Classification of fermionic topological orders from congruence representations

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Two-dimensional topologically ordered states such as fractional quantum Hall fluids host anyonic excitations, which are relevant for realizing fault-tolerant topological quantum computers. Classification and characterization of topological orders have been intensely pursued in both the condensed matter and mathematics literature. These topological orders can be bosonic or fermionic depending on whether the system hosts fundamental fermionic excitations or not. In particular, emergent topological orders in usual solid state systems are fermionic topological orders because the electron is a fermion. Recently, bosonic topological orders have been extensively completely classified up to rank 6 using representation theory. Inspired by their method, we provide in this paper a systematic method to classify the fermionic topological orders by explicitly building their modular data, which encodes the self and mutual statistics between anyons. Our construction of the modular data relies on the fact that the modular data of a fermionic topological order forms a projective representation of the Γ_θ subgroup of the modular group $SL_2(\mathbb{Z})$. We carry out the classification up to rank 10 and obtain both unitary and nonunitary modular data. This includes all previously known unitary modular data, and also two new classes of modular data of rank 10. We also determine the chiral central charges (mod $\frac{1}{2}$) via a novel method, which does not require the explicit computation of modular extensions.

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I. INTRODUCTION

Topological orders are gapped systems beyond the Landau symmetry-breaking paradigm, characterized by long-ranged entanglement of the ground state, topological ground state degeneracy, and nontrivial statistics between emergent topological excitations [1–6]. In (2+1)D in particular, quasiparticle excitations can have statistics other than bosonic or fermionic, in which case they are called *anyons*. The emergence of anyons in (2+1)D and their physical properties have been extensively investigated both in condensed matter physics and high energy physics [7–10]. For example, they are at the heart of the physics of important condensed matter systems such as fractional quantum Hall states and gapped spin liquids [2,11–17]. In addition to being of theoretical interest, the physics of anyons is highly relevant for the realization of the fault-tolerant quantum computation, i.e., topological quantum computation [8,14,18,19]. In spite of the strong interest in the study of anyons, a complete classification of possible topological orders hosting anyons has proven elusive.

The bulk topological properties of (2+1)D topological orders can be completely characterized by the fusion and braiding properties of its anyonic excitations [6,7,21]. Mathematically, the types of anyons together with their fusion and braiding properties form a structure known as a braided fusion category (BFC) [6,22,23]. When the fundamental degrees of freedom in the theory are bosonic (we may nevertheless get emergent fermions), the corresponding mathematical structure is called a modular tensor category (MTC), but when the fundamental degrees of freedom contain fermions [24,25], we get a supermodular tensor category (super-MTC) [22,26]. (We note that there also exists a different formalism of *super pivotal categories* for studying fermionic topoloigcal orders [27,28].) While such bosonic and fermionic theories are intimately related via a process called modular extension [22,29], an intrinsically fermionic classification for fermionic topological orders is both conceptually illuminating and computationally more efficient. Indeed, for well-known condensed matter systems built out of elections, such as the Laughlin state at filling $\nu = 1/3$, or more generally the odd K matrix Abelian fractional quantum Hall states [30,31], an intrinsically fermionic description as a super-MTC is more natural. Super-MTC can also be *symmetry-enriched*, and their anomalies will be related to (3+1)D fermionic symmetry-protected topological phases [32]. Hence, the study of super-MTCs is of broad interest.

We note that there are different physical topological orders which share the same BFC data, related by stacking with invertible topological orders. Invertible topological orders are systems with no nontrivial bulk topological excitations or ground state degeneracy but which nevertheless cannot be

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deformed smoothly to the trivial product state, and support gapless edge modes (see, e.g., Refs. [16,22] for a review of the concept). This means they carry a chiral central charge c but do not affect the bulk anyon data. For fermionic topological orders, all invertible topoloigcal orders (in the absence of symmetry) are stacks of the $p+ip$ superconductor, which carry Majorana edge modes giving $c=\frac{1}{2}$ [21,33]. Thus we can fully specify a 2 + 1D fermionic topological order via $(\mathcal{C}, c)$, where $\mathcal{C}$ is the super-MTC representing its anyon data and c tells us how many layers of the $p+ip$ superconductor are present [22].

In spite of the interest in BFCs, a direct classification of the defining data of BFCs is known to be prohibitively difficult because of huge gauge redundancies. Thus attempts at classification have instead focused on the so-called modular data (MD) of BFCs, which consist of the so-called S and T matrices. The S matrix encodes the mutual statistics of the anyons, while the T matrix encodes their self-statistics. The dimension of these matrices—or, equivalently, the number of distinct types of anyons—is called the *rank* of the MD (or of the corresponding BFC). For example, the topological properties of the Kitaev toric code are described by a rank 4 MTC, while the Ising MTC is rank 3 [7,14]. Although the MD do not provide a complete classification of BFCs, known examples of BFCs which cannot be distinguished by their MD only occur at very high rank [34,35], and since the MD are gauge invariant they are much more amenable to classification. See Refs. [36–40] for the previous efforts in this line of thinking.

Recently, Ref. [20] has introduced a method which uses representations of $SL_2(\mathbb{Z})$ to classify the MD of MTCs, i.e., bosonic topological orders, and used it to classify MD up to rank 6. They make use of the fact that every MD given by a pair (S, T) forms a projective *congruence* representation of $SL_2(\mathbb{Z})$ [41]. Since every congruence representation of $SL_2(\mathbb{Z})$ can be constructed explicitly [42,43], this gives us a list of candidates from which valid MD can be constructed, and leads to the most complete classification of bosonic topological orders so far obtained.

There have also been attempts to classify super-MTCs, which characterize fermionic topological orders. The fusion rules of unitary super-MTCs have been completely classified up to rank 6 [39] and partially for rank 8 [40], while their explicit MD have been partially classified in Ref. [22].

We go beyond these results and obtain a classification of fermionic MD up to rank 10. Our classification is complete up to some "unresolved" cases (details will be explained in Sec. III B 1). We recover all previously known unitary MD [22,39,40] and obtain nonunitary MD, which had previously not been classified. Furthermore, we discover two completely new classes of MD with a previously unknown fusion rule. The new types of MD are primitive in that they are not obtained from stacking other theories, and they contain large fusion coefficients $\hat{N}_{ij}^k = 3$ or 4, and large total quantum dimensions, larger than that of any MD discovered by Ref. [6].

Moreover, our method allows us to identify the central charge modulo $\frac{1}{2}$ of all these theories, without having to compute their modular extensions. It had been known that, in principle, a super-MTC determines the chiral central charge mod $\frac{1}{2}$ (stacking with fermionic invertible topological orders can change c by multiples of $\frac{1}{2}$ without affecting the super-MTC data, so c only has meaning modulo $\frac{1}{2}$ for a given super-MTC), but due to a lack of an explicit formula relating the MD to the central charge in the fermionic case, previous results could only identify c through the bosonic MTC obtained by modular extension [22]. Since modular extensions are in general difficult to compute for a given super-MTC, it is advantageous to be able to identify c without explicit reference to modular extensions. Our classification method based on congruence representation is in fact able to do this, as will be explained in Sec. III B 3.

The starting point of our method is the result of Ref. [44] that the MD of the fermionic quotient of a super-MTC form projective congruence representations of Γ_θ, a subgroup of $SL_2(\mathbb{Z})$ generated by [26] $\mathfrak{s} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\mathfrak{t}^2$, where $\mathfrak{t} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Thus, if we have a list of congruence representations of Γ_θ, we could hope to do something similar to the bosonic classification of MD carried out in Ref. [20]. We do exactly that, by first obtaining the list of Γ_θ congruence representations using representation theory, and then constructing and checking potential MD from the representations.

In Sec. II, we introduce super-MTCs and some known facts about them which shall be relevant for our classification procedure. In Sec. III, we state the necessary theorems which allow us to obtain all congruence representations of Γ_θ (Sec. III A), and explain how to construct MD from Γ_θ representations (Sec. III B). We present and compare our results, which classify fermionic MD up to rank 10, to previous results in Sec. III C.

II. CATEGORICAL DESCRIPTION OF FERMIONIC TOPOLOGICAL ORDERS

A. Fermionic topological orders, supermodular tensor categories, and spin modular tensor categories

Fermionic topological orders are zero temperature phases beyond Landau's symmetry breaking paradigm, realized in a fermionic many-body system [22]. In (2+1)D, fermionic topological orders (up to invertible topological orders) are characterized by the fusion rules and braiding statistics of emergent pointlike excitations (anyons) together with the fundamental fermionic excitation. The fusion and braiding of these excitations form a categorical structure, known as a super-MTC [22,26].

A super-MTC is a ribbon fusion category with its nontrivial transparent object isomorphic to the local fermion object f. We refer to Ref. [26] for details of super-MTCs, and only introduce some key properties necessary for our purposes. Physically, "transparent" means that f has trivial mutual statistics with any other pointlike excitations. The simple objects of a super-MTC always come in pairs, as for any anyon a, $f \otimes a$ is a distinct object. Hence the rank, or the number of anyons, of a super-MTC is always even. The subcategory of transparent objects (the Müger center) of a super-MTC is sVec, the category of supervector spaces. The trivial super-MTC is equivalent to sVec, and we shall denote it as $\mathcal{F}_0$.

The full physical data of a fermionic topological order (up to stacking with invertible bosonic topological orders) is specified by a *spin modular tensor category* (spin MTC), which is a *modular extension* of a super-MTC [22,26]. A spin MTC is simply a regular MTC which contains a distinguished excitation f which is fermionic ($d_f = 1$, $\theta_f = -1$). Restricting to anyons which have trivial double braiding with f (including f itself), we obtain a super-MTC. For example, the Laughlin fractional quantum Hall states at filling fraction $\nu = \frac{1}{m}$ with odd m are described by the $U(1)_{4m}$ MTC, which has anyons labeled by $l = 0, 1, \cdots, 4m-1$. The $l = 2m$ anyon is the distinguished fermion, and the corresponding super-MTC consists of even labels l [26].

Conversely, given a super-MTC $\mathcal{B}$, we can add anyons which braid nontrivially with the fermion f to build a spin MTC (with nondegenerate S matrix) $\mathcal{M}$, and $\mathcal{M}$ is called a modular extension of $\mathcal{B}$. If $\mathcal{M}$ has smallest possible total quantum dimension $D^2_{\mathcal{M}} = 2D^2_{\mathcal{B}}$, it is called a *minimal* modular extension. In the sequel, modular extension will always mean minimal modular extension unless stated otherwise.

Given a super-MTC, a modular extension always exists [45], and there are always 16 different modular extensions [29]. In other words, a super-MTC does not uniquely determine its modular extension. However, the modular extensions can be distinguished by their *central charge* c. The 16 different modular extensions will have different c mod 8, with c differing by multiples of 1/2. Thus, instead of specifying a spin MTC, we may instead specify the same physical data by specifying a super-MTC together with c mod 8.

B. Fermionic modular data and congruence representations of Γ_θ

As for MTCs, a full characterization of super-MTCs requires gauge-dependent data called R and F tensors. The MD S and T matrix are gauge-invariant and much easier to classify, while they give only a partial characterization: there may be multiple inequivalent fusion categories with the same MD [34] and even if we find candidate MD which satisfy the necessary conditions explained below, it remains to explicitly construct and prove the existence of a fusion category which gives rise to such MD. However, the MD capture a large part of the physical properties of interest [22], and the conditions are stringent enough that they allow us to narrow down the list of candidates considerably.

The S matrix of a super-MTC is always degenerate. However, it is known that the MD of a super-MTC always admit a tensor decomposition [22,26]

$$S = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} \otimes \hat{S}, \quad T = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \otimes \hat{T} \tag{1}$$

where $\hat{S}$ is unitary. Given T, $\hat{T}$ is not well-defined, but $\hat{T}^2$ is. We can always uniquely determine $(\hat{S}, \hat{T}^2)$ in terms of (S, T) and vice versa, so we refer to them interchangeably as fermionic MD, and also simply as MD when no confusion with the bosonic case should arise.

The decomposition in Eq. (1) allows us to make use of $\hat{S}$ and $\hat{T}^2$, which are unitary matrices. These together generate a projective representation of a subgroup Γ_θ of $\mathrm{SL}_2(\mathbb{Z})$ [22,26]. The fact only $\hat{T}^2$ is well-defined reflects the fact that the topological spins of anyons are defined modulo $\frac{1}{2}$ due to existence of the local fermion f. Mathematically, it is because Γ_θ is generated by $\mathfrak{s}$ and $\mathfrak{t}^2$.

In Ref. [44], it was shown that any projective Γ_θ representation arising from a super-MTC (assuming that the super-MTC admits a modular extension) is a *congruence* representation, i.e., its kernel contains a principal congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$. The definition and properties of congruence representations will be detailed in Appendix C. In the sequel, any representation we mention will be assumed to be congruence unless otherwise stated.

Physically, this Γ_θ representation contains information about the NS-NS sector states of the theory. In the bosonic case, topological orders have a Hilbert space of states on the torus, whose dimension corresponds to the number of anyon types. The S and T matrices describe how these states transform under modular transformations of the torus, which form an $\mathrm{SL}_2(\mathbb{Z})$ representation. Similarly, fermionic topological orders carry a space of states on the torus, but the theory is now sensitive to the spin structure. The torus with NS-NS spin structure carries a number of states corresponding to half the rank of the super-MTC, and under modular transformations of the torus which preserve the NS-NS spin structure (which forms Γ_θ), they transform precisely as a Γ_θ representation given by $\hat{S}$ and $\hat{T}^2$ [46].

For the full physical data including those of the R-R sector (the NS-R and R-NS sectors can actually be obtained from the NS-NS sector through modular transformations), we need the modular extension. As discussed in Sec. II A, the modular extension can be specified by giving the central charge mod 8. The advantage of this approach is that instead of working with higher rank $\mathrm{SL}_2(\mathbb{Z})$ representations, we can work with Γ_θ representations of much lower rank. A modular extension $\mathcal{M}$ of a super-MTC $\mathcal{B}$ satisfies $\frac{3}{2}\,\mathrm{rank}\,\mathcal{B} \leqslant \mathrm{rank}\,\mathcal{M} \leqslant 2\,\mathrm{rank}\,\mathcal{B}$ [26]. Moreover, the important information about the modular data of the super-MTC $\mathcal{B}$ is actually captured by the Γ_θ representation formed by $\hat{S}$ and $\hat{T}^2$, whose dimension is $\frac{1}{2}\,\mathrm{rank}\,\mathcal{B}$. For example, consider super-MTCs of rank 10. The corresponding spin MTCs will have rank between 15 and 20; however, the corresponding Γ_θ representation is merely of dimension 5. Thus, describing fermionic topological orders as a pair $(\mathcal{B}, c)$ of a super-MTC and central charge mod 8, as opposed to describing them with a spin MTC $\mathcal{M}$, greatly facilitates their classification.

III. CLASSIFICATION OF FERMIONIC TOPOLOGICAL ORDERS

A. Classification of congruence representations

A complete list of irreducible representations of the modular group $\mathrm{SL}_2(\mathbb{Z})$, organized either by level or by dimension, can be obtained from Ref. [43]. In this section, we explain how to obtain the representations of Γ_θ from those of $\mathrm{SL}_2(\mathbb{Z})$.

1. Representations of a subgroup

Consider a finite group G and its subgroup $H < G$. Suppose we have a representation of G, denoted by R. Then we can obtain a representation of H, denoted by $\mathrm{Res}^G_H R$, by *restriction*, which simply means that we limit ourselves to $R(h)$

such that $h \in H$. If $\mathrm{Res}^G_H R$ of an irreducible representation R is again irreducible, both R and $\mathrm{Res}^G_H R$ are of the same dimension and we say that $\mathrm{Res}^G_H R$ is *extendable*. In other words, any irreducible representation of H which is extendable can be obtained by restriction.

On the other hand, given any representation π of H, we can construct an *induced representation* $\mathrm{Ind}^G_H\pi$ of G (this is unique for a given π). While not every irreducible representation of H is extendable, every representation of H can be induced to a representation of G. Restriction and induction are "adjoint" to each other due to a property known as *Frobenius reciprocity*. Roughly speaking, Frobenius reciprocity states that the induced representation of π decomposes as a direct sum of irreducible representations R_i of G, where each irreducible representation appears with the multiplicity m_i equal to the number of times its restriction to H contains π. As a corollary, every irreducible representation of H is contained in the restriction of some irreducible representation of G. Therefore, by Frobenius reciprocity, we can obtain every irreducible representation of Γ_θ from restriction of irreducible representations of $\mathrm{SL}_2(\mathbb{Z})$. Readers interested in mathematical details are invited to Appendix D.

An explicit description of induced representations is given as follows. Let $\pi : H \to \mathrm{GL}(V)$ be a representation of $H < G$ on a vector space V. Let $N = [H : G]$ be the index of H in G and $\{g_i\}_{i=1}^N$ be the full set of representatives of left cosets in G/H. The G representation $\mathrm{Ind}^G_H\pi$ acts on the vector space $W = \bigoplus_{i=1}^N g_iV$, i.e., N copies of V. For any $g \in G$, its action on W is given by the following. First, for each coset g_i, $g \cdot g_i = g_{j(i)}h_i$ for some (possibly different) coset corresponding to $g_{j(i)}$ and $h_i \in H$. Once we have fixed the set $\{g_i\}_{i=1}^n$ of coset representatives, the decomposition is unique. Second, the g-action permutes the cosets according to $g_i \mapsto g_{j(i)}$. Moreover, on each subspace g_iV, h_i acts by $\pi(h_i)$.

2. Computation of congruence representations of $\boldsymbol{\Gamma_\theta}$

How do these results apply to the case at hand? Both $\mathrm{SL}_2(\mathbb{Z})$ and Γ_θ are infinite, noncompact groups. However, $\mathrm{SL}_2(\mathbb{Z}_n)$ and $\Gamma_\theta/\Gamma(n)$ are finite groups for every n, and $\Gamma_\theta/\Gamma(n) < \mathrm{SL}_2(\mathbb{Z}_n)$. Here, $\Gamma(n)$ is the level-n principal congruence subgroup. Note that n is always an even number because Γ_θ itself is a level-2 congruence subgroup.

