


Time evolution of spread complexity and statistics of work done in quantum quenchesKuntal Pal,^{*} Kunal Pal[†],[‡] Ankit Gill,[‡] and Tapobrata Sarkar[§]
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We relate the probability distribution of the work done on a statistical system under a sudden quench to the Lanczos coefficients corresponding to evolution under the postquench Hamiltonian. Using the general relation between the moments and the cumulants of the probability distribution, we show that the Lanczos coefficients can be identified with physical quantities associated with the distribution, e.g., the average work done on the system and its variance, as well as the higher order cumulants. In a sense this gives an interpretation of the Lanczos coefficients in terms of experimentally measurable quantities. Consequently, our approach provides a way towards understanding spread complexity, a quantity that measures the spread of an initial state with time in the Krylov basis generated by the postquench Hamiltonian, from a thermodynamical perspective. We illustrate these relations with two examples. The first one involves quench done on a harmonic chain with periodic boundary conditions and with nearest neighbor interactions. As a second example, we consider mass quench in a free bosonic field theory in d spatial dimensions in the limit of large system size. In both cases, we find out the time evolution of the spread complexity after the quench and relate the Lanczos coefficients with the cumulants of the distribution of the work done on the system.

DOI: [10.1103/PhysRevB.108.104311](https://doi.org/10.1103/PhysRevB.108.104311)**I. INTRODUCTION**

Physical quantities that are used to probe nonequilibrium scenarios such as a sudden quench of a quantum mechanical system [1] are broadly categorized into two different sets. The first includes different correlation functions [2,3], entanglement entropy [4], and out of time order correlators, as well as the more recently introduced notions of complexity (studied in the context of quantum quenches in, e.g., Refs. [5–8]). Time evolution of these quantities after a quench generally show some characteristic behavior at early as well as late times and these can also be used to detect criticality in many-body quantum systems that exhibit quantum phase transitions. The second set is related to thermodynamics. Indeed, it has been established that the problem of quantum quenches can be viewed as a thermodynamic transformation [9,10]. This fact offers a new way of looking at the physics after a quantum quench in terms of quantities that are commonly used to characterize standard thermodynamic processes, such as the heat and entropy generated, as well as the work done on the system [11].

In this work, our main focus will be a new measure of the complexity of constructing a target wave function starting from a reference one, namely the spread complexity (SC), introduced in [12]. The complexity of a state is broadly a measure of the minimum number of basis gates one requires to construct that state, starting from a given reference state.

There are various notions of this measure, such as the Nielsen complexity (NC) [13–19], Fubini-Study complexity (FSC) [20], bi-invariant complexity [21], complexity from covariance matrix [22], complexity from information geometry [23], path integral approach to circuit optimization [24], possible extensions to conformal field theories [25,26], etc. (see the recent review [27] and references therein). These studies have gained a lot of attention in recent literature due to usefulness of various notions' complexity as probes of quantum phase transitions [28–39] and quantum quenches [6,24,40–45], as well as indicators of quantum chaotic evolution [46–48].

In a similar vein, the SC of a state under the evolution by a unitary operator measures the spreading of the wave function on the Hilbert space [12]. Intuitively the more it spreads over the corresponding Hilbert space, the more difficult it is to construct that state. The central idea of finding the SC of a state lies in the construction of the Krylov basis by the Hermitian operator that generates the flow, which is done by the well-known Lanczos algorithm of constructing a tridiagonal form of a given matrix [49–51]. This process takes the autocorrelation function of the final state and the initial state as an input and gives two sets of coefficients, known as the Lanczos coefficients (LC), as outputs. Using the discretized version of the Schrödinger equation on the Krylov basis, it is then possible to obtain the SC as the minimum value of the associated cost for the Krylov basis, as was proven in [12]. There is a lot of recent attention on various aspects of the SC and its corresponding operator version (Krylov complexity); see, for example, works related to quantum phase transition [7,8,52], operator scrambling [53,54], conformal field theory [55–57], open systems [58–60], as a tool of probing delocalization properties in nonchaotic quantum systems [61], and other related contexts [57,62–79].

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In this paper our goal is to relate two of these quantities from the above mentioned different sets, thereby offering a new way of interpreting the evolution of a system after a quantum quench. In particular, the two apparently distinct quantities that we consider in this paper are (a) the LC used to study the SC and (b) various cumulants of the probability distribution of the work done on the system by suddenly changing its parameters. That these two sets of numbers are related to each other can, in some sense, be “guessed” by noting that the characteristic function (CF) associated with the distribution of the work done is related to the complex conjugate of the autocorrelation function—a quantity whose moments contain all the information about the LC. In this paper we make this relation precise and quantify this with two examples.

The rest of the paper is organized as follows. In Sec. II, we first briefly review the two quantities mentioned above and then obtain a relation between the LC and the cumulants of the probability distribution by using Faà di Bruno’s formula. We show that these quantities are related to each other via the Bell polynomials. In Sec. III, we apply this formalism to the time evolution of the SC after a single sudden quench of the parameters of a harmonic chain with periodic boundary conditions and provide a physical interpretation of the LC when a critical quench is considered. Section IV elaborates on our second example—the mass quench of a noninteracting bosonic model in d -spatial dimensions in the limit of infinite linear size of the system. Section V discusses the main outcomes of our analysis.

II. STATISTICS OF WORK AND THE LANCZOS COEFFICIENTS IN QUANTUM QUENCHES

In this section we first briefly review the formalism of the statistics of work done under a sudden quantum quench and the Lanczos algorithm of obtaining the LC from the autocorrelation function of a time-evolved state.

A. Statistics of the work done under a quantum quench

It is well known that a quantum quench can be viewed as a thermodynamic transformation [9,10]. Thus, apart from the quantities related to time evolution of quantum correlation functions and quantum information theoretic quantities (such as the entanglement entropy or complexity), a fundamental way to characterize quantum quench is to consider the statistics of the work done on a quantum system when its parameters are changed suddenly [11]. The work done on the system through a given quench protocol is defined as the difference between the internal energies before and after the quench. That we need a probability distribution function to quantify the work done on the system can be understood from the fact that due to the sudden change of the parameters of the system Hamiltonian, even if we consider different realizations of the same quench protocol, the measurement of the work done will yield different results, i.e., it will show fluctuations.

To mathematically quantify the distribution of the work done, we consider the simplest single quench protocol, where the parameters of a quantum system, collectively denoted by λ , are changed suddenly to a new set of values $\tilde{\lambda}$ at an

instant of time $t = t_0$ (we will usually take $t_0 = 0$). We denote the energy eigenstate before and after the quench as $|n(\lambda)\rangle$ and $|\tilde{n}(\tilde{\lambda})\rangle$, which correspond to energies $E_n(\lambda)$ and $\tilde{E}_n(\tilde{\lambda})$, respectively. The Hamiltonians before and after the quench are denoted by H_0 and \tilde{H} .

Now, if energy measurements before and after the quench give the results E_m and \tilde{E}_n , respectively, then the probability distribution of the work done, $W = \tilde{E}_n - E_m$, is given by [80,81]

$$P(W) = \sum_{m,n} |\langle m|\tilde{n}\rangle|^2 \delta(W - (\tilde{E}_n - E_m)). \quad (1)$$

For the quench protocols considered in this paper, we shall take the state before the quench to be the ground state of the Hamiltonian. Hence the above distribution reduces to

$$P(W) = \sum_n |\langle 0|\tilde{n}\rangle|^2 \delta(W - (\tilde{E}_n - E_0)). \quad (2)$$

Once we know the distribution of the work, it is useful to first consider its CF defined as the Fourier transform

$$G(t) = \langle e^{-iWt} \rangle = \int e^{-iWt} P(W) dW. \quad (3)$$

The importance of this quantity in the context of quantum quenches, as established in [81], is that the CF is actually the correlation function

$$\begin{aligned} G(t) &= \langle 0|e^{iH_0(\lambda)t} e^{-i\tilde{H}(\tilde{\lambda})t} |0\rangle = e^{iE_0(\lambda)t} \langle 0|e^{-i\tilde{H}(\tilde{\lambda})t} |0\rangle \\ &= e^{iE_0(\lambda)t} \langle \psi_0|\Psi(t)\rangle, \end{aligned} \quad (4)$$

where $|\psi_0\rangle = |0\rangle$ is the state before the quench at $t = 0$. Now it is easy to see that this quantity is just the conjugate of the Loschmidt echo studied extensively in the context of the quantum quenches and quantum chaos [82–85]. Furthermore, we can see that it is related to the complex conjugate of the autocorrelation function, apart from a trivial phase factor. This is the first indication of a connection between the LC and the CF, which we elaborate on below.

