i^{p_1 h^{(2)}}, \quad \mu_{xt}(h) = i^{p_2 h^{(1)}}. \tag{H8}$$

In addition to these projective representations of $Z_2 \times Z_2$, we have the respective ordinary representations

$$\mu_1(h) = (-1)^{h^{(1)}}, \ \ \mu_2(h) = (-1)^{h^{(2)}}, \ \ \mu_{12}(h) = (-1)^{h^{(1)} + h^{(2)}}. \tag{H9}$$

If we label the anyons in $Z_2 \times Z_2$ TQD by $e^{(1)}$, $e^{(2)}$, $m^{(1)}$, and $m^{(2)}$, where $e^{(i)}$ denote the elementary charges (chargeons) and $m^{(i)}$ denote the elementary fluxes, then charges are given by the ordinary representation of $Z_2 \times Z_2$ and the fluxes are given by the projective representations of $Z_2 \times Z_2$.

Then the general formula for calculating the braiding phase between anyons a and b is

$$B(a, b) = \mu_a(\text{flux}(b))\mu_b(\text{flux}(a)). \tag{H10}$$

We list the braiding phases between the various elementary charges and fluxes as follows:

$$\begin{aligned} &B(e^{(1)}, m^{(1)}) = \mu_1(x)\mu_x(1) = -1, \quad B(e^{(1)}, m^{(2)}) = \mu_1(t)\mu_t(1) = 1, \\ &B(e^{(2)}, m^{(1)}) = \mu_2(x)\mu_x(1) = 1, \quad B(e^{(2)}, m^{(2)}) = \mu_2(t)\mu_t(1) = -1, \\ &B(m^{(1)}, m^{(1)}) = \mu_x(x)^2 = i^{2p_2} = (-1)^{p_2}, \quad B(m^{(1)}, m^{(2)}) = \mu_x(t)\mu_t(x) = i^{p_1}, \\ &B(m^{(2)}, m^{(2)}) = \mu_t(t)^2 = i^{2p_1} = (-1)^{p_1}. \end{aligned} \tag{H11}$$

APPENDIX I: FUSION RULE IN SET FROM GAUGING S_3 SPT

We write the element in S_3 as $\tilde{g} = (G, g) \equiv x^G a^g$. We construct a representative of 3-cocycle in $H^3(S_3, U(1))$ as follows:

$$\begin{aligned} \omega(\tilde{g}, \tilde{h}, \tilde{l}) = \exp\bigg\{ &\frac{2\pi i p_1}{9} g(-1)^{H+L}(h(-1)^L + l \\ &- [h(-1)^L + l]_3) + \pi i p_2 GHL \bigg\}, \end{aligned} \tag{I1}$$

where $p_1 = 0, 1, 2$, and $p_2, = 0, 1$. Gauging the normal subgroup Z_3 of a S_3-SPT state results in a state in an SET state that has the same anyon theory as $D^\nu(Z_3)$, where

$$\nu(g, h, l) = \exp\left\{\frac{2\pi i p_1}{9} g(h + l - [h + l]_3)\right\} \tag{I2}$$

is the restriction of ω on Z_3. Different values of p_1 correspond to different Z_3-TQD models. The symmetry action takes an anyon to its inverse, which is nontrivial. In sector $\mathcal{C}_x$, there is only one object of quantum dimension 3. We name it 0_x. By dimension counting, we can write a fusion rule of the form

$$0_x \times 0_x = \sum_{i=1}^{9} a_i, \tag{I3}$$

where $a_i \in \mathcal{C}$ are Abelian anyons. Let $b_1 = 1$, $b_2 = a$, and $b_3 = a^2$. One can write a matrix-valued operator on an open ribbon as

$$\left(H_l^x\right)_{ii'} = H_l^{b_i x b_i^{-1}, b_i b_{i'}^{-1}} \epsilon_{b_i x b_i^{-1}}\left(b_i b_{i'}^{-1}\right), \tag{I4}$$

where the matrix indices are $i, i' = 1, 2, 3$. As we conjectured, the object $0_x \times 0_x$ should be created by the operator $(H_l^x)^{\otimes 2}$ on the endpoint of l. Let $\chi \equiv \exp\{\frac{2\pi i}{9}\}$, then a calculation similar to that in Appendix F gives us