Given an $\mathrm{SL}_2(\mathbb{Z})$ representation R of level n, we can restrict it to Γ_θ straightforwardly. Denote the restricted representation by $R|_{\Gamma_\theta}$. Since $\ker R \leqslant \Gamma(n)$, $\ker R|_{\Gamma_\theta} \leqslant \Gamma_\theta \cap \Gamma(n)$. As previously mentioned, $\Gamma(n) < \Gamma_\theta$ for every even n, thus $\Gamma_\theta \cap \Gamma(n) = \Gamma(n)$ if n is even and $\Gamma(2n)$ if n is odd. On the other hand, $\ker R|_{\Gamma_\theta}$ cannot contain $\Gamma(n')$ for $n' < n$; if it were the case, we could think of $R|_{\Gamma_\theta}$ as a representation of $\Gamma_\theta/\Gamma(n')$ and induce it to a representation of $\mathrm{SL}_2(\mathbb{Z}_{n'})$. Frobenius reciprocity ensures the induced representation contains R, but this contradicts the fact that R is of level n. Hence the level of $R|_{\Gamma_\theta}$ is n (if n is even) or $2n$ (if n is odd).

As a consequence, every irreducible representation of $\Gamma_\theta/\Gamma(n)$ (where n is always even) can be obtained from the decomposition into irreducible representations of the restriction of irreducible representations of $\mathrm{SL}_2(\mathbb{Z}_n)$ and $\mathrm{SL}_2(\mathbb{Z}_{n/2})$. In other words, every congruence irreducible representation of Γ_θ can be obtained from restricting and decomposing the congruence irreducible representations of $\mathrm{SL}_2(\mathbb{Z}_n)$. This is the key result which enables us to obtain the full list of congruence representations of Γ_θ up to a given dimension.

In order to facilitate the computation, we make explicit use of induction. For simplicity, we shall speak of the induction from Γ_θ to $\mathrm{SL}_2(\mathbb{Z})$ in the sequel, but technically this should always be understood as an induction from $\Gamma_\theta/\Gamma(n)$ to $\mathrm{SL}_2(\mathbb{Z}_n)$, which are both finite groups. Note that $N = [\Gamma_\theta : \mathrm{SL}_2(\mathbb{Z})] = [\Gamma_\theta/\Gamma(n) : \mathrm{SL}_2(\mathbb{Z}_n)] = 3$ [47], so that if we start with a d-dimensional representation of $\Gamma_\theta/\Gamma(n)$, the dimension of the induced representation is always $3d$.

A choice of left coset representatives of $\Gamma_\theta < \mathrm{SL}_2(\mathbb{Z})$ is given by $\{1, \mathfrak{t}, \mathfrak{st}\}$. Let us denote the Γ_θ representation by $\rho : \Gamma_\theta \to \mathrm{GL}(V)$, and its induced representation by $R : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{GL}(W)$. If the action of ρ on V is represented by the matrices $\rho(\mathfrak{s}) = \mathfrak{S}$ and $\rho(\mathfrak{t}^2) = \mathfrak{T}^2$, then the action of R on $W = V \oplus \mathfrak{t}V \oplus \mathfrak{st}V$ is given by the matrices $R(\mathfrak{s}) = \mathcal{S}$ and $R(\mathfrak{t}) = \mathcal{T}$ which take the block form

$$\mathcal{S} = \begin{pmatrix} \mathfrak{S} & 0 & 0 \\ 0 & 0 & \mathfrak{S}^2 \\ 0 & \mathbb{1} & 0 \end{pmatrix}, \quad \mathcal{T} = \begin{pmatrix} 0 & \mathfrak{T}^2 & 0 \\ \mathbb{1} & 0 & 0 \\ 0 & 0 & (\mathfrak{S}\mathfrak{T}^2)^{-1} \end{pmatrix}. \quad (2)$$

This explicit form of the induced representation allows us to efficiently compute d-dimensional irreducible representations of Γ_θ coming from $3d$-dimensional irreducible representations of $\mathrm{SL}_2(\mathbb{Z})$, using the reverse induction formula detailed in Appendix B. The restriction to Γ_θ means the generators are now

$$\mathcal{S} = \begin{pmatrix} \mathfrak{S} & 0 & 0 \\ 0 & 0 & \mathfrak{S}^2 \\ 0 & \mathbb{1} & 0 \end{pmatrix}, \quad \mathcal{T}^2 = \begin{pmatrix} \mathfrak{T}^2 & 0 & 0 \\ 0 & \mathfrak{T}^2 & 0 \\ 0 & 0 & (\mathfrak{S}\mathfrak{T}^2)^{-2} \end{pmatrix}. \quad (3)$$

Note that the restriction of the induced representation indeed contains the original representation of Γ_θ in the first block.

Frobenius reciprocity heavily constrains which irreducible representations of $\mathrm{SL}_2(\mathbb{Z})$ give rise to an irreducible representation of Γ_θ of a given dimension d. Consider a d-dimensional irreducible representation ρ of Γ_θ. Its induced representation takes the form

$$\mathrm{Ind}\rho = \bigoplus_i m_i R_i \quad (4)$$

for some irreducible representations R of $\mathrm{SL}_2(\mathbb{Z})$, which in turn satisfy

$$\mathrm{Res}R_i = m_i\rho \oplus \cdots \quad (5)$$

for each i. Since $\dim \mathrm{Ind}\rho = 3d$, $\mathrm{Ind}\rho$ (if it is not $3d$-dimensional) can only decompose as $3d = d + d + d$ (in which case ρ is extendable), or $(d + a) + (2d - a)$ for some $0 \leqslant a \leqslant d$. The $(d + a)$-dimensional irreducible representation R_{d+a} satisfies $\mathrm{Res}R_{d+a} = \rho \oplus \sigma \oplus \cdots$ where σ can at most be a-dimensional. $\mathrm{Ind}\sigma$ should in turn contain R_{d+a}, so we need

$$3a \geqslant \dim \mathrm{Ind}\sigma \geqslant \dim R_{d+a} = d + a. \quad (6)$$

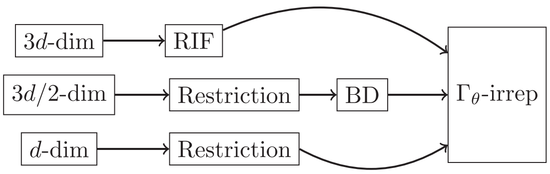

FIG. 1. Flowchart of computation of Γ_θ representations.

This translates to $a \geqslant d/2$, or

$$d + a \geqslant \tfrac{3}{2}d. \tag{7}$$

On the other hand, if $a > d/2$, then $2d - a < 3d/2$, so the $(2d - a)$-dimensional irreducible representation R_{2d-a} would not satisfy the above requirements. Hence we need exactly $a = d/2$, or, in other words,

$$d + a = \tfrac{3}{2}d. \tag{8}$$

Thus, for any given ρ, either its induced representation is a $3d$-dimensional irreducible representation of $\mathrm{SL}_2(\mathbb{Z})$, or it decomposes into two $3d/2$-dimensional irreducible representations of $\mathrm{SL}_2(\mathbb{Z})$. When d is odd, the latter possibility is precluded as $3d/2$ is not an integer.

For example, if we are interested in obtaining four-dimensional irreducible representations of Γ_θ, we should look at the restrictions of (i) four-dimensional $\mathrm{SL}_2(\mathbb{Z})$-irreps (these give rise to extendable irreps), (ii) six-dimensional $\mathrm{SL}_2(\mathbb{Z})$-irreps, and (iii) 12-dimensional $\mathrm{SL}_2(\mathbb{Z})$-irreps and obtain their irreducible components.

When the spectrum of $\mathcal{T}$ is degenerate, the reverse induction formula is difficult to be utilized due to the freedom of orthogonal transformation in the degenerate subspace. Fortunately, at least for $d \leqslant 5$, the $3d$-dimensional irreducible representations of $\mathrm{SL}_2(\mathbb{Z})$ which have degenerate $\mathcal{T}$ are not induced representations. However, $(3d/2 + 3d/2)$-dimensional representations should always have degenerate $\mathcal{T}$; otherwise, they cannot be valid MD. Thus we directly block-diagonalize the restricted representations (i.e., $\mathcal{S}$ and $\mathcal{T}^2$) simultaneously to get irreducible representations of Γ_θ. Schematic flowchart of Γ_θ representation computing process is shown in Fig. 1.

We compute all congruence irreducible representations of Γ_θ up to dimension 5. The number of irreducible representations for each dimension is shown in Table IV. For comparison, the number of irreducible representations of $\mathrm{SL}_2(\mathbb{Z})$ for each dimension is shown as well.

B. Construction of fermionic modular data

Once we have obtained all irreducible representations of Γ_θ from those of $\mathrm{SL}_2(\mathbb{Z})$, using methods outlined in Sec. III A 2, we can use them to construct candidate MD. As in bosonic case [20], we first construct a Γ_θ representation ρ_{isum} for a given dimension d as a direct sum of irreducible Γ_θ representations. Then, by applying an orthogonal transformation U, we put it into a specific basis which makes it a candidate for an MD. After obtaining the list of candidates, we can check the necessary conditions for being a valid MD of a super-MTC, such as the Verlinde formula and the Frobenius-Schur indicator condition. In this work, we carry this program out up to dimension 5 (which corresponds to super-MTCs of rank 10).

1. Basis transformation and resolved representations

The MD $(\hat{S}, \hat{T}^2)$ are basis-dependent quantities, and even if $(\hat{S}, \hat{T}^2)$ form a reducible representation, the corresponding super-MTC may be indecomposable. Thus, if we are interseted in fermionic MD $(\hat{S}, \hat{T}^2)$ of dimension d, we need to look at all d-dimensional representations of Γ_θ (including reducible ones), in all possible valid bases. Following Ref. [20], we denote by ρ_{isum} the direct sum of irreducible representations in the basis coming from our list of symmetric irreducible representations Γ_θ (in their case ρ_{isum} denotes a direct sum of irreducible $\mathrm{SL}_2(\mathbb{Z})$ representations), and $\rho = U\rho_{\text{isum}}U^{-1}$ the basis-changed version, which is a candidate for the MD. More precisely, ρ is a linear lift of the projective representation formed by $(\hat{S}, \hat{T}^2)$.

There are several conditions for a valid basis. First, $\rho(\mathfrak{t}^2)$ should be diagonal. As our irreducible representations of Γ_θ are all in this form, this condition is automatically satisfied by any ρ_{isum}. In orther to preserve this under a transformation $U\rho_{\text{isum}}U^{-1}$, U can only act block-diagonally, where each block corresponds to a degenerate subspace of the eigenvalues of $\rho(\mathfrak{t}^2)$. Second, we require that $\rho(\mathfrak{s})$ is symmetric. As ρ_{isum} is always symmetric, we need U to be an orthogonal matrix. (To make each step clear, we think of U as a combination of a signed diagonal matrix V and an orthogonal matrix U_0, i.e., $U = VU_0$.) Lastly, $\rho(\mathfrak{s})$ should not have zeros in the first row (or, equivalenly, the column, since it is symmetric), i.e., $\forall i$, $\rho(\mathfrak{s})_{1i} \neq 0$, corresponding to the fact that quantum dimensions cannot be zero. This leads to the $\mathfrak{t}^2$-spectrum condition, which states that whenever ρ_{isum} is a direct sum, the $\mathfrak{t}^2$-spectrum of each direct summand should have nonempty overlap [37]. (In Ref. [37], authors deal with $\mathrm{SL}_2(\mathbb{Z})$ representations and hence the $\mathfrak{t}$-spectrum, rather than the $\mathfrak{t}^2$-spectrum, but the idea is the same.)

Accordingly, for a given dimension, we build ρ_{isum} and organize them into types, according to how much overlap their $\mathfrak{t}$-spectra have. For example, for dimension 4, we consider the following types of ρ_{isum}: 4-d irreps, (3+1)-d type (2), (2+2)-d type (2), (2+2)-d type (2,2), (2+1+1)-d type (2,2), and (2+1+1)-d type (3,1). Here, type (a, b) denotes that the eigenvalues of $\rho_{\text{isum}}(\mathfrak{t}^2)$ (hence $\rho(\mathfrak{t}^2)$) overlap in sets of sizes a and b. For example, if the four eigenvalues are $\{1, 1, 1, -1\}$, the representation is of type (3,1). In dimension 5, (1+1+1+1+1)-d type (5) does not yield any valid MD. We prove in Appendix G that a direct sum of one-dimensional representations can only give rise to split (hence nonprimitive) super-MTCs.

We observe that when ρ_{isum} is irreducible, the $\mathfrak{t}^2$-spectrum is nondegenerate, at least up to dimension 5, so there is no further possibility of orthogonal transformation U_0 available. In such a case we simply perform all possible signed diagonal transformations, $V\rho_{\text{isum}}V^{-1}$, which gives us the candidate ρ. On the other hand, when ρ_{isum} is a direct sum, the $\mathfrak{t}^2$-spectrum is degenerate, and in each degenerate eigenspace of dimension d_θ (corresponding to the topological spin θ) we can perform an orthogonal transformation of dimension d_θ.

The possible orthogonal transformations are in fact heavily constrained for the so-called "resolved representations" [20], for which the degenerate eigenspace can be "resolved" (i.e., the degeneracy lifted) by the set of matrices

$$H(a) = \rho(\mathfrak{s})^2\rho(\mathfrak{t})^{-(a-1)}\rho(\mathfrak{s})(\rho(\mathfrak{t})^2\rho(\mathfrak{s}))^{\bar{a}-1}\rho(\mathfrak{t})^{-(a-1)}\rho(\mathfrak{s}) \tag{9}$$

where a is an element of $\mathbb{Z}_n^\times$ which satisfies $\theta^{a^2} = \theta$. Here, n is the level of the representation ρ, and $\bar{a}$ is the inverse of a modulo n, i.e., $a\bar{a} \equiv 1 \mod n$. Due to a theorem related to Galois conjugation [48], each $H(a)$ should be a signed permutation for a valid ρ. The theorem is proved for the bosonic case, but using the existence of a modular extension, we can extend the result to fermionic case. The fact that $H(a)$ has to be a signed permutation matrix after the orthogonal transforamtion U_0 places severe constraints on what U_0 can be for resolved representations. For details, see Sec. C.1 of Ref. [20]. We apply their logic to the fermionic case, and find that we need only consider the follwoing orthogonal transformations for the resolved degenerate eigenspaces (as mentioned above and in Appendix G, we need not consider a five-dimensional eigenspace). For a two-dimensional subspace,

$$\begin{pmatrix}\cos\phi & -\sin\phi\\ \sin\phi & \cos\phi\end{pmatrix} \tag{10}$$

with $\phi = 0, \pi/4, -\pi/4$. For a three-dimensional subspace,

$$\begin{pmatrix}\cos\phi & -\sin\phi & 0\\ \sin\phi & \cos\phi & 0\\ 0 & 0 & 1\end{pmatrix}, \quad \begin{pmatrix}\cos\phi & 0 & -\sin\phi\\ 0 & 1 & 0\\ \sin\phi & 0 & \cos\phi\end{pmatrix}, \quad \begin{pmatrix}1 & 0 & 0\\ 0 & \cos\phi & -\sin\phi\\ 0 & \sin\phi & \cos\phi\end{pmatrix} \tag{11}$$

with $\phi = 0, \pi/4, -\pi/4$. For a four-dimensional subspace,

$$\begin{gathered}\begin{pmatrix}\cos\phi & -\sin\phi & 0 & 0\\ \sin\phi & \cos\phi & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}, \quad \begin{pmatrix}\cos\phi & 0 & -\sin\phi & 0\\ 0 & 1 & 0 & 0\\ \sin\phi & 0 & \cos\phi & 0\\ 0 & 0 & 0 & 1\end{pmatrix}, \quad \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & \cos\phi & -\sin\phi & 0\\ 0 & \sin\phi & \cos\phi & 0\\ 0 & 0 & 0 & 1\end{pmatrix},\\ \begin{pmatrix}\cos\phi & 0 & 0 & -\sin\phi\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ \sin\phi & 0 & 0 & \cos\phi\end{pmatrix}, \quad \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & \cos\phi & 0 & -\sin\phi\\ 0 & 0 & 1 & 0\\ 0 & \sin\phi & 0 & \cos\phi\end{pmatrix}, \quad \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & \cos\phi & -\sin\phi\\ 0 & 0 & \sin\phi & \cos\phi\end{pmatrix}\end{gathered} \tag{12}$$

with $\phi = 0, \pi/4, -\pi/4$.

Hence, for resolved representations, there are only a discrete set of possible candidates for MD. The vast majority of known valid MD come from resolved representations—in fact, up to rank 8, for which there is a more or less complete classification for unitary super-MTCs, all but one of them come from resolved representations (the one exception corresponds to the toric code stacked with the trivial fermionic theory, $4_0^B \boxtimes \mathcal{F}_0$). For rank 10, a few of the known unitary super-MTCs are obtained from unresolved representations. We discuss how we obtained them, as well as their nonunitary versions, and our general (though incomplete) strategy for dealing with unresolved representations, in Appendix E.

2. From linear to projective representations and modular data

Once we obtain the candidate linear representations $\rho = U\rho_{\text{isum}}U^{-1}$, we can easily construct the MD by

$$\hat{S} = \frac{|\rho(\mathfrak{s})_{11}|}{\rho(\mathfrak{s})_{11}}\rho(\mathfrak{s})_{11}, \quad \hat{T}^2 = \frac{\rho(\mathfrak{t}^2)}{\rho(\mathfrak{t}^2)_{11}} \tag{13}$$

where the vacuum corresponds to the first index. The MD $(\hat{S}, \hat{T}^2)$ now only satisfy the relations of the congruence representation projectively, and the level may change.

The linear representation ρ may be thought of as a lift of the projective representation formed by $\hat{S}$ and $\hat{T}^2$ to a linear representation. If every projective representation of Γ_θ formed by MD admits such a linear lift, then we can claim that our search for MD is complete, since we begin with a complete list of linear representations of a given dimension. For $\mathrm{SL}_2(\mathbb{Z})$, the existence of linear lifts of the projective reprsentations formed by bosonic MD is guaranteed [48]. We state a similar theorem for fermionic MD and Γ_θ.

Theorem III.1. Suppose $\tilde{\rho}$ is a projective representation of Γ_θ formed by the fermionic MD $(\hat{S}, \hat{T}^2)$ of a super-MTC $\mathcal{B}$, i.e., $\tilde{\rho}(\mathfrak{s}) = \hat{S}$ and $\tilde{\rho}(\mathfrak{t}^2) = \hat{T}^2$. Then, $\tilde{\rho}$ always admits a lift to a linear congruence representation of Γ_θ.

For brevity, we detail the proof of theorem III.1 in Appendix F.