For future references, at this point it is useful to define the cumulants ($\beta_n, n \geq 1$) of the distribution as expansions of the logarithm of the CF¹

$$\ln(G(t)) = \sum_{n=1}^{\infty} \frac{(-it)^n}{n!} \beta_n, \quad \beta_n = (-i)^{-n} \left. \frac{d^n \ln G(t)}{dt^n} \right|_{t=0}. \quad (5)$$

Furthermore, we can also write down the expansion of the CF in terms of the moments (M_n) of the distribution

$$G(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} M_n. \quad (6)$$

Note that, since $G(t = 0) = 1$, here we have $M_0 = 1$ by definition. Furthermore, the way we have defined the coefficients, M_n ’s are actually related to the average of W^n with powers of i , i.e., $M_n = (-i)^n \int W^n P(W) dW$. Using this along with the

¹To be consistent with the notion of the quench and spread complexity literature, we have defined the cumulant expansion with factors of i .

expression in Eq. (2) for a quench from the ground state, we obtain the following meaningful expression for the moments in terms of products of the energy difference between energy eigenstates before and after the quench and their overlap:

$$M_n = (-i)^n \sum_j | \langle 0 | \tilde{j} \rangle |^2 (\tilde{E}_j - E_0)^n. \quad (7)$$

Here, $|\tilde{j}\rangle$ represents an eigenstate of a postquench Hamiltonian with energy \tilde{E}_j . Therefore, we see that the moments depend on the overlap between the initial state (here, the ground state of the prequench Hamiltonian) and the eigenstates of the postquench Hamiltonian, as well as the spacing between energy levels of the postquench Hamiltonian with that of the ground state energy of the initial Hamiltonian H_0 . In Appendix A we illustrate an example of this formula, where we obtain the moments for the case of a quench in a single harmonic oscillator.

It is well known that the cumulants of a probability distribution are related to its central moments. For example, the first cumulant (β_1) is the mean of the distribution (here $\langle W \rangle$), the second cumulant is its variance σ^2 (i.e., the second central moment; here $\langle W^2 \rangle - \langle W \rangle^2$), and the third cumulant is equal to the third central moment, etc. All the other higher cumulants are actually polynomial functions of the central moments with integer coefficients.

The behavior of these quantities—the probability distribution of the work done (PDWD) $P(W)$, CF $G(t)$, and the moments—have been studied after sudden quenches in systems which show quantum phase transitions, e.g., in [11] and [86], by taking the Ising chain and the Dicke model as prototypical examples. In these references it was established that the moments of the PDWD show diverging behavior when the system is quenched through the critical points of the systems. See also the recent works of [87,88], where the authors have studied the work distribution of a quench across a quantum phase transition and found universal scaling relations in such cases.

In this paper, we relate the moments of the distributions with the LC and study the properties of these moments (and the cumulants) of the distribution when there is a zero mode present in the dispersion relation, as well as in the limit of infinite system size.

B. Lanczos coefficients and the Krylov basis construction

We now briefly review the Lanczos algorithm of constructing the Krylov basis and the definition of the SC of an arbitrary initial state under Hamiltonian evolution.

The central idea behind the construction of the Krylov basis is to write the Hamiltonian in the tridiagonal basis in the Lanczos algorithm. In this construction, a new basis is defined from the old one as follows:

$$|K_{n+1}\rangle = \frac{1}{b_{n+1}} [(H - a_n)|K_n\rangle - b_n|K_{n-1}\rangle]. \quad (8)$$

We take $|K_0\rangle = |\psi(0)\rangle$, i.e., the algorithm starts from the reference state. The computation of the coefficients a_n, b_n (known as the LC) plays a key role in implementing the Lanczos algorithm. It is important to note that information about the LC is also encoded in the so called “return

amplitude,” which is defined as the overlap between the state at any particular value of the circuit parameter t and the initial state, i.e.,

$$\mathcal{S}(t) = \langle \psi(t) | \psi(0) \rangle. \quad (9)$$

Once we have constructed the Krylov basis for the Hamiltonian evolution, we can expand the desired state in this basis as

$$|\psi(t)\rangle = \sum_n \phi_n(t) |K_n\rangle. \quad (10)$$

It can be shown that the expansion coefficients $\phi_n(t)$ satisfy the following discrete Schrödinger equation:

$$i\dot{\phi}_n(t) = a_n\phi_n(t) + b_n\phi_{n-1}(t) + b_{n+1}\phi_{n+1}(t), \quad (11)$$

where a dot denotes a derivative with respect to t . It was recently proved that the Krylov basis as defined above minimizes the cost function $\mathcal{C}_B(t) = \sum_n n |\langle \psi(t) | B_n \rangle|^2$, which measures the spreading of the state under the desired evolution [12]. Here, $|B_n\rangle$ is the particular basis which we use to evaluate the spreading. We can write the above cost in the Krylov basis as

$$\mathcal{C}(t) = \sum_n n |\phi_n(t)|^2. \quad (12)$$

This is the definition of the SC.

Next we briefly describe how the LC can be calculated from the return amplitude and, subsequently, how the $\phi_n(t)$ are obtained by solving Eq. (11). For calculating the LC from the return amplitude, we first need to find the even and odd moments from the expansion

$$\mathcal{S}(t) = \sum_n M_n^* \frac{t^n}{n!}. \quad (13)$$

Here, M_n^* 's are the expansion coefficients of the return amplitude. After knowing the moments, we can find the full sets of a_n 's and b_n 's using the standard recursion methods available in the literature for dynamics under Hermitian evolution [50,51] that was recently extended for the case of open systems in [59], which we briefly recall below.

To construct the full set of orthonormal Krylov basis on the Hilbert space, we start from the given state $|\psi(0)\rangle$, i.e., this is the first Krylov state $|K_0\rangle = |\psi(0)\rangle$. Then the recursion relation of Eq. (8) implies that the next basis is $|K_1\rangle = \frac{1}{b_1} [(H - a_0)|K_0\rangle]$. Here we have used the fact that $b_0 = 0$. The condition that this state $|K_1\rangle$ is orthogonal to the previous state $|K_0\rangle$ fixes the unknown coefficient a_0 to be equal to $\langle K_0 | H | K_0 \rangle$. The other coefficient b_1 ensures the normalization of this state. We continue this recursive process to construct the full set of basis and the general coefficients are given as

$$a_n = \langle K_n | H | K_n \rangle, \quad (14)$$

while b_n 's fix the normalization at each step. However, in practice, where the above process does not terminate after the first few steps, it is more useful to implement the Lanczos algorithm by means of two sets of auxiliary matrices $L_k^{(n)}$ and $M_k^{(n)}$ constructed from the moments M_n^* of the return amplitude defined in Eq. (13). The recursion relations then can

be written down in terms of those $L_k^{(n)}$'s and $M_k^{(n)}$'s and finally the LC are obtained as $b_n = \sqrt{M_n^{(n)}}$ and $a_n = -L_n^{(n)}$ with the initial conditions properly chosen ($b_0 = 0$) [12,50]. This is a standard procedure and we refer the reader to [50] for details. Once we have the full set of LC, we have all the information that is required to find the coefficients ϕ_n , by solving the discrete Schrödinger equation in Eq. (11), and calculate the spread complexity as a function of time.