$$\begin{aligned}
\left(H_l^x\right)^{\otimes 2} &= \begin{pmatrix} H_l^{x,1} & \epsilon_x(a^2)H_l^{x,a^2} & \epsilon_x(a)H_l^{x,a} \\ \epsilon_{xa}(a)H_l^{xa,a} & H_l^{xa,1} & \epsilon_{xa}(a^2)H_l^{xa,a^2} \\ \epsilon_{xa^2}(a^2)H_l^{xa^2,a^2} & \epsilon_{xa^2}(a)H_l^{xa^2,a} & H_l^{xa^2,1} \end{pmatrix}^{\otimes 2} \\
&= \begin{pmatrix} H_l^{x,1} & \chi^{-2p_1}H_l^{x,a^2} & \chi^{p_1}H_l^{x,a} \\ \chi^{-p_1}H_l^{xa,a} & H_l^{xa,1} & H_l^{xa,a^2} \\ \chi^{-4p_1}H_l^{xa^2,a^2} & H_l^{xa^2,a} & H_l^{xa^2,1} \end{pmatrix}^{\otimes 2} \\
&= \begin{pmatrix} H_l^{1,1} & \chi^{-4p_1}H_l^{1,a^2} & \chi^{2p_1}H_l^{1,a} \\ \chi^{-5p_1}H_l^{1,a} & H_l^{1,1} & \chi^{-3p_1}H_l^{1,a^2} \\ \chi^{-2p_1}H_l^{1,a^2} & \chi^{-6p_1}H_l^{1,a} & H_l^{1,1} \end{pmatrix} \oplus \begin{pmatrix} H_l^{a,1} & \chi^{-2p_1}H_l^{a,a^2} & \chi^{-3p_1}H_l^{a,a} \\ \chi^{-4p_1}H_l^{a,a} & H_l^{a,1} & \chi^{2p_1}H_l^{a,a^2} \\ \chi^{-3p_1}H_l^{a,a^2} & \chi^{p_1}H_l^{a,a} & H_l^{a,1} \end{pmatrix} \\
&\quad \oplus \begin{pmatrix} H_l^{a^2,1} & \chi^{-3p_1}H_l^{a^2,a^2} & \chi^{-2p_1}H_l^{a^2,a} \\ H_l^{a^2,a} & H_l^{a^2,1} & \chi^{-2p_1}H_l^{a^2,a^2} \\ \chi^{-p_1}H_l^{a^2,a^2} & \chi^{-p_1}H_l^{a^2,a} & H_l^{a^2,1} \end{pmatrix} \\
&= \begin{pmatrix} H_l^{1,1} & \chi^{-4p_1}H_l^{1,a^2} & \chi^{2p_1}H_l^{1,a} \\ \chi^{-5p_1}H_l^{1,a} & H_l^{1,1} & \chi^{-3p_1}H_l^{1,a^2} \\ \chi^{-2p_1}H_l^{1,a^2} & \chi^{-6p_1}H_l^{1,a} & H_l^{1,1} \end{pmatrix} \oplus \begin{pmatrix} H_l^{a,1} & \chi^{-4p_1}\epsilon_a(a^2)H_l^{a,a^2} & \chi^{-4p_1}\epsilon_a(a)H_l^{a,a} \\ \chi^{-5p_1}\epsilon_a(a)H_l^{a,a} & H_l^{a,1} & \epsilon_a(a^2)H_l^{a,a^2} \\ \chi^{-5p_1}\epsilon_a(a^2)H_l^{a,a^2} & \epsilon_a(a)H_l^{a,a} & H_l^{a,1} \end{pmatrix} \\
&\quad \oplus \begin{pmatrix} H_l^{a^2,1} & \chi^{-7p_1}\epsilon_{a^2}(a^2)H_l^{a^2,a^2} & \chi^{-4p_1}\epsilon_{a^2}(a)H_l^{a^2,a} \\ \chi^{-2p_1}\epsilon_{a^2}(a)H_l^{a^2,a} & H_l^{a^2,1} & \chi^{-6p_1}\epsilon_{a^2}(a^2)H_l^{a^2,a^2} \\ \chi^{-5p_1}\epsilon_{a^2}(a^2)H_l^{a^2,a^2} & \chi^{-3p_1}\epsilon_{a^2}(a)H_l^{a^2,a} & H_l^{a^2,1} \end{pmatrix} \\
&\simeq \left(H_l^{1,1} + H_l^{1,a} + H_l^{1,a^2}\right) \oplus \left(H_l^{1,1} + \chi^3 H_l^{1,a} + \chi^6 H_l^{1,a^2}\right) \oplus \left(H_l^{1,1} + \chi^{-3}H_l^{1,a} + \chi^{-6}H_l^{1,a^2}\right) \\
&\quad \oplus \left(H_l^{a,1} + \epsilon_a(a)H_l^{a,a} + \epsilon_a(a^2)H_l^{a,a^2}\right) \oplus \left(H_l^{a,1} + \chi^3\epsilon_a(a)H_l^{a,a} + \chi^6\epsilon_a(a^2)H_l^{a,a^2}\right) \\
&\quad \oplus \left(H_l^{a,1} + \chi^{-3}\epsilon_a(a)H_l^{a,a} + \chi^{-6}\epsilon_a(a^2)H_l^{a,a^2}\right) \oplus \left(H_l^{a^2,1} + \epsilon_{a^2}(a)H_l^{a^2,a} + \epsilon_{a^2}(a^2)H_l^{a^2,a^2}\right) \\
&\quad \oplus \left(H_l^{a^2,1} + \chi^3\epsilon_{a^2}(a)H_l^{a^2,a} + \chi^6\epsilon_{a^2}(a^2)H_l^{a^2,a^2}\right) \oplus \left(H_l^{a^2,1} + \chi^{-3}\epsilon_{a^2}(a)H_l^{a^2,a} + \chi^{-6}\epsilon_{a^2}(a^2)H_l^{a^2,a^2}\right).
\end{aligned} \tag{I5}$$

