Energy transport across two interacting quantum baths without quasiparticles

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Energy transport in quantum many-body systems with well defined quasiparticles has recently attracted interest across different fields, including out of equilibrium conformal field theories, one-dimensional quantum lattice models, and holographic matter. Here we study energy transport between two interacting quantum baths without quasiparticles made by two Sachdev-Ye-Kitaev (SYK) models at temperatures $T_L \neq T_R$ and connected by a Fermi-liquid system. We obtain an exact expression for the nonequilibrium energy current, valid in the limit of large bath and system size and for any system-bath coupling V. We show that the peculiar criticality of the SYK baths has direct consequences on the thermal conductance, which above a temperature $T^*(V) \sim V^4$ is parametrically enhanced with respect to the linear-T behavior expected in systems with quasiparticles. Interestingly, below $T^*(V)$ the linear thermal conductance behavior is restored, yet transport is not due to quasiparticles. Rather the system gets strongly renormalized by the bath and becomes non-Fermi liquid and maximally chaotic. Finally, we discuss the full nonequilibrium energy current and show that its form is compatible with the structure $\mathcal{J} = \Phi(T_L) - \Phi(T_R)$, with $\Phi(T) \sim T^\gamma$ and power law crossing over from $\gamma = 3/2$ to $\gamma = 2$ below T^*.

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I. INTRODUCTION

Nonequilibrium heat and energy transport phenomena in strongly interacting quantum matter are attracting interest across condensed matter, atomic physics, statistical mechanics and high-energy physics. From one side new experimental platforms to explore quantum heat transport in mesoscopic systems [1], ultracold atomic gases [2], or strongly correlated quantum materials [3,4] have raised new interest on this topic. In parallel, fresh theoretical understanding on quantum many-body systems far from equilibrium has brought forward new results on energy transport and surprising universalities [5,6]. A well established paradigm concerns systems in which energy transport is due to well defined quasiparticles such as mesoscopic systems made by ballistic channels, leading to the quantum of thermal conductance recently measured experimentally [7]. For one-dimensional integrable quantum many-body systems, where quasiparticles scatter elastically, several results have been obtained concerning linear response energy transport, both in quantum lattice models [8] and in Luttinger liquids [9,10] as well as on the full nonequilibrium energy current [11–14]. The latter was found to display a universal form, predicted by out of equilibrium conformal field theories and related to the Stefan-Boltzmann law [5,15,16]. Deviations due to irrelevant operators have been also actively discussed [12,17].

The general transport behavior of strongly coupled quantum matter which lacks any quasiparticle is on the other hand much less understood. The Sachdev-Ye-Kitaev (SYK) model [18–20] has emerged in recent years as paradigmatic model for non-Fermi liquids (NFLs) [21–25], featuring a peculiar criticality associated to an emergent conformal invariance and which leads to maximal chaos [26,27]. Understanding transport properties of models in the SYK family and their crossover to more conventional Fermi liquid (FL) behavior can open new windows in our understanding of exotic phases of matter such as planckian metals [24,28].

In this paper, we study the nonequilibrium energy transport between two maximally chaotic reservoirs, described by the SYK model, in equilibrium at different temperatures and connected by tunnel coupling to a FL quantum dot. We note that in the literature the interest has been focused on the study of energy transport and thermal conductivity of SYK-like systems coupled to FL contacts (leads) [29–31] or in arrays of SYK dots [21,29,32–34]. Here instead we discuss the role of interactions and maximal chaos in the reservoirs and compare it to the case in which well defined quasiparticles are present both in the system and in the environments. We note that conventional quantum transport settings involve environments which are described as a collection of harmonic excitations or quasiparticles. Our focus here is therefore to understand the consequences for energy transport of a quantum bath that lacks any coherent quasiparticle excitation and is described by an SYK model, as a simple solvable realization of a NFL. We expect our results therefore to not apply to other interacting quantum baths for which the low-energy description of physical degrees of freedom remains FL, even though these excitations fractionalize into partons which are described by SYK-like models [35].

To attack this problem, we derive an exact formula for the energy current through the system which takes the form of a generalized Meir-Wingreen formula [36] for interacting reservoirs. We discuss the linear transport regime and show how the NFL nature of the bath leaves clear fingerprints in the thermal conductance, which is parametrically enhanced by a weak coupling V to the SYK environment and at low temperature crosses over to a linear temperature scaling. Interestingly we show that the system at low temperature is not a FL, despite the linear thermal conductance, but rather is strongly renormalized by the SYK quantum bath, leading to anomalous

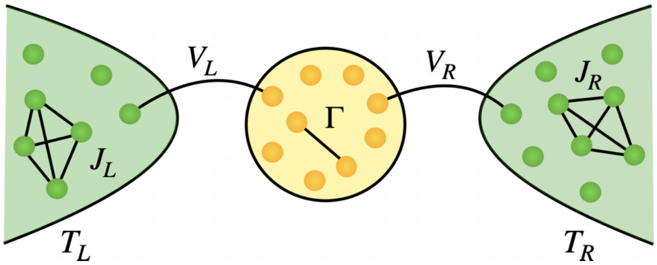

FIG. 1. Sketch of the setup: two SYK models in equilibrium at different temperatures $T_L \neq T_R$ are connected through a Fermi liquid dot with random all to all couplings.

spectral function and maximal chaos. Furthermore, we compute the full out of equilibrium energy current and provide evidence that it takes the form $\mathcal{J} = \Phi(T_L) - \Phi(T_R)$, with $\Phi(T) = T^{\gamma}$ and a power-law exponent crossing over from $\gamma = 3/2$ to $\gamma = 2$, as the temperature goes below a crossover scale $T^*(V) \sim V^4$.

The manuscript is organized as follows. In Sec. II, we introduce the SYK thermal transport setup focus of this chapter. In Sec. III, we derive the exact formula for the energy current through the system in terms of Green's functions. We use this formula to discuss linear transport in Sec. IV and nonlinear effects in Sec. V. Finally, in Sec. VI, we draw our conclusions.

II. SYK THERMAL TRANSPORT SETUP

We consider the transport setup represented in Fig. 1 where two sets of M randomly interacting Majorana fermions ψ_a^{α} ($a = 1, \dots, M$, $\alpha = L, R$) described by the SYK_4 model in equilibrium at temperatures T_L, T_R, are suddenly connected by an island made of N noninteracting Majorana fermions χ_i ($i = 1, \dots, N$) with random hoppings, described by the SYK_2 model. The total Hamiltonian is

$$H = \sum_{\alpha=L,R} H_4^{\alpha} + H^S + \sum_{\alpha=L,R} H^{S\alpha}, \tag{1}$$

where H_4^{α} with $\alpha = L/R$ describes the left/right SYK_4 reservoirs with Hamiltonian

$$H_4^{\alpha} = -\frac{1}{4!} \sum_{a,b,c,d=1}^{M} J_{abcd}\, \psi_a^{\alpha} \psi_b^{\alpha} \psi_c^{\alpha} \psi_d^{\alpha}. \tag{2}$$

Here H^S describes the island of noninteracting Majorana fermions with SYK_2 Hamiltonian

$$H^S = \frac{i}{2} \sum_{i,j=1}^{N} \Gamma_{ij}\, \chi_i \chi_j \tag{3}$$

and the remaining terms describe a linear coupling between reservoir and island

$$H^{S\alpha} = i \sum_{i=1}^{N} \sum_{a=1}^{M} V_{ia}\, \chi_i \psi_a^{\alpha}. \tag{4}$$

The couplings entering the Hamiltonian, $J_{abcd}, \Gamma_{ij}, V_{ia}$ are all independent Gaussian random variables with zero mean and variance respectively $\overline{J_{abcd}^2} = \frac{3!J^2}{M^3}$, $\overline{V_{ia}^2} = \frac{V^2}{M}$, and $\overline{\Gamma_{ij}^2} = \frac{\Gamma^2}{N}$. We consider the two reservoirs to be identical and equally coupled to the system and set $J_L = J_R = J$ and $V_L = V_R = V$ in the following. We emphasize that the choice of H^S to be noninteracting is made for the sake of highlighting the transport anomalies due to the interacting and maximally chaotic reservoirs, but can be relaxed as we will discuss later on.

