


Proximity-induced equilibrium supercurrent and perfect superconducting diode effect due to band asymmetry

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(Received 25 September 2022; revised 22 June 2023; accepted 31 August 2023; published 18 September 2023)

We theoretically investigate the consequences of proximity-induced conventional superconductivity in metals that break time-reversal and inversion symmetries through their energy dispersion. We discover behaviors impossible in an isolated superconductor such as an equilibrium supercurrent that apparently violates a no-go theorem and, at suitable topological defects, nonconservation of electric charge reminiscent of the chiral anomaly. The equilibrium supercurrent is expected to be trainable by a helical electromagnetic field in the normal state. Remarkably, if the band asymmetry exceeds the critical current of the parent superconductor in appropriate units, we predict a perfect superconducting diode effect with diode coefficient unity. We propose toroidal metals such as UNi₄B and metals with directional scalar spin chiral order as potential platforms.

DOI: [10.1103/PhysRevB.108.094513](https://doi.org/10.1103/PhysRevB.108.094513)

I. INTRODUCTION

Nonreciprocal phenomena in superconductors (SCs) have a long history. Among diode-like systems, early examples included amplification of the luminescence of light-emitting diodes when the diode was attached to a SC [1,2]. More recently, the asymmetry in the current-voltage characteristics of noncentrosymmetric metals under a magnetic field was seen to be enhanced if the metal turned superconducting [3–5]. Recent theoretical and experimental breakthroughs in the theory and realization of superconducting and Josephson diodes [3–26], which carry immense technological potential by avoiding the enormous heating losses of semiconductor diodes, have driven fervent activity in the field. These diodes are characterized by unequal critical supercurrents in opposite directions, resulting in Ohmic and dissipationless transport, respectively, for current magnitudes between the two critical currents. Such diode effects are intimately connected to the exotic Fulde-Ferrell superconductivity, defined by finite momentum Cooper pairs in the ground state [7,13,14,18,23,27]. Another exotic nonreciprocal phenomenon entails the existence of spontaneous supercurrents in a preferred direction through Josephson junctions [26,28–52] and SCs with spin-orbit coupling in proximity to magnetism [53–58]. While details vary, all the above approaches rely crucially on one principle: broken \mathcal{T} and \mathcal{I} symmetries. Violation of these symmetries results in other peculiar phenomena, such as unusual vortex dynamics in noncentrosymmetric SCs [59–62].

In this paper, we revisit the problem of nonreciprocity in superconducting systems and explore it in a minimal scenario. Specifically, we consider metals with an asymmetric dispersion $\varepsilon_k \neq \varepsilon_{-k}$ and no Berry phases, proximity couple them to a conventional s -wave SC, and focus on a uniform system without any Josephson junctions. We dub metals with $\varepsilon_k \neq \varepsilon_{-k}$ *band asymmetric metals* (BAMs) and refer to BAMs

that acquire conventional superconductivity as band asymmetric superconductors (BASCs). Since \mathbf{k} is inequivalent to $-\mathbf{k}$ in BAM, intrinsic pairing tendencies in them, if any, are expected to be towards exotic Fulde-Ferrell superconductivity built from finite momentum Cooper pairs [63]. On the other hand, band asymmetry eliminates a Cooper instability at weak interactions, so a more practical route to superconductivity in BAMs may be extrinsic. We show that even this minimal setup leads to strange behaviors impossible in isolated SCs, namely, (i) an apparent violation of a basic no-go theorem due to an equilibrium current density, (ii) a perfect superconducting diode effect (SDE) without fine-tuning, and (iii) topological defects that violate charge conservation.

II. EQUILIBRIUM SUPERCURRENT

We first derive the equilibrium current I^{eq} in a one-dimensional (1D) BAM. Generalization to higher dimensions is straightforward. While spontaneous supercurrents have been studied before [26,28–58], their significance with respect to basic quantum mechanics has not been appreciated, which we do here. In particular, we show how it naively violates a theorem by Bloch that forbids current densities in the thermodynamic limit in arbitrary systems of interacting fermions [64–68], and then resolve the paradox.