The nine parts in the last line after the decomposition are the nine ribbon operators in $D^{\nu}(Z_3)$ that create the anyons $1, e, e^2, m, em, e^2m, m^2, em^2$, and e^2m^2. Therefore, one can conclude the fusion rule from the above calculation:

$$0_x \times 0_x = 1 + e + e^2 + m + em + e^2m + m^2 + em^2 + e^2m^2. \tag{I6}$$

APPENDIX J: AN ALTERNATIVE N-STEP GAUGING VIA MEASUREMENT

In this Appendix, we give an alternative N-step gauging procedure. Following the two-step gauging, we generalize the procedure to N-step gauging (a similar method was proposed by Refs. [42,43] for solvable groups) for a group G that satisfies a criterion. The N-steps correspond to the N factors of Abelian groups of G.

Let us consider a group G with the following property: there exist a sequence of groups $N_0, N_1, \dots N_n$ Abelian and another sequence $M_0, M_1, \dots M_n$ such that

$$\begin{aligned}
&N_0 \equiv \{e\}, \quad M_0 = G, \\
&N_1 \lhd G, \quad M_1 = \frac{G}{N_1}, \\
&N_2 \lhd M_1, \quad M_2 = \frac{M_1}{N_2}, \\
&\quad\vdots \\
&N_n \lhd M_{n-1}, \quad M_n = \frac{M_{n-1}}{N_n} = e.
\end{aligned} \tag{J1}$$

If the group G satisfies this property, then we say it admits a sequential normal subgroups. We will prove in Appendix K that admitting sequential normal subgroups is equivalent to the group being solvable. Given G admits a sequential normal subgroup, we get a sequence of short exact sequences

$$1 \longrightarrow N_1 \xrightarrow{i_1} G \xrightarrow{\pi_1} M_1 \longrightarrow 1, \tag{J2a}$$

$$1 \longrightarrow N_2 \xrightarrow{i_2} M_1 \xrightarrow{\pi_2} M_2 \longrightarrow 1, \tag{J2b}$$

$$\vdots \tag{J2c}$$

$$1 \longrightarrow N_{n-1} \xrightarrow{i_{n-1}} M_{n-2} \xrightarrow{\pi_n} M_{n-1} \longrightarrow 1, \tag{J2d}$$

$$1 \longrightarrow N_n \xrightarrow{i_n} M_{n-1} \longrightarrow 1, \tag{J2e}$$

where i_k is an inclusion map and π_k is a projection map. Now we choose a sequence of lifts

$$M_1 \xrightarrow{s_1} G, \tag{J3a}$$

$$M_2 \xrightarrow{s_2} M_1, \tag{J3b}$$

$$\vdots \tag{J3c}$$

$$M_{n-1} \xrightarrow{s_{n-2}} M_n, \tag{J3d}$$

which will be fixed throughout this Appendix. With these lifts we can embed each of the normal groups N_k as a set in G. If $n \in N_k$, then $s_1(s_2(\ldots s_{k-1}(i_k(n)))) \in G$ is an embedding of $n \in N_k$ in G. To simplify notation, we denote $s_1 \circ s_2 \circ \ldots \circ s_k$ by $\tilde{s}_k$. Then $\tilde{s}_k : M_k \longrightarrow G$. Using this notation, a general $g \in G$ can be written as

$$g = \tilde{s}_{n-1}\big(i_n\big(a_n^{i_n}\big)\big)\ldots\tilde{s}_1\big(i_2\big(a_2^{i_2}\big)\big)i_1\big(a_1^{i_1}\big), \tag{J4}$$

where $a_j \in N_j$ is a generator for each Abelian normal subgroup. The notation $a_j^{i_j}$ is a shorthand for the product of generators for each cyclic subgroup of the Abelian group, i.e., $a_j^{i_j} = \prod_{k=1}^{l}(a_j^k)^{i_j^k}$ where a_j^k for $k = 1, \ldots, l$ are the generators for the l cyclic factors.

Claim J.1. The representation of $g \in G$ given in Eq. (J4) is unique. If $g = \tilde{s}_{n-1}(i_n(a_n^{i_n}))\ldots i_1(a_1^{i_1}) = \tilde{s}_{n-1}(i_n(a_n^{i'_n}))\ldots i_1(a_1^{i'_1})$, then $i_n = i'_n, \ldots, i_1 = i'_1$.