Finally, we will compare this transport setting to the more conventional case of FL reservoirs described by the SYK_2 model,

$$H_2^{\alpha} = \frac{i}{2} \sum_{a,b=1}^{M} J_{ab}\, \psi_a^{\alpha} \psi_b^{\alpha}, \tag{5}$$

coupled to the same system Hamiltonian (3) through the linear term in Eq. (4). As before, the coupling entering the SYK_2 reservoirs are all independent Gaussian random variables with zero mean and variance respectively $\overline{J_{ab}^2} = \frac{J_2^2}{M}$, with $J_2 = J_L = J_R = J$ playing the role of unit of energy in the following. In this setting with FL baths, one expects ballistic energy transport due to quasiparticles, as we will indeed show later. We note that in the literature related models have appeared discussing the effect of coupling one (or multiple) noninteracting bath to the SYK_4 model and also studying transport [37–41].

The model introduced in this section, for both SYK_2 and SYK_4 types of baths, is exactly solvable using Keldysh techniques in the limit $N, M \to \infty$ at fixed $p \equiv N/M$. Here we will focus on the energy transport, namely on the stationary state current that sets at long times through the two reservoirs when $T_L \neq T_R$.

Keldysh formalism and Schwinger-Dyson equations

In this section, we use Keldysh formalism to derive the exact Schwinger-Dyson equations for the single-particle Green's functions of system and baths in the large N, M limit. We write down the partition function on the closed-time Keldysh contour $Z = \int \mathcal{D}[\chi, \psi L, \psi^R] e^{iS[\chi,\psi^L,\psi^R]}$ with the action $S[\chi, \psi^L, \psi^R]$

$$\begin{aligned} S[\chi, \psi^L, \psi^R] = \int_{-\infty}^{+\infty} \sum_{s=\pm} s \Bigg\{ & \frac{i}{2} \sum_{i,j=1}^{N} \chi_i^s(t) \partial_t \chi_i^s(t) + \frac{i}{2} \sum_{i,j=1}^{M} \sum_{\alpha=L,R} \psi_a^{\alpha,s}(t) \partial_t \psi_b^{\alpha,s}(t) - \frac{i}{2} \sum_{i,j=1}^{N} \Gamma_{ij}\, \chi_i^s \chi_i^s \\ & - \frac{i^{\frac{q_B}{2}}}{q_B!} \sum_{a_1,\cdots,a_{q_B}=1}^{M} \sum_{\alpha=L,R} J_{a_1,\cdots,a_{q_B}} \psi_{a_1}^{\alpha,s} \cdots \psi_{a_{q_B}}^{\alpha,s} - i\,\theta(t) \sum_{i=1}^{N} \sum_{a=1}^{M} \sum_{\alpha=L,R} V_{ia}\, \chi_i^s \psi_a^{\alpha,s} \Bigg\}. \end{aligned} \tag{6}$$

Here $s = \pm$ denotes the upper and lower branches of the closed-time contour and $\alpha = L, R$ the bath fermions. To obtain a more compact notation able to describe at once both types of environments (SYK_2 and SYK_4) we have introduced the parameter

$q_B = 2, 4$ for the SYK_2 and SYK_4 reservoirs, respectively. After averaging the partition function over the disorder we can rewrite the action in terms of the bilocal fields

$$G_S^{ss'}(t_1,t_2) = -\frac{i}{N}\sum_{i=1}^{N}\left\langle \chi_i^s(t_1)\chi_i^{s'}(t_2)\right\rangle \equiv \begin{pmatrix} G_S^T(t,t') & G_S^<(t,t') \\ G_S^>(t,t') & G_S^{\tilde{T}}(t,t') \end{pmatrix}_{ss'} \tag{7}$$

$$G_\alpha^{ss'}(t_1,t_2) = -\frac{i}{M}\sum_{a=1}^{M}\left\langle \psi_a^{\alpha,s}(t_1)\psi_a^{\alpha,s'}(t_2)\right\rangle \equiv \begin{pmatrix} G_\alpha^T(t,t') & G_\alpha^<(t,t') \\ G_\alpha^>(t,t') & G_\alpha^{\tilde{T}}(t,t') \end{pmatrix}_{ss'} \quad \alpha = L, R, \tag{8}$$

which describe the single-particle Green's functions of Majorana fields for the system and the bath respectively, and with the corresponding Lagrange multipliers

$$\Sigma_S^{ss'}(t_1,t_2) = \begin{pmatrix} \Sigma_S^T(t,t') & -\Sigma_S^<(t,t') \\ -\Sigma_S^>(t,t') & \Sigma_S^{\tilde{T}}(t,t') \end{pmatrix}_{ss'}, \quad \Sigma_\alpha^{ss'}(t_1,t_2) = \begin{pmatrix} \Sigma_\alpha^T(t,t') & -\Sigma_\alpha^<(t,t') \\ -\Sigma_\alpha^>(t,t') & \Sigma_\alpha^{\tilde{T}}(t,t') \end{pmatrix}_{ss'}. \tag{9}$$

After integrating over the fermions χ, ψ^L and ψ^R we get an effective action $S_{\rm eff}$ written only in terms of the fields G and Σ

$$\begin{aligned} S_{\rm eff}[G,\Sigma] = & -i\frac{N}{2}\mathrm{Tr}\ln\left[-i\,\hat{G}_{0,S}^{-1} + i\hat{\Sigma}_S\right] - i\,\frac{M}{2}\mathrm{Tr}\ln\left[-i\hat{G}_{0,L}^{-1} + i\hat{\Sigma}_L\right] - i\,\frac{M}{2}\mathrm{Tr}\ln\left[-i\hat{G}_{0,R}^{-1} + i\hat{\Sigma}_R\right] \\ & + i\,\frac{N}{2}\int dt\;dt'\;\sum_{ss'} ss'\left(-\frac{\Gamma^2}{2}G_S^{ss'}(t,t')^2 + G_S^{ss'}(t,t')\Sigma_S^{ss'}(t,t')\right) \\ & + i\,\frac{M}{2}\int dt\;dt'\;\sum_{\alpha=L,R}\sum_{ss'} ss'\left(i^{q_B}\frac{J^2}{q_B}G_\alpha^{ss'}(t,t')^{q_B} + G_\alpha^{ss'}(t,t')\Sigma_\alpha^{ss'}(t,t')\right) \\ & - i\,\frac{N}{2}\int dt\;dt'\;\sum_{\alpha=L,R}\sum_{ss'} ss'\,\theta(t)\theta(t')V_\alpha^2 G_S^{ss'}(t,t')\,G_\alpha^{ss'}(t,t'). \end{aligned} \tag{10}$$