After obtaining the candidate $(\hat{S}, \hat{T}^2)$ via Eq. (13), we check whether they are valid using the Verlinde formula

$$\hat{N}_k^{ij} = \sum_{l\in\Pi_0} \frac{\hat{S}_{il}\hat{S}_{jl}\hat{S}_{kl}^*}{\hat{S}_{1l}} \tag{14}$$

where the nonnegative integer fusion coefficients $\hat{N}_k^{ij}$ form a fusion ring, and the Frobenius-Schur indicator condition

$$\pm 1 = \nu_2(a) = \frac{2}{D^2} \sum_{j,k\in\Pi_0} \hat{N}_a^{jk} d_j d_k \left(\frac{\theta_j}{\theta_k}\right)^2 \qquad (15)$$

for any self-dual anyon a (i.e., an anyon which satisfies $\bar{a} = a$). Here, Π_0 is the label set of anyons, $d_i = \hat{S}_{1i}$ is the quantum dimension of anyon i, $\theta_i = \hat{T}_{ii} = e^{2i\pi s_i}$ where s_i is the topological spin of anyon i, and $D = \sqrt{\sum_i d_i^2}$ is the total quantum dimension.

In addition, we check the balancing equation (A3). While the Verlinde formula and the Frobenius-Schur indicator conditions can be checked in terms of the fusion rules $\hat{N}_{ij}^k$ of the fermionic quotient, to check the balancing equation we need the full fusion rules N_{ij}^k. Checking the balancing equation then is really a question of asking: given the $\hat{N}_{ij}^k$, obtained from our candidate $\hat{S}$, can we construct N_{ij}^k satisfying Eq. (A3) such that the balancing equation is satisfied? We find that, sometimes there are two very similar MD (with identical lists of spins and quantum dimensions) with minor differences in some of the entries of $\hat{S}$, and that only one of the $\hat{S}$ is consistent with the balancing equation. Hence the balancing equation helps us pin down the correct $\hat{S}$ matrix.

3. Central charge from linear representations

A strength of our approach is that we can determine the central charge of the resulting super-MTC, which is defined modulo 1/2. For bosonic MTCs, whose central charge is defined modulo 8, the approach of congruence representations confers no additional advantage as it is straightforward to determine the central charge from the modular matrices S and T via $(ST)^3 = e^{2i\pi c/8} S^2$. In the fermionic case, where S is degenerate, and $(\hat{S}, \hat{T}^2)$ only form a projective representation of Γ_θ rather than $\mathrm{SL}_2(\mathbb{Z})$, it is impossible to determine c from the given MD by themselves using only the group relations of Γ_θ.

Rather, for super-MTCs, c is defined in terms of the central charge of the modular extensions [22]. While the central charge of each modular extension is defined modulo 8, there are 16 different modular extensions for a given super-MTC (as a consequence of theorem 5.4 of Ref. [29]) with their central charges differing by multiples of 1/2 [26], so c is defined modulo 1/2 for a super-MTC. This means that, in order to compute the central charge of a super-MTC, we first need to compute (one of) the modular extensions. The modular extensions are bosonic MTCs of much higher rank (see lemma 4.2 of Ref. [39] for an explicit bound on the rank), and their computation is a highly nontrivial task.

Our approach, which begins first with linear representations and then constructs the projective representations, allows us to determine the central charge of the super-MTCs we obtain without having to compute their modular extensions. The key idea is that the central charge of the modular extensions is involved in the lift of the fermionic MD to a linear representation. (See Appendix F.)

For each MD $(\hat{S}, \hat{T}^2)$, if $\rho(\mathfrak{t}^2) = e^{-2i\pi c/12} \hat{T}^2$ furnishes a linear lift, then $e^{-2i\pi (c+m/2)/12} \hat{T}^2$ also furnishes a linear lift. Hence there are at least 24 different linear representations (up to tensor product with one-dimensional representations, which does not affect the central charge) for a given MD. In our classification process, we start with a complete list of linear representations which can potentially yield valid MD. Thus our list of linear representations must include these linear lifts coming from the existence of minimal modular extensions, i.e., for every projective representation formed by a given MD, there are at least 24 different linear representations which all lead to it. If there are exactly 24, their c should differ by multiples of 1/2, and this fixes the c of the super-MTC modulo 1/2.

More concretely, consider a particular pair $(\hat{S}, \hat{T}^2)$. We keep track of which linear representations ρ_α gave raise to this MD. These ρ_α differ from one another by a phase of $\rho_\alpha(\mathfrak{t}^2)$. If we find that there are 24 such ρ_α with $\rho_\alpha(\mathfrak{t}^2) = e^{2i\pi\alpha/12} \rho_0(\mathfrak{t}^2)$, where ρ_0 is a chosen reference representation, and α come in steps of 1/2, then we can fix c modulo 1/2. For every MD, we obtain, this has been the case, enabling us to determine c modulo 1/2.

C. Comparison to previous results

The results of the classification are summarized in Tables I–IX. Let us compare our results to previous results in the literature. First, previous results (for any rank) were limited to unitary MD, but we obtain both unitary and nonunitary MD. We find that for every non-Abelian fusion rule, there are both unitary and nonunitary MD realizing it (for comparison, in the bosonic case, every non-Abelian fusion rule up to rank 5 has both a unitary and nonunitary realization [20,49]). We expect that the nonunitary MD are related by Galois conjugation to the unitary MD. Moreover, in the unitary case, we recover all previously known MD [22].

In addition, we obtain two completely new fusion rules of rank 10, and unitary and nonunitary MD realizing it (Table IX). The unitary MD for these fusion rules have total quantum dimension $D^2 = 472.379$ or 475.151, which are much larger than any previously known total quantum dimension for rank 10 [22]. The new MD are non-Abelian and primitive, and do not fall into (the fermion condensation of) any known series of MTCs [50]. One may ask whether these new MD are in fact realizable by a super-MTC. The answer is yes. First, in a work to be published [51], we compute the minimal modular extensions (on the level of bosonic modular data) for these MD. If we assume that those bosonic MD are realizable by MTCs, these fermionic MD are also realizable by super-MTCs through fermion condensation. Moreover, recently, inspired by the arXiv version of the present paper, Ref. [52] has used the Drinfeld centers of near-group fusion categories to construct super-MTC realizing these new MD. Their construction is explicit for representative cases, and others are believed to be closely related to these via Galois conjugation. Hence it is reasonable to believe that these new MD all realizable. These new MD involve largest fusion coefficient $\hat{N}_k^{ij} = 3$ or 4. For comparison, all previously known examples of rank 10 MD had the bound $\hat{N}_k^{ij} \leqslant 2$.

The classification of rank 8 fusion rules by Ref. [40] had to place some bounds on the fusion coefficients in certain cases, though the bounds are very generous ($\hat{N}_k^{ij} \leqslant 14$ or $\hat{N}_k^{ij} \leqslant 21$). Our method places no such bound on the fusion coefficients, and yet do not find new fusion rules of rank 8; this is evidence for the results of Ref. [40] being complete.

TABLE I. List of rank 4 fermionic MD. Shaded data are of nonunitary MD. MD in the same box share the same (fermionic quotient) fusion rule. They may be related by Galois conjugation. For simplicity of notation, we have introduced $\zeta_n^m = \frac{\sin[\pi(m+1)/(n+2)]}{\sin[\pi/(n+2)]}$ and $\chi_n^m = m + \sqrt{n}$. In the last column, we comment on whether the MD is obtained from stacking, or is primitive. Data of non-Abelian bosonic MD are retrieved from Supplementary Material of Ref. [20], and given the notation $\text{Rank}_{\#}^{B}$, while Abelian MD, which are taken from Table 11 of Ref. [6] and specified by Rank_c^B. $\text{Rank}_c^{B,*}$ refer to nonunitary analogues of the unitary Abelian MD. We also note the cases where the primitive MD are obtained from fermion condensation ("f.c.") of known affine Lie algebra constructions.

#	c	D^2	Quantum dimensions	Topological spins	Comments
1	0	4	1,1,1,1	$0, \frac{1}{2}, \frac{1}{4}, -\frac{1}{4}$	$\mathcal{F}_0 \boxtimes 2_1^B$
2	0	4	$1, 1, -1, -1$	$0, \frac{1}{2}, \frac{1}{4}, -\frac{1}{4}$	$\mathcal{F}_0 \boxtimes 2_1^{B,*}$
3	$\frac{1}{5}$	7.2360	$1, 1, \zeta_3^1, \zeta_3^1$	$0, \frac{1}{2}, \frac{1}{10}, -\frac{2}{5}$	$\mathcal{F}_0 \boxtimes 2_{\#2}^B$
4	$-\frac{1}{5}$	7.2360	$1, 1, \zeta_3^1, \zeta_3^1$	$0, \frac{1}{2}, -\frac{1}{10}, \frac{2}{5}$	$\mathcal{F}_0 \boxtimes 2_{\#1}^B$
5	$\frac{1}{10}$	2.762	$1, 1, -\frac{1}{\zeta_3^1}, -\frac{1}{\zeta_3^1}$	$0, \frac{1}{2}, -\frac{1}{5}, \frac{3}{10}$	$\mathcal{F}_0 \boxtimes 2_{\#4}^B$
6	$-\frac{1}{10}$	2.762	$1, 1, -\frac{1}{\zeta_3^1}, -\frac{1}{\zeta_3^1}$	$0, \frac{1}{2}, \frac{1}{5}, -\frac{3}{10}$	$\mathcal{F}_0 \boxtimes 2_{\#3}^B$
7	$\frac{1}{4}$	13.656	$1, 1, \chi_2^1, \chi_2^1$	$0, \frac{1}{2}, \frac{1}{4}, -\frac{1}{4}$	Primitive: f.c. of $(A_1)_6$
8	$\frac{1}{4}$	2.343	$1, 1, -\frac{1}{\chi_2^1}, -\frac{1}{\chi_2^1}$	$0, \frac{1}{2}, \frac{1}{4}, -\frac{1}{4}$	Primitive

The arXiv version of Ref. [22] listed some MD which did not have valid modular extensions, colored in red. These do not appear in our classification as they do not form congruence representations. Our method automatically excludes such spurious MD without having to independently check the existence of modular extensions. Moreover, as noted in Sec. III B 3, we find central charges modulo 1/2 for every MD we obtain. Previously, the central charge data was missing for several MD of rank 8 and rank 10 in Ref. [22].

IV. CONCLUSION

In this paper, we have detailed a procedure to classify the MD of super-MTCs using congruence representations, and have provided a full classification of both unitary and nonunitary MD up to rank 10. The classification is complete up to potential new MD coming from unresolved representations. Our result includes every unitary MD hitherto obtained, and also includes nonunitary MD. We also find new primitive MD of rank 10 with completely new fusion rules.

TABLE II. List of rank 6 fermionic MD.

#	c	D^2	Quantum dimensions	Topological spins	Comments
1	0	6	1,1,1,1,1,1	$0, \frac{1}{2}, \frac{1}{6}, -\frac{1}{3}, \frac{1}{6}, -\frac{1}{3}$	$\mathcal{F}_0 \boxtimes 3_2^B$
2	0	6	1,1,1,1,1,1	$0, \frac{1}{2}, -\frac{1}{6}, \frac{1}{3}, -\frac{1}{6}, \frac{1}{3}$	$\mathcal{F}_0 \boxtimes 3_{-2}^B$
3	0	8	$1, 1, 1, 1, \sqrt{2}, \sqrt{2}$	$0, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{16}, -\frac{7}{16}$	$\mathcal{F}_0 \boxtimes 3_{\#7}^B = \mathcal{F}_0 \boxtimes 3_{\#9}^B$
4	0	8	$1, 1, 1, 1, \sqrt{2}, \sqrt{2}$	$0, \frac{1}{2}, 0, \frac{1}{2}, -\frac{1}{16}, \frac{7}{16}$	$\mathcal{F}_0 \boxtimes 3_{\#8}^B = \mathcal{F}_0 \boxtimes 3_{\#10}^B$
5	0	8	$1, 1, 1, 1, \sqrt{2}, \sqrt{2}$	$0, \frac{1}{2}, 0, \frac{1}{2}, \frac{3}{16}, -\frac{5}{16}$	$\mathcal{F}_0 \boxtimes 3_{\#15}^B = \mathcal{F}_0 \boxtimes 3_{\#17}^B$
6	0	8	$1, 1, 1, 1, \sqrt{2}, \sqrt{2}$	$0, \frac{1}{2}, 0, \frac{1}{2}, -\frac{3}{16}, \frac{5}{16}$	$\mathcal{F}_0 \boxtimes 3_{\#16}^B = \mathcal{F}_0 \boxtimes 3_{\#18}^B$
7	0	8	$1, 1, 1, 1, -\sqrt{2}, -\sqrt{2}$	$0, \frac{1}{2}, 0, \frac{1}{2}, -\frac{3}{16}, \frac{5}{16}$	$\mathcal{F}_0 \boxtimes 3_{\#12}^B = \mathcal{F}_0 \boxtimes 3_{\#14}^B$
8	0	8	$1, 1, 1, 1, -\sqrt{2}, -\sqrt{2}$	$0, \frac{1}{2}, 0, \frac{1}{2}, -\frac{1}{16}, \frac{7}{16}$	$\mathcal{F}_0 \boxtimes 3_{\#20}^B = \mathcal{F}_0 \boxtimes 3_{\#22}^B$
9	0	8	$1, 1, 1, 1, -\sqrt{2}, -\sqrt{2}$	$0, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{16}, -\frac{7}{16}$	$\mathcal{F}_0 \boxtimes 3_{\#19}^B = \mathcal{F}_0 \boxtimes 3_{\#21}^B$
10	0	8	$1, 1, 1, 1, -\sqrt{2}, -\sqrt{2}$	$0, \frac{1}{2}, 0, \frac{1}{2}, \frac{3}{16}, -\frac{5}{16}$	$\mathcal{F}_0 \boxtimes 3_{\#11}^B = \mathcal{F}_0 \boxtimes 3_{\#13}^B$
11	$\frac{1}{7}$	18.591	$1, 1, \zeta_5^1, \zeta_5^1, \zeta_5^2, \zeta_5^2$	$0, \frac{1}{2}, -\frac{1}{7}, \frac{5}{14}, -\frac{3}{14}, \frac{2}{7}$	$\mathcal{F}_0 \boxtimes 3_{\#2}^B$
12	$-\frac{1}{7}$	18.591	$1, 1, \zeta_5^1, \zeta_5^1, \zeta_5^2, \zeta_5^2$	$0, \frac{1}{2}, \frac{1}{7}, -\frac{5}{14}, \frac{3}{14}, -\frac{2}{7}$	$\mathcal{F}_0 \boxtimes 3_{\#1}^B$
13	$-\frac{3}{14}$	5.724	$1, 1, -\frac{\zeta_5^2}{\zeta_5^1}, -\frac{\zeta_5^2}{\zeta_5^1}, \frac{1}{\zeta_5^1}, \frac{1}{\zeta_5^1}$	$0, \frac{1}{2}, \frac{3}{14}, -\frac{2}{7}, \frac{1}{14}, -\frac{3}{7}$	$\mathcal{F}_0 \boxtimes 3_{\#5}^B$
14	$\frac{1}{14}$	3.682	$1, 1, -\frac{\zeta_5^1}{\zeta_5^2}, -\frac{\zeta_5^1}{\zeta_5^2}, \frac{1}{\zeta_5^2}, \frac{1}{\zeta_5^2}$	$0, \frac{1}{2}, \frac{1}{7}, -\frac{5}{14}, -\frac{1}{14}, \frac{3}{7}$	$\mathcal{F}_0 \boxtimes 3_{\#3}^B$
15	$-\frac{1}{14}$	3.682	$1, 1, -\frac{\zeta_5^1}{\zeta_5^2}, -\frac{\zeta_5^1}{\zeta_5^2}, \frac{1}{\zeta_5^2}, \frac{1}{\zeta_5^2}$	$0, \frac{1}{2}, -\frac{1}{7}, \frac{5}{14}, \frac{1}{14}, -\frac{3}{7}$	$\mathcal{F}_0 \boxtimes 3_{\#6}^B$
16	$\frac{3}{14}$	5.724	$1, 1, -\frac{\zeta_5^2}{\zeta_5^1}, -\frac{\zeta_5^2}{\zeta_5^1}, \frac{1}{\zeta_5^1}, \frac{1}{\zeta_5^1}$	$0, \frac{1}{2}, -\frac{3}{14}, \frac{2}{7}, -\frac{1}{14}, \frac{3}{7}$	$\mathcal{F}_0 \boxtimes 3_{\#4}^B$
17	0	44.784	$1, 1, \chi_3^1, \chi_3^1, \chi_3^2, \chi_3^2$	$0, \frac{1}{2}, -\frac{1}{6}, \frac{1}{3}, 0, \frac{1}{2}$	Primitive: f.c. of $(A_1)_{-10}$
18	0	44.784	$1, 1, \chi_3^1, \chi_3^1, \chi_3^2, \chi_3^2$	$0, \frac{1}{2}, \frac{1}{6}, -\frac{1}{3}, 0, \frac{1}{2}$	Primitive: f.c. of $(A_1)_{10}$
19	0	3.2154	$1, 1, -\frac{\chi_3^1}{\chi_3^2}, -\frac{\chi_3^1}{\chi_3^2}, \frac{1}{\chi_3^2}, \frac{1}{\chi_3^2}$	$0, \frac{1}{2}, -\frac{1}{6}, \frac{1}{3}, 0, \frac{1}{2}$	Primitive
20	0	3.2154	$1, 1, -\frac{\chi_3^1}{\chi_3^2}, -\frac{\chi_3^1}{\chi_3^2}, \frac{1}{\chi_3^2}, \frac{1}{\chi_3^2}$	$0, \frac{1}{2}, \frac{1}{6}, -\frac{1}{3}, 0, \frac{1}{2}$	Primitive

TABLE III. List of rank 8 fermionic MD.