We assume a single band with degeneracy g ; for spin-degenerate bands, $g = 2$. The Bloch Hamiltonian for such a BAM is

$$H_{\text{BAM}} = \int_k \sum_{n=1}^g c_{kn}^\dagger c_{kn} \varepsilon_k, \quad (1)$$

where $\int_k \equiv \int \frac{dk}{2\pi}$. Let us deposit the BAM wire on a conventional s -wave SC with zero Cooper pair momentum, as sketched in Fig. 1. The BAM will develop



FIG. 1. Depositing a BAM wire on a conventional SC will generate an equilibrium current I^{eq} , a SDE in general, and a perfect SDE with unit diode coefficient if the band asymmetry exceeds a threshold determined by the critical Cooper pair momentum of the parent SC. (See text for details.)

conventional superconductivity too via the proximity effect. The Bogoliubov-deGennes Hamiltonian in the basis $\Psi_k = (c_k^T, \mathcal{T}c_k^\dagger \mathcal{T}^{-1})^T$ is $H_{\text{BdG}} = \frac{1}{2} \sum_k \Psi_k^\dagger (H_k^\Delta \otimes \mathbb{I}_g) \Psi_k$, where

$$H_k^\Delta = \begin{pmatrix} \varepsilon_k & \Delta_0^* \\ \Delta_0 & -\varepsilon_{-k} \end{pmatrix} \quad (2)$$

and \mathbb{I}_g is a $g \times g$ identity matrix. I^{eq} is given by

$$I^{\text{eq}} = \int_k \text{Tr} \{ j_k [f(H_k^\Delta) - f(H_k^0)] \}, \quad (3)$$

where $j_k = \frac{e}{2} \begin{pmatrix} v_k & 0 \\ 0 & -v_{-k} \end{pmatrix} \otimes \mathbb{I}_g$ is the current operator, $f(X) = [e^{X/T} + 1]^{-1}$, and we have set $\hbar = k_B = 1$. We have explicitly subtracted a spurious current due to Hilbert space doubling that captures the current carried by the filled bands when $\Delta = 0$. This current vanishes in general lattice models and in continuum models with a symmetric dispersion. However, in an asymmetric continuum, it is nonzero, regularization dependent, and can even diverge. For weak pairing, we find

$$I^{\text{eq}} \approx ge|\Delta_0|^2 \int_k \frac{v_{-k}}{(\varepsilon_k + \varepsilon_{-k})^2} \tanh \left[\frac{\varepsilon_k}{2T} \right] \quad (4)$$

to leading order in Δ_0 . I^{eq} is generically nonzero as long as $\varepsilon_k \neq \varepsilon_{-k}$. To gain more insight into this result, suppose the BAM has Fermi momenta K_i and Fermi velocities v_i . Linearizing the dispersion as $\varepsilon_{K_i+p} \approx v_i p$, $\varepsilon_{-K_i+p} \approx \varepsilon_{-K_i}$ and assuming $|\varepsilon_{-K_i}| \gg \Delta$, where Δ is an energy cutoff, gives

$$I^{\text{eq}} \approx \frac{ge|\Delta_0|^2 \Lambda^2}{2\pi} \sum_i \frac{v_{-K_i}}{|v_i| \varepsilon_{-K_i}^3} \quad (5)$$

for $T \rightarrow 0$. If we assume $\sum_i \frac{\Lambda^2 v_{-K_i}}{|v_i| \varepsilon_{-K_i}^3} \sim 10^{-9}/\text{eV}$, which amounts to a one-part-per-million band asymmetry if Δ and ε_{-K_i} are each $O(\text{meV})$ and all Fermi velocities are of the same order, then $\Delta_0 \sim 1\text{K}$ gives a large $I^{\text{eq}} \sim 10\text{mA}$, which should be detectable via the magnetic fields it creates.