Proof. The proof follows by applying the projections $\tilde{\pi}_k := \pi_1 \circ \pi_2 \circ \ldots \circ \pi_k$ for $k = 1, \ldots, n-1$. First apply $\tilde{\pi}_{n-1}$ to g. This gives $i_n = i'_n$. Then apply $\tilde{\pi}_{n-2}$ which gives $i_{n-1} = i'_{n-1}$. Proceeding similarly, at kth step apply $\tilde{\pi}_{n-k}$ to get $i_{n-k+1} = i'_{n-k+1}$. At $(n-1)$th step we get $i_2 = i'_2$. This automatically fixes $i_1 = i'_1$ proving the unique representation of g. ■

To simplify the notation in the remaining part of this Appendix, we omit writing the lifts explicitly and write $a_k^{i_k} \equiv \tilde{s}_{k-1}(i_k(a_k^{i_k}))$. Hence,

$$g = a_n^{i_n}\ldots a_1^{i_1}. \tag{J5}$$

Let $h \in G$. Similarly, we can write $h = a_n^{\bar{i}_n}\ldots a_1^{\bar{i}_1}$. Using this notation, we can write down the group multiplication as

$$gh^{-1} = a_n^{i_n}\ldots a_1^{i_1}a_1^{-\bar{i}_1}\ldots a_n^{-\bar{i}_n}. \tag{J6}$$

We will use Eq. (J6) to implement the N-step gauging procedure. We will gauge the G DOFs on the vertices of the lattice sequentially in N-steps. The complete procedure for N-step gauging is as follows:

(1) *Include ancillas.* Add ancillas in the state $|e\rangle$, where $e \in G$ is the identity element, on the edges between the vertices.

(2) *Entangle gauge and matter DOFs.* Apply the following two controlled-shift operators with controls c_1, c_2 on neighboring vertices (oriented as $c_2 \to c_1$) and target t on the in-between ancilla,

$$U_{N_1} = \sum_{g_1,g_2\in G}\sum_{g_3\in G}|g_1, g_2\rangle_{c_1,c_2}\langle g_1, g_2| \otimes |N_1(g_1)g_3N_1(g_2)^{-1}\rangle_e\langle g_3|. \tag{J7}$$

Here $N_1(g)$ is the part of the decomposition g which lies in N_1; when $g = a_n^{i_n}\ldots a_1^{i_1}$ with $a_j \in N_j$, then $N_1(g) = a_1^{i_1}$.

(3) *Measure X_{N_1} on matter DOFs and correct the z_{N_1} factors.* Define: $X_{N_1} = \{X_1, \ldots, X_{n_{m_1}}\}$ for $N_1 = \prod_{k=1}^{m_1}\mathbb{Z}_{n_k}$, where X_j denote $j \times j$ Pauli X matrix. Following the same define: $z_{N_1} = \prod_{k=1}^{m_1} z_{n_k}$, where z_{n_k} is the phase factor coming from measuring X_{n_k} on the vertex. The value of z_{n_k} is same as acting Pauli Z_{n_k} operator on the vertex before measurement. After measurement, with the outcome being $X_{N_1} = \{\omega_1^{-p_1}, \ldots, \omega_1^{-p_{m_1}}\}$ (ω_k being n_kth root of unity), there is a corresponding phase factor $\prod_{k=1}^{m_1} z_{n_k}^{-p_k}$. Using the transmutation rule for each of the phase terms in the product z_{N_1},

$$Z_{n_k}(N_1(g_2))Z_{n_k}(N_1(g_1)N_1(g_2)^{-1}) = Z_{n_k}(N_1(g_1)). \tag{J8}$$

One can correct all those factors by moving them to a single vertex, resulting in an M_1 SET ground state.

(4) *Repeat the procedure of entangling gauge and matter DOFs for M_1 DOFs on the vertices.* Apply the following unitary as before,

$$U_{N_2} = \sum_{g_1,g_2\in M_1}\sum_{g_3\in G}|g_1, g_2\rangle_{c_1,c_2}\langle g_1, g_2| \otimes |N_2(g_1)g_3N_2(g_2)^{-1}\rangle_e\langle g_3|. \tag{J9}$$

(5) *Measure X_{N_2} on matter DOFs and correct the z_{N_2} factors.* Define: $X_{N_2} = \{X_1, \ldots, X_{n_{m_2}}\}$ for $N_2 = \prod_{k=1}^{m_2}\mathbb{Z}_{n_k}$, where X_j denote $j \times j$ Pauli X matrix. Following the same define: $z_{N_2} = \prod_{k=1}^{m_2} z_{n_k}$, where z_{n_k} is the phase factor coming from measuring X_{n_k} on the vertex. The value of z_{n_k} is same as acting Pauli Z_{n_k} operator on the vertex before measurement. After measurement, with the outcome being $X_{N_2} = \{\omega_1^{-p_1}, \ldots, \omega_1^{-p_{m_2}}\}$ (ω_k being n_kth root of unity), there is a corresponding phase factor $\prod_{k=1}^{m_2} z_{n_k}^{-p_k}$. Using the transmutation rule for each of the phase terms in the product z_{N_2},

$$Z_{n_k}(N_2(g_2))Z_{n_k}(N_2(g_1)N_2(g_2)^{-1}\phi^{N_2(g_2)}(g_e)) = Z_{n_k}(N_2(g_1)). \tag{J10}$$

One can correct all those factors by moving them to a single vertex, resulting in an M_2 SET ground state. (Note that $Z_{n_k}(\phi^{N_2(g_2)}(g_3)) = 1$. This is because $\phi^{N_2(g_2)}(g_3) \in N_1$ for $g_3 \in N_1$ and hence has no component in N_2.)