#	c	D^2	Quantum dimensions	Topological spins	Comments
1	0	8	1,1,1,1,1,1,1,1	$0, \frac{1}{2}, 0, \frac{1}{2}, 0, \frac{1}{2}, 0, \frac{1}{2}$	$\mathcal{F}_0 \boxtimes 4_0^{B,a}$
2	0	8	$1, 1, 1, 1, -1, -1, -1, -1$	$0, \frac{1}{2}, 0, \frac{1}{2}, 0, \frac{1}{2}, 0, \frac{1}{2}$	$\mathcal{F}_0 \boxtimes 4_0^{B,a,*}$
3	0	8	1,1,1,1,1,1,1,1	$0, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{8}, -\frac{3}{8}, \frac{1}{8}, -\frac{3}{8}$	$\mathcal{F}_0 \boxtimes 4_1^{B}$
4	0	8	1,1,1,1,1,1,1,1	$0, \frac{1}{2}, 0, \frac{1}{2}, -\frac{1}{8}, \frac{3}{8}, -\frac{1}{8}, \frac{3}{8}$	$\mathcal{F}_0 \boxtimes 4_3^{B}$
5	0	8	$1, 1, -1, -1, -1, -1, 1, 1$	$0, \frac{1}{2}, 0, \frac{1}{2}, -\frac{1}{8}, \frac{3}{8}, -\frac{1}{8}, \frac{3}{8}$	$\mathcal{F}_0 \boxtimes 4_3^{B,*}$
6	0	8	$1, 1, -1, -1, -1, -1, 1, 1$	$0, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{8}, -\frac{3}{8}, \frac{1}{8}, -\frac{3}{8}$	$\mathcal{F}_0 \boxtimes 4_1^{B,*}$
7	0	8	1,1,1,1,1,1,1,1	$0, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, -\frac{1}{4}$	$\mathcal{F}_0 \boxtimes 4_0^{B,b}$
8	0	8	$1, 1, -1, -1, -1, -1, 1, 1$	$0, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, -\frac{1}{4}$	$\mathcal{F}_0 \boxtimes 4_0^{B,b,*1}$
9	0	8	$1, 1, 1, 1, -1, -1, -1, -1$	$0, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, -\frac{1}{4}$	$\mathcal{F}_0 \boxtimes 4_0^{B,b,*2}$
10	$\frac{1}{5}$	14.472	$1, 1, 1, 1, \zeta_3^1, \zeta_3^1, \zeta_3^1, \zeta_3^1$	$0, \frac{1}{2}, \frac{1}{4}, -\frac{1}{4}, \frac{1}{10}, -\frac{2}{5}, -\frac{3}{20}, \frac{7}{20}$	$\mathcal{F}_0 \boxtimes 4_{\#18}^{B} = \mathcal{F}_0 \boxtimes 4_{\#20}^{B}$
11	$-\frac{1}{5}$	14.472	$1, 1, 1, 1, \zeta_3^1, \zeta_3^1, \zeta_3^1, \zeta_3^1$	$0, \frac{1}{2}, \frac{1}{4}, -\frac{1}{4}, \frac{3}{20}, -\frac{7}{20}, -\frac{1}{10}, \frac{2}{5}$	$\mathcal{F}_0 \boxtimes 4_{\#17}^{B} = \mathcal{F}_0 \boxtimes 4_{\#19}^{B}$
12	$-\frac{1}{5}$	14.472	$1, 1, -1, -1, -\zeta_3^1, -\zeta_3^1, \zeta_3^1, \zeta_3^1$	$0, \frac{1}{2}, \frac{1}{4}, -\frac{1}{4}, \frac{3}{20}, -\frac{7}{20}, -\frac{1}{10}, \frac{2}{5}$	$\mathcal{F}_0 \boxtimes 4_{\#26}^{B} = \mathcal{F}_0 \boxtimes 4_{\#27}^{B}$
13	$\frac{1}{10}$	5.528	$1, 1, -1, -1, -\frac{1}{\zeta_3^1}, -\frac{1}{\zeta_3^1}, \frac{1}{\zeta_3^1}, \frac{1}{\zeta_3^1}$	$0, \frac{1}{2}, \frac{1}{4}, -\frac{1}{4}, -\frac{1}{5}, \frac{3}{10}, \frac{1}{20}, -\frac{9}{20}$	$\mathcal{F}_0 \boxtimes 4_{\#25}^{B} = \mathcal{F}_0 \boxtimes 4_{\#29}^{B}$
14	$\frac{1}{10}$	5.528	$1, 1, 1, 1, -\frac{1}{\zeta_3^1}, -\frac{1}{\zeta_3^1}, -\frac{1}{\zeta_3^1}, -\frac{1}{\zeta_3^1}$	$0, \frac{1}{2}, \frac{1}{4}, -\frac{1}{4}, \frac{1}{20}, -\frac{9}{20}, -\frac{1}{5}, \frac{3}{10}$	$\mathcal{F}_0 \boxtimes 4_{\#22}^{B} = \mathcal{F}_0 \boxtimes 4_{\#24}^{B}$
15	$-\frac{1}{10}$	5.528	$1, 1, -1, -1, -\frac{1}{\zeta_3^1}, -\frac{1}{\zeta_3^1}, \frac{1}{\zeta_3^1}, \frac{1}{\zeta_3^1}$	$0, \frac{1}{2}, \frac{1}{4}, -\frac{1}{4}, \frac{1}{5}, -\frac{3}{10}, -\frac{1}{20}, \frac{9}{20}$	$\mathcal{F}_0 \boxtimes 4_{\#28}^{B} = \mathcal{F}_0 \boxtimes 4_{\#32}^{B}$
16	$-\frac{1}{10}$	5.528	$1, 1, 1, 1, -\frac{1}{\zeta_3^1}, -\frac{1}{\zeta_3^1}, -\frac{1}{\zeta_3^1}, -\frac{1}{\zeta_3^1}$	$0, \frac{1}{2}, \frac{1}{4}, -\frac{1}{4}, \frac{1}{5}, -\frac{3}{10}, -\frac{1}{20}, \frac{9}{20}$	$\mathcal{F}_0 \boxtimes 4_{\#21}^{B} = \mathcal{F}_0 \boxtimes 4_{\#23}^{B}$
17	$\frac{1}{5}$	14.472	$1, 1, -1, -1, -\zeta_3^1, -\zeta_3^1, \zeta_3^1, \zeta_3^1$	$0, \frac{1}{2}, \frac{1}{4}, -\frac{1}{4}, -\frac{3}{20}, \frac{7}{20}, \frac{1}{10}, -\frac{2}{5}$	$\mathcal{F}_0 \boxtimes 4_{\#30}^{B} = \mathcal{F}_0 \boxtimes 4_{\#31}^{B}$
18	0	24	$1, 1, 1, 1, 2, 2, \sqrt{6}, \sqrt{6}$	$0, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{6}, -\frac{1}{3}, \frac{1}{16}, -\frac{7}{16}$	Primitive
19	0	24	$1, 1, 1, 1, 2, 2, \sqrt{6}, \sqrt{6}$	$0, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{6}, -\frac{1}{3}, -\frac{1}{16}, \frac{7}{16}$	Primitive
20	0	24	$1, 1, 1, 1, 2, 2, \sqrt{6}, \sqrt{6}$	$0, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{6}, -\frac{1}{3}, \frac{3}{16}, -\frac{5}{16}$	Primitive: f.c. of $(D_6)_2$
21	0	24	$1, 1, 1, 1, 2, 2, \sqrt{6}, \sqrt{6}$	$0, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{6}, -\frac{1}{3}, -\frac{3}{16}, \frac{5}{16}$	Primitive
22	0	24	$1, 1, 1, 1, 2, 2, \sqrt{6}, \sqrt{6}$	$0, \frac{1}{2}, 0, \frac{1}{2}, -\frac{1}{6}, \frac{1}{3}, \frac{1}{16}, -\frac{7}{16}$	Primitive
23	0	24	$1, 1, 1, 1, 2, 2, \sqrt{6}, \sqrt{6}$	$0, \frac{1}{2}, 0, \frac{1}{2}, -\frac{1}{6}, \frac{1}{3}, -\frac{1}{16}, \frac{7}{16}$	Primitive
24	0	24	$1, 1, 1, 1, 2, 2, \sqrt{6}, \sqrt{6}$	$0, \frac{1}{2}, 0, \frac{1}{2}, -\frac{1}{6}, \frac{1}{3}, \frac{3}{16}, -\frac{5}{16}$	Primitive
25	0	24	$1, 1, 1, 1, 2, 2, \sqrt{6}, \sqrt{6}$	$0, \frac{1}{2}, 0, \frac{1}{2}, -\frac{1}{6}, \frac{1}{3}, -\frac{3}{16}, \frac{5}{16}$	Primitive: f.c. of $(D_6)_{-2}$
26	0	24	$1, 1, 1, 1, 2, 2, -\sqrt{6}, -\sqrt{6}$	$0, \frac{1}{2}, 0, \frac{1}{2}, -\frac{1}{6}, \frac{1}{3}, -\frac{1}{16}, \frac{7}{16}$	Primitive
27	0	24	$1, 1, 1, 1, 2, 2, -\sqrt{6}, -\sqrt{6}$	$0, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{6}, -\frac{1}{3}, -\frac{1}{16}, \frac{7}{16}$	Primitive
28	0	24	$1, 1, 1, 1, 2, 2, -\sqrt{6}, -\sqrt{6}$	$0, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{6}, -\frac{1}{3}, -\frac{3}{16}, \frac{5}{16}$	Primitive
29	0	24	$1, 1, 1, 1, 2, 2, -\sqrt{6}, -\sqrt{6}$	$0, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{6}, -\frac{1}{3}, \frac{3}{16}, -\frac{5}{16}$	Primitive
30	0	24	$1, 1, 1, 1, 2, 2, -\sqrt{6}, -\sqrt{6}$	$0, \frac{1}{2}, 0, \frac{1}{2}, -\frac{1}{6}, \frac{1}{3}, \frac{1}{16}, -\frac{7}{16}$	Primitive
31	0	24	$1, 1, 1, 1, 2, 2, -\sqrt{6}, -\sqrt{6}$	$0, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{6}, -\frac{1}{3}, \frac{1}{16}, -\frac{7}{16}$	Primitive
32	0	24	$1, 1, 1, 1, 2, 2, -\sqrt{6}, -\sqrt{6}$	$0, \frac{1}{2}, 0, \frac{1}{2}, -\frac{1}{6}, \frac{1}{3}, -\frac{3}{16}, \frac{5}{16}$	Primitive
33	0	24	$1, 1, 1, 1, 2, 2, -\sqrt{6}, -\sqrt{6}$	$0, \frac{1}{2}, 0, \frac{1}{2}, -\frac{1}{6}, \frac{1}{3}, \frac{3}{16}, -\frac{5}{16}$	Primitive

The generalization to higher rank should be straightforward. There are several advantages to our approach compared to other approaches [22,39,40]: (1) we do not need to place any bound on either the fusion coefficients or the total quantum dimension, (2) by treating unitary and nonunitary MD on an equal footing, we can easily obtain nonunitary as well as unitary MD, (3) we can determine the central charge without having to compute the modular extensions explicitly, and (4) spurious MD which do not admit a modular extension (see arXiv version of Ref. [22]) are automatically excluded.

TABLE IV. Number of congruence irreducible representations of $SL_2(\mathbb{Z})$ and Γ_θ.

d	1	2	3	4	5
$SL_2(\mathbb{Z})$	12	54	136	180	36
Γ_θ	96	600	416	2436	288

Another advantage is that it allows to focus on nonsplit super-MTCs. In this paper, we have included split super-MTCs as well as nonsplit super-MTCs to illustrate the power of this approach, but a classification of split super-MTCs are redundant since they follow trivially from the classification of MTCs. By excluding representations $\rho_{\rm isum}$ which are projectively extendable, we can automatically get rid of this redundancy and obtain a only the MD of nonsplit super-MTCs. A weakness of our approach is that it is difficult to handle unresolved representations. In practice, a judicious choice of orthogonal transformations allows us to obtain some valid MD even from unresolved representations. Thus we cannot claim the completeness of our classification. We leave

TABLE V. List of rank 8 fermionic MD. *Continued.* In the Comments, $4^F_{\#n}$ refers to the *n*th entry of our rank 4 fermionic MD table I.

#	c	D^2	Quantum dimensions	Topological spins	Comments
34	$-\frac{1}{10}$	26.180	$1, 1, \zeta_3^1, \zeta_3^1, \zeta_3^1, \zeta_3^1, \zeta_8^2, \zeta_8^2$	$0, \frac{1}{2}, \frac{1}{10}, -\frac{2}{5}, \frac{1}{10}, -\frac{2}{5}, \frac{1}{5}, -\frac{3}{10}$	$\mathcal{F}_0 \boxtimes 4^B_{\#2}$
35	0	26.180	$1, 1, \zeta_3^1, \zeta_3^1, \zeta_3^1, \zeta_3^1, \zeta_8^2, \zeta_8^2$	$0, \frac{1}{2}, \frac{1}{10}, -\frac{2}{5}, -\frac{1}{10}, \frac{2}{5}, 0, \frac{1}{2}$	$\mathcal{F}_0 \boxtimes 4^B_{\#5}$
36	$\frac{1}{10}$	26.180	$1, 1, \zeta_3^1, \zeta_3^1, \zeta_3^1, \zeta_3^1, \zeta_8^2, \zeta_8^2$	$0, \frac{1}{2}, -\frac{1}{10}, \frac{2}{5}, -\frac{1}{10}, \frac{2}{5}, -\frac{1}{5}, \frac{3}{10}$	$\mathcal{F}_0 \boxtimes 4^B_{\#1}$
37	0	3.820	$1, 1, -\frac{1}{\zeta_3^1}, -\frac{1}{\zeta_3^1}, -\frac{1}{\zeta_3^1}, -\frac{1}{\zeta_3^1}, \frac{1}{\zeta_8^2}, \frac{1}{\zeta_8^2}$	$0, \frac{1}{2}, \frac{1}{5}, -\frac{3}{10}, -\frac{1}{5}, \frac{3}{10}, 0, \frac{1}{2}$	$\mathcal{F}_0 \boxtimes 4^B_{\#6}$
38	$-\frac{1}{5}$	3.820	$1, 1, -\frac{1}{\zeta_3^1}, -\frac{1}{\zeta_3^1}, -\frac{1}{\zeta_3^1}, -\frac{1}{\zeta_3^1}, \frac{1}{\zeta_8^2}, \frac{1}{\zeta_8^2}$	$0, \frac{1}{2}, \frac{1}{5}, -\frac{3}{10}, \frac{1}{5}, -\frac{3}{10}, -\frac{1}{10}, \frac{2}{5}$	$\mathcal{F}_0 \boxtimes 4^B_{\#3}$
39	$\frac{1}{5}$	3.820	$1, 1, -\frac{1}{\zeta_3^1}, -\frac{1}{\zeta_3^1}, -\frac{1}{\zeta_3^1}, -\frac{1}{\zeta_3^1}, \frac{1}{\zeta_8^2}, \frac{1}{\zeta_8^2}$	$0, \frac{1}{2}, -\frac{1}{5}, \frac{3}{10}, -\frac{1}{5}, \frac{3}{10}, \frac{1}{10}, -\frac{2}{5}$	$\mathcal{F}_0 \boxtimes 4^B_{\#4}$
40	$-\frac{1}{5}$	10	$1, 1, -1, -1, -\frac{1}{\zeta_3^1}, -\frac{1}{\zeta_3^1}, \zeta_3^1, \zeta_3^1$	$0, \frac{1}{2}, -\frac{1}{10}, \frac{2}{5}, -\frac{1}{5}, \frac{3}{10}, \frac{1}{10}, -\frac{2}{5}$	$\mathcal{F}_0 \boxtimes 4^B_{\#10}$
41	$-\frac{1}{10}$	10	$1, 1, -1, -1, -\frac{1}{\zeta_3^1}, -\frac{1}{\zeta_3^1}, \zeta_3^1, \zeta_3^1$	$0, \frac{1}{2}, -\frac{1}{5}, \frac{3}{10}, \frac{1}{5}, -\frac{3}{10}, \frac{1}{10}, -\frac{2}{5}$	$\mathcal{F}_0 \boxtimes 4^B_{\#9}$
42	$\frac{1}{10}$	10	$1, 1, -1, -1, -\frac{1}{\zeta_3^1}, -\frac{1}{\zeta_3^1}, \zeta_3^1, \zeta_3^1$	$0, \frac{1}{2}, \frac{1}{5}, -\frac{3}{10}, -\frac{1}{5}, \frac{3}{10}, -\frac{1}{10}, \frac{2}{5}$	$\mathcal{F}_0 \boxtimes 4^B_{\#8}$
43	$\frac{1}{5}$	10	$1, 1, -1, -1, -\frac{1}{\zeta_3^1}, -\frac{1}{\zeta_3^1}, \zeta_3^1, \zeta_3^1$	$0, \frac{1}{2}, \frac{1}{10}, -\frac{2}{5}, \frac{1}{5}, -\frac{3}{10}, -\frac{1}{10}, \frac{2}{5}$	$\mathcal{F}_0 \boxtimes 4^B_{\#7}$
44	$\frac{1}{4}$	27.313	$1, 1, 1, 1, \chi_2^1, \chi_2^1, \chi_2^1, \chi_2^1$	$0, \frac{1}{2}, \frac{1}{4}, -\frac{1}{4}, 0, \frac{1}{2}, \frac{1}{4}, -\frac{1}{4}$	$4^F_{\#7} \boxtimes 2^B_1$
45	$\frac{1}{4}$	27.313	$1, 1, -1, -1, -\chi_2^1, -\chi_2^1, \chi_2^1, \chi_2^1$	$0, \frac{1}{2}, \frac{1}{4}, -\frac{1}{4}, 0, \frac{1}{2}, \frac{1}{4}, -\frac{1}{4}$	$4^F_{\#7} \boxtimes 2^{B,*}_1$
46	$\frac{1}{4}$	4.6863	$1, 1, -1, -1, -\frac{1}{\chi_2^1}, -\frac{1}{\chi_2^1}, \frac{1}{\chi_2^1}, \frac{1}{\chi_2^1}$	$0, \frac{1}{2}, \frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, 0, \frac{1}{2}$	$4^F_{\#8} \boxtimes 2^B_1$
47	$\frac{1}{4}$	4.6863	$1, 1, 1, 1, -\frac{1}{\chi_2^1}, -\frac{1}{\chi_2^1}, -\frac{1}{\chi_2^1}, -\frac{1}{\chi_2^1}$	$0, \frac{1}{2}, \frac{1}{4}, -\frac{1}{4}, 0, \frac{1}{2}, \frac{1}{4}, -\frac{1}{4}$	$4^F_{\#8} \boxtimes 2^{B,*}_1$
48	$\frac{1}{6}$	38.468	$1, 1, \zeta_7^1, \zeta_7^1, \zeta_7^2, \zeta_7^2, \zeta_7^3, \zeta_7^3$	$0, \frac{1}{2}, \frac{1}{6}, -\frac{1}{3}, -\frac{2}{9}, \frac{5}{18}, -\frac{1}{6}, \frac{1}{3}$	$\mathcal{F}_0 \boxtimes 4^B_{\#12}$
49	$-\frac{1}{6}$	38.468	$1, 1, \zeta_7^1, \zeta_7^1, \zeta_7^2, \zeta_7^2, \zeta_7^3, \zeta_7^3$	$0, \frac{1}{2}, -\frac{1}{6}, \frac{1}{3}, \frac{2}{9}, -\frac{5}{18}, \frac{1}{6}, -\frac{1}{3}$	$\mathcal{F}_0 \boxtimes 4^B_{\#11}$
50	$-\frac{1}{6}$	10.890	$1, 1, -\frac{\zeta_7^3}{\zeta_7^1}, -\frac{\zeta_7^3}{\zeta_7^1}, -\frac{\zeta_7^3}{\zeta_7^1}, -\frac{\zeta_7^3}{\zeta_7^1}, \frac{\zeta_7^2}{\zeta_7^1}, \frac{\zeta_7^2}{\zeta_7^1}$	$0, \frac{1}{2}, -\frac{1}{6}, \frac{1}{3}, \frac{1}{6}, -\frac{1}{3}, \frac{1}{18}, -\frac{4}{9}$	$\mathcal{F}_0 \boxtimes 4^B_{\#15}$
51	$-\frac{1}{6}$	4.640	$1, 1, -\frac{\zeta_7^2}{\zeta_7^3}, -\frac{\zeta_7^2}{\zeta_7^3}, -\frac{1}{\zeta_7^3}, -\frac{1}{\zeta_7^3}, \frac{\zeta_7^1}{\zeta_7^3}, \frac{\zeta_7^1}{\zeta_7^3}$	$0, \frac{1}{2}, -\frac{1}{9}, \frac{7}{18}, -\frac{1}{6}, \frac{1}{3}, \frac{1}{6}, -\frac{1}{3}$	$\mathcal{F}_0 \boxtimes 4^B_{\#16}$
52	$\frac{1}{6}$	10.890	$1, 1, -\frac{\zeta_7^3}{\zeta_7^1}, -\frac{\zeta_7^3}{\zeta_7^1}, -\frac{\zeta_7^3}{\zeta_7^1}, -\frac{\zeta_7^3}{\zeta_7^1}, \frac{\zeta_7^2}{\zeta_7^1}, \frac{\zeta_7^2}{\zeta_7^1}$	$0, \frac{1}{2}, \frac{1}{6}, -\frac{1}{3}, -\frac{1}{6}, \frac{1}{3}, -\frac{1}{18}, \frac{4}{9}$	$\mathcal{F}_0 \boxtimes 4^B_{\#14}$
53	$\frac{1}{6}$	4.640	$1, 1, -\frac{\zeta_7^2}{\zeta_7^3}, -\frac{\zeta_7^2}{\zeta_7^3}, -\frac{1}{\zeta_7^3}, -\frac{1}{\zeta_7^3}, \frac{\zeta_7^1}{\zeta_7^3}, \frac{\zeta_7^1}{\zeta_7^3}$	$0, \frac{1}{2}, \frac{1}{9}, -\frac{7}{18}, \frac{1}{6}, -\frac{1}{3}, -\frac{1}{6}, \frac{1}{3}$	$\mathcal{F}_0 \boxtimes 4^B_{\#13}$