The saddle point of the action $S_{\rm eff}$ in the large N, M limit gives us the Schwinger-Dyson equations

$$\left[\hat{G}_0^{-1} - \hat{\Sigma}_S\right]\circ\hat{G}_S = 1, \quad \left[\hat{G}_0^{-1} - \hat{\Sigma}_{L,R}\right]\circ\hat{G}_{L,R} = 1, \tag{11}$$

where the symbol $\circ$ stands for time convolution

$$A\circ B = \int_{-\infty}^{+\infty} dt A(t_1,t)B(t,t_2)$$

and the self-energies read

$$\Sigma_S^{ss'}(t,t') = ss'\Gamma^2 G_S^{ss'}(t,t') + ss'V_R^2\,\theta(t)\theta(t')\,G_R^{ss'}(t,t') + ss'V_L^2\,\theta(t)\theta(t')\,G_L^{ss'}(t,t'), \tag{12}$$

$$\Sigma_{L,R}^{ss'}(t,t') = -i^{q_B}ss'J^2 G_{L,R}^{ss'}(t,t')^{q_B-1} + p\,ss'V_{L,R}^2\,\theta(t)\theta(t')\,G_S^{ss'}(t,t'), \tag{13}$$

where $[\hat{G}_0^{-1}]^{ss'}(t,t') = is\delta_{ss'}\delta(t-t')\partial_t$ is the free Majorana Green's function and we have introduced the ratio $p = N/M$. We note that in general the baths Green's functions are coupled to the system's one due to the term in Eq. (13) which describes the feedback of the system on the environment and which vanishes in the limit of infinite bath $p \to 0$. As we are going to discuss in the next section, this feedback is crucial in order to generate a finite contribution to the energy current between the interacting baths. Rather than working in the $s, s' = \pm$ basis it is convenient to introduce the retarded, advanced and Keldysh Green's functions

$$G_S^R(t,t') = \theta(t-t')(G_S^>(t,t') - G_S^<(t,t')), \tag{14}$$

$$G_S^A(t,t') = \theta(t'-t)(G_S^<(t,t') - G_S^>(t,t')), \tag{15}$$

$$G_S^K(t,t') = G_S^>(t,t') + G_S^<(t,t'), \tag{16}$$

and likewise for the self-energies. We can perform a rotation to the retarded, advanced and Keldysh basis by multiplying the Dyson equation (11) on the left and on the right by the unitary matrix U

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \tag{17}$$

which gives

$$\begin{pmatrix} 0 & \left[G_0^A\right]^{-1}-\Sigma^A \\ \left[G_0^R\right]^{-1}-\Sigma^R & -\Sigma^K \end{pmatrix} \circ \begin{pmatrix} G^K & G^R \\ G^A & 0 \end{pmatrix} = \mathbf{1} \quad (18)$$

from which we can read out the three Dyson equations on G_S^R, G_S^A and G_S^K

$$\left(\left[G_0^R\right]^{-1}-\Sigma^R\right)\circ G^R = 1, \quad (19)$$

$$\left(\left[G_0^A\right]^{-1}-\Sigma^A\right)\circ G^A = 1, \quad (20)$$

$$\left(\left[G_0^R\right]^{-1}-\Sigma^R\right)\circ G^K = \Sigma^K \circ G^A. \quad (21)$$

Finally, it can be shown that the first Dyson equation (11) can be recast into a more convenient form known as the Kadanoff-Baym equations

$$i\partial_{t_1} G_S^{>,<}(t_1,t_2) = \left(\Sigma_S^R \circ G_S^{>,<} + \Sigma_S^{>,<}\circ G_S^A\right), \quad (22)$$

$$-i\partial_{t_2} G_S^{>,<}(t_1,t_2) = \left(G_S^R \circ \Sigma_S^{>,<} + G_S^{>,<}\circ \Sigma_S^A\right), \quad (23)$$

and likewise for the left and right baths.

III. FORMULA FOR THE ENERGY CURRENT

Despite our model is fully interacting, the exact solvablity of the SYK$_4$ model allows us to obtain an exact formula for the energy current flowing from one reservoir to the other. In particular, using Keldysh techniques, we can compute the current $\mathcal{J}_\alpha = \overline{\dot{E}_\alpha}(t) = i\overline{\langle [H,H_\alpha]\rangle}(t)$ from the lead $\alpha = L, R$, where $\langle\cdots\rangle$ is the average over the Keldysh action while the overline represents average over all disordered couplings. To this extent, we first evaluate the rate of energy flow across the left bath

$$\mathcal{J}_L \equiv \dot{E}_L = \overline{\frac{d}{dt}\langle H_L(t)\rangle}. \quad (24)$$

We can proceed in two ways, either taking the time derivative first and then average over disorder, or do it in the opposite order. Both ways lead to the same result. We chose to take the disorder average first and compute $E_L(t)$. In Keldysh formalism, the expectation value of an operator $\mathcal{O}$ can be obtained by introducing a generating functional $Z[\eta]$

$$\langle \mathcal{O}(t)\rangle = \frac{i}{2}\lim_{\eta\to 0}\frac{\delta Z[\eta]}{\delta\eta(t)}, \quad (25)$$

where $Z[\eta]$ is the partition function corresponding to the action (6) except that we shift the Hamiltonian in the Keldysh action $S[\chi,\psi^L,\psi^R]$ by $H\to H+\eta(t)\mathcal{O}$ on the upper branch and by $H\to H-\eta(t)\mathcal{O}$ on the lower branch of the time contour. Averaging over the disorder and following the same steps as in the derivation of the Schwinger-Dyson equation, we get

$$E_L(t) = -M\, i^{q_B+1}\frac{J^2}{q_B}\int_{-\infty}^{t} dt'[G_L^>(t,t')^{q_B} - G_L^<(t,t')^{q_B}], \quad (26)$$

where $q_B = 2$ for the SYK$_2$ bath and $q_B = 4$ for the SYK$_4$ bath. Taking the derivative with respect to time, we obtain

$$\dot{E}_L(t) = -M\, i^{q_B+1}J^2\int_{-\infty}^{t} dt'[G_L^>(t,t')^{q_B-1}\partial_t G_L^>(t,t') - G_L^<(t,t')^{q_B-1}\partial_t G_L^<(t,t')]. \quad (27)$$

Then we use the expression of the self-energy of the bath, Eq. (13), to get

$$\dot{E}_L(t) = iM\int_{-\infty}^{t} dt'[\Sigma_L^>(t,t')\partial_t G_L^>(t,t') - \Sigma_L^<(t,t')\partial_t G_L^<(t,t')] - ipMV^2\int_{-\infty}^{t} dt'[G_S^>(t,t')\partial_t G_L^>(t,t') - G_S^<(t,t')\partial_t G_L^<(t,t')]. \quad (28)$$

Using the Kadanoff-Baym equation for $G_L^{>,<}(t,t')$ to replace $\partial_t G_L^{>,<}(t,t')$ one can show that the first integral is zero. Thus the time derivative of the energy of the L bath is

$$\dot{E}_L(t) = -iNV^2\int_{-\infty}^{t} dt'[G_S^>(t,t')\partial_t G_L^>(t,t') - G_S^<(t,t')\partial_t G_L^<(t,t')]. \quad (29)$$

We see therefore that the presence of a finite energy flow from the bath is an effect of the feedback term between system and bath encoded in the last term of Eq. (13). Furthermore, notice that this expression is independent of q_B: it takes the same form for the noninteracting ($q_B = 2$) as well as the interacting ($q_B = 4$) reservoir.