(6) *At lth step of gauging process, again repeat the procedure of entangling gauge and matter DOFs for M_{l-1} DOFs on the vertices.* Apply the following unitary:

$$U_{N_l} = \sum_{g_1,g_2\in M_{l-1}}\sum_{g_3\in G}|g_1, g_2\rangle_{c_1,c_2}\langle g_1, g_2| \otimes |N_l(g_1)g_3N_l(g_2)^{-1}\rangle_e\langle g_3|. \tag{J11}$$

(7) *Measure X_{N_l} on matter DOFs and correct the z_{N_l} factors.* Define: $X_{N_l} = \{X_1, \ldots, X_{n_{m_l}}\}$ for $N_l = \prod_{k=1}^{m_l}\mathbb{Z}_{n_k}$, where

X_j denote $j \times j$ Pauli X matrix. Following the same define: $z_{N_l} = \prod_{k=1}^{m_l} z_{n_k}$, where z_{n_k} is the phase factor coming from measuring X_{n_k} on the vertex. The value of z_{n_k} is same as acting Pauli Z_{n_k} operator on the vertex before measurement. After measurement, with the outcome being $X_{N_l} = \{\omega_1^{-p_1}, \dots, \omega_1^{-p_{m_l}}\}$ (ω_k being n_kth root of unity), there is a corresponding phase factor $\prod_{k=1}^{m_l} z_{n_k}^{-p_k}$. Using the transmutation rule for each of the phase terms in the product z_{N_l},

$$Z_{n_k}(N_l(g_2))Z_{n_k}(N_l(g_1)N_l(g_2)^{-1}\phi^{N_l(g_2)}(g_e)) = Z_{n_k}(N_l(g_1)), \tag{J12}$$

one can correct all those factors by moving them to a single vertex, resulting in an M_l SET ground state. (Note that $Z_{n_k}(\phi^{N_l(g_2)}(g_3)) = 1$. This is because $\phi^{N_l(g_2)}(g_3) \in (...(N_1 \otimes N_2) \otimes .... \otimes N_{l-1})$ for $g_3 \in (...(N_1 \otimes N_2) \otimes ... \otimes N_{l-1})$ and hence has no component in N_l. The notation $G \otimes H$ is a short hand for group extension of G by H.)

(8) *Repeat this process for all till the last normal subgroup N_n.*

The groups formed from extensions are

$$\begin{aligned} G_l &\equiv ((N_1 \otimes N_2) \otimes N_3)... \otimes N_l \\ &\equiv \left\{\tilde{s}_{l-1}\left(a_l^{i_l}\right)...\tilde{s}_1\left(a_2^{i_2}\right)i\left(a_1^{i_1}\right)\middle|a_l^{i_l} \in N_l, \dots, a_1^{i_1} \in N_1\right\}. \end{aligned} \tag{J13}$$

Claim J.2. G_l is a subgroup of G.

Proof. We prove this by induction. First we prove G_2 is subgroup of G. $G_2 = N_1 \otimes N_2$. Suppose $g_1 = s(q_1)i(n_1) \in G_2$ and $g_2 = s(q_2)i(n_2) \in G_2$, then $g_1g_2 = s(q_1)i(n_1)s(q_2)i(n_2) = s(q_1)s(q_2)s(q_2)^{-1}(i(n_1))s(q_2)i(n_2) = s(q_1q_2)\omega(q_1, q_2)\phi^{q_2^{-1}}(i(n_1))i(n_2)$. Hence, $g_1g_2 \in G_2$. Assume G_{l-1} is a subgroup of G. We prove G_l is a subgroup of G. Suppose $g_1 = \tilde{s}_{l-1}(a_l^{\bar{i}_l})...\tilde{s}_1(a_2^{\bar{i}_2})i(a_1^{\bar{i}_1}) \in G_l$, $g_2 = \tilde{s}_{l-1}(a_l^{i_l})...\tilde{s}_1(a_2^{i_2})i(a_1^{i_1}) \in G_l$. Let us define

$$\begin{aligned} b_1 &= \tilde{s}_{l-1}\left(a_l^{i_l}\right)...\tilde{s}_1\left(a_2^{i_2}\right) \\ b_2 &= \tilde{s}_{l-1}\left(a_l^{i_l}\right)...\tilde{s}_2\left(a_3^{i_3}\right) \\ &\vdots \\ b_l &= \tilde{s}_{l-1}\left(a_l^{i_l}\right). \end{aligned} \tag{J14}$$