TABLE VI. List of rank 8 fermionic MD. *Continued.*

#	c	D^2	Quantum dimensions	Topological spins	Comments
54	$-\frac{1}{20}$	49.410	$1, 1, \zeta_3^1\zeta_6^2, \zeta_3^1\zeta_6^2, \zeta_6^2, \zeta_6^2, \zeta_3^1, \zeta_3^1$	$0, \frac{1}{2}, -\frac{3}{20}, \frac{7}{20}, \frac{1}{4}, -\frac{1}{4}, \frac{1}{10}, -\frac{2}{5}$	$4^F_{\#7} \boxtimes 2^B_{\#2}$
55	$\frac{1}{20}$	49.410	$1, 1, \zeta_3^1\zeta_6^2, \zeta_3^1\zeta_6^2, \zeta_6^2, \zeta_6^2, \zeta_3^1, \zeta_3^1$	$0, \frac{1}{2}, \frac{3}{20}, -\frac{7}{20}, \frac{1}{4}, -\frac{1}{4}, -\frac{1}{10}, \frac{2}{5}$	$4^F_{\#7} \boxtimes 2_{\#1}$
56	$-\frac{1}{20}$	8.478	$1, 1, -\frac{\zeta_3^1}{\zeta_6^2}, -\frac{\zeta_3^1}{\zeta_6^2}, -\frac{1}{\zeta_6^2}, -\frac{1}{\zeta_6^2}, \zeta_3^1, \zeta_3^1$	$0, \frac{1}{2}, \frac{3}{20}, -\frac{7}{20}, \frac{1}{4}, -\frac{1}{4}, -\frac{1}{10}, \frac{2}{5}$	$4^F_{\#8} \boxtimes 2^B_{\#1}$
57	$-\frac{3}{20}$	18.873	$1, 1, -\frac{\zeta_6^2}{\zeta_3^1}, -\frac{\zeta_6^2}{\zeta_3^1}, \zeta_6^2, \zeta_6^2, -\frac{1}{\zeta_3^1}, -\frac{1}{\zeta_3^1}$	$0, \frac{1}{2}, \frac{1}{20}, -\frac{9}{20}, \frac{1}{4}, -\frac{1}{4}, -\frac{1}{5}, \frac{3}{10}$	$4^F_{\#7} \boxtimes 2^B_{\#4}$
58	$-\frac{3}{20}$	3.2381	$1, 1, \frac{1}{\zeta_3^1\zeta_6^2}, \frac{1}{\zeta_3^1\zeta_6^2}, -\frac{1}{\zeta_6^2}, -\frac{1}{\zeta_6^2}, -\frac{1}{\zeta_3^1}, -\frac{1}{\zeta_3^1}$	$0, \frac{1}{2}, \frac{1}{20}, -\frac{9}{20}, \frac{1}{4}, -\frac{1}{4}, -\frac{1}{5}, \frac{3}{10}$	$4^F_{\#8} \boxtimes 2^B_{\#4}$
59	$\frac{3}{20}$	18.873	$1, 1, -\frac{\zeta_6^2}{\zeta_3^1}, -\frac{\zeta_6^2}{\zeta_3^1}, \zeta_6^2, \zeta_6^2, -\frac{1}{\zeta_3^1}, -\frac{1}{\zeta_3^1}$	$0, \frac{1}{2}, -\frac{1}{20}, \frac{9}{20}, \frac{1}{4}, -\frac{1}{4}, \frac{1}{5}, -\frac{3}{10}$	$4^F_{\#7} \boxtimes 2^B_{\#3}$
60	$\frac{3}{20}$	3.2381	$1, 1, \frac{1}{\zeta_3^1\zeta_6^2}, \frac{1}{\zeta_3^1\zeta_6^2}, -\frac{1}{\zeta_6^2}, -\frac{1}{\zeta_6^2}, -\frac{1}{\zeta_3^1}, -\frac{1}{\zeta_3^1}$	$0, \frac{1}{2}, -\frac{1}{20}, \frac{9}{20}, \frac{1}{4}, -\frac{1}{4}, \frac{1}{5}, -\frac{3}{10}$	$4^F_{\#8} \boxtimes 2^B_{\#3}$
61	$-\frac{1}{20}$	8.478	$1, 1, -\frac{\zeta_3^1}{\zeta_6^2}, -\frac{\zeta_3^1}{\zeta_6^2}, -\frac{1}{\zeta_6^2}, -\frac{1}{\zeta_6^2}, \zeta_3^1, \zeta_3^1$	$0, \frac{1}{2}, -\frac{3}{20}, \frac{7}{20}, \frac{1}{4}, -\frac{1}{4}, \frac{1}{10}, -\frac{2}{5}$	$4^F_{\#8} \boxtimes 2^B_{\#2}$
62	0	93.254	$1, 1, \chi_2^1, \chi_2^1, \chi_2^1, \chi_2^1, \chi_8^3, \chi_8^3$	$0, \frac{1}{2}, \frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, 0, \frac{1}{2}$	$4^F_{\#7} \boxtimes_{\mathcal{F}_0} 4^F_{\#7}$
63	0	16	$1, 1, -1, -1, -\frac{1}{\chi_2^1}, -\frac{1}{\chi_2^1}, \chi_2^1, \chi_2^1$	$0, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, -\frac{1}{4}$	$4^F_{\#7} \boxtimes_{\mathcal{F}_0} 4^F_{\#8}$
64	0	2.7452	$1, 1, \frac{1}{(\chi_2^1)^2}, \frac{1}{(\chi_2^1)^2}, -\frac{1}{\chi_2^1}, -\frac{1}{\chi_2^1}, -\frac{1}{\chi_2^1}, -\frac{1}{\chi_2^1}$	$0, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, -\frac{1}{4}$	$4^F_{\#8} \boxtimes_{\mathcal{F}_0} 4^F_{\#8}$
65	$-\frac{1}{8}$	105.09	$1, 1, \zeta_{14}^2, \zeta_{14}^2, \zeta_{14}^4, \zeta_{14}^4, \zeta_{14}^6, \zeta_{14}^6$	$0, \frac{1}{2}, -\frac{1}{8}, \frac{3}{8}, \frac{1}{8}, -\frac{3}{8}, \frac{1}{4}, -\frac{1}{4}$	Primitive: f.c. of $(A_1)_{-14}$
66	$\frac{1}{8}$	105.09	$1, 1, \zeta_{14}^2, \zeta_{14}^2, \zeta_{14}^4, \zeta_{14}^4, \zeta_{14}^6, \zeta_{14}^6$	$0, \frac{1}{2}, \frac{1}{8}, -\frac{3}{8}, -\frac{1}{8}, \frac{3}{8}, \frac{1}{4}, -\frac{1}{4}$	Primitive: f.c. of $(A_1)_{14}$
67	$\frac{1}{8}$	12.959	$1, 1, -\frac{\zeta_{14}^4}{\zeta_{14}^2}, -\frac{\zeta_{14}^4}{\zeta_{14}^2}, \frac{1}{\zeta_{14}^2}, \frac{1}{\zeta_{14}^2}, \frac{\zeta_{14}^6}{\zeta_{14}^2}, \frac{\zeta_{14}^6}{\zeta_{14}^2}$	$0, \frac{1}{2}, \frac{1}{4}, -\frac{1}{4}, -\frac{1}{8}, \frac{3}{8}, \frac{1}{8}, -\frac{3}{8}$	Primitive
68	$-\frac{1}{8}$	12.959	$1, 1, -\frac{\zeta_{14}^4}{\zeta_{14}^2}, -\frac{\zeta_{14}^4}{\zeta_{14}^2}, \frac{1}{\zeta_{14}^2}, \frac{1}{\zeta_{14}^2}, \frac{\zeta_{14}^6}{\zeta_{14}^2}, \frac{\zeta_{14}^6}{\zeta_{14}^2}$	$0, \frac{1}{2}, \frac{1}{4}, -\frac{1}{4}, \frac{1}{8}, -\frac{3}{8}, -\frac{1}{8}, \frac{3}{8}$	Primitive
69	$\frac{1}{8}$	5.7859	$1, 1, -\frac{\zeta_{14}^6}{\zeta_{14}^4}, -\frac{\zeta_{14}^6}{\zeta_{14}^4}, \frac{1}{\zeta_{14}^4}, \frac{1}{\zeta_{14}^4}, \frac{\zeta_{14}^2}{\zeta_{14}^4}, \frac{\zeta_{14}^2}{\zeta_{14}^4}$	$0, \frac{1}{2}, -\frac{1}{8}, \frac{3}{8}, \frac{1}{8}, -\frac{3}{8}, \frac{1}{4}, -\frac{1}{4}$	Primitive
70	$-\frac{1}{8}$	5.7859	$1, 1, -\frac{\zeta_{14}^6}{\zeta_{14}^4}, -\frac{\zeta_{14}^6}{\zeta_{14}^4}, \frac{1}{\zeta_{14}^4}, \frac{1}{\zeta_{14}^4}, \frac{\zeta_{14}^2}{\zeta_{14}^4}, \frac{\zeta_{14}^2}{\zeta_{14}^4}$	$0, \frac{1}{2}, \frac{1}{8}, -\frac{3}{8}, -\frac{1}{8}, \frac{3}{8}, \frac{1}{4}, -\frac{1}{4}$	Primitive
71	$-\frac{1}{8}$	4.1583	$1, 1, -\frac{\zeta_{14}^4}{\zeta_{14}^6}, -\frac{\zeta_{14}^4}{\zeta_{14}^6}, -\frac{1}{\zeta_{14}^6}, -\frac{1}{\zeta_{14}^6}, \frac{\zeta_{14}^2}{\zeta_{14}^6}, \frac{\zeta_{14}^2}{\zeta_{14}^6}$	$0, \frac{1}{2}, -\frac{1}{8}, \frac{3}{8}, \frac{1}{4}, -\frac{1}{4}, \frac{1}{8}, -\frac{3}{8}$	Primitive
72	$\frac{1}{8}$	4.1583	$1, 1, -\frac{\zeta_{14}^4}{\zeta_{14}^6}, -\frac{\zeta_{14}^4}{\zeta_{14}^6}, -\frac{1}{\zeta_{14}^6}, -\frac{1}{\zeta_{14}^6}, \frac{\zeta_{14}^2}{\zeta_{14}^6}, \frac{\zeta_{14}^2}{\zeta_{14}^6}$	$0, \frac{1}{2}, \frac{1}{8}, -\frac{3}{8}, \frac{1}{4}, -\frac{1}{4}, -\frac{1}{8}, \frac{3}{8}$	Primitive

TABLE VII. List of rank 10 fermionic MD.

#	c	D^2	Quantum dimensions	Topological spins	Comments
1	0	10	1,1,1,1,1,1,1,1,1,1	$0, \frac{1}{2}, \frac{1}{10}, -\frac{2}{5}, \frac{1}{10}, -\frac{2}{5}, \frac{2}{5}, -\frac{1}{10}, \frac{2}{5}, -\frac{1}{10}$	$\mathcal{F}_0 \boxtimes 5_4^B$
2	0	10	1,1,1,1,1,1,1,1,1,1	$0, \frac{1}{2}, \frac{1}{5}, -\frac{3}{10}, \frac{1}{5}, -\frac{3}{10}, \frac{3}{10}, -\frac{1}{5}, \frac{3}{10}, -\frac{1}{5}$	$\mathcal{F}_0 \boxtimes 5_0^B$
3	0	24	$1, 1, 1, 1, \sqrt{3}, \sqrt{3}, \sqrt{3}, \sqrt{3}, 2, 2$	$0, \frac{1}{2}, 0, \frac{1}{2}, 0, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{6}, -\frac{1}{3}$	Primitive: f.c. of $(A_1)_{-4} \boxtimes U(1)_4$
4	0	24	$1, 1, 1, 1, \sqrt{3}, \sqrt{3}, \sqrt{3}, \sqrt{3}, 2, 2$	$0, \frac{1}{2}, 0, \frac{1}{2}, 0, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{3}, -\frac{1}{6}$	Primitive: f.c. of $(A_1)_4 \boxtimes U(1)_{-4}$
5	0	24	$1, 1, 1, 1, -\sqrt{3}, -\sqrt{3}, -\sqrt{3}, -\sqrt{3}, 2, 2$	$0, \frac{1}{2}, 0, \frac{1}{2}, 0, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{6}, -\frac{1}{3}$	Primitive
6	0	24	$1, 1, 1, 1, -\sqrt{3}, -\sqrt{3}, -\sqrt{3}, -\sqrt{3}, 2, 2$	$0, \frac{1}{2}, 0, \frac{1}{2}, 0, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{3}, -\frac{1}{6}$	Primitive
7	0	24	$1, 1, 1, 1, \sqrt{3}, \sqrt{3}, \sqrt{3}, \sqrt{3}, 2, 2$	$0, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, \frac{1}{6}, -\frac{1}{3}$	Primitive: f.c. of $(A_1)_{-4} \boxtimes 4_3^B$
8	0	24	$1, 1, 1, 1, \sqrt{3}, \sqrt{3}, \sqrt{3}, \sqrt{3}, 2, 2$	$0, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, \frac{1}{3}, -\frac{1}{6}$	Primitive: f.c. of $(A_1)_4 \boxtimes 4_{-3}^B$
9	0	24	$1, 1, 1, 1, -\sqrt{3}, -\sqrt{3}, -\sqrt{3}, -\sqrt{3}, 2, 2$	$0, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, \frac{1}{6}, -\frac{1}{3}$	Primitive
10	0	24	$1, 1, 1, 1, -\sqrt{3}, -\sqrt{3}, -\sqrt{3}, -\sqrt{3}, 2, 2$	$0, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, \frac{1}{3}, -\frac{1}{6}$	Primitive
11	0	24	$1, 1, 1, 1, 2, 2, \sqrt{3}, \sqrt{3}, \sqrt{3}, \sqrt{3}$	$0, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{6}, -\frac{1}{3}, \frac{1}{8}, -\frac{3}{8}, \frac{1}{8}, -\frac{3}{8}$	$\mathcal{F}_0 \boxtimes 5_{\#5}^B$
12	0	24	$1, 1, 1, 1, 2, 2, \sqrt{3}, \sqrt{3}, \sqrt{3}, \sqrt{3}$	$0, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{6}, -\frac{1}{3}, \frac{3}{8}, -\frac{1}{8}, \frac{3}{8}, -\frac{1}{8}$	$\mathcal{F}_0 \boxtimes 5_{\#2}^B$
13	0	24	$1, 1, 1, 1, 2, 2, \sqrt{3}, \sqrt{3}, \sqrt{3}, \sqrt{3}$	$0, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{3}, -\frac{1}{6}, \frac{1}{8}, -\frac{3}{8}, \frac{1}{8}, -\frac{3}{8}$	$\mathcal{F}_0 \boxtimes 5_{\#1}^B$
14	0	24	$1, 1, 1, 1, 2, 2, \sqrt{3}, \sqrt{3}, \sqrt{3}, \sqrt{3}$	$0, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{3}, -\frac{1}{6}, \frac{3}{8}, -\frac{1}{8}, \frac{3}{8}, -\frac{1}{8}$	$\mathcal{F}_0 \boxtimes 5_{\#6}^B$
15	0	24	$1, 1, 1, 1, 2, 2, -\sqrt{3}, -\sqrt{3}, -\sqrt{3}, -\sqrt{3}$	$0, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{6}, -\frac{1}{3}, \frac{1}{8}, -\frac{3}{8}, \frac{1}{8}, -\frac{3}{8}$	$\mathcal{F}_0 \boxtimes 5_{\#4}^B$
16	0	24	$1, 1, 1, 1, 2, 2, -\sqrt{3}, -\sqrt{3}, -\sqrt{3}, -\sqrt{3}$	$0, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{6}, -\frac{1}{3}, \frac{3}{8}, -\frac{1}{8}, \frac{3}{8}, -\frac{1}{8}$	$\mathcal{F}_0 \boxtimes 5_{\#8}^B$
17	0	24	$1, 1, 1, 1, 2, 2, -\sqrt{3}, -\sqrt{3}, -\sqrt{3}, -\sqrt{3}$	$0, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{3}, -\frac{1}{6}, \frac{1}{8}, -\frac{3}{8}, \frac{1}{8}, -\frac{3}{8}$	$\mathcal{F}_0 \boxtimes 5_{\#7}^B$
18	0	24	$1, 1, 1, 1, 2, 2, -\sqrt{3}, -\sqrt{3}, -\sqrt{3}, -\sqrt{3}$	$0, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{3}, -\frac{1}{6}, \frac{3}{8}, -\frac{1}{8}, \frac{3}{8}, -\frac{1}{8}$	$\mathcal{F}_0 \boxtimes 5_{\#3}^B$
19	0	40	$1, 1, 1, 1, 2, 2, 2, 2, \sqrt{10}, \sqrt{10}$	$0, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{10}, -\frac{2}{5}, \frac{2}{5}, -\frac{1}{10}, \frac{1}{16}, -\frac{7}{16}$	Primitive
20	0	40	$1, 1, 1, 1, 2, 2, 2, 2, \sqrt{10}, \sqrt{10}$	$0, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{10}, -\frac{2}{5}, \frac{2}{5}, -\frac{1}{10}, \frac{3}{16}, -\frac{5}{16}$	Primitive: f.c. of $(D_{10})_2$
21	0	40	$1, 1, 1, 1, 2, 2, 2, 2, \sqrt{10}, \sqrt{10}$	$0, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{10}, -\frac{2}{5}, \frac{2}{5}, -\frac{1}{10}, \frac{5}{16}, -\frac{3}{16}$	Primitive: f.c. of $(D_{10})_{-2}$
22	0	40	$1, 1, 1, 1, 2, 2, 2, 2, \sqrt{10}, \sqrt{10}$	$0, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{10}, -\frac{2}{5}, \frac{2}{5}, -\frac{1}{10}, \frac{7}{16}, -\frac{1}{16}$	Primitive
23	0	40	$1, 1, 1, 1, 2, 2, 2, 2, \sqrt{10}, \sqrt{10}$	$0, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{5}, -\frac{3}{10}, \frac{3}{10}, -\frac{1}{5}, \frac{1}{16}, -\frac{7}{16}$	Primitive
24	0	40	$1, 1, 1, 1, 2, 2, 2, 2, \sqrt{10}, \sqrt{10}$	$0, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{5}, -\frac{3}{10}, \frac{3}{10}, -\frac{1}{5}, \frac{3}{16}, -\frac{5}{16}$	Primitive
25	0	40	$1, 1, 1, 1, 2, 2, 2, 2, \sqrt{10}, \sqrt{10}$	$0, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{5}, -\frac{3}{10}, \frac{3}{10}, -\frac{1}{5}, \frac{5}{16}, -\frac{3}{16}$	Primitive
26	0	40	$1, 1, 1, 1, 2, 2, 2, 2, \sqrt{10}, \sqrt{10}$	$0, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{5}, -\frac{3}{10}, \frac{3}{10}, -\frac{1}{5}, \frac{7}{16}, -\frac{1}{16}$	Primitive
27	0	40	$1, 1, 1, 1, 2, 2, 2, 2, -\sqrt{10}, -\sqrt{10}$	$0, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{10}, -\frac{2}{5}, \frac{2}{5}, -\frac{1}{10}, \frac{1}{16}, -\frac{7}{16}$	Primitive
28	0	40	$1, 1, 1, 1, 2, 2, 2, 2, -\sqrt{10}, -\sqrt{10}$	$0, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{10}, -\frac{2}{5}, \frac{2}{5}, -\frac{1}{10}, \frac{3}{16}, -\frac{5}{16}$	Primitive
29	0	40	$1, 1, 1, 1, 2, 2, 2, 2, -\sqrt{10}, -\sqrt{10}$	$0, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{10}, -\frac{2}{5}, \frac{2}{5}, -\frac{1}{10}, \frac{5}{16}, -\frac{3}{16}$	Primitive
30	0	40	$1, 1, 1, 1, 2, 2, 2, 2, -\sqrt{10}, -\sqrt{10}$	$0, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{10}, -\frac{2}{5}, \frac{2}{5}, -\frac{1}{10}, \frac{7}{16}, -\frac{1}{16}$	Primitive
31	0	40	$1, 1, 1, 1, 2, 2, 2, 2, -\sqrt{10}, -\sqrt{10}$	$0, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{5}, -\frac{3}{10}, \frac{3}{10}, -\frac{1}{5}, \frac{1}{16}, -\frac{7}{16}$	Primitive
32	0	40	$1, 1, 1, 1, 2, 2, 2, 2, -\sqrt{10}, -\sqrt{10}$	$0, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{5}, -\frac{3}{10}, \frac{3}{10}, -\frac{1}{5}, \frac{3}{16}, -\frac{5}{16}$	Primitive
33	0	40	$1, 1, 1, 1, 2, 2, 2, 2, -\sqrt{10}, -\sqrt{10}$	$0, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{5}, -\frac{3}{10}, \frac{3}{10}, -\frac{1}{5}, \frac{5}{16}, -\frac{3}{16}$	Primitive
34	0	40	$1, 1, 1, 1, 2, 2, 2, 2, -\sqrt{10}, -\sqrt{10}$	$0, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{5}, -\frac{3}{10}, \frac{3}{10}, -\frac{1}{5}, \frac{7}{16}, -\frac{1}{16}$	Primitive