Now we look at the long time limit. We assume that the system reaches a nonequilibrium steady state with a finite energy current and therefore that the two-point functions are time translational invariant $G(t,t') = G(t-t')$. Then we can introduce the Fourier transform of the Green's functions

$$G(\omega) = \int dt e^{i\omega t}G(t), \qquad G(t) = \int \frac{d\omega}{2\pi}e^{-i\omega t}G(\omega). \quad (30)$$

After some simple manipulations we can rewrite $\dot{E}_L(t)$ as a single integral over frequencies

$$\mathcal{J}_L = -N\,V^2\int\frac{d\omega}{2\pi}\,\omega\, G_L^<(\omega)G_S^>(\omega). \quad (31)$$

Of course a similar expression is found for $\mathcal{J}_R$. The nonequilibrium energy current between the two reservoirs

$\mathcal{J} \equiv (\mathcal{J}_L - \mathcal{J}_R)/2$ is then given by

$$\mathcal{J} = -\frac{NV^2}{2}\int \frac{d\omega}{2\pi}\,\omega\,(G_L^<(\omega) - G_R^<(\omega))G_S^>(\omega). \tag{32}$$

Several remarks are in order concerning Eq. (32), which is one of our main result. First, the expression for the energy current in Eq. (32) is an exact result in the large N, M limit, at fixed ratio $p = N/M$, and it is nonperturbative in the system-bath coupling V. In this respect the Green's functions entering the energy current $\mathcal{J}$ are those fully renormalized by the system-bath interaction (see Appendix). The structure of Eq. (32) is reminiscent of the Meir-Wingreen formula usually describing transport between two noninteracting reservoirs connected by an interacting intermediate region [36]. This analogy becomes more transparent in the limit $p \ll 1$, corresponding to a bath which is parametrically larger than the system, as we are going to consider here.

With respect to the Meir-Wingreen formula, our result can account for both noninteracting or fully interacting and maximally chaotic reservoirs linearly coupled to a central system. In fact, as the demonstration has shown, it is valid for any random q_B-body interactions of the SYK type in the reservoirs. Furthermore, the derivation of Eq. (32) did not use at any step the expression of the self-energy of the central system, meaning that in fact it still holds if the system interacts with a general q_S-body SYK interaction.

Since we are interested in the regime $p \ll 1$, we can evaluate the Green's functions of system and baths entering the expression of the energy current at $p = 0$. This amounts to disregard the feedback term and consider as self energy of the baths the expression

$$\Sigma_{L,R}^{>,<}(t,t') = -i^{q_B}J^2 G_{L,R}^{>,<}(t,t')^{q_B-1}. \tag{33}$$

Thus we can consider that the $\psi^{L,R}$ fermions are isolated and are not affected by the contact with the small central system. As an immediate consequence we can safely assume that the left and right baths are in thermal equilibrium and satisfy the fluctuation-dissipation theorem at temperature T_L, T_R respectively

$$G_L^<(\omega) = iA_L(\omega)f_L(\omega), \quad G_R^<(\omega) = iA_R(\omega)f_R(\omega), \tag{34}$$

where $f_L(\omega)$ and $f_R(\omega)$ are the Fermi-Dirac distributions of the left and right baths, respectively. Thus we finally arrive at the expression of the current $\mathcal{J}$

$$\mathcal{J} = -\frac{iNV^2}{2}\int \frac{d\omega}{2\pi}\,\omega\,(A_L(\omega)f_L(\omega) - A_R(\omega)f_R(\omega))G_S^>(\omega). \tag{35}$$

Lastly, we show in Appendix that $G_S^>(\omega)$ satisfies a nonequilibrium FDT-like relation

$$G_S^>(\omega) = -iA_S(\omega)f_S(-\omega), \tag{36}$$

where $f_S(\omega)$ is an average of the left and right Fermi-Dirac distributions

$$f_S(\omega) = \frac{f_L(\omega)A_L(\omega) + f_R(\omega)A_R(\omega)}{A_L(\omega) + A_R(\omega)}. \tag{37}$$

Note that if $T_L = T_R = T$, $f_S(\omega)$ reduces to the Fermi-Dirac distribution $f_{\rm eq}(\omega)$ at temperature T.

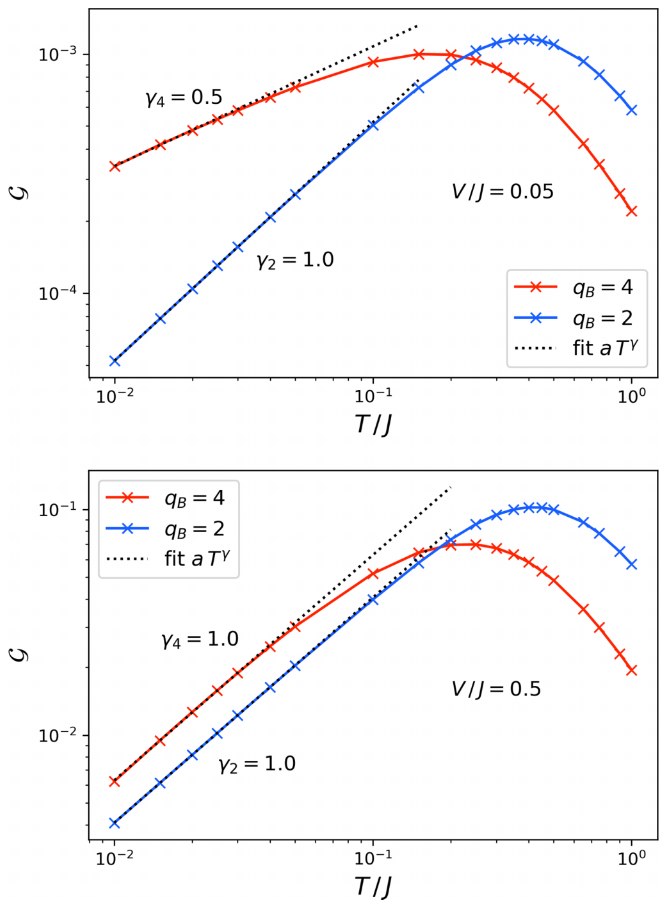

FIG. 2. Linear thermal conductance $\mathcal{G}(T)$ at weak (top) and strong (bottom) system-bath coupling, for both conventional SYK_2 corresponding to $q_B = 2$ and the SYK_4 bath corresponding to $q_B = 4$.

IV. LINEAR ENERGY TRANSPORT

We start our discussion of energy transport from the linear regime, corresponding to two temperatures differing by a small amount $\Delta T \to 0$, i.e., $T_{L,R} = T \pm \Delta T/2$. In this case from Eq. (35), we can obtain the thermal conductance $\mathcal{G}(T) \equiv \mathcal{J}/\Delta T$ as

$$\mathcal{G}(T) = \frac{NV^2}{2}\int \frac{d\omega}{2\pi}\,\omega A_S^{\rm eq}(\omega)f_{\rm eq}(-\omega)\frac{\partial}{\partial T}\big(A_B^{\rm eq}(\omega)f_{\rm eq}(\omega)\big). \tag{38}$$

In this expression, all quantities are evaluated at thermal equilibrium with temperature T. In particular $f_{\rm eq}(\omega)$ is the equilibrium Fermi-Dirac distribution while $A_B^{\rm eq}, A_S^{\rm eq}$ are the equilibrium spectral density of the bath and the system coupled to it. These can be obtained numerically by solving the equilibrium Dyson equation of the reservoirs (see Sec. II), and using the fact that we can write down a closed form expression for the system Green's function given the Green's function of the bath (see Appendix A). We note that in the limit $p \ll 1$ that we consider here the system is strongly renormalized by the baths, which instead are not affected by the feedback of the system since their size is parametrically larger.