The above current seems to contradict a seminal theorem by Bloch, which states that the ground or equilibrium state of a generic, interacting fermionic system cannot carry a current density [64–68]. In particular, a recent refinement of the theorem showed that the current density along x is bounded as $|J_x| < O(L_x^{-1})$, where L_x is the linear dimension in the x direction [69]. Historically, Bloch's theorem helped prove that persistent currents in isolated superconducting and

metallic [70,71] rings necessarily occur in excited states and are stabilized by the quantization of magnetic flux piercing the ring. Thus, the persistent currents there have a long lifetime that is limited only by the probability of spontaneous or stimulated emission that relaxes them to the ground state. In contrast, BASCs clearly carry a ground-state current with a truly infinite lifetime, apparently evading Bloch's theorem. The spontaneous supercurrents described in Refs. [53–58] are special cases of I^{eq} . However, I^{eq} differs fundamentally from spontaneous currents in \mathcal{T} and \mathcal{I} , breaking Josephson junctions that crucially rely on the presence of a junction and decay exponentially with junction thickness [26,28–52] while I^{eq} is independent of the length of the BAM wire.

The resolution to the paradox lies in the observation that Bloch's theorem explicitly assumes charge conservation whereas the BASC can freely exchange pairs of electrons with the parent SC. Viewed differently, the BAM-plus-SC system conserves charge, obeys Bloch's theorem, and indeed has a vanishing current density in the thermodynamic limit. However, the BASC alone can host a nonzero current density, which physically corresponds to a surface current for the combined system and is not suppressed by Bloch's theorem. Yet another interpretation of the result is that the superconducting instability of an isolated BAM is towards a finite momentum state. In other words, a $q \neq 0$ pairing state minimizes the Ginzburg-Landau free energy or, equivalently, solves the superconducting mean-field equations self-consistently, of the isolated BAM wire with suitable interactions at low temperatures. Then, the induced $q = 0$ pairing state can be viewed as an excited state of an isolated superconducting BAM and is therefore not restricted by Bloch's theorem.

It is instructive to contrast the above current with topological boundary phenomena. In particular, topological condensed matter physics is rife with phenomena that are forbidden in isolated systems, but occur robustly on the boundaries of topological phases. There, the violation of the relevant no-go theorems on one boundary is cured by the opposite boundary. From this perspective, the above current is a *nontopological* phenomenon that is forbidden in an isolated SC, but occurs robustly on the surface of a conventional SC. Here, the apparent violation of the relevant no-go theorem is rectified by the parent SC that acts as an infinite reservoir of Cooper pairs.

III. PERFECT SDE

We now argue that the above system realizes a perfect superconducting diode for large enough band asymmetry with a diode coefficient at its theoretical maximum, $\eta = 1$, while small band asymmetry still results in a nonzero η . Unlike, for instance, Ref. [23], where perfect diode behavior requires fine-tuning to a tricritical point, the perfect behavior here appears immediately once the band asymmetry exceeds a threshold. This remarkable behavior directly aligns with the central pursuit of the field of achieving a large η . While experimental nonidealities such as contact resistance will undoubtedly reduce η in our proposal, the fact that the ideal scenario predicts $\eta = 1$ without fine-tuning is exciting. In comparison, the largest η experimentally achieved so far is $\eta \approx 0.35$ in a heterostructure of β -Sn superconducting nanowires embedded in α -Sn Dirac semimetal [72].

The proof of the perfect SDE is as follows. The ground state supports a nonzero current I^{eq} through the BAM wire mediated by $q = 0$ Cooper pairs in the parent SC. Thus, driving a different dissipationless current through the BAM wire will require $q \neq 0$ Cooper pairs. Explicitly, the generalizations of Eqs. (2) and (3) are

$$H_k^\Delta(q) = \begin{pmatrix} \varepsilon_{k+q} & \Delta_q^* \\ \Delta_q & -\varepsilon_{-k} \end{pmatrix}, \quad (6)$$

$$I(q) = \int_k \text{Tr}\{j_k(q)(f[H_k^\Delta(q)] - f[H_k^0(q)])\}, \quad (7)$$

where Δ_q is the pairing amplitude associated with momentum- q Cooper pairs and $j_k(q) = \frac{e}{2} \begin{pmatrix} v_{k+q} & 0 \\ 0 & -v_{-k} \end{pmatrix} \otimes \mathbb{I}_g$. Since the parent SC is conventional, we expect $|\Delta_q| = |\Delta_{-q}|$ and $\Delta_q = 0$ when $|q|$ exceeds a critical value q_c . For weak pairing, the generalization of Eq. (4) is

$$\begin{aligned} I