a complete treatment of unresolved representations to a future work.

Congruence representations also appear in rational conformal field theories (RCFTs): it is known that characters of RCFTs transform as representations of $SL_2(\mathbb{Z})$, and that these representations are congruence [48]. Reference [53] has used this idea to classify characters of bosonic RCFTs, and Ref. [54] has used congruence representations of subgroups of $SL_2(\mathbb{Z})$ to carry out a similar classification program for fermionic RCFTs. It would be interesting to flesh out the bulk-boundary relation between super-MTCs and fermionic RCFTs, along the lines of Refs. [55–57] which dealt with the bosonic case. In a future work [51], we will make this connection by explicitly computing the modular extensions of super-MTCs.

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TABLE VIII. List of rank 10 fermionic MD. *Continued*.

#	c	D^2	Quantum dimensions	Topological spins	Comments
35	$\frac{1}{22}$	69.2929	$1, 1, \zeta_9^1, \zeta_9^1, \zeta_9^2, \zeta_9^2, \zeta_9^3, \zeta_9^3, \zeta_9^4, \zeta_9^4$	$0, \frac{1}{2}, \frac{2}{11}, -\frac{7}{22}, \frac{7}{22}, -\frac{2}{11}, \frac{9}{22}, -\frac{1}{11}, \frac{5}{11}, -\frac{1}{22}$	$\mathcal{F}_0 \boxtimes 5^B_{\#15}$
36	$-\frac{1}{22}$	69.2929	$1, 1, \zeta_9^4, \zeta_9^4, \zeta_9^3, \zeta_9^3, \zeta_9^2, \zeta_9^2, \zeta_9^1, \zeta_9^1$	$0, \frac{1}{2}, \frac{1}{22}, -\frac{5}{11}, \frac{1}{11}, -\frac{9}{22}, \frac{2}{11}, -\frac{7}{22}, \frac{7}{22}, -\frac{2}{11}$	$\mathcal{F}_0 \boxtimes 5^B_{\#16}$
37	$\frac{5}{22}$	5.6137	$1, 1, -\frac{\zeta_9^1}{\zeta_9^4}, -\frac{\zeta_9^1}{\zeta_9^4}, -\frac{\zeta_9^3}{\zeta_9^4}, -\frac{\zeta_9^3}{\zeta_9^4}, \frac{\zeta_9^2}{\zeta_9^4}, \frac{\zeta_9^2}{\zeta_9^4}, \frac{1}{\zeta_9^4}, \frac{1}{\zeta_9^4}$	$0, \frac{1}{2}, \frac{1}{22}, -\frac{5}{11}, \frac{1}{11}, -\frac{9}{22}, \frac{3}{11}, -\frac{5}{22}, \frac{9}{22}, -\frac{1}{11}$	$\mathcal{F}_0 \boxtimes 5^B_{\#23}$
38	$-\frac{5}{22}$	5.6137	$1, 1, \frac{1}{\zeta_9^4}, \frac{1}{\zeta_9^4}, \frac{\zeta_9^2}{\zeta_9^4}, \frac{\zeta_9^2}{\zeta_9^4}, -\frac{\zeta_9^3}{\zeta_9^4}, -\frac{\zeta_9^3}{\zeta_9^4}, -\frac{\zeta_9^1}{\zeta_9^4}, -\frac{\zeta_9^1}{\zeta_9^4}$	$0, \frac{1}{2}, \frac{1}{11}, -\frac{9}{22}, \frac{5}{22}, -\frac{3}{11}, \frac{9}{22}, -\frac{1}{11}, \frac{5}{11}, -\frac{1}{22}$	$\mathcal{F}_0 \boxtimes 5^B_{\#18}$
39	$\frac{2}{11}$	6.64709	$1, 1, \frac{\zeta_9^4}{\zeta_9^3}, \frac{\zeta_9^4}{\zeta_9^3}, -\frac{\zeta_9^2}{\zeta_9^3}, -\frac{\zeta_9^2}{\zeta_9^3}, -\frac{1}{\zeta_9^3}, -\frac{1}{\zeta_9^3}, -\frac{\zeta_9^1}{\zeta_9^3}, -\frac{\zeta_9^1}{\zeta_9^3}$	$0, \frac{1}{2}, \frac{3}{22}, -\frac{4}{11}, \frac{5}{22}, -\frac{3}{11}, \frac{3}{11}, -\frac{5}{22}, \frac{7}{22}, -\frac{2}{11}$	$\mathcal{F}_0 \boxtimes 5^B_{\#22}$
40	$-\frac{2}{11}$	6.64709	$1, 1, -\frac{\zeta_9^1}{\zeta_9^3}, -\frac{\zeta_9^1}{\zeta_9^3}, -\frac{1}{\zeta_9^3}, -\frac{1}{\zeta_9^3}, -\frac{\zeta_9^2}{\zeta_9^3}, -\frac{\zeta_9^2}{\zeta_9^3}, \frac{\zeta_9^4}{\zeta_9^3}, \frac{\zeta_9^4}{\zeta_9^3}$	$0, \frac{1}{2}, \frac{2}{11}, -\frac{7}{22}, \frac{5}{22}, -\frac{3}{11}, \frac{3}{11}, -\frac{5}{22}, \frac{4}{11}, -\frac{3}{22}$	$\mathcal{F}_0 \boxtimes 5^B_{\#19}$
41	$\frac{3}{22}$	9.62957	$1, 1, \frac{\zeta_9^4}{\zeta_9^2}, \frac{\zeta_9^4}{\zeta_9^2}, -\frac{1}{\zeta_9^2}, -\frac{1}{\zeta_9^2}, -\frac{\zeta_9^3}{\zeta_9^2}, -\frac{\zeta_9^3}{\zeta_9^2}, \frac{\zeta_9^1}{\zeta_9^2}, \frac{\zeta_9^1}{\zeta_9^2}$	$0, \frac{1}{2}, \frac{1}{22}, -\frac{5}{11}, \frac{5}{22}, -\frac{3}{11}, \frac{4}{11}, -\frac{3}{22}, \frac{5}{11}, -\frac{1}{22}$	$\mathcal{F}_0 \boxtimes 5^B_{\#20}$
42	$-\frac{3}{22}$	9.62957	$1, 1, \frac{\zeta_9^1}{\zeta_9^2}, \frac{\zeta_9^1}{\zeta_9^2}, -\frac{\zeta_9^3}{\zeta_9^2}, -\frac{\zeta_9^3}{\zeta_9^2}, -\frac{1}{\zeta_9^2}, -\frac{1}{\zeta_9^2}, \frac{\zeta_9^4}{\zeta_9^2}, \frac{\zeta_9^4}{\zeta_9^2}$	$0, \frac{1}{2}, \frac{1}{22}, -\frac{5}{11}, \frac{3}{22}, -\frac{4}{11}, \frac{3}{11}, -\frac{5}{22}, \frac{5}{11}, -\frac{1}{22}$	$\mathcal{F}_0 \boxtimes 5^B_{\#21}$
43	$\frac{1}{11}$	18.8168	$1, 1, \frac{\zeta_9^4}{\zeta_9^1}, \frac{\zeta_9^4}{\zeta_9^1}, -\frac{\zeta_9^2}{\zeta_9^1}, -\frac{\zeta_9^2}{\zeta_9^1}, -\frac{\zeta_9^3}{\zeta_9^1}, -\frac{\zeta_9^3}{\zeta_9^1}, \frac{1}{\zeta_9^1}, \frac{1}{\zeta_9^1}$	$0, \frac{1}{2}, \frac{3}{22}, -\frac{4}{11}, \frac{7}{22}, -\frac{2}{11}, \frac{4}{11}, -\frac{3}{22}, \frac{9}{22}, -\frac{1}{11}$	$\mathcal{F}_0 \boxtimes 5^B_{\#24}$
44	$-\frac{1}{11}$	18.8168	$1, 1, \frac{1}{\zeta_9^1}, \frac{1}{\zeta_9^1}, -\frac{\zeta_9^3}{\zeta_9^1}, -\frac{\zeta_9^3}{\zeta_9^1}, -\frac{\zeta_9^2}{\zeta_9^1}, -\frac{\zeta_9^2}{\zeta_9^1}, \frac{\zeta_9^4}{\zeta_9^1}, \frac{\zeta_9^4}{\zeta_9^1}$	$0, \frac{1}{2}, \frac{1}{11}, -\frac{9}{22}, \frac{3}{22}, -\frac{4}{11}, \frac{2}{11}, -\frac{7}{22}, \frac{4}{11}, -\frac{3}{22}$	$\mathcal{F}_0 \boxtimes 5^B_{\#17}$
45	$\frac{1}{14}$	70.6848	$1, 1, \zeta_{12}^2, \zeta_{12}^2, \zeta_5^2, \zeta_5^2, \zeta_5^2, \zeta_5^2, \zeta_{12}^4, \zeta_{12}^4$	$0, \frac{1}{2}, \frac{1}{7}, -\frac{5}{14}, \frac{5}{14}, -\frac{1}{7}, \frac{5}{14}, -\frac{1}{7}, \frac{3}{7}, -\frac{1}{14}$	$\mathcal{F}_0 \boxtimes 5^B_{\#10}$
46	$-\frac{1}{14}$	70.6848	$1, 1, \zeta_{12}^4, \zeta_{12}^4, \zeta_5^2, \zeta_5^2, \zeta_5^2, \zeta_5^2, \zeta_{12}^2, \zeta_{12}^2$	$0, \frac{1}{2}, \frac{1}{14}, -\frac{3}{7}, \frac{1}{7}, -\frac{5}{14}, \frac{1}{7}, -\frac{5}{14}, \frac{5}{14}, -\frac{1}{7}$	$\mathcal{F}_0 \boxtimes 5^B_{\#9}$
47	$\frac{1}{7}$	4.3117	$1, 1, \frac{\zeta_5^2}{\zeta_{12}^4}, \frac{\zeta_5^2}{\zeta_{12}^4}, \frac{\zeta_5^2}{\zeta_{12}^4}, \frac{\zeta_5^2}{\zeta_{12}^4}, -\frac{1}{\zeta_{12}^4}, -\frac{1}{\zeta_{12}^4}, -\frac{\zeta_{12}^2}{\zeta_{12}^4}, -\frac{\zeta_{12}^2}{\zeta_{12}^4}$	$0, \frac{1}{2}, \frac{3}{14}, -\frac{2}{7}, \frac{3}{14}, -\frac{2}{7}, \frac{2}{7}, -\frac{3}{14}, \frac{5}{14}, -\frac{1}{7}$	$\mathcal{F}_0 \boxtimes 5^B_{\#14}$
48	$-\frac{1}{7}$	4.3117	$1, 1, -\frac{\zeta_{12}^2}{\zeta_{12}^4}, -\frac{\zeta_{12}^2}{\zeta_{12}^4}, -\frac{1}{\zeta_{12}^4}, -\frac{1}{\zeta_{12}^4}, \frac{\zeta_5^2}{\zeta_{12}^4}, \frac{\zeta_5^2}{\zeta_{12}^4}, \frac{\zeta_5^2}{\zeta_{12}^4}, \frac{\zeta_5^2}{\zeta_{12}^4}$	$0, \frac{1}{2}, \frac{1}{7}, -\frac{5}{14}, \frac{3}{14}, -\frac{2}{7}, \frac{2}{7}, -\frac{3}{14}, \frac{2}{7}, -\frac{3}{14}$	$\mathcal{F}_0 \boxtimes 5^B_{\#11}$
49	$\frac{3}{14}$	9.00346	$1, 1, -\frac{\zeta_5^2}{\zeta_{12}^2}, -\frac{\zeta_5^2}{\zeta_{12}^2}, -\frac{\zeta_5^2}{\zeta_{12}^2}, -\frac{\zeta_5^2}{\zeta_{12}^2}, -\frac{1}{\zeta_{12}^2}, -\frac{1}{\zeta_{12}^2}, \frac{\zeta_{12}^4}{\zeta_{12}^2}, \frac{\zeta_{12}^4}{\zeta_{12}^2}$	$0, \frac{1}{2}, \frac{1}{14}, -\frac{3}{7}, \frac{1}{14}, -\frac{3}{7}, \frac{2}{7}, -\frac{3}{14}, \frac{3}{7}, -\frac{1}{14}$	$\mathcal{F}_0 \boxtimes 5^B_{\#12}$
50	$-\frac{3}{14}$	9.00346	$1, 1, \frac{\zeta_{12}^4}{\zeta_{12}^2}, \frac{\zeta_{12}^4}{\zeta_{12}^2}, -\frac{1}{\zeta_{12}^2}, -\frac{1}{\zeta_{12}^2}, -\frac{\zeta_5^2}{\zeta_{12}^2}, -\frac{\zeta_5^2}{\zeta_{12}^2}, -\frac{\zeta_5^2}{\zeta_{12}^2}, -\frac{\zeta_5^2}{\zeta_{12}^2}$	$0, \frac{1}{2}, \frac{1}{14}, -\frac{3}{7}, \frac{3}{14}, -\frac{2}{7}, \frac{3}{7}, -\frac{1}{14}, \frac{3}{7}, -\frac{1}{14}$	$\mathcal{F}_0 \boxtimes 5^B_{\#13}$
51	$\frac{1}{5}$	204.317	$1, 1, \zeta_{18}^8, \zeta_{18}^8, \zeta_{18}^6, \zeta_{18}^6, \zeta_{18}^2, \zeta_{18}^2, \zeta_{18}^4, \zeta_{18}^4$	$0, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{10}, -\frac{2}{5}, \frac{1}{10}, -\frac{2}{5}, \frac{3}{10}, -\frac{1}{5}$	Primitive: f.c. of $(A_1)_{18}$
52	$-\frac{1}{5}$	204.317	$1, 1, \zeta_{18}^8, \zeta_{18}^8, \zeta_{18}^4, \zeta_{18}^4, \zeta_{18}^6, \zeta_{18}^6, \zeta_{18}^2, \zeta_{18}^2$	$0, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{5}, -\frac{3}{10}, \frac{2}{5}, -\frac{1}{10}, \frac{2}{5}, -\frac{1}{10}$	Primitive: f.c. of $(A_1)_{-18}$
53	$\frac{1}{5}$	5.12543	$1, 1, \frac{1}{\zeta_{18}^8}, \frac{1}{\zeta_{18}^8}, -\frac{\zeta_{18}^2}{\zeta_{18}^8}, -\frac{\zeta_{18}^2}{\zeta_{18}^8}, -\frac{\zeta_{18}^6}{\zeta_{18}^8}, -\frac{\zeta_{18}^6}{\zeta_{18}^8}, \frac{\zeta_{18}^4}{\zeta_{18}^8}, \frac{\zeta_{18}^4}{\zeta_{18}^8}$	$0, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{10}, -\frac{2}{5}, \frac{1}{10}, -\frac{2}{5}, \frac{3}{10}, -\frac{1}{5}$	Primitive
54	$-\frac{1}{5}$	5.12543	$1, 1, \frac{1}{\zeta_{18}^8}, \frac{1}{\zeta_{18}^8}, \frac{\zeta_{18}^4}{\zeta_{18}^8}, \frac{\zeta_{18}^4}{\zeta_{18}^8}, -\frac{\zeta_{18}^2}{\zeta_{18}^8}, -\frac{\zeta_{18}^2}{\zeta_{18}^8}, -\frac{\zeta_{18}^6}{\zeta_{18}^8}, -\frac{\zeta_{18}^6}{\zeta_{18}^8}$	$0, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{5}, -\frac{3}{10}, \frac{2}{5}, -\frac{1}{10}, \frac{2}{5}, -\frac{1}{10}$	Primitive
55	$\frac{1}{10}$	6.29808	$1, 1, -\frac{\zeta_{18}^2}{\zeta_{18}^6}, -\frac{\zeta_{18}^2}{\zeta_{18}^6}, \frac{\zeta_{18}^8}{\zeta_{18}^6}, \frac{\zeta_{18}^8}{\zeta_{18}^6}, -\frac{1}{\zeta_{18}^6}, -\frac{1}{\zeta_{18}^6}, -\frac{\zeta_{18}^4}{\zeta_{18}^6}, -\frac{\zeta_{18}^4}{\zeta_{18}^6}$	$0, \frac{1}{2}, 0, \frac{1}{2}, \frac{3}{10}, -\frac{1}{5}, \frac{3}{10}, -\frac{1}{5}, \frac{2}{5}, -\frac{1}{10}$	Primitive
56	$-\frac{1}{10}$	6.29808	$1, 1, -\frac{\zeta_{18}^2}{\zeta_{18}^6}, -\frac{\zeta_{18}^2}{\zeta_{18}^6}, -\frac{\zeta_{18}^4}{\zeta_{18}^6}, -\frac{\zeta_{18}^4}{\zeta_{18}^6}, \frac{\zeta_{18}^8}{\zeta_{18}^6}, \frac{\zeta_{18}^8}{\zeta_{18}^6}, -\frac{1}{\zeta_{18}^6}, -\frac{1}{\zeta_{18}^6}$	$0, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{10}, -\frac{2}{5}, \frac{1}{5}, -\frac{3}{10}, \frac{1}{5}, -\frac{3}{10}$	Primitive
57	$\frac{1}{10}$	24.2592	$1, 1, -\frac{\zeta_{18}^6}{\zeta_{18}^2}, -\frac{\zeta_{18}^6}{\zeta_{18}^2}, -\frac{1}{\zeta_{18}^2}, -\frac{1}{\zeta_{18}^2}, \frac{\zeta_{18}^8}{\zeta_{18}^2}, \frac{\zeta_{18}^8}{\zeta_{18}^2}, \frac{\zeta_{18}^4}{\zeta_{18}^2}, \frac{\zeta_{18}^4}{\zeta_{18}^2}$	$0, \frac{1}{2}, 0, \frac{1}{2}, \frac{3}{10}, -\frac{1}{5}, \frac{3}{10}, -\frac{1}{5}, \frac{2}{5}, -\frac{1}{10}$	Primitive
58	$-\frac{1}{10}$	24.2592	$1, 1, -\frac{\zeta_{18}^6}{\zeta_{18}^2}, -\frac{\zeta_{18}^6}{\zeta_{18}^2}, \frac{\zeta_{18}^4}{\zeta_{18}^2}, \frac{\zeta_{18}^4}{\zeta_{18}^2}, -\frac{1}{\zeta_{18}^2}, -\frac{1}{\zeta_{18}^2}, \frac{\zeta_{18}^8}{\zeta_{18}^2}, \frac{\zeta_{18}^8}{\zeta_{18}^2}$	$0, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{10}, -\frac{2}{5}, \frac{1}{5}, -\frac{3}{10}, \frac{1}{5}, -\frac{3}{10}$	Primitive