In Fig. 2, we plot the thermal conductance $\mathcal{G}(T)$ as a function of temperature, for both the SYK_2 and the SYK_4 baths and for two different values of the system-bath coupling V. We first note that for a FL bath such as SYK_2 the conductance shows a linear scaling with temperature, independently on the value of V (see the blue curves on the left and right panels $q_B = 2$). This result is expected for gapless systems with well defined quasiparticles [1] and can be obtained from Eq. (38) by considering that in the low-energy limit $\omega, T \ll J$ the bath spectral density is $A_B(\omega) \simeq 2/J$ and the system spectral density becomes flat $A_S^{\rm eq}(\omega) \simeq 2/\tilde{\Gamma}$ but with a modified coupling constant [see Eq. (B2) in Appendix], so that we obtain

$$\mathcal{G}(T) = \frac{NV^2}{\pi\tilde{\Gamma}J}\int d\omega\,\frac{\omega^2}{T^2}\frac{e^{\frac{\omega}{T}}}{\left(e^{-\frac{\omega}{T}}+1\right)\left(e^{\frac{\omega}{T}}+1\right)^2} = N\frac{\pi}{6}\frac{V^2}{\tilde{\Gamma}J}T, \tag{39}$$

where we used the integral $\int_{-\infty}^{\infty} du\, u^2 \frac{e^u}{(e^{-u}+1)(e^u+1)^2} = \frac{\pi^2}{6}$. Putting back the physical dimensions and writing explicitly $\hbar$ and k_B, we get

$$\mathcal{G}(T) = N\frac{V^2}{\tilde{\Gamma}J}\frac{\pi^2 k_B^2}{3h}T = N\frac{V^2}{\tilde{\Gamma}J}\mathcal{G}_Q \quad (q_B = 2) \tag{40}$$

with $\mathcal{G}_Q = \frac{\pi^2 k_B^2}{3h}T$ the quantum of thermal conductance, which as we recalled above corresponds to ballistic transport, i.e., a probability of transmission across the channel equal to 1. We can interpret the factor $V^2/(\tilde{\Gamma}J)$ as a typical probability of transmission of an energy carrier from the left reservoir to the right reservoir.

We now turn to the interacting SYK_4 reservoirs for which, on the contrary, the thermal conductance $\mathcal{G}(T)$ shows a nontrivial dependence on the system-bath coupling V. In particular, as we see in Fig. 2 (red curves on the left and right panels, for $q_B = 4$), for weak coupling $V/J = 0.05$ and low-to-intermediate temperatures the thermal conductance shows a $\sqrt{T}$ scaling, i.e., the NFL nature of the bath and its absence of coherent quasiparticle excitations leads to an enhanced thermal conductance with respect to a noninteracting bath of quasiparticles.

We can understand the origin of this effect by considering the structure of bath and system spectral functions in the low-frequency conformal limit $\omega, T \ll J$. In fact we know that for the SYK_4 baths the spectral density is peaked at the origin and is given by the expression

$$A_B^{\rm eq}(\omega) = 2\left(\frac{\pi}{J^2}\right)^{1/4}\frac{1}{\sqrt{2\pi T}}\mathrm{Re}\left(\frac{\Gamma\left(\frac{1}{4}-\frac{i\omega}{2\pi T}\right)}{\Gamma\left(\frac{3}{4}-\frac{i\omega}{2\pi T}\right)}\right). \tag{41}$$

Using this expression of the spectral density, we can write

$$\frac{\partial}{\partial T}\left(A_B^{\rm eq}(\omega)f_{\rm eq}(\omega)\right) = 2\left(\frac{\pi}{J^2}\right)^{1/4}\frac{1}{\sqrt{2\pi T}}\frac{1}{T}\psi\left(\frac{\omega}{T}\right), \tag{42}$$

where

$$\psi(u) = -\frac{1}{2}\frac{1}{e^u+1}\mathrm{Re}\left(\frac{\Gamma\left(\frac{1}{4}-i\frac{u}{2\pi}\right)}{\Gamma\left(\frac{3}{4}-i\frac{u}{2\pi}\right)}\right) - u\frac{d}{du}\left[\frac{1}{e^u+1}\mathrm{Re}\left(\frac{\Gamma\left(\frac{1}{4}-i\frac{u}{2\pi}\right)}{\Gamma\left(\frac{3}{4}-i\frac{u}{2\pi}\right)}\right)\right]. \tag{43}$$

Then the thermal conductance takes the form

$$\mathcal{G}_4(T) = \frac{NV^2}{2\pi}T\int\frac{d\omega}{T}\frac{\omega}{T}\frac{\psi\left(\frac{\omega}{T}\right)}{e^{-\frac{\omega}{T}}+1}\left(\frac{\pi}{J^2}\right)^{1/4}\frac{A_S^{\rm eq}(\omega)}{\sqrt{2\pi T}}. \tag{44}$$

When the tunnel coupling V is weak the spectral density of the system remains close to the isolated one, at least for not too low temperatures, and the Dyson equation can be solved perturbatively in powers of V^2/Γ^2. As the energy current has an overall V^2 factor we can keep the spectral density of the system at order 0 in V^2/Γ^2, so $A_S^{\rm eq} \simeq 2/\Gamma$. Plugging this ansatz and Eq. (41) in the expression for the conductance, we get

$$\mathcal{G}(T) = NI_4\frac{V^2}{\Gamma\sqrt{J}}\sqrt{T} \qquad (q_B = 4) \tag{45}$$

with I_4 a numerical prefactor given by

$$I_4 = \left(\frac{1}{4\pi^5}\right)^{\frac{1}{4}}\int_{-\infty}^{\infty} du\,\frac{u}{e^{-u}+1}\psi(u) \simeq 0.36. \tag{46}$$

Similar scaling have been reported for linear thermal transport of an SYK_4 model coupled to FL baths [30]. We emphasize here that the anomalous temperature scaling is a direct consequence of the NFL nature of the two reservoirs, whose enhanced density of states leads to an increased thermal conductance as compared to the noninteracting SYK_2 case, i.e., $\frac{\mathcal{G}_4}{\mathcal{G}_2} \sim \sqrt{\frac{J}{T}} \gg 1$.

The parametrically large enhancement in the thermal conductance does not however survive up to strong system-bath couplings, as we see in Fig. 2 (bottom panel, for $q_B = 4$) where for $V/J = 0.5$ the thermal conductance crosses over to a linear temperature scaling. To see how this comes about we note that for larger couplings V the spectral function of the system becomes dressed by the NFL bath and develops a dip at small frequency [42]. In particular at zero temperature it scales like $\propto \sqrt{\omega}$, a behavior reminiscent of the zero-bias anomaly in one-dimensional disordered interacting conductors [43–45]. In the limit $V \gg V^*(T) \equiv (\Gamma^2 JT)^{1/4}$, which also defines a low-temperature scale $T^*(V)$, we can obtain an analytic expression for the system spectral function (see Appendix) which reads

$$A_S^{\rm eq}(\omega) = \frac{1}{V^2}\left(\frac{J^2}{\pi}\right)^{1/4}\sqrt{2\pi T}\,\mathrm{Re}\left(\frac{\Gamma\left(\frac{3}{4}-\frac{i\omega}{2\pi T}\right)}{\Gamma\left(\frac{1}{4}-\frac{i\omega}{2\pi T}\right)}\right). \tag{47}$$