Then

$$\begin{aligned} g_1g_2 = &\tilde{s}_{l-1}\left(a_l^{\bar{i}_l}\right)\tilde{s}_{l-1}\left(a_l^{i_l}\right)\phi^{b_l^{-1}}\left(\tilde{s}_{l-2}\left(a_{l-1}^{\bar{i}_{l-1}}\right)\right)\tilde{s}_{l-2} \\ &\times \left(a_{l-1}^{i_{l-1}}\right)...\phi^{b_1^{-1}}\left(i\left(a_1^{\bar{i}_1}\right)\right)i\left(a_1^{i_1}\right). \end{aligned} \tag{J15}$$

Only the first two terms in the above equation lies in N_l. The remaining terms lie in G_{l-1}. Hence, they can be expressed as

$$\begin{aligned} &\phi^{b_l^{-1}}\left(\tilde{s}_{l-2}\left(a_{l-1}^{\bar{i}_{l-1}}\right)\right)\tilde{s}_{l-2}\left(a_{l-1}^{i_l}\right)...\phi^{b_1^{-1}}\left(i\left(a_1^{\bar{i}_1}\right)\right)i\left(a_1^{i_1}\right) \\ &\quad = \tilde{s}_{l-2}\left(a_{l-1}^{i'_{l-1}}\right)....i\left(a_1^{i'_1}\right). \end{aligned} \tag{J16}$$

Now we prove that $\tilde{s}_{l-1}(a_l^{\bar{i}_l})\tilde{s}_{l-1}(a_l^{i_l}) = \tilde{s}_{l-1}(a_l^{\bar{i}_l+i_l})h$ where $h \in G_{l-1}$. Then by induction hypothesis,

$$g_1g_2 = \tilde{s}_{l-1}(a_l^{\bar{i}_l+i_l})\tilde{s}_{l-2}(a_{l-1}^{i''_{l-1}})...\tilde{s}_1(a_2^{i''_2})i(a_1^{i''_1}), \tag{J17}$$

which shows that $g_1g_2 \in G_l$. First note that applying the relation $s(a)s(b) = s(ab)\omega(a, b)$, we get

$$\begin{aligned} \tilde{s}_{l-1}\left(a_l^{\bar{i}_l}\right)\tilde{s}_{l-1}\left(a_l^{i_l}\right) &= s\left(\tilde{s}_{l-2}\left(a_l^{\bar{i}_l}\right)\tilde{s}_{l-2}\left(a_l^{i_l}\right)\right)\omega\left(\tilde{s}_{l-2}\left(a_l^{\bar{i}_l}\right), \tilde{s}_{l-2}\left(a_l^{i_l}\right)\right), \\ &= s\left(s\left(\tilde{s}_{l-3}\left(a_l^{\bar{i}_l}\right)\tilde{s}_{l-3}\left(a_l^{i_l}\right)\right)\omega\left(\tilde{s}_{l-3}\left(a_l^{\bar{i}_l}\right), \tilde{s}_{l-3}\left(a_l^{i_l}\right)\right)\right)\omega\left(\tilde{s}_{l-2}\left(a_l^{\bar{i}_l}\right), \tilde{s}_{l-2}\left(a_l^{i_l}\right)\right), \\ &\vdots \end{aligned} \tag{J18}$$

One can write this equation in short hand as

$$\begin{aligned} \tilde{s}_{l-1}\left(a_l^{\bar{i}_l}\right)\tilde{s}_{l-1}\left(a_l^{i_l}\right) &= s(M_1)N_1, \\ &= s(s(M_2)N_2)N_1, \\ &= s(s(s(M_3)N_3)N_2)N_1, \\ &\vdots \\ &= s(...(s(M_{l-1})N_{l-1})...)N_1, \end{aligned} \tag{J19}$$

where N_l denote the terms coming from ω factors in Eq. (J18) and M_r denote $\tilde{s}_{l-1-r}(a_l^{\bar{i}_l})\tilde{s}_{l-1-r}(a_l^{i_l})$, M_{l-1} denote $a_l^{\bar{i}_l+i_l}$. Now applying the relation $s(a)s(b)\omega(a, b)^{-1} = s(ab)$, we get

$$\tilde{s}_{l-1}\left(a_l^{\bar{i}_l+i_l}\right) \times \text{ (terms in } G_{l-1}). \tag{J20}$$

By induction hypothesis, terms in G_{l-1} can be written as $\tilde{s}_{l-2}(a_{l-1}^{\bar{i}_{l-1}})\tilde{s}_{l-3}(a_{l-2}^{\bar{i}_{l-2}})...\tilde{s}_1(a_1^{\bar{i}_1})i(a_1^{\bar{i}_1})$. Combining the terms in Eq. (J16), we get the desired decomposition of g_1g_2 as in Eq. (J17) which prove $g_1g_2 \in G_l$. One can show if $g \in G_l$, $g^{-1} \in G_l$ by applying $\tilde{\pi}_{l-1}, \tilde{\pi}_{l-2}, \dots$ upto π on g^{-1}. This will give the explicit decomposition of g^{-1} using each of the normal subgroups N_i for $i = 1, \dots, l$. ■

Claim J.3. G_l is normal in G.