C. Relation between the Lanczos coefficients and the cumulants

From the discussions of the previous two subsections, it should be clear that we can relate the LC associated with the Lanczos algorithm (and the Krylov basis) with fundamental physical quantities characterizing a sudden quench as a thermodynamic transformation—such as the average work, its variance, and higher order cumulants. To establish such a relation, we need to relate the cumulants of the expansion (which have information of the average work, variance, etc.) with the moments (from which we can obtain the LC). This can be achieved using Faà di Bruno's formula, which generalizes the chain rule to the higher derivatives (see, e.g., Ref. [89]). Here we briefly outline the derivation of this relation to use the notation consistent with the ones used above and the rest of this section. For details of the derivation, we refer to, e.g., Refs. [90,91].

First we define the partial Bell polynomials as the coefficients in the expansion of the following generating function of two variables [92]:

$$\begin{aligned} \Phi(t, u) &= \exp\left(u \sum_{m=1}^{\infty} g_m \frac{t^m}{m!}\right) \\ &= 1 + \sum_{n=1}^{\infty} \sum_{k=1}^n B_{n,k}(g_1, \dots, g_{n-k+1}) u^k \frac{t^n}{n!}. \end{aligned} \quad (15)$$

The partial Bell polynomial $B_{n,k}$ is a homogeneous polynomial of degree k and weight n in the expansion coefficients g_m . Evaluating this at $u = 1$ we get the definition of the complete Bell polynomials Y_{g_1, \dots, g_n} as

$$\begin{aligned} \Phi(t, 1) &= \exp\left(\sum_{m=1}^{\infty} g_m \frac{t^m}{m!}\right) \\ &= 1 + \sum_{n=1}^{\infty} Y_n(g_1, \dots, g_n) \frac{t^n}{n!}, \end{aligned} \quad (16)$$

so that the complete polynomials are related to the partial polynomials through

$$Y_n(g_1, \dots, g_n) = \sum_{k=1}^n B_{n,k}(g_1, \dots, g_{n-k+1}). \quad (17)$$

Explicit expressions for both the partial and complete Bell polynomials are known which are written as a sum over all the partitions of n into k non-negative parts and all the partitions of n into arbitrarily many non-negative parts, respectively [91].

Now using Faà di Bruno's formula we can obtain

$$\begin{aligned} \ln\left(\sum_{n=0}^{\infty} M_n \frac{t^n}{n!}\right) &= \ln M_0 + \sum_{n=1}^{\infty} \sum_{k=1}^n (-1)^{k-1} (k-1)! M_0^{-k} B_{n,k} \\ &\quad \times (M_1, \dots, M_{n-k+1}) \frac{t^n}{n!}, \end{aligned} \quad (18)$$

where we have assumed $M_0 > 0$. In fact, since we are considering the expansion of the CF around the start of the quench at $t = 0$, here $M_0 = 1$ and the first term in the above expansion vanishes. We can now compare the right hand side with the cumulant expansion in Eq. (5) to obtain the desired relation between the cumulant of the PDWD (β_n) and the moments M_n , which carry the information of the LC as

$$\beta_n = \sum_{k=1}^n \frac{(-1)^{k-1}}{(-i)^n} (k-1)! B_{n,k}(M_1, \dots, M_{n-k+1}). \quad (19)$$

This formula will be used in the next section to relate the average and variance of the work done on the system through a quench and the LC (such as a_0 and b_1 , etc.) in a simple manner.

It is also useful to obtain the inverse of the above relation. This can be straightforwardly obtained using the definition of the complete Bell polynomials given in Eq. (16) to be

$$M_n = (-i)^n Y_n(\beta_1, \dots, \beta_n). \quad (20)$$

The well known tabulated expressions for the Bell polynomials can now be used to explicitly relate these two types of expansion coefficients.

Once we know these moments in terms of the cumulants, they can be used, following the procedure outlined in the previous subsection, to obtain the LC. For example, we have the following relations between a_0 and the average work and b_1 and the variance of the work done:

$$a_0 = iM_1 = \langle W \rangle, \quad b_1^2 = -M_2 + M_1^2 = \langle W^2 \rangle - \langle W \rangle^2. \quad (21)$$

Similarly, it is possible to write the coefficient a_1 in terms of $\langle W^3 \rangle$, variance, and the average of the work done, and it is given by

$$a_1 = \frac{\langle W^3 \rangle - \langle W \rangle^3}{\langle W^2 \rangle - \langle W \rangle^2} - 2\langle W \rangle. \quad (22)$$

See Appendix B for a brief derivation of the last two equations. Other LC can similarly be written in terms of averages of $\langle W^n \rangle$; however, their expressions become increasingly cumbersome and hence we avoid writing them here.

Since we have established that all the LC can be related to the moments and cumulants of the PDWD, at this point, it is important to comment on the measurements of PDWDs in experimental setups. It is clear from its definition in Eq. (1) that, to measure the work distribution, one needs to perform two projective energy measurements on the system—one before the quench and the other after it. It is well known that it is difficult to perform reliable projective measurements of energy in many-body quantum systems. However, for relatively simple systems, there exist pioneering experiments where the quantum work statistics have been measured. Examples of

such systems include a spin-1/2 system undergoing closed nonadiabatic evolution (which can be realized in NMR setups) [93], driven oscillator systems which can be used to describe the dynamics of trapped ions [94], or ultracold atoms [95]. Since the relations between the LC and the work distribution we have obtained are very general (valid for any evolution with any generic time-independent Hamiltonian), our results can be applied to these cases as well. Therefore, in a sense, relations of the kind given in Eqs. (21) and (22) can be thought of as providing an interpretation of LC in terms of experimentally measurable quantities.

Before concluding this section, here we mention an important relation between a universal behavior of the survival probability at short times after quench ($t \ll \sigma^{-1}$) and b_1 , where σ is the variance of the PDWD under the quench (see below for its definition). The expression for the survival probability, defined as the modulus square of the overlap between the initial state and the time-evolved state, is given by

$$\mathcal{P}(t) = |\langle \psi(0) | \psi(t) \rangle|^2 = \left| \sum_n \langle 0 | \tilde{n} \rangle|^2 e^{-i\tilde{E}_n t} \right|^2. \quad (23)$$

Now from Eq. (7), we first write down the expressions for the average and the variance of the PDWD under quench in terms of the overlaps of pre- and postquench energy eigenstates:

$$\langle W \rangle = \sum_j |\langle 0 | \tilde{j} \rangle|^2 (\tilde{E}_j - E_0), \quad (24)$$

$$\sigma^2 = \sum_j |\langle 0 | \tilde{j} \rangle|^2 (\tilde{E}_j - E_0 - \langle W \rangle)^2. \quad (25)$$

Next, we expand the above expression for the survival probability and for times $t \ll \sigma^{-1}$ obtain [96]

$$\mathcal{P}(t) \approx 1 - \sigma^2 t^2 \approx 1 - b_1^2 t^2. \quad (26)$$

Therefore, as is well known, for a very short time after a quench, the survival probability shows universal quadratic decay in t [96]. Importantly, here we see that the rate of decay of the survival probability is actually determined by b_1 . Since the time behavior shown by the survival probability is universal, this observation shows the important role played by b_1 in the early time evolution of a quenched quantum many-body system.

D. Lanczos coefficients in quench of a general system of length L in d dimensions

Before moving on to describe particular examples of the formalism discussed till now, in this subsection we consider a fairly general case of quench in a closed system placed in a d -dimensional box of length L with periodic boundary conditions. We assume that a sudden quench is performed in such a system prepared in an initial state at zero temperature. Work statistics and its CF for quenches done in such scenarios have been studied previously in [97–99].