The suppressed spectral density of the system renormalized by our NFL bath leads to a suppression of thermal conductance and restores a linear temperature scaling, as it would be for the FL leads

$$\mathcal{G}(T) = I_4' NT \sim N\mathcal{G}_Q(T) \qquad (q_B = 4) \tag{48}$$

with I_4' a numerical prefactor which is

$$I_4' = \frac{1}{2\pi}\int_{-\infty}^{\infty} du\,\frac{u}{e^{-u}+1}\,\mathrm{Re}\left(\frac{\Gamma\left(\frac{3}{4}-i\frac{u}{2\pi}\right)}{\Gamma\left(\frac{1}{4}-i\frac{u}{2\pi}\right)}\right)\psi(u) \simeq 0.16. \tag{49}$$

This similarity is however only superficial, as energy transport in this regime is not due to quasiparticles. The linear T scaling arises in fact from a subtle cancellation between the enhanced spectral density of the SYK_4 baths and the

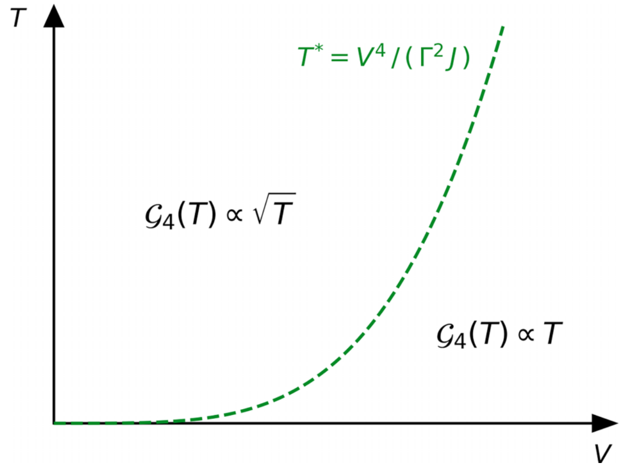

FIG. 3. Transport phase diagram as a function of temperature T and system-bath coupling V. Above the crossover temperature $T^*(V) = V^4/(\Gamma^2 J)$, the thermal conductance $\mathcal{G}(T)$ is enhanced for the SYK_4 bath through the $\sqrt{T}$ scaling, while at low temperatures or strong-coupling the linear behavior is restored.

suppressed density of state of the system and reminds other situations in which violation of a FL scaling leads to a thermal conductance linear in temperature [46]. To further appreciate the physics behind this result it is interesting to comment on the relation between transport and chaos in our system. In fact the calculation of the out-of-time-order correlators for the model in Eq. (1) in the large N, M limit shows [37] that below $T^*(V)$, when the system is dressed by the SYK bath and the thermal conductance is linear, the Lyapunov exponent saturates the bound on chaos. In other words, the phase at strong coupling and low-temperature provides an example where transport is suppressed by the anomalous spectral properties of the bath while chaos is enhanced, pointing towards different mechanisms controlling these two processes. We emphasize that while chaos estimators such as Lyapunov exponents are not straightforward to access experimentally, the renormalization of the system density of states due to the coupling to a NFL bath, that we have discussed above can be considered as a smoking gun of this phase that distinguishes it from the case of a FL quasiparticle bath, despite the same scaling of the thermal conductance.

We summarize the linear transport regime of our model in Fig. 3. We note that the crossover scale $T^*(V)$ is strongly dependent on system-bath coupling and controls also the regime of validity of Eq. (45), setting a low-temperature scale below which the $\sqrt{T}$ scaling crosses over to the linear one. Yet at weak coupling V this scale is parametrically small [corresponding to a temperature $T^* \sim 10^{-5}$ for the parameters in Eq. (45)] leaving a broad range of temperatures where the enhanced conductance is visible.

V. NONLINEAR ENERGY TRANSPORT

Finally, we discuss the full nonequilibrium energy current $\mathcal{J}$ as a function of the two temperatures T_L, T_R and beyond the linear response regime.

We first look at the $q_B = 2$ case corresponding to noninteracting reservoirs with well-defined quasiparticles, for which we can proceed analytically. In this case, the spectral densities of the reservoirs are independent of temperature so $A_L(\omega) = A_R(\omega) = A_2(\omega)$ with $A_2(\omega)$ the SYK_2 semi-circle spectral density. Also, the distribution function of the system reduces to $f_S(\omega) = (f_L(\omega) + f_R(\omega))/2$. Thus the energy current can be rewritten as

$$\mathcal{J} = -\frac{NV^2}{4} \int \frac{d\omega}{2\pi}\, \omega\, (f_L(\omega) - f_R(\omega)) \times (f_L(-\omega) + f_R(-\omega)) A_2(\omega) A_S(\omega). \quad (50)$$

The Fermi-Dirac distribution varies from 0 to 1 on an interval of the order of the temperature around $\omega = 0$, so $f_L(\omega) - f_R(\omega)$ is nonzero only on an interval of width $\max(T_L, T_R)$ around $\omega = 0$. Therefore, in the low temperature limit $T_L, T_R \ll \Gamma, J$, we can use the low energy expressions of the bath spectral density $A_2(\omega) \simeq 2/J$ and of the system spectral density $A_S(\omega) \simeq 2/\tilde{\Gamma}$, where $\tilde{\Gamma}$ is a renormalized hopping which accounts for the coupling between the system and the two reservoirs (see Appendix). Making a change of variable $\omega \to -\omega$ in the integral and using $f(-\omega) = 1 - f(\omega)$, the energy current becomes

$$\mathcal{J} = -N \frac{V^2}{\tilde{\Gamma} J} \int \frac{d\omega}{2\pi}\, \omega\, (f_L(\omega)^2 - f_R(\omega)^2). \quad (51)$$

By splitting the two integrals we see that the energy current takes the functional form $\mathcal{J} = \Phi(T_L) - \Phi(T_R)$, with $\Phi(T) \sim T^2$ at low temperatures $T \ll J$, both at weak and strong coupling V. This result resembles the one obtained in out of equilibrium CFT [5,16] mentioned above.

The situation is richer for SYK_4 baths, as we see in Fig. 4 where we plot the energy current $\mathcal{J}$ as a function of T_L, for different values of T_R and for weak (upper panel) and strong (lower panel) system-bath coupling V. In both cases, we see that the effect of changing T_R is to induce a rigid shift of the current, suggesting that a functional form of the type $\mathcal{J} = \Phi(T_L) - \Phi(T_R)$ is still compatible with the data. In the inset, we show that indeed all the curves collapse on a single one once we compensate for the vertical shit. This result is non trivial since the spectral function of the system is strongly renormalized by the bath and acquires a rich temperature dependence. Here, we ca not use the picture of thermal quasi-particles emitted by the two sources to explain this functional form. As we show in the insets of Fig. 4, for weak system-bath coupling, we find $\Phi(T) = T^{3/2}$, a power-law behavior which, while compatible with the thermal conductance discussed earlier, extends for a temperature range well above the linear regime implying a modified Stefan-Boltzmann scaling. For large system-bath coupling on the other hand, or for low enough average temperature $(T_L + T_R)/2$, we see from the inset in the bottom panel that the conventional scaling is recovered $\Phi(T) = T^2$, which in this context however does not signal the presence of well-defined quasiparticles.