TABLE IX. New class of rank 10 primitive fermionic MD.

#	c	D^2	Quantum dimensions	Topological spins	Comments
59	0	472.379	$1, 1, \chi_{15}^4, \chi_{15}^4, \chi_{15}^5, \chi_{15}^5, \chi_{15}^3, \chi_{15}^3, \chi_{15}^3, \chi_{15}^3$	$0, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{6}, -\frac{1}{3}, \frac{1}{10}, -\frac{2}{5}, \frac{2}{5}, -\frac{1}{10}$	Primitive, $\hat{N} \leqslant 3$
60	0	472.379	$1, 1, \chi_{15}^4, \chi_{15}^4, \chi_{15}^5, \chi_{15}^5, \chi_{15}^3, \chi_{15}^3, \chi_{15}^3, \chi_{15}^3$	$0, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{6}, -\frac{1}{3}, \frac{1}{5}, -\frac{3}{10}, \frac{3}{10}, -\frac{1}{5}$	Primitive, $\hat{N} \leqslant 3$
61	0	472.379	$1, 1, \chi_{15}^4, \chi_{15}^4, \chi_{15}^5, \chi_{15}^5, \chi_{15}^3, \chi_{15}^3, \chi_{15}^3, \chi_{15}^3$	$0, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{3}, -\frac{1}{6}, \frac{1}{10}, -\frac{2}{5}, \frac{2}{5}, -\frac{1}{10}$	Primitive, $\hat{N} \leqslant 3$
62	0	472.379	$1, 1, \chi_{15}^4, \chi_{15}^4, \chi_{15}^5, \chi_{15}^5, \chi_{15}^3, \chi_{15}^3, \chi_{15}^3, \chi_{15}^3$	$0, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{3}, -\frac{1}{6}, \frac{1}{5}, -\frac{3}{10}, \frac{3}{10}, -\frac{1}{5}$	Primitive, $\hat{N} \leqslant 3$
63	0	7.621	$1, 1, \frac{1}{\chi_{15}^4}, \frac{1}{\chi_{15}^4}, \frac{\chi_{15}^5}{\chi_{15}^4}, \frac{\chi_{15}^5}{\chi_{15}^4}, -\frac{\chi_{15}^3}{\chi_{15}^4}, -\frac{\chi_{15}^3}{\chi_{15}^4}, -\frac{\chi_{15}^3}{\chi_{15}^4}, -\frac{\chi_{15}^3}{\chi_{15}^4}$	$0, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{6}, -\frac{1}{3}, \frac{1}{10}, -\frac{2}{5}, \frac{2}{5}, -\frac{1}{10}$	Primitive, $\hat{N} \leqslant 3$
64	0	7.621	$1, 1, \frac{1}{\chi_{15}^4}, \frac{1}{\chi_{15}^4}, \frac{\chi_{15}^5}{\chi_{15}^4}, \frac{\chi_{15}^5}{\chi_{15}^4}, -\frac{\chi_{15}^3}{\chi_{15}^4}, -\frac{\chi_{15}^3}{\chi_{15}^4}, -\frac{\chi_{15}^3}{\chi_{15}^4}, -\frac{\chi_{15}^3}{\chi_{15}^4}$	$0, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{6}, -\frac{1}{3}, \frac{1}{5}, -\frac{3}{10}, \frac{3}{10}, -\frac{1}{5}$	Primitive, $\hat{N} \leqslant 3$
65	0	7.621	$1, 1, \frac{1}{\chi_{15}^4}, \frac{1}{\chi_{15}^4}, \frac{\chi_{15}^5}{\chi_{15}^4}, \frac{\chi_{15}^5}{\chi_{15}^4}, -\frac{\chi_{15}^3}{\chi_{15}^4}, -\frac{\chi_{15}^3}{\chi_{15}^4}, -\frac{\chi_{15}^3}{\chi_{15}^4}, -\frac{\chi_{15}^3}{\chi_{15}^4}$	$0, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{3}, -\frac{1}{6}, \frac{1}{10}, -\frac{2}{5}, \frac{2}{5}, -\frac{1}{10}$	Primitive, $\hat{N} \leqslant 3$
66	0	7.621	$1, 1, \frac{1}{\chi_{15}^4}, \frac{1}{\chi_{15}^4}, \frac{\chi_{15}^5}{\chi_{15}^4}, \frac{\chi_{15}^5}{\chi_{15}^4}, -\frac{\chi_{15}^3}{\chi_{15}^4}, -\frac{\chi_{15}^3}{\chi_{15}^4}, -\frac{\chi_{15}^3}{\chi_{15}^4}, -\frac{\chi_{15}^3}{\chi_{15}^4}$	$0, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{3}, -\frac{1}{6}, \frac{1}{5}, -\frac{3}{10}, \frac{3}{10}, -\frac{1}{5}$	Primitive, $\hat{N} \leqslant 3$
67	0	475.151	$1, 1, \chi_{24}^5, \chi_{24}^5, \chi_6^3, \chi_6^3, \chi_6^3, \chi_6^3, \chi_{24}^4, \chi_{24}^4$	$0, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, \frac{1}{6}, -\frac{1}{3}$	Primitive, $\hat{N} \leqslant 4$
68	0	475.151	$1, 1, \chi_{24}^5, \chi_{24}^5, \chi_6^3, \chi_6^3, \chi_6^3, \chi_6^3, \chi_{24}^4, \chi_{24}^4$	$0, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, \frac{1}{3}, -\frac{1}{6}$	Primitive, $\hat{N} \leqslant 4$
69	0	4.84898	$1, 1, \frac{1}{\chi_{24}^5}, \frac{1}{\chi_{24}^5}, \frac{\chi_6^3}{\chi_{24}^5}, \frac{\chi_6^3}{\chi_{24}^5}, \frac{\chi_6^3}{\chi_{24}^5}, \frac{\chi_6^3}{\chi_{24}^5}, -\frac{\chi_{24}^4}{\chi_{24}^5}, -\frac{\chi_{24}^4}{\chi_{24}^5}$	$0, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, \frac{1}{6}, -\frac{1}{3}$	Primitive, $\hat{N} \leqslant 4$
70	0	4.84898	$1, 1, \frac{1}{\chi_{24}^5}, \frac{1}{\chi_{24}^5}, \frac{\chi_6^3}{\chi_{24}^5}, \frac{\chi_6^3}{\chi_{24}^5}, \frac{\chi_6^3}{\chi_{24}^5}, \frac{\chi_6^3}{\chi_{24}^5}, -\frac{\chi_{24}^4}{\chi_{24}^5}, -\frac{\chi_{24}^4}{\chi_{24}^5}$	$0, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, \frac{1}{3}, -\frac{1}{6}$	Primitive, $\hat{N} \leqslant 4$

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APPENDIX A: ALGEBRAIC STRUCTURE OF ANYONS

Anyons in a topologically ordered system are characterized by their fusion and braiding data. In terms of mathematics, these data are encoded in braided fusion categories (BFCs). For bosonic topological orders, relevant BFCs are modualr tensor categories (MTCs) [6,7]. In contrast, for fermionic topological orders, so-called super-modular tensor categories (super-MTCs) play the role [22,26]. Because of its relevance to our work, we here briefly review some important concepts of super-MTCs.

A super-MTC, as in MTCs, has gauge-invariant data called S and T matrices. Each element S_{ij} gives us the information of mutual statistics of anyons labeled by i and j, while $T_{ij} = \delta_{ij}e^{2i\pi s_i}$ encodes the self-statistics of an anyon i. Here, s_i is called the topological spin of anyon i and defined modulo 1.

The consistency of fusion of anyons translates as a fusion ring given by

$$i \otimes j = \sum_{k\in\Pi} N_k^{ij} k, \tag{A1}$$

where N_k^{ij} are the fusion coefficients and Π is the label set of simple objects, i.e., anyons. The fusion coefficients N_k^{ij} and the S matrix are related by the Verlinde formula [22]

$$\sum_{k\in\Pi} N_k^{ij} S_{kl} = \frac{S_{il}S_{jl}}{S_{1l}}. \tag{A2}$$

Elements in the first column of the S matrix correspond to the quantum dimension of anyons, $d_i = S_{i1}/S_{11}$, where the index 1 corresponds to the vacuum. The total quantum dimension is given by $D^2 = \sum_{i\in\Pi} d_i^2$. These data satisfy the *balancing equation* [23]

$$S_{ij} = \frac{1}{D}\sum_{k\in\Pi} N_k^{ij}\frac{\theta_k}{\theta_i\theta_j} d_k. \tag{A3}$$

Since the simple objects of a super-MTC always come in pairs related by fusion with f, i.e., a and $a\otimes f \equiv a^f$, we can decompose the set of simple objects into two

$$S = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\otimes \hat{S}, \quad T = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\otimes \hat{T}. \tag{A4}$$

After this decomposition, we obtain the *fermionic quotient* $\mathcal{B}_0$ of a super-MTC $\mathcal{B}$, which is a fusion category with half the number of simple objects as $\mathcal{B}$. While the decomposition is not canonical, the properties which follow will not depend on the choice [26]. The pair $(\hat{S}, \hat{T})$ can now be thought of as the MD of the fermionic quotient $\mathcal{B}_0$. The anyons of $\mathcal{B}_0$ form a fusion ring among themselves

$$i \otimes j = \sum_{k\in\Pi_0} \hat{N}_k^{ij} d_k, \tag{A5}$$

where Π_0 is the label set of simple objects of $\mathcal{B}_0$, and $\hat{N}_k^{ij}$, satisfy

$$\hat{N}_k^{ij} = N_k^{ij} + N_{k^f}^{ij}. \tag{A6}$$

Not all $(\hat{S}, \hat{T})$ can describe a valid super-MTC. There are necessary conditions which $(\hat{S}, \hat{T})$ need to satisfy if they are to describe a valid super-MTC [22,26,39]: the Verlinde formula

$$\hat{N}_k^{ij} = \frac{2}{D}\sum_{l\in\Pi_0}\frac{\hat{S}_{il}\hat{S}_{jl}\hat{S}_{kl}^*}{\hat{S}_{1l}} \tag{A7}$$

and the Frobenius-Schur indicator condition

$$\pm 1 = \nu_2(a) = \frac{2}{D^2}\sum_{j,k\in\Pi_0}\hat{N}_a^{jk} d_j d_k \left(\frac{\theta_j}{\theta_k}\right)^2 \tag{A8}$$

for any self-dual anyon a.

APPENDIX B: REVERSE INDUCTION FORMULA

Suppose we have a $3d$-dimensional symmetric representation of $\mathrm{SL}_2(\mathbb{Z})$ given by $(\mathcal{S}, \mathcal{T})$. We assume that the spectrum of T is nondegenerate. If the representation is an induced representation of some d-dimensional representation of Γ_θ given by $(\mathfrak{S}, \mathfrak{T}^2)$, then there exists a $3d \times 3d$ unitary matrix $\mathcal{U}$ such that

$$\begin{aligned}\mathcal{U}\mathcal{S}\mathcal{U}^{-1} &= \begin{pmatrix}\mathfrak{S} & 0 & 0\\ 0 & 0 & \mathfrak{S}^2\\ 0 & \mathbb{1} & 0\end{pmatrix},\\ \mathcal{U}\mathcal{T}\mathcal{U}^{-1} &= \begin{pmatrix}0 & \mathfrak{T}^2 & 0\\ \mathbb{1} & 0 & 0\\ 0 & 0 & (\mathfrak{S}\mathfrak{T}^2)^{-1}\end{pmatrix}.\end{aligned} \tag{B1}$$

To find such $\mathcal{U}$, first we re-arrange T via a permutation matrix P so that

$$P\mathcal{T}P^{-1} = \begin{pmatrix}-\mathfrak{T} & 0 & 0\\ 0 & \mathfrak{T} & 0\\ 0 & 0 & \mathfrak{T}'\end{pmatrix}. \tag{B2}$$

Second, we introduce

$$U = \begin{pmatrix}-\mathbb{1} & \mathbb{1} & 0\\ \mathfrak{T}^{-1} & \mathfrak{T}^{-1} & 0\\ 0 & 0 & C\end{pmatrix}, \tag{B3}$$

where $C\mathfrak{T}'C^{-1} = (\mathfrak{S}\mathfrak{T}^2)^{-1}$, then Eq. (B1) is satisfied for $\mathcal{U} = UP$. Note that the U in Eq. (B3) confines $\mathfrak{S}$ to a symmetric matrix. In addition, we have freedom of signed diagonal conjugation before conjugating with U. We denote the signed diagonal matrix by D. As a result, the transformation (B1) can be implemented by $\mathcal{U} = UDP$. We can obtain a d-dimensional representation of Γ_θ given by $(\mathfrak{S}, \mathfrak{T}^2)$ from $3d$-dimensional $(\mathcal{S}, \mathcal{T})$ via

$$\mathcal{U}\mathcal{S}\mathcal{U}^{-1} = \begin{pmatrix}\mathfrak{S} & 0 & 0\\ 0 & 0 & \mathfrak{S}^2\\ 0 & \mathbb{1} & 0\end{pmatrix}, \quad P\mathcal{T}P^{-1} = \begin{pmatrix}-\mathfrak{T} & 0 & 0\\ 0 & \mathfrak{T} & 0\\ 0 & 0 & \mathfrak{T}'\end{pmatrix}. \tag{B4}$$

To efficiently implement above formula on a computer, we further simplify the procedure. Let

$$PSP^{-1} = \begin{pmatrix} \mathcal{S}_{11} & \mathcal{S}_{12} & \mathcal{S}_{13} \\ \mathcal{S}_{21} & \mathcal{S}_{22} & \mathcal{S}_{23} \\ \mathcal{S}_{31} & \mathcal{S}_{32} & \mathcal{S}_{33} \end{pmatrix}, \quad D = \mathrm{diag}(a, b, c), \tag{B5}$$

where each $\mathcal{S}_{ij}$ is a $d \times d$ matrix satisfying $\mathcal{S}_{ji} = \mathcal{S}_{ij}^T$ for all i, j, and a, b, c are $d \times d$ signed diagonal matrices. Explicit calculation yields

$$\mathcal{USU}^{-1} = \frac{1}{2}\begin{pmatrix} \mathcal{S}_{11}^{aa} + \mathcal{S}_{21}^{ba} + \mathcal{S}_{12}^{ab} + \mathcal{S}_{22}^{bb} & \left(-\mathcal{S}_{11}^{aa} - \mathcal{S}_{21}^{ba} + \mathcal{S}_{12}^{ab} + \mathcal{S}_{22}^{bb}\right)\mathfrak{T} & * \\ -\mathfrak{T}^{-1}\left(\mathcal{S}_{11}^{aa} - \mathcal{S}_{21}^{ba} + \mathcal{S}_{12}^{ab} - \mathcal{S}_{22}^{bb}\right) & -\mathfrak{T}^{-1}\left(\mathcal{S}_{11}^{aa} - \mathcal{S}_{21}^{ba} - \mathcal{S}_{12}^{ab} + \mathcal{S}_{22}^{bb}\right)\mathfrak{T} & * \\ * & * & * \end{pmatrix}, \tag{B6}$$

where $\mathcal{S}^{ab} = a\mathcal{S}b^{-1}$ (same for similar notations) and irrelevant blocks are denoted by $*$ for simplicity. Comparing Eqs. (B4) and (B6), we notice that

$$\mathfrak{S} = 2\mathcal{S}_{11}^{aa}, \quad \mathcal{S}_{11}^{aa} = \mathcal{S}_{12}^{ab} = \mathcal{S}_{22}^{bb}. \tag{B7}$$

Therefore, for given $(\mathcal{S}, \mathcal{T})$, we first find all permutation matrices P satisfying Eq. (B2), and permute $(\mathcal{S}, \mathcal{T})$ by them. Then, for each permutation, we check if Eq. (B7) is satisfied for some signed diagonal matrices a, b. If Eq. (B7) is satisfied, we store $(\mathfrak{S}, \mathfrak{T}^2)$.