Proof. Let $g \in G$. $g = \tilde{s}_{n-1}(a_n^{i_n})...\tilde{s}_1(a_2^{i_2})i(a_1^{i_1})$. If $k \in G_l$, then let $k = \tilde{s}_{l-1}(a_l^{\bar{i}_l})...\tilde{s}_1(a_2^{\bar{i}_2})i(a_1^{\bar{i}_1})$. Then $gkg^{-1} = \tilde{s}_{n-1}(a_n^{i_n})...\tilde{s}_1(a_2^{i_2})i(a_1^{i_1})\tilde{s}_{l-1}(a_l^{\bar{i}_l})...\tilde{s}_1(a_2^{\bar{i}_2})i(a_1^{\bar{i}_1})i(a_1^{i_1})^{-1}\tilde{s}_1(a_2^{i_2})^{-1} \dots \tilde{s}_{n-1}(a_n^{i_n})^{-1}$. Applying $\tilde{\pi}_i$ for $i = n-1, \dots, l$, we see that $\tilde{\pi}_i(gkg^{-1}) = 1$. This shows that $gkg^{-1} \in G_l$. Hence, G_l is normal in G. ■

APPENDIX K: PROOF OF SOLVABLE EQUIVALENT TO ADMITTING SEQUENTIAL NORMAL ABELIAN SUBGROUPS

In this Appendix, we prove that the assumption about the group G we used in Appendix J is equivalent to the assumption that G is a solvable group.

Definition 1. A derived series of a finite group G is a sequence of normal subgroups normal inside the previous one

$$G \rhd G^{(1)} \rhd G^{(2)} \rhd G^{(3)} .... \rhd G^{(n)} \rhd e, \tag{K1}$$

for some n such that the quotient groups $G/G^{(1)}$, $G^{(1)}/G^{(2)}, \dots, G^{(n)}/e$ are all Abelian.

Definition 2. A finite group G is solvable if it admits a derived series.

Proposition K.1. If G is solvable, then it admits the following sequence:

$$G \triangleright G^1 \triangleright G^2 \triangleright \ldots \triangleright G^m \triangleright e, \tag{K2}$$

for some m which is called the derived length of the group G. Here $G^{i+1} = [G^i, G^i]$ is the commutator subgroup of G^i.

Proof. Note that the commutator subgroup of a group G is the smallest normal subgroup in G such that $G/[G, G]$ is Abelian. Hence, we have $G^1 \subset G^{(1)}$. Now we have $G^2 = [G^1, G^1] \subset [G^{(1)}, G^{(1)}] \subset G^{(2)}$. Inductively we can assume that $G^k \subset G^{(k)}$. Then $G^{k+1} = [G^k, G^k] \subset [G^{(k)}, G^{(k)}] \subset G^{(k+1)}$. Hence, $G^k \subset G^{(k)}\ \forall k \in \{1, 2, \ldots, m\}$. This clearly says that the sequence of commutator groups does not terminate. If it would have terminated at G^{k+1}, then $G^{k+2} = [G^{k+1}, G^{k+1}] = G^{k+1}$. One can repeat this to argue $G^{k+1} \subset G^{(k+1)}$, $G^{k+1} \subset G^{(k+2)}, \ldots$ $G^{k+1} \subset G^{(n)}$. But G^{k+1} is non-Abelian group and $G^{(n)}$ is Abelian. Hence, we cannot have $G^{k+1} \subset G^{(n)}$, contradiction. So the sequence of commutator subgroups does not terminate and we have the sequence. ■

Definition 3. We say a finite group G admits a sequential normal subgroups if it satisfies the following property:

$$\begin{aligned} &N_1 \triangleleft G, N_1 \text{ Abelian} \quad M_1 = G/N_1, \\ &N_2 \triangleleft M_1, N_2 \text{ Abelian} \quad M_2 = M_1/N_2, \\ &\vdots \\ &N_n \triangleleft M_{n-1}, N_n \text{ Abelian} \quad M_n = M_{n-1}/N_n = e. \end{aligned} \tag{K3}$$

Claim K.1. Suppose the finite group G is solvable then it admits sequential normal subgroups.