VI. CONCLUSIONS

In this work, we studied the energy transport between two strongly interacting quantum baths described by the maximally chaotic SYK_4 model, coupled through an SYK_2 system. In particular we have focused on understanding how the

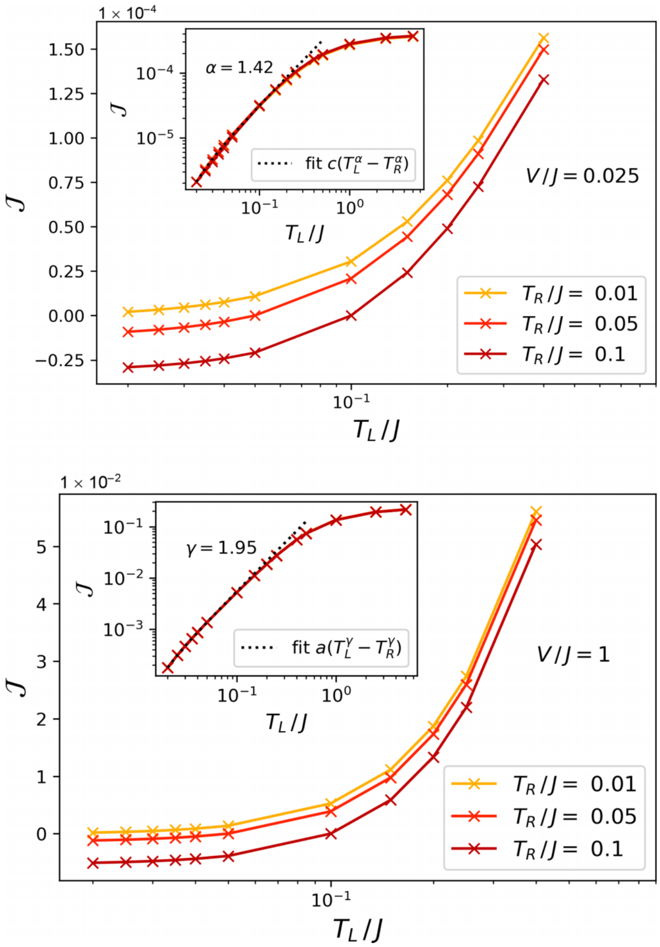

FIG. 4. Nonlinear energy current $\mathcal{J}$ as a function of T_L and different T_R for weak (top) and strong (bottom) couplings.

absence of quasiparticles in the bath affects the energy transport, both in the linear and nonlinear regimes.

We have obtained an exact formula for the energy current in this setting, which is valid in the large N, M limit at fixed ratio and arbitrary system-bath coupling. We have shown that the quantum-critical nature of the SYK baths has direct consequences on energy transport. The thermal conductance shows a $\sqrt{T}$ scaling above a temperature T^* and crosses over to a linear-T behavior at low temperatures, even though the system becomes non-Fermi liquid and maximal chaotic due to the coupling with the bath. We show that the full nonequilibrium energy current takes the form $\mathcal{J} = \Phi(T_L) - \Phi(T_R)$, with $\Phi(T) \sim T^\gamma$ and a power-law exponent γ crossing over from $\gamma = 3/2$ to $\gamma = 2$ below T^*. Future directions include considering the charged SYK model to discuss thermoelecricity and the full counting statistics of energy current.

ACKNOWLEDGMENTS

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APPENDIX A: FORMAL SOLUTION OF DYSON EQUATION FOR SYSTEM GREEN'S FUNCTION

In this Appendix, we show how to get an exact expression of $G_S^{\gtrless}(\omega)$ in terms of the left and right baths Green's functions. We assume that the system is in a stationary state, possibly nonequilibrium, and that two-point functions are time translational invariant. In particular, the greater/lesser self-energy of the system is

$$\Sigma_S^{>,<}(t) = \Gamma^2\, G_S^{>,<}(t) + V_L^2\, G_L^{>,<}(t) + V_R^2\, G_R^{>,<}(t). \quad \text{(A1)}$$

Here $\Sigma_S^{>,<}$ is a linear function of the Green's functions, so the retarded self-energy $\Sigma_S^R(t) = \theta(t)(\Sigma_S^>(t) - \Sigma_S^<(t))$ takes exactly the same form

$$\Sigma_S^R(t) = \Gamma^2 G_S^R(t) + V_L^2 G_L^R(t) + V_R^2 G_R^R(t). \quad \text{(A2)}$$

Taking the Fourier transform of this equation and plugging $\Sigma_S^R(\omega)$ into the Dyson equation $G_S^R(\omega)^{-1} = \omega - \Sigma_S^R(\omega)$, we arrive at the quadratic equation on $G_S^R(\omega)$

$$\Gamma^2 G_S^R(\omega)^2 - \left[\omega - V_L^2 G_L^R(\omega) - V_R^2 G_R^R(\omega)\right] G_S^R(\omega) + 1 = 0. \quad \text{(A3)}$$

The solution to this equation is

$$G_S^R(\omega) = \frac{1}{2\Gamma^2}[\omega - S^R(\omega) - \delta(\omega)], \quad \text{(A4)}$$

where we called $S^R(\omega) \equiv V_L^2 G_L^R(\omega) + V_R^2 G_R^R(\omega)$ and $\delta(\omega) = x(\omega) + iy(\omega)$ is given by

$$x = \text{sign}(\omega - \text{Re}S^R(\omega))\sqrt{\frac{1}{2}(\sqrt{B^2 + C^2} + B)},$$
$$y = \sqrt{\frac{1}{2}(\sqrt{B^2 + C^2} - B)}, \quad \text{(A5)}$$

where

$$B = \omega^2 - 4\Gamma^2 - 2\omega \text{Re}S^R(\omega) + \text{Re}^2 S^R(\omega) - \text{Im}^2 S^R(\omega), \quad \text{(A6)}$$

$$C = -2\omega \text{Im}S^R(\omega) + 2\text{Re}S^R(\omega)\text{Im}S^R(\omega). \quad \text{(A7)}$$

We continue by deriving an expression for the Keldysh Green's function of the system. We write equation (21) in Fourier space and use equation (A1) to get an expression of $\Sigma_S^K(\omega) = \Sigma_S^>(\omega) + \Sigma_S^<(\omega)$

$$G_S^K(\omega) = G_S^R(\omega)\Sigma_S^K(\omega)G_S^A(\omega),$$
$$\Sigma_S^K(\omega) = \Gamma^2 G_S^K(\omega) + V_L^2 G_L^K(\omega) + V_R^2 G_R^K(\omega). \quad \text{(A8)}$$

Combining these two equations, we get

$$G_S^K(\omega) = \frac{G_S^R(\omega)G_S^A(\omega)}{1 - \Gamma^2 G_S^R(\omega)G_S^A(\omega)}\left(V_L^2 G_L^K(\omega) + V_R^2 G_R^K(\omega)\right)$$
$$= \frac{A_S(\omega)}{V_L^2 A_L(\omega) + V_R^2 A_R(\omega)}\left(V_L^2 G_L^K(\omega) + V_R^2 G_R^K(\omega)\right), \quad \text{(A9)}$$