In such a quench scenario, the expressions for the PDWD and its CF are the ones given in Eqs. (2) and (4), respectively. Now we consider performing a Wick rotation from time t to the imaginary time $-i\tau$ on the amplitude $\langle 0 | e^{-i\tilde{H}t} | 0 \rangle$. The transformed amplitude $\langle 0 | e^{-\tilde{H}\tau} | 0 \rangle$ can be thought of as the partition function of a $(d+1)$ dimensional classical system

with Hamiltonian \tilde{H} , defined on a strip of thickness τ [97]. The boundaries of the strip are described by the boundary states $|0\rangle$. Using techniques used in the studies of the critical Casimir effect, it is possible to evaluate partition functions of such geometries, so that continuing back to real time $\tau = it$, the expression for the CF can be conveniently written as the sum of three terms

$$\ln G(t) = -L^d [i f_b t + 2 f_s + f_c(it)]. \quad (27)$$

Here, in the first term, which is linearly dependent on time, $f_b = \frac{\tilde{E}_0 - E_0}{L^d}$ is the difference between the ground state energies of the Hamiltonians after and before the quench, per unit volume. Clearly, this term comes from a linearly time dependent overall phase factor to the CF. On the other hand, the second term, which is the surface free energy associated with each of the two boundaries of the strip geometry, is time independent. This quantity is actually related to the fidelity between the ground states of the Hamiltonians before and after the quench through the relation $f_s = -L^d \ln |\langle \tilde{0} | 0 \rangle|$ [97,98]. The third term in Eq. (27) has nontrivial time dependence and represents an effective interaction between two boundaries.

Using the general formula for the CF written above, let us now calculate the cumulants and hence relate the corresponding LC to the probability distribution. We have the expression for the cumulants as

$$\beta_n = \begin{cases} -iL^d [i f_b + \frac{df_c(it)}{dt} \Big|_{t=0}] & \text{for } n = 1, \\ -i^n L^d \frac{d^n f_c(it)}{dt^n} \Big|_{t=0} & \text{for } n \neq 1. \end{cases} \quad (28)$$

Since, in the second term, the contribution of the surface free energy of the boundary is time independent, it does not contribute to the expressions for the moments. Thus the ground state fidelity or its higher order derivatives, which are widely used to study quantum quenches and quantum phase transitions, are insensitive to the behavior of the LC in a quench problem. Furthermore, the first term has only an additive contribution in β_1 , i.e., the first LC, a_0 . In most cases, we can omit the first term since one usually measures the work done W starting from $\delta E_{gs} = \tilde{E}_0 - E_0$ [98]. In fact, most of the literature on the Lanczos algorithm which uses the autocorrelation function [the complex conjugate of the CF defined in Eq. (4)] neglects the constant phase factor in the definition of the CF in Eq. (4). In the example of quench in a harmonic chain considered in the next section we will also neglect this phase factor.

Thus the cumulants, moments of the PDWD, and hence the LC of the quench problem under consideration are determined by the time dependence of the third term in the expression for the CF, namely, $f_c(it)$. If we now assume that the time derivative of this function is zero for $n = k+1$, then $\beta_{k+1} = 0$, and as we can see from the explicit expressions for the complete Bell polynomials $Y_n(\beta_1, \dots, \beta_n)$, all the higher order moments are determined by the first β_k cumulants. In Sec. IV, we will discuss the limit $L \rightarrow \infty$ of the quench problem considered in this section and explicit expressions for the time dependence of the function f_c will be obtained to find out the cumulants and moments of the work distribution.

III. LANCZOS COEFFICIENTS, SPREAD COMPLEXITY, AND STATISTICS OF WORK IN QUENCH OF A HARMONIC CHAIN

In this section we first discuss the time evolution of the SC after a single sudden quench of the parameters of a harmonic chain with periodic boundary conditions. Then we study the relation between the PDWD under the quench and the corresponding LC, with particular emphasis given on the case of the critical quench, i.e., when the final (or the initial) frequency of the oscillators of the harmonic chain vanishes.

The protocol for the quench we consider is the following. At $t = 0$, we change the initial Hamiltonian (denoted as H_0) to a new one H_1 , which has different values of the frequency and interaction strengths than H_0 . Subsequent evolution of the system is governed by the new Hamiltonian H_1 . We calculate the SC by taking the state at $t = 0$ as the initial state, i.e., the first state of the Krylov basis, and the target state as the time-evolved state after the quench. For our purposes, in this section we assume that the reference state, i.e., the state before the quench, is the ground state of the initial Hamiltonian H_0 .

We denote the Hamiltonian before the quench (i.e., for $t < 0$) as H_0 , which is given by

$$H_0 = \frac{1}{2} \left[\sum_{j=1}^N (p_{0j}^2 + \lambda_0^2 x_{0j}^2) + \sum_{j=1}^N k_0 (x_{0j} - x_{0(j+1)})^2 \right] \\ = \frac{1}{2} \left[\sum_{j=1}^N p_{0j}^2 + X_0^T \cdot \mathcal{K}_0 \cdot X_0 \right].$$

Here λ_0 is the frequency of each oscillator before the quench and k_0 is the nearest neighbor interaction strength. Furthermore, $X_0 = (x_{01}, x_{02}, \dots, x_{0N})^T$ denotes the column matrix for the collective position of each oscillator and \mathcal{K}_0 is a real symmetric matrix whose eigenvalues are denoted as ω_{0j} . It is assumed that periodic boundary conditions are imposed on the chain. We diagonalize this Hamiltonian by performing an orthogonal transformation U_0 , which changes the coordinates to $Y_0 = (y_{01}, y_{02}, \dots, y_{0N})^T = U_0 X_0$. Denoting the transformed momenta as P_{0k} , the diagonal form for the Hamiltonian is therefore given by

$$H_0(y_{0k}, P_{0k}) = \frac{1}{2} \sum_{k=1}^N [P_{0k}^2 + \omega_{0k}^2 y_{0k}^2] \\ = \sum_{k=1}^N \omega_{0k} \left[a_{0k}^\dagger a_{0k} + \frac{1}{2} \right] = \sum_{k=1}^N H_{0k}(y_{0k}, P_{0k}, t), \quad (29)$$

where $\omega_{0k}^2 = \lambda_0^2 + 2k_0[1 - \cos(\frac{2\pi k}{N})]$ is the k th normal mode frequency, and in the second equality we have introduced the usual creation and annihilation operators for the individual modes. We assume that, by the quench, only the frequency of all the oscillators λ_0 are changed simultaneously to a new value λ_1 , while keeping the interaction strength fixed. After the quench, the Hamiltonian of the harmonic chain can similarly be diagonalized in terms of a new set of creation and

annihilation operators as

$$H_1(y_{1k}, P_{1k}) = \frac{1}{2} \sum_{k=1}^N [P_{1k}^2 + \omega_{1k}^2 y_{1k}^2] \\ = \sum_{k=1}^N \omega_{1k} \left[a_{1k}^\dagger a_{1k} + \frac{1}{2} \right] = \sum_{k=1}^N H_{1k}(y_{1k}, P_{1k}), \quad (30)$$

where the expressions for the normal mode frequencies are now given by

$$\omega_{1k}^2 = \lambda_1^2 + 2k_1 \left[1 - \cos\left(\frac{2\pi k}{N}\right) \right], \quad (31)$$

where k_1 is the nearest neighbor interaction strength after the quench, and here we assume that it is fixed, i.e., $k_1 = k_0$ for the quench protocol we consider.

The time-evolved state at an arbitrary time t after the quench is given by

$$|\Psi(t)\rangle = e^{-iH_1 t} |\Psi_0\rangle = e^{-iH_1 t} |K_0\rangle. \quad (32)$$

Here $|\Psi_0\rangle$ is the state before the first quench and, as we have mentioned before, this is the first state of the Krylov basis. Here we take the ground state of the initial Hamiltonian H_0 to be the initial state $|\Psi_0\rangle = |0\rangle = \prod_{k=1}^N |0\rangle_k$.

Since we have separated the Hamiltonian into individual modes [see Eqs. (29) and (30)], we can write the time-evolved state after the quench as a product of each individual time-evolved mode

$$|\Psi(t)\rangle_k = e^{-iH_{1k} t} |0\rangle_k, \quad (33)$$

so that the SC of the state $|\Psi(t)\rangle$ is the sum of individual SCs of N such modes. Below we illustrate the calculation of SC of an individual time-evolved mode $|\Psi(t)\rangle_k$.

A. Autocorrelation function

Since the annihilation and creation operators before the quench, i.e., (a_{0k}, a_{0k}^\dagger) , respectively, are related to the operators after the quench (a_{1k}, a_{1k}^\dagger) through Bogoliubov transformations, the postquench Hamiltonian H_1 can be expressed in the following way in terms of the prequench operators as [8,41]

$$H_1 = 2 \sum_{k=1}^N \omega_{1k} [\mathcal{U}_{1k} \mathcal{V}_{1k} \mathbf{K}_k^+ + (\mathcal{U}_{1k}^2 + \mathcal{V}_{1k}^2) \mathbf{K}_k^0 + \mathcal{U}_{1k} \mathcal{V}_{1k} \mathbf{K}_k^-], \quad (34)$$

where the Bogoliubov coefficients \mathcal{U}_{jk} and \mathcal{V}_{jk} are given by

$$\mathcal{U}_{1k} = \frac{\omega_{1k} + \omega_{0k}}{2\sqrt{\omega_{1k}\omega_{0k}}}, \quad \mathcal{V}_{1k} = \frac{\omega_{1k} - \omega_{0k}}{2\sqrt{\omega_{1k}\omega_{0k}}}. \quad (35)$$

Derivations of the above two equations have been provided in many references dealing with quenches of harmonic oscillators; see, e.g., Refs. [8,41]. For details of these derivations, we refer the reader to these works and omit them here for brevity.

The operators $\mathbf{K}_k^{(+,-)}$ defined above are related to the creation and annihilation operators before the quench through

the following relations:

$$\begin{aligned}\mathbf{K}_k^+ &= \frac{1}{2}a_{0k}^\dagger a_{0k}^\dagger, & \mathbf{K}_k^0 &= \frac{1}{4}(a_{0k}^\dagger a_{0k} + a_{0k} a_{0k}^\dagger), \\ \mathbf{K}_k^- &= \frac{1}{2}a_{0k} a_{0k}.\end{aligned}\quad (36)$$

Utilizing the standard commutation relations for the bosonic operators we can see that operators $\mathbf{K}_k^{(+,-,0)}$ provide a single-mode bosonic representation of the $su(1, 1)$ Lie algebra; thus they satisfy the following commutation relations:

$$[\mathbf{K}_k^+, \mathbf{K}_k^-] = -2\mathbf{K}_k^0, \quad [\mathbf{K}_k^0, \mathbf{K}_k^\pm] = \pm\mathbf{K}_k^\pm. \quad (37)$$

The Casimir operator corresponding to the algebra, defined through the relation

$$\mathbf{K}^2 = \mathbf{K}_0^2 - \frac{1}{2}(\mathbf{K}_+ \mathbf{K}_- + \mathbf{K}_- \mathbf{K}_+), \quad (38)$$

commutes with all three generators and satisfies the eigenvalue equation

$$\mathbf{K}^2 |j, h\rangle = h(h-1) |j, h\rangle. \quad (39)$$

The constant h is known as the Bargmann index of the algebra and j takes values $0, 1, 2, \dots$. For the single-mode bosonic representation of the $su(1, 1)$ Lie algebra, h can take values $1/4$ or $3/4$ (see [100]). In this paper we take h to be $1/4$, for which the basis corresponding to a unitary irreducible representation of $su(1, 1)$ algebra is the set of states with an even number of bosons. Furthermore, the operations of the generators \mathbf{K}_i on the states $|j, h\rangle$ are given by standard formulas; see, e.g., Ref. [100].

From the identification that the operators $\mathbf{K}_k^{(+,-,0)}$ satisfy a $su(1, 1)$ algebra, we see that the Hamiltonian after the quench is actually an element of this algebra. Thus the time-evolved state is a generalized coherent state (CS) associated with the $SU(1, 1)$ group. Hence the autocorrelation function (for each mode), given by

$$\mathcal{S}_k(t) = {}_k\langle \Psi(t) | \Psi(t=0) \rangle_k = {}_k\langle 0 | e^{iH_k t} | 0 \rangle_k, \quad (40)$$

can be thought of as an overlap of a $SU(1, 1)$ CS with the ground state before the quench [101]. Our first goal in the rest of this section is to quantify the spread of the time-evolved CS with respect to the state before quench in terms of Krylov basis elements.

We first obtain an analytical formula for the autocorrelation function defined above. This is conveniently done by using the standard decomposition formula for the $su(1, 1)$ algebra (see, e.g., Ref. [102] for derivations of a collection of such well known relations). Using such decomposition formulas, we obtain the expression for a time-evolved state to be

$$\begin{aligned}|\Psi(t)\rangle_k &= \exp[A_{1k}^+(t)\mathbf{K}_k^+] \exp[\ln(A_{1k}^0(t))\mathbf{K}_k^0] \\ &\quad \times \exp[A_{1k}^-(t)\mathbf{K}_k^-] |0\rangle_k.\end{aligned}\quad (41)$$

The expressions for time-dependent functions $A_1^0(t)$ and $A_1^+(t)$ appearing in the time-evolved state above are given by²

$$\begin{aligned}A_1^0(t) &= [\cos(\omega_1 t) + i\Omega_1 \sin(\omega_1 t)]^{-2} = f_1(t)^{-2}, \\ A_1^+(t) &= i\tilde{\Omega}_1 \sin(\omega_1 t) f_1(t)^{-1},\end{aligned}\quad (42)$$

²Since we are working with single mode wave functions, from now on we remove the mode index k for the rest of this section, unless otherwise specified explicitly. The total complexity is given by the sum over all the modes.

where we have defined

$$\Omega_1 = \frac{\omega_0^2 + \omega_1^2}{2\omega_0\omega_1}, \quad \tilde{\Omega}_1 = \frac{\omega_0^2 - \omega_1^2}{2\omega_0\omega_1}. \quad (43)$$

The time-evolved state given in Eq. (41) can therefore be written in the form

$$\begin{aligned}|\Psi(t)\rangle &= (A_1^0(t))^{1/4} \sum_{j=0}^{\infty} \frac{(A_1^+(t))^j}{j!} (\mathbf{K}^+)^j |0\rangle \\ &= (A_1^0(t))^{1/4} \sum_{j=0}^{\infty} \sqrt{\frac{\Gamma(j+\frac{1}{2})}{j!\sqrt{\pi}}} (A_1^+(t))^j |j, h\rangle,\end{aligned}\quad (44)$$

where in the last expression we have used the fact that here $h = 1/4$ and the states $|j, h\rangle$ have been defined in Eq. (39). From this expression, it is then straightforward to obtain the autocorrelation function to be

$$\mathcal{S}(t) = \langle \Psi(t) | \Psi(t=0) \rangle = (A_1^0(t))^{*1/4}, \quad (45)$$

with the expression for the time-dependent function $A_1^0(t)$ given in Eq. (42) above. Note that to obtain this autocorrelation function, we have neglected a phase factor which corresponds to the difference between the ground state energies before and after the quench. This factor is usually neglected in the discussion of the characteristics function and the corresponding PDWD, as we have mentioned before. Furthermore, using this autocorrelation function, or using the coherent state method of [12,103], we obtain the LC to be

$$\begin{aligned}a_n &= (2n + \frac{1}{2})\Omega_1 \omega_1, \quad n = 0, 1, 2, \dots, \\ b_l &= \frac{1}{2}\sqrt{2(2l^2 - l)}\tilde{\Omega}_1 \omega_1, \quad l = 1, 2, 3, \dots\end{aligned}\quad (46)$$

B. Evolution of the spread complexity

Now since the postquenched time-evolved state is a $SU(1, 1)$ CS, to determine the expansion coefficients $\phi_n(t)$ required in the computation of the SC, we can use the geometric method proposed in [103] and [12], respectively, in the context of the Krylov and spread complexity. Using the procedure explained in these references, we obtain the exact expressions for the expansion coefficients to be

$$\begin{aligned}\phi_n(t) &= \mathcal{N}_n \phi_0(t) (A_1^+(t))^n, \quad \text{with } \phi_0(t) = (A_1^0(t))^{1/4}, \\ \mathcal{N}_n &= \sqrt{\frac{\Gamma(n+\frac{1}{2})}{n!\sqrt{\pi}}}.\end{aligned}\quad (47)$$

The sum in the SC expression is performed exactly and the final expression for SC of a single mode is given by (here we have restored the mode index k to emphasize that this expression represents the SC of a single mode evolution)

$$\mathcal{C}_k(t) = \frac{|\phi_{1k}(t)|^2}{(1 - \mathcal{F}_k(t))^{3/2}}, \quad \text{where } \mathcal{F}_k(t) = |A_{1k}^+(t)|^2. \quad (48)$$

Using the expression for the $A_{1k}^+(t)$ given in Eq. (42), we obtain the following simplified expression for the k th mode contribution to the total SC:

$$\mathcal{C}_k(t) = \frac{1}{2}\tilde{\Omega}_{1k}^2 \sin^2(\omega_{1k}t) = \frac{(\omega_{0k}^2 - \omega_{1k}^2)^2}{8\omega_{0k}^2 \omega_{1k}^2} \sin^2(\omega_{1k}t). \quad (49)$$

C. Lanczos coefficients and the cumulants of the distribution of the work done

We now illustrate the general relation between the LC and the cumulants derived in Sec. II for the single global quench of the harmonic chain considered in this section. In fact, since in this case the analytical expression for the autocorrelation function is relatively simple [see Eq. (45)], we can derive the first few relations between the LC and the cumulants directly without using the general relations in Eqs. (19) and (20).

First consider the cumulant β_1 which, from Eqs. (45) and (42), is obtained to be equal to $\beta_1 = \frac{1}{2}\omega_1\Omega_1$. Now from the expressions for a_n given in Eq. (46) we see that this is exactly equal to the first of a_n 's, i.e., $\beta_1 = a_0 = \langle W \rangle$.³ Similarly, we calculate the second cumulant from the expression for the autocorrelation function to be equal to $\beta_2 = \frac{1}{2}(\Omega_1^2 - 1)\omega_1^2 = \frac{1}{2}\tilde{\Omega}_1^2\omega_1^2$. On the other hand, from the expression for the coefficients b_n , we see that $b_1^2 = \beta_2 = \langle W^2 \rangle - \langle W \rangle^2$. These two relations indicate an alternative interpretation for the LC in terms of the moments of the thermodynamic quantity—the work W done on the system through the quench. In fact the first element of the first set of LC a_0 is equal to the average work done on the system through the process of sudden quench, while the first element of the second set of LC, i.e., b_1 , is the standard deviation of this work done from the average value. Similarly, it is possible to write all the higher LC as polynomials of the moments of the distribution of the work done with integer coefficients. If we want to obtain the higher order relations, it is useful to directly use the general formula Eqs. (19) and (20) given in the previous section.

Interpretation for critical quenches

To understand the above relations between the LC and various moments of the PDWD more clearly, we consider a special case of the single quench scenario, namely, we consider a critical quench where the final (or the initial) value of the frequency of the oscillators vanishes. As discussed previously in [8,42,45], when one considers a critical quench, both the Nielsen complexity and the SC show characteristic behaviors different from noncritical quenches. In particular, the divergence of the complexity at late times can be attributed to the presence of the zero modes originated through the critical quench of the system [42]. Here we discuss its connection with the divergence of the average work done on the system in such critical quenches.

First we write down the contribution of the N th mode towards the total SC. From Eq. (49), we obtain this contribution to be (the subscript N refers to the fact that these quantities correspond to mode number N)

$$C_N(t) = \frac{1}{2}\tilde{\Omega}_{1N}^2 \sin^2(\omega_{1N}t) = \frac{(\omega_{0N}^2 - \omega_{1N}^2)^2}{8\omega_{0N}^2\omega_{1N}^2} \sin^2(\omega_{1N}t), \quad (50)$$

³Note that here $\langle W \rangle$ is actually $\langle W_k \rangle$, i.e., work done on an individual mode. Since the harmonic chain is diagonalized in normal modes, the total work is the sum of works done on these individual modes. This is true for higher moments as well.

where ω_{0N} and ω_{1N} are the N th mode normal frequencies before and after the quench.

We first consider the case when the frequency λ_1 of the individual oscillators vanishes after the quench. In this case it is easy to see that the contribution of the N th mode (which is in fact a zero mode for the critical quench) grows quadratically with time, i.e.,

$$C_N(t)|_{\lambda_1 \rightarrow 0} = \frac{1}{8}\lambda_0^2 t^2. \quad (51)$$

On the other hand, in the opposite case when the frequency before the quench is zero, we see that the SC is divergent at all times. These behaviors of the SC for critical quenches can be understood from the point of view of the average work done on the system due to the quench. First we note that, when the frequency before the quench vanishes, all the LC are divergent [see Eq. (46)]. Next, from the identifications $a_0 = \langle W \rangle = \frac{1}{2}\omega_1\Omega_1$ and $b_1^2 = \langle W^2 \rangle - \langle W \rangle^2 = \frac{1}{2}\tilde{\Omega}_1^2\omega_1^2$, we see that in this case the average work as well as its variance diverge. This divergent behavior of these two quantities is very similar to what is observed when a quantum system is quenched from criticality [86].

The divergence of the average work done on the system (and the higher cumulants) is explained by observing that, when there is a zero mode present in the system before the quench, it corresponds to a free particle and the spread of the initial wave function of this free particle in an $su(1, 1)$ basis is infinity—and hence the SC diverges. Instead, when the system is quenched in such a way that the frequency after the quench is zero it does not result in any divergence in the average work done on the system, since in this case the zero mode (i.e., the free particle) results from an initially localized harmonic oscillator. Therefore, we conclude that when there is a zero mode present in the system before a sudden quench it corresponds to a divergent average work as well as a divergent SC. We also note that the growth rate of the zero mode complexity when $\lambda_1 = 0$ is proportional to the square of the initial frequency, whereas the average work corresponding to that mode is also proportional to the initial frequency. Hence, if we consider two different critical quench protocols where the initial frequencies are different, then the protocol with higher frequency will correspond to greater average work and higher growth rate of the SC. This discussion thus provides a direct connection between the growth of SC in critical quenches and the average work done on the system.

Next, we graphically study the evolution of the SC for critical quench when the frequency after the quench vanishes. In Fig. 1, we have plotted the contributions from zero mode (denoted by C_0) and the sum of the rest of the modes (denoted by C_r) towards the SC separately. We see that, for early times, just after the quench, the contribution of the zero mode towards the total complexity is smaller than some of the other modes. However, after a particular value of time (which depends on the initial value of the frequency) the zero mode complexity becomes equal to the sum of other mode contributions and continues to grow with time (whereas the total complexity of the nonzero modes oscillates with time). The quadratic growth of the zero mode SC is therefore responsible for the overall quadratic growth of the SC with time.

Before concluding this section, we notice an important point regarding the nature of the time-evolved wave

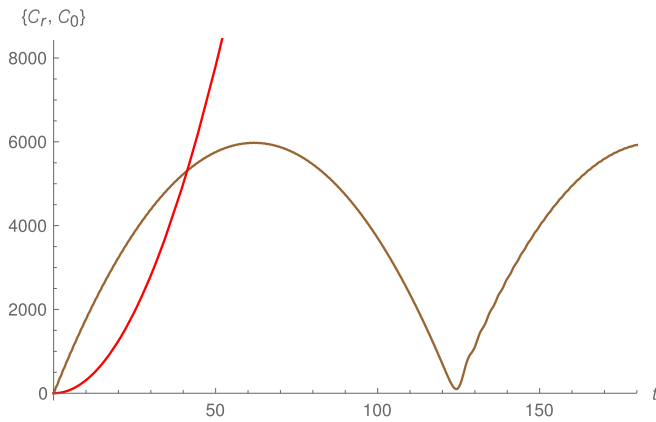


FIG. 1. Time evolution of the SC of the zero mode (red) and the sum of the nonzero mode contributions (brown). Here $\lambda_0 = 5$, $k = 2$, and $N = 350$. At large times the contribution of the zero mode always dominates.

function, the CS, and the associated LC. For the problem considered in this section—quantum quench in a harmonic chain—the time-evolved state is a generalized $SU(1, 1)$ CS. Hence the associated Krylov basis is infinite dimensional and we have an infinite number of LC a_n and b_n . Here we have explicitly related only the first few LC with the cumulants of the work distribution. On the other hand, when the time-evolved state belongs to a finite dimensional group, such as $SU(2)$, there are only finite numbers of Krylov basis elements and an associated finite number of nonzero LC. In that case, it is possible to obtain explicit relations between all the LC with the cumulants of the work distribution.

IV. MASS QUENCH OF A BOSONIC SCALAR FIELD IN D DIMENSIONS—LANCZOS COEFFICIENTS AND COMPLEXITY

In this section, we consider the mass quench of a noninteracting bosonic model in d -spatial dimensions and study corresponding cumulants of the work distribution, the LC, and the relation between them in the limit that the linear size of the system $L \rightarrow \infty$. In this case, the general formula for the CF discussed in Sec. II D is applied and the explicit time dependence of the function f_C can be obtained.

The Hamiltonian of the system under consideration is that of a bosonic field ϕ of mass m and is given by the following expression when written in a diagonalized form in terms of individual momentum modes [97,98]:

$$H(m) = \frac{1}{2} \int_k [\pi_k \pi_{-k} + \omega_k^2(m) \phi_k \phi_{-k}], \quad (52)$$

where the modes ϕ_k and their conjugates satisfy the commutation relations $[\phi_k, \pi_{k'}] = i(2\pi)^d \delta_{k,k'}$ and $\omega_k(m) = \sqrt{k^2 + m^2}$. Furthermore, for the continuum model, the integral is over all the d dimensional space $\int_k = \frac{\Omega_d}{(2\pi)^d} \int_0^\infty dk k^{d-1}$. The above Hamiltonian can be obtained, e.g., in the small interaction limit of the sine-Gordon model—one of the most popular models used to study nonlinearly interacting quantum systems.

We consider a mass quench in the model in Eq. (52), where we change the mass m_0 of the field to a new value m_1 suddenly at $t = 0$. This quench corresponds to a change in the dispersion relation, so that the mode frequencies before and after the quench are ω_{0k} and ω_{1k} , respectively. The CF corresponding to each mode (with momentum k) was obtained in [98], so that the CF is given by

$$G(t) = \prod_k G_k(t) = \prod_k e^{\frac{i}{2}(\omega_{0k} - \omega_{1k})} \sqrt{\frac{1 - \eta_k^2}{1 - \eta_k^2 e^{-2i\omega_{1k}t}}},$$

$$\text{where } \eta_k = \frac{\omega_{0k} - \omega_{1k}}{\omega_{0k} + \omega_{1k}}. \quad (53)$$

Notice that, apart from an unimportant phase factor, this CF for individual momentum modes is the same as the one we have derived in Eq. (45) for a harmonic chain. The modulus squared of the two expressions are therefore identical.

Now taking the $L \rightarrow \infty$ limit of the logarithm of the CF and replacing the sum over all the momentum modes with an integral over k , we obtain

$$\begin{aligned} \frac{\ln G(t)}{L^d} = & -\frac{it}{2} \int_k (\omega_{1k} - \omega_{0k}) + \frac{1}{2} \int_k (1 - \eta_k^2) \\ & - \frac{1}{2} \int_k (1 - \eta_k^2 e^{-2i\omega_{1k}t}). \end{aligned} \quad (54)$$

Comparing this with the general expression for the CF for a general d -dimensional system of length L given in Eq. (4), we can easily identify the expressions for the three functions f_b , f_s , and f_c . As mentioned before, in the calculation of the cumulants and the moments, the contribution of the first term coming from the phase factor in the CF will be neglected.

The expressions for the first three cumulants calculated from the above CF [see Eq. (28)] have been obtained in [98] for different spatial dimensions. For some values of d , these expressions are UV divergent. The average of the work done on the system through the quench is expressed as $\langle W \rangle = \langle 0|H(m) - H(m_0)|0 \rangle$ which, apart from a trivial shift by an amount equal to the ground state energy of the prequench Hamiltonian, is just the first LC, i.e., a_0 . Thus, once again, like the case of the harmonic chain considered in the previous section, for the bosonic field quench setup, the LC a_0 is the average work done on the system due to the quench. This is true for this model for any value of the space dimension d .

Next, as it was noticed in [98], since $\ln G(t)$ is an extensive quantity (it is proportional to L^d), all the cumulants calculated from it using the formula Eq. (4) are also extensive. Thus we can define the probability distribution for the intensive work $w = W/L^d$ and, since all the higher order cumulants apart from β_1 and β_2 go to zero as $O(\frac{1}{L^{2d}})$ or faster in the limit $L \rightarrow \infty$, the distribution $P(w)$ is Gaussian, with mean $\beta_1 = \gamma_1$ and variance $\beta_2 = \gamma_2/L^d$, where γ_1 and γ_2 are constants, independent of L . The CF $\tilde{G}(t)$ corresponding to $P(w)$ is also Gaussian and can thus be written as

$$\tilde{G}(t) = \exp \left[-it\gamma_1 - \frac{t^2}{2L^d} \gamma_2 + O\left(\frac{1}{L^{2d}}\right) \right]. \quad (55)$$

Now, using this form for the CF and the explicit expressions for the complete Bell polynomials [92] in Eq. (20), we can obtain the moments of the distribution of the intensive

work done on the system. To obtain a compact expression for these moments in the $L \rightarrow \infty$ limit, it is instructive to first consider expressions for the Bell polynomials Y_n as a function of $\beta_1, \beta_2, \dots, \beta_n$. For example, we have $Y_4(\beta_1, \beta_2, \beta_3, \beta_4) = \beta_1^4 + 6\beta_1^2\beta_2 + 4\beta_1\beta_3 + 3\beta_2^2 + \beta_4$ and $Y_5(\beta_1, \beta_2, \beta_3, \beta_4, \beta_5) = \beta_1^5 + 10\beta_2\beta_1^3 + 15\beta_2^2\beta_1 + 10\beta_3^3\beta_1^2 + 10\beta_3\beta_2 + 5\beta_4\beta_1 + \beta_5$. From these two expressions, as well as the expressions for the higher order Bell polynomials, we see that there are only two terms which are either $O(1)$ or $O(1/L^d)$, i.e., survive in the limit $L \rightarrow \infty$. The first one is equal to β_1^n and the second one is proportional to β_2 and has coefficient $\mathcal{N}_{n-2}\beta_1^{n-2}$ for a sequence of positive numbers \mathcal{N}_n . Apart from these two terms, all the other terms in the expression for the complete Bell polynomials Y_n are $O(\frac{1}{L^{2d}})$ or smaller and hence it is possible to neglect them in the $L \rightarrow \infty$ limit.

Therefore, from Eq. (20) we have the expression for the moments to be

$$M_n = (-i)^n \left[\gamma_1^n + \frac{\mathcal{N}_{n-2}}{L^d} \gamma_1^{n-2} \gamma_2 \right], \quad (56)$$

where the expression for the polynomial \mathcal{N}_n is given by $\mathcal{N}_n = \frac{1}{2}(n^2 + 2n + 3)$.

Next, to find out the LC corresponding to these moments and the autocorrelation function $\tilde{G}(t)^*$ [with $\tilde{G}(t)$ given in Eq. (55) above], we note that, as observed in [103] and [12], the Gaussian CF corresponds to a particle moving in the Weyl-Heisenberg group with the Hamiltonian of the form

$$H = \Lambda(a^\dagger + a) + \Omega a^\dagger a + \delta \mathbf{I}, \quad (57)$$

where Λ , Ω , and δ are constants. When $\Omega \ll 1$, so that in the CF contributions of $O(\Omega^2)$ and higher can be neglected, then the CF (of the particle moving in the Weyl-Heisenberg group), which is equal to the coefficient of $|K_0\rangle$ in the expansion of the time-evolved state in terms of the Krylov basis, is a Gaussian function of time of the form given in Eq. (55). In our case, we can identify, $\delta = \gamma_1$ and $\Lambda = \sqrt{\frac{\gamma_2}{L^d}}$. Furthermore, here $\Omega \approx O(\frac{1}{L^{2d}})$ and hence in the limit $L \rightarrow \infty$, $\Omega \rightarrow 0$. In this limit, it is possible to write the LC approximately as [12,103]

$$a_n \approx \gamma_1, \quad b_n = \sqrt{\frac{\gamma_2 n}{L^d}}. \quad (58)$$

In the exact expressions for the LC a_n 's for a particle moving in the Weyl-Heisenberg group, there is an additional additive term, which increases with n . However, since this term is proportional to Ω , we have neglected it here and the a_n coefficients are just constants. This term will contribute only at the $n \rightarrow \infty$ limit.

Now the SC of this bosonic field system after the mass quench in the limit $L \rightarrow \infty$ can also be obtained by taking the $\Omega \rightarrow 0$ limit of the general formula for the SC for a particle moving in the Weyl-Heisenberg group. We obtain the corresponding expression for the SC to be

$$\mathcal{C}(t)|_{L \rightarrow \infty} \approx \Lambda^2 t^2 = \frac{\gamma_2}{L^d} t^2. \quad (59)$$

This is the leading contribution to the SC, with all the subleading contributions being $O(\frac{1}{L^{2d}})$ and smaller, and thus SC grows quadratically with time after quench in noninteracting bosonic

field theory. However, since γ_2 is finite and we are considering the limit $L \rightarrow \infty$, this quadratic growth will be apparent only when we study the system a long time after the quench. Furthermore, for higher spatial dimension of the system, this growth may be smaller than that with lower spatial dimension, even at late times. This example therefore makes it clear that, whenever the PDWD of a system due to a sudden quench is Gaussian, the associated CF is also a Gaussian function of time, so that the corresponding SC grows quadratically with time, with the coefficient of the growth being proportional to the variance of the distribution. This conclusion is true irrespective of the details of the system under consideration.

This example also nicely illustrates the usefulness of connecting the SC with the PDWD. To reiterate, due to the close relationship between the CF and work distribution, the former is fixed whenever the distribution of work under a quench is specified. Hence the LC and the SC obtained from the CF are also determined and can only have the same form for different systems that might have similar work distribution under a quench. Furthermore, this simple example of quench in a noninteracting bosonic field theory can be used as the starting point for studying the SC evolution in quenches of general interacting field theories. Presumably, in the presence of interaction, one needs to consider work distribution which deviates from the Gaussian one for the noninteracting case and SC will grow with a more intricate pattern compared to the simple quadratic growth obtained here. This is an interesting problem and we leave it for a future study.

V. CONCLUSIONS AND DISCUSSIONS

In this paper, we have provided a generic relation between the statistics of work done on a quantum system under a sudden quantum quench and the LC associated with the Krylov basis constructed using the postquench Hamiltonian. By using the relation between moments of the autocorrelation function and the corresponding cumulants of the probability distribution, we have shown that it is possible to express the LC in terms of the physically measurable quantities, such as the average, variance, and higher order cumulants of the work done on the system through the quench. We believe that this should be an important step towards understanding the significance of these coefficients, specifically in a quench scenario and circuit evolution in general.

We have applied our findings to two realistic examples, with the first being the time evolution of the SC in a quenched harmonic chain with nearest neighbor interaction. We have shown that a_0 is equal to the average of the work done on the system, while b_1 represents the standard deviation of this work from the corresponding average. Using this observation, we can explain the fact that SC for time evolution under a critically quenched Hamiltonian diverges at all times, since the corresponding average work diverges as well.

Similarly, for the second example we consider—a mass quench in a bosonic scalar field theory in the limit of large system size—we verified the same relation between the LC and cumulants of the work done on the system. Since, in this limit, the probability distribution is Gaussian, the SC is seen to be growing quadratically with time. As we have discussed,

this feature shown by the SC is true whenever this distribution and hence the CF is Gaussian.

We conclude by pointing out a few potential future applications of the results presented here. First, as discussed in the Introduction, our goal is to provide a unified description of observables studied in quantum quenches by relating quantities from two different sets of such observables. Here we have illustrated one such connection between a thermodynamic quantity (work distribution) and an information theoretic measure (complexity of spread of a time-evolved state) by relating the moments of the work distribution with the LC corresponding to the postquench Hamiltonian evolution. As an application, these relations can be useful to understand the characteristics' features of SC evolution and the pattern of the LC in a chaotic system [12], as well as the phenomena of information scrambling in such systems. Furthermore, it will be interesting to see whether similar relations can be established between other information theoretic quantities which are commonly studied in quenches, such as the entanglement entropy or the out-of-time order correlator, and thermodynamic quantities, e.g., the entropy or the heat generated, etc.

Secondly, in this paper we have considered sudden quenches in bosonic systems which can be diagonalized via normal modes and, therefore, are noninteracting in nature. As a future application, one can consider quenches in realistic interacting many-body quantum systems, and quenches in fermionic field theories, and see whether the kind of relations between the work statistics and the LC that we have established here are also valid in more general quench scenarios as well [104]. This should further our understanding of the SC and LC in terms of thermodynamic quantities.

Finally, the results presented in this paper offer the first steps of understanding the significance of the LC, as well as the SC, for time evolution after a sudden quantum quench. Though here we have considered time evolution after a quantum quench only, a similar construction can be envisaged for any general circuit evolution as well. In this case, the relation between LC and the quantities analogous to the average and variance of the work done can shed light on understanding the link between the Krylov basis construction, the SC, and the geometric formulation of circuit complexity [13–16]. We hope to report on this in the near future [105].

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APPENDIX A: MOMENTS FOR QUENCH IN A SINGLE HARMONIC OSCILLATOR

In this Appendix we illustrate the use of the general formula for the moments in Eq. (7) for quench in a single harmonic oscillator with frequency ω_0 to a new frequency ω . This can then be easily generalized to the case of the quench in a harmonic chain discussed in Sec. III.

To use the formula in Eq. (7) we need to evaluate the overlap between the ground state before the quench and an arbitrary number state $|\tilde{n}\rangle$ after the quench. For a quench in a harmonic oscillator, this is given by [98]

$$\langle \tilde{n}|0\rangle = \begin{cases} \left[\frac{(-\eta)^n (n-1)!!}{\mathcal{U}} \right]^{1/2} & \text{for } n = 0, 2, 4, \dots, \\ 0 & \text{for } n = 1, 3, 5, \dots \end{cases} \quad (\text{A1})$$

Here we have defined⁴

$$\eta = \frac{\omega_0 - \omega}{\omega_0 + \omega}, \quad \mathcal{U} = \frac{\omega_0 + \omega}{2\sqrt{\omega_0\omega}}. \quad (\text{A2})$$

Substituting this overlap in Eq. (7), we have the expression for the moments of the work distribution for a quench in a single harmonic oscillator as

$$M_n = (-i)^n \sum_{j=\text{even}} \left[\frac{(\eta)^j (j-1)!!}{\mathcal{U}} \right] \left[\left(j + \frac{1}{2} \right) \omega - \frac{1}{2} \omega_0 \right]^n. \quad (\text{A3})$$

APPENDIX B: DERIVATION OF THE EXPRESSION FOR a_1

In this Appendix we briefly describe the derivation of the expressions in Eqs. (21) and (22) for the first three LC in terms of averages of various powers of the work done W . First, we write down the expressions for a_0 , a_1 , and b_1 in terms of the moments of the work distribution:

$$a_0 = iM_1, \quad b_1^2 = -M_2 + M_1^2, \quad a_1 = \frac{iM_1^3 - iM_3}{M_1^2 - M_2} - 2iM_1. \quad (\text{B1})$$

We have taken these expressions from well known tabulated relations between the moments of the LC (see, e.g., Table 4.2 on p. 37 of [50]). Now, from Eq. (3), since $G(t) = \langle e^{-iWt} \rangle$, we get the expressions for the first three moments to be

$$M_1 = -i\langle W \rangle, \quad M_2 = -\langle W^2 \rangle, \quad M_3 = i\langle W^3 \rangle. \quad (\text{B2})$$

Substituting these in Eq. (B1) above, we get the relations given in Eqs. (21) and (22). An entirely similar procedure can be used to obtain the expression for other LCs in terms of averages of various powers of the work done using the tabulated relations between the LC and the moments of the CF.

⁴For convenience here we assume that $\omega_0 < \omega$, i.e., η is negative.

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