Proof. From proposition K.1 we see that G admits the sequence

$$G \triangleright G^1 \triangleright G^2 \triangleright \ldots \triangleright G^m \triangleright e, \tag{K4}$$

where $G^{i+1} = [G^i, G^i]$ is the commutator subgroup. First we prove that $G^k \triangleleft G\ \forall k \in \{1, 2, \ldots, m\}$. This we prove by induction on k. Clearly, $G^1 \triangleleft G$. Assuming $G^{k-1} \triangleleft G$, we need to prove $G^k \triangleleft G$. $G^k = [G^{k-1}, G^{k-1}]$. Hence, G^k is generated by elements of the form $ghg^{-1}h^{-1}$ where $g, h \in G^{k-1}$. Now one can write $kghg^{-1}h^{-1}k^{-1} = kgk^{-1}khk^{-1}kg^{-1}k^{-1}kh^{-1}k^{-1}$. Since $G^{k-1} \triangleleft G$, $kgk^{-1} \in G^k\ \ \forall g \in G^k$ and $k \in G$. So $kgk^{-1}khk^{-1}kg^{-1}k^{-1}kh^{-1}k^{-1} \in [G^{k-1}, G^{k-1}] = G^k$. So we see that $G^k \triangleleft G$.

Now we choose

$$\begin{aligned} N_1 = G^m \triangleleft G, N_1 \text{ Abelian} \quad & M_1 = G/N_1, \\ N_2 = G^{m-1}/G^m \triangleleft G/G^m, N_2 \text{ Abelian} \quad & M_2 = M_1/N_2, \\ & \vdots \\ N_{m+1} = G/G^1, N_{m+1} \text{ Abelian} \quad & M_{m+1} = M_m/N_{m+1} = e. \end{aligned} \tag{K5}$$

■

Claim K.2. If the finite group G admits a sequential normal subgroups, then G is solvable.

Proof. Suppose G is not solvable, assuming it admits sequential normal subgroups. Then the derived series of commutator subgroup terminate

$$G \triangleright G^1 \triangleright G^2 \triangleright \ldots \triangleright G^k \tag{K6}$$

for some k, where $G^{i+1} = [G^i, G^i]$. Now one could consider the series

$$\begin{aligned} &M^1 \triangleright M^{1(1)} \triangleright M^{1(2)} \triangleright \ldots \\ &M^2 \triangleright M^{2(1)} \triangleright M^{2(2)} \triangleright \ldots \\ &\vdots \\ &M^l \triangleright M^{l(1)} \triangleright M^{l(2)} \triangleright \ldots \\ &M^{n-1} \triangleright M^{(n-1)(1)} \triangleright M^{(n-1)(2)} \triangleright \ldots, \end{aligned} \tag{K7}$$

where $M^{j(l+1)} = [M^{j(l)}, M^{j(l)}]$ is the commutator subgroup and $M^{j(0)} \equiv M^j$. Now let us look at the following proposition.

Proposition K.2. If the derived series of commutator subgroups of G terminates, then so does for $M^1, M^2, \ldots, M^{n-1}$.

Proof. Consider $[M^1, M^1] = [G/N_1, G/N_1]$. It is generated by $gN_1g'N_1g^{-1}N_1g'^{-1}N_1 = gg'g^{-1}g'^{-1}N_1$. We know that $gg'g^{-1}g'^{-1} \in G^1$. However, $gN_1 = N_1$ if and only if $g \in N_1 \cap G^1$. Hence, we find $[G/N_1, G/N_1] \cong G^1/(N_1 \cap G^1)$. Repeating this we find $[G^1/(N_1 \cap G^1), G^1/(N_1 \cap G^1)] = G^2/(N_1 \cap G^2)$ and so on. Hence, the derived series for commutator subgroups for M^1 is given by

$$\begin{aligned} &G/N_1 \triangleright G^1/(N_1 \cap G^1) \triangleright G^2/(N_1 \cap G^2) \triangleright \ldots \\ &\triangleright G^k/(N_1 \cap G^k). \end{aligned} \tag{K8}$$

This terminates since $[G^k, G^k] = G^k$ and hence $[G^k/(N_1 \cap G^k), G^k/(N_1 \cap G^k)] = G^k/(N_1 \cap G^k)$. A similar argument shows that all other derived series terminates. ■

Now we have the following terminating derived series

$$\begin{gathered} G \triangleright G^1 \triangleright \ldots \triangleright G^k, \\ G/N_1 \triangleright G^1/(N_1 \cap G^1) \triangleright \ldots \triangleright G^k/(N_1 \cap G^k), \\ M^1/N_2 \triangleright M^{1(1)}/(N_2 \cap M^{1(1)}) \triangleright \ldots \triangleright M^{1(k)}/(N_2 \cap M^{1(k)}), \\ \vdots \\ M^l/N_{l+1} \triangleright M^{l(1)}/(N_{l+1} \cap M^{l(1)}) \triangleright \ldots \triangleright M^{l(k)}/(N_{l+1} \cap M^{l(k)}), \\ \vdots \\ M^{n-2}/N_{n-1} \triangleright M^{(n-2)(1)}/(N_{n-1} \cap M^{(n-2)(1)}) \triangleright \ldots \triangleright M^{(n-2)(k)}/(N_{n-1} \cap M^{(n-2)(k)}). \end{gathered} \tag{K9}$$

Since the last series is $M^{n-1} = N^n \triangleright e$, it terminates in length 1. This is a contradiction. Hence, G is solvable. ■

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