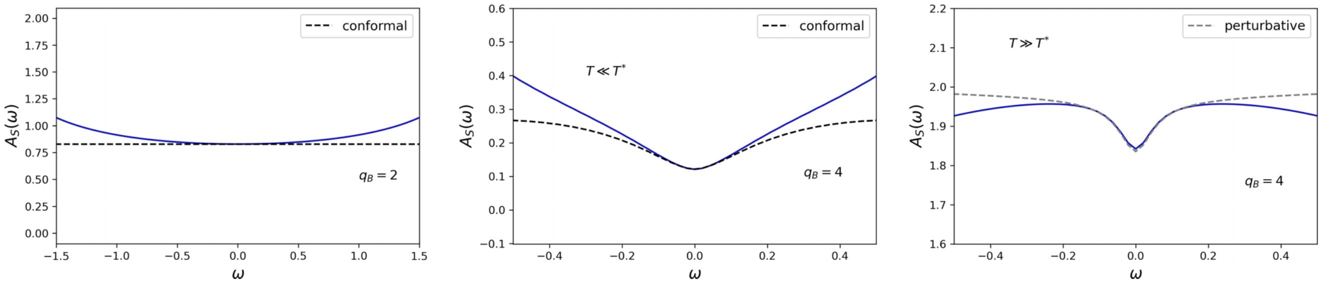

FIG. 5. Spectral functions of the system renormalized by the bath. For the three plots $\Gamma = J_L = J_R = 1$, $T_L = 0.05$, and $T_R = 0.03$ (a) Noninteracting SYK_2 reservoirs $q_B = 2$. The black dashed line is the low energy conformal solution (B2). Here $V_L = V_R = 1$. (b) Strongly interacting SYK_4 reservoirs for $V_L = V_R = 1$. In this case, $T^* \simeq 1 \gg T_L, T_R$ and the conformal solution (B3) correclty reproduces the low energy behavior (dotted black line). (c) Strongly interacting SYK_4 reservoirs for $V_L = V_R = 0.1$. Now $T^* \simeq 10^{-4} \ll T_L, T_R$ and one must use the pertubative expansion (B5) to find the low energy behavior of $A_s(\omega)$ (dotted gray line).

where in the second line, we used the two Dyson equation $G_S^{R,A}(\omega)^{-1} = \omega - \Sigma_S^{R,A}(\omega)$ to rewrite the first factor. Assuming that the two baths satisfy FDT

$$G_L^K(\omega) = -iA_L(\omega)\tanh\left(\frac{\beta_L\omega}{2}\right), \tag{A10}$$

$$G_R^K(\omega) = -iA_R(\omega)\tanh\left(\frac{\beta_R\omega}{2}\right), \tag{A11}$$

we arrive at

$$G_S^K(\omega) = -iA_S(\omega)\,\frac{V_L^2A_L(\omega)\tanh\left(\frac{\beta_L\omega}{2}\right) + V_R^2A_R(\omega)\tanh\left(\frac{\beta_R\omega}{2}\right)}{V_L^2A_L(\omega) + V_R^2A_R(\omega)}. \tag{A12}$$

From this we finally get for the lesser component $G_S^>(\omega) = (G_S^K(\omega) - iA_S(\omega))/2$

$$G_S^>(\omega) = -iA_S(\omega)f_S(-\omega),$$
$$f_S(\omega) = \frac{f_L(\omega)A_L(\omega) + f_R(\omega)A_R(\omega)}{A_L(\omega) + A_R(\omega)}, \tag{A13}$$

where we assumed $V_L = V_R = V$ for simplicity. This form of $G_S^>(\omega)$ reminds of FDT and indeed in the case where $J_L = J_R = J$ and $T_L = T_R = T$, $f_S(\omega)$ reduces to the Fermi-Dirac distribution $f_{\rm eq}(\omega)$ at temperature T. This suggests to interpret $f_S(\omega)$ as the steady-state distribution of the system coupled to the reservoirs. Notice that in the case of the noninteracting SYK_2 reservoirs, $A_L(\omega) = A_R(\omega)$ (assuming $J_L = J_R$) and f_S is simply

$$f_S(\omega) = \frac{f_L(\omega) + f_R(\omega)}{2}, \quad q_B = 2. \tag{A14}$$

APPENDIX B: ANALYTIC EXPRESSION FOR SYSTEM GREEN'S FUNCTIONS IN THE CONFORMAL LIMIT

In this Appendix, we obtain an analytic expression for the system Green's functions in the low-energy conformal limit, using the well known conformal expressions for the SYK_2 and SYK_4 baths. We start from the former, i.e., $q_B = 2$, and neglect in Eq. (A3) the term ω in the low-energy limit. We also replace $G_L^R(\omega)$ and $G_R^R(\omega)$ by their conformal expressions $G_L^R(\omega) \simeq -i/J_L$ and $G_R^R(\omega) \simeq -i/J_R$ which yields

$$\Gamma^2 G_S^R(\omega)^2 - i\left(\frac{V_L^2}{J_L} + \frac{V_R^2}{J_R}\right)G_S^R(\omega) + 1 = 0. \tag{B1}$$

The solution to this equation is $G_S^R(\omega) = -\frac{i}{\tilde{\Gamma}}$ with

$$\tilde{\Gamma} = \frac{\Gamma}{\sqrt{1 + \left(\frac{V_L^2}{2\Gamma J_L} + \frac{V_R^2}{2\Gamma J_R}\right)^2} - \left(\frac{V_L^2}{2\Gamma J_L} + \frac{V_R^2}{2\Gamma J_R}\right)}. \tag{B2}$$

As we see from this result, the effect of the SYK_2 baths is just to dress the coupling constant of the SYK_2 χ fermions. This is consistent with the fact that with our Hamiltonian V_L and V_R are marginal perturbations with respect to the SYK_2 fixed point. Thus the scaling dimension of the χ fermions remains $\Delta_\chi = 1/2$, see Fig. 5.

We now consider the SYK_4 baths and, as done before, we neglect the term ω in the Dyson equation (A3). Besides, now with the SYK_4 baths, V_L and V_R are relevant perturbations with respect to the SYK_2 fixed point and the χ fermions acquire the scaling dimension $\Delta_\chi = 3/4$. Thus, to find the low energy behavior of $G_S^R(\omega)$, we can try to neglect also the term $\Gamma^2 G_S^R(\omega)$ in the Dyson equation and we get

$$G_S^R(\omega) = \frac{-1}{V_L^2G_L^R(\omega) + V_R^2G_R^R(\omega)}. \tag{B3}$$

We recall that in the conformal limit $G^R_{L,R}(\omega)$ are given by

$$G^R_{L,R}(\omega) = -i\left(\frac{\pi}{J^2_{L,R}}\right)^{1/4} \frac{1}{\sqrt{2\pi T_{L,R}}} \frac{\Gamma\left(\frac{1}{4} - i\frac{\omega}{2\pi T_{L,R}}\right)}{\Gamma\left(\frac{3}{4} - i\frac{\omega}{2\pi T_{L,R}}\right)}. \quad \text{(B4)}$$

This solution for $G^R_S(\omega)$ can only hold if $V_L^2 G_L^R(\omega) + V_R^2 G_R^R(\omega) \gg \Gamma^2 G_S^R(\omega)$. In the simple case where $J_L = J_R = J$, $V_L = V_R = V$, and $T_L = T_R = T$, this imposes the condition $V \gg V^*(T) \equiv (J\Gamma^2 T)^{1/4}$ or equivalently $T \ll T^* = V^4/(\Gamma^2 J)$. If $T^* \ll T \ll J$, we can estimate $G^R_S(\omega)$ by treating the term in V^2 as a small perturbation with respect to the pure SYK_2 solution and we get to first order in V^2/Γ^2

$$G^R_S(\omega) = -\frac{i}{\Gamma} - \frac{V_L^2 G_L^R(\omega) + V_R^2 G_R^R(\omega)}{2\Gamma^2}, \quad T^* \ll T \ll J. \quad \text{(B5)}$$

which compares well with the numerical solution, see Fig. 5.

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