APPENDIX C: CONGRUENCE REPRESENTATIONS OF Γ_θ

The subgroup $\Gamma_\theta < \mathrm{SL}_2(\mathbb{Z})$ is defined as

$$\Gamma_\theta = \left\{ \begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) | \gamma_{11}\gamma_{21} \equiv \gamma_{12}\gamma_{22} \equiv 0 \mod 2 \right\}. \tag{C1}$$

Based on the connection to the MD of super-MTCs, we are interested in *congruence* representations of Γ_θ, rather than general representations.

A congruence representation of $\mathrm{SL}_2(\mathbb{Z})$ is a representation whose kernel contains the principal congruence subgroup

$$\Gamma(n) = \{ \gamma \in \mathrm{SL}_2(\mathbb{Z}) \mid \gamma \equiv \mathbb{1} \mod n \} \tag{C2}$$

for some positive integer n. The smallest such n is called the *level* of the congruence representation. In other words, a congruence representation of level n is a representation of a level-n congruence subgroup. (A congruence subgroup of level n is a subgroup which have $\Gamma(n)$ as its subgroup.) A congruence representation of Γ_θ is defined in the same way, i.e., a representation ρ such that $\Gamma(n) < \ker \rho$ for some positive integer n. It is noteworthy that for any congruence representation of Γ_θ, n is always even since Γ_θ itself is a level-2 congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$.

While the only relations among generators, $\mathfrak{s}$ and $\mathfrak{t}^2$, of Γ_θ are $\mathfrak{s}^4 = \mathbb{1}$ and $\mathfrak{s}^2\mathfrak{t}^2 = \mathfrak{t}^2\mathfrak{s}^2$, the generators satisfy much more relations in $\Gamma_\theta/\Gamma(n)$, and any congruence representation of Γ_θ needs its representation matrices to satisfy these relations. The precise number and content of these relations depend on the level n, since these relations are extra conditions which a representation ρ needs to satisfy in order to be a congruence conditions of Γ_θ representations.

We start from theorem 1 of Ref. [58], which lists the congruence conditions for representations of $\mathrm{SL}_2(\mathbb{Z})$. The expressions used there to write the conditions involve odd powers of $\rho(\mathfrak{t}) = \mathfrak{T}$, which are ill-defined for Γ_θ, and are unusable for our purposes. However, the conditions themselves are well-defined for Γ_θ. Thus we need only re-write the expressions in terms of elements of Γ_θ. We have rewritten the expressions so that the conditions are written in terms of $\rho(\mathfrak{s}) = \mathfrak{S}$ and $\rho(\mathfrak{t}^2) = \mathfrak{T}^2$ only. Note that the level n is always even for a Γ_θ representation.

Below, we list the congruence conditions for Γ_θ representations, generated by $\mathfrak{S}$ and $\mathfrak{T}^2$.

(1) $\mathfrak{T}$ forms an orbit of order n under matrix multiplication, i.e., $\mathfrak{T}^n = \mathbb{1}$.

(2) For $a, b \in \mathbb{Z}_n^\times$, let $H(a) = \rho(\left(\begin{smallmatrix} a & 0 \\ 0 & \bar{a} \end{smallmatrix}\right))$ where $\bar{a}$ is the multiplicative inverse of a modulo n. Note that $\left(\begin{smallmatrix} a & 0 \\ 0 & \bar{a} \end{smallmatrix}\right)$ is indeed an element of $\Gamma_\theta/\Gamma(n)$. Then,

$$\begin{aligned} H(-1) &= \mathfrak{S}^2, \\ H(a)H(b) &= H(ab), \\ \mathfrak{S}H(a) &= H(\bar{a})\mathfrak{S}, \\ H(a) &= \mathfrak{S}^2\mathfrak{T}^{a^2-a}\mathfrak{S}\mathfrak{T}^{-(\bar{a}-1)}\mathfrak{S}(\mathfrak{T}^2\mathfrak{S})^{a-1}. \end{aligned} \tag{C3}$$

(3) $(\mathfrak{S}\mathfrak{T}^2)^n = \mathbb{1}$. This condition is not independent of the above conditions, but provides a simple check in many cases.

APPENDIX D: MORE ON FROBENIUS RECIPROCITY

In this section, we formally state Frobenius reciprocity and prove its corollary.

Theorem D.1. (Frobenius reciprocity) Given a subgroup $H < G$ and an irreducible representation π of H, the induced representation of π decomposes as a direct sum of irreducible G representations R_i, where each irreducible representation appears with multiplicity m_i equal to the number of times its restriction to H contains π. In other words, $\mathrm{Ind}_H^G\pi = \bigoplus_i m_iR_i$ such that $\mathrm{Res}_H^G R_i = m_i\pi_i \oplus \cdots$.

Frobenius reciprocity allows us to obtain every irreducible representation of H from restriction of irreducible representations of G.

Corollary D.1. Every irreducible representation of H is contained in the restriction of some irreducible representation R of G, i.e., $\mathrm{Res}_H^G R = m\pi \oplus \cdots$ where m is the multiplicity of R.

Proof. Every irreducible representation π of H has an induced representation $\mathrm{Ind}_H^G\pi$, which is a representation of G, and this decomposes as $\mathrm{Ind}_H^G\pi = \bigoplus_i m_iR_i$ where R_i are irreducible representations of G. By Frobenius reciprocity, $\mathrm{Res}^G_H R_i = m_i \pi \oplus \cdots$ so these R_i are precisely those irreducible representations of G whose restriction to H contains π. ■

APPENDIX E: UNRESOLVED REPRESENTATIONS

In Sec. III B 1, we saw that the possible orthogonal basis transformations U_0 are constrained to a finite set for resolved representations. For unresolved representations, we have a continuum of potential orthogonal transformations we need to apply and then check.

In practice, however, all known cases are obtained from $\pi/4$ and $-\pi/4$ rotations, and we expect all valid MD will be obtained from orthogonal transformations involving simple angles, since the resulting $\hat{S}$ matrix must solve the Verlinde formula condition that $\hat{S}$ must diagonalize the fusion matrices, which are nonnegative integer matrices. Hence, we check the following set of angles for unresolved type (2), type (2,2), and type (3) representations: $\{\pm\pi/4, \pm\pi/6, \pm\pi/3\}$. For type (3), we allow the combination of all such rotations along all three axes of rotations, but again along every possible axes of rotation.

We find that, among dimension 5 reducible representations, those of (2+2+1)-d type (2,2) and (2+1+1+1)-d type (4) yield valid MD. Among dimension 4 reducible representations, those of (1+1+1+1)-d type (4) yield valid MD. Every unitary MD obtained this way had previously been obtained [22], though we also obtain the nonunitary MD with the same fusion rules. In every case, the valid MD is obtained from a $\pi/4$ or $-\pi/4$ orthogonal transformation. [For type (4), we make two such orthogonal transformations along different axes.]

For unresolved type (2) representations, there is only a single parameter ϕ for the orthogonal transformation, since we only have a rotation matrix on a two-dimensional subspace. In this case, we use Mathematica to directly solve for this unknown parameter given that all fusion coefficients (which depends on ϕ) must be non-negative integers. We find that there is no solution, meaning we can definitively claim that unresolved type (2) representations do not yield valid MD. (Technically, because of numerical issues we need to specify some upper bound for the fusion coefficients, and in this case we only check up to $\hat{N}^{ij}_k \leqslant 7$. However, we believe this should be sufficient, since the largest known fusion coefficient from valid MD is 4.)

Thus our classification is complete for type (2) unresolved representations (with the bound $\hat{N}^{ij}_k \leqslant 7$); for other types, our classification may be incomplete and there may exist valid MD we have missed. However, up to rank 10, only a small minority of known MD come from unresolved representations, and in those few cases they are all obtained by orthogonal transformations involving only the angles $\pm\pi/4$, so in practice it is unlikely that we have missed very many.

APPENDIX F: PROOF OF THEOREM III.1

In this section, we prove theorem III.1 in Sec. III B 2.

Proof. The super-MTC $\mathcal{B}$ admits a minimal modular extension $\mathcal{M}$. According to Sec. 3.1 of Ref. [44], we can choose a particular basis so that S and T^2 of $\mathcal{M}$ take the block-diagonal form

$$S = \begin{pmatrix} \hat{S} & 0 & 0 & 0 & 0 \\ 0 & 0 & 2A & \sqrt{2}X & 0 \\ 0 & 2A^T & 0 & 0 & 0 \\ 0 & \sqrt{2}X^T & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & B \end{pmatrix}, \quad T^2 = \begin{pmatrix} \hat{T}^2 & 0 & 0 & 0 & 0 \\ 0 & \hat{T}^2 & 0 & 0 & 0 \\ 0 & 0 & \hat{T}_v^2 & 0 & 0 \\ 0 & 0 & 0 & T_\sigma^2 & 0 \\ 0 & 0 & 0 & 0 & \hat{T}_v^2 \end{pmatrix}, \tag{F1}$$

where $\hat{S}$ and $\hat{T}^2$ are the MD of $\mathcal{B}$. (The other matrices such as A and X are not relevant for our purposes.) This means that the projective $\mathrm{SL}_2(\mathbb{Z})$ representation formed by S and T, after restriction to Γ_θ, becomes reducible. (We are here interested only in representation-theoretic properties and are thus free to choose a basis.) More precisely, if we denote by $\tilde{\Phi}$ the projective $\mathrm{SL}_2(\mathbb{Z})$ representation formed by S and T, we have

$$\tilde{\Phi}|_{\Gamma_\theta} = \tilde{\rho} \oplus \tilde{\Phi}' \tag{F2}$$

where $\tilde{\Phi}'$ is the remaining part.

Let $N = \operatorname{ord} T$. By theorem II of Ref. [48], $\tilde{\Phi}$ always admits a lift to a linear congruence representation Φ of level n such that $N|n|12N$, which takes the form

$$\begin{aligned} \Phi(\mathfrak{s}) &= \tilde{\Phi}(\mathfrak{s}), \\ \Phi(\mathfrak{t}) &= e^{-2i\pi c/12}\tilde{\Phi}(\mathfrak{t}), \end{aligned} \tag{F3}$$

where c is determined modulo 8 by $\tilde{\Phi}$. (We can also always take the tensor product of this linear representation with one-dimensional representations of $\mathrm{SL}_2(\mathbb{Z})$, but this does not affect our argument.) The restriction of Φ to Γ_θ then takes the form

$$\begin{aligned} \tilde{\Phi}|_{\Gamma_\theta}(\mathfrak{s}) &= \hat{S} \oplus \cdots, \\ \tilde{\Phi}|_{\Gamma_\theta}(\mathfrak{t}^2) &= e^{-2i\pi c/12}\hat{T}^2 \oplus \cdots. \end{aligned} \tag{F4}$$

Since $\Phi|_{\Gamma_\theta}$ is a linear representation of Γ_θ, its direct summand ρ given by

$$\begin{aligned} \rho(\mathfrak{s}) &= \hat{S}, \\ \rho(\mathfrak{t}^2) &= e^{-2i\pi c/12}\hat{T}^2 \end{aligned} \tag{F5}$$

must also be a linear representation of Γ_θ. Moreover, since the level of Φ is n, we note that

$$\ker \rho \geqslant \ker \Phi|_{\Gamma_\theta} \geqslant \Gamma_\theta \cap \Gamma(n). \tag{F6}$$

Since N is always even (because the list of simple objects of $\mathcal{M}$ includes fermions, of spin 1/2), n is also even, so $\Gamma_\theta \cap \Gamma(n) = \Gamma(n)$. Thus we have

$$\ker \rho \geqslant \Gamma(n), \tag{F7}$$

i.e., the linear lift ρ obtained by attaching a phase $e^{-2i\pi c/12}$ to $\hat{T}$, where c is the central charge of one of the modular extensions, is congruence. ■

APPENDIX G: MODULAR DATA FROM A DIRECT SUM OF 1-DIMENSIONAL REPRESENTATIONS

In Sec. III B 1, we have mentioned that a direct sum of five one-dimensional representations of Γ_θ do not give rise to any valid MD. Let us call representations which are a direct sum of one-dimensional representations "1d-sum representations," and super-MTCs arising from them as "1d-sum super-MTCs." Here we prove that no 1d-sum representations of dimension 5 give rise to valid super-MTCs. We also prove that 1d-sum super-MTCs are always split and Abelian.

Let $\rho = \bigoplus_{i=1}^{d} \chi_i$ for some one-dimensional representations χ_i. By the t^2-spectrum criterion (see section III B 1), $\rho(t^2)$ must be proportional to the identity. The resulting MD will be $\hat{T}^2 = \mathbb{1}$, and T will consist of 1s and -1s. Thus the resulting super-MTC will have $\operatorname{ord} T = 2$. Spherical fusion categories (of which super-MTCs are a special case) satisfying this condition have been classified in Ref. [59]. In particular, they find that any such spherical fusion category is pointed (Abelian). On the other hand, any pointed super-MTC is split (see Proposition 2.1 of Ref. [40]). Thus there is no need for extra classification of 1d-sum super-MTCs, as all of them come from stacking bosonic theories with $\mathcal{F}_0$.

For the specific case of dimension 5, we can simply look at the known bosonic classification in, say, Ref. [20], and see that there is no rank 5 MTC whose T matrix consists exclusively of 1s and -1s. This proves the assertion in section III B 1 that there are no rank 10 1d-sum super-MTCs. On the other hand, in the bosonic classification of rank 4 MTCs there is a well-known MTC whose T matrix consists exclusively of 1s and -1s: the toric code theory. Hence in rank 8 the toric code theory stacked with $\mathcal{F}_0$ is a 1d-sum super-MTC.

APPENDIX H: EXPLICIT MODULAR DATA OF NEW CLASSES

We present the explicit data of the new classes of rank 10 MD we have found. We show only one representative from each class. The MD of the first class is

$$\hat{S} = \frac{1}{\sqrt{30\chi_{15}^4}} \begin{pmatrix} 1 & \chi_{15}^4 & \chi_{15}^5 & \chi_{15}^3 & \chi_{15}^3 \\ \chi_{15}^4 & 1 & \chi_{15}^5 & -\chi_{15}^3 & -\chi_{15}^3 \\ \chi_{15}^5 & \chi_{15}^5 & -\chi_{15}^5 & 0 & 0 \\ \chi_{15}^3 & -\chi_{15}^3 & 0 & \frac{\chi_5^1\chi_{15}^3}{2} & -\frac{2\sqrt{30\chi_{15}^4}}{\chi_5^5} \\ \chi_{15}^3 & -\chi_{15}^3 & 0 & -\frac{2\sqrt{30\chi_{15}^4}}{\chi_5^5} & \frac{\chi_5^1\chi_{15}^3}{2} \end{pmatrix}, \quad \hat{T}^2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & e^{2i\pi/3} & 0 & 0 \\ 0 & 0 & 0 & e^{4i\pi/5} & 0 \\ 0 & 0 & 0 & 0 & e^{-4i\pi/5} \end{pmatrix}, \tag{H1}$$

where $\chi_n^m = m + \sqrt{n}$ and the total quantum dimensions is $D^2 = 472.379$. The MD of the second class is

$$\hat{S} = \frac{1}{2\sqrt{6\chi_{24}^5}} \begin{pmatrix} 1 & \chi_{24}^5 & \chi_6^3 & \chi_6^3 & \chi_{24}^4 \\ \chi_{24}^5 & 1 & \chi_6^3 & -\chi_6^3 & -\chi_{24}^4 \\ \chi_6^3 & \chi_6^3 & -\chi_6^3 - i\sqrt{6\chi_{24}^5} & -\chi_6^3 + i\sqrt{6\chi_{24}^5} & 0 \\ \chi_6^3 & \chi_6^3 & -\chi_6^3 + i\sqrt{6\chi_{24}^5} & -\chi_6^3 - i\sqrt{6\chi_{24}^5} & 0 \\ \chi_{24}^4 & -\chi_{24}^4 & 0 & 0 & \chi_{24}^4 \end{pmatrix}, \quad \hat{T}^2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & e^{2i\pi/3} \end{pmatrix}, \tag{H2}$$

where the total quantum dimensions is $D^2 = 475.151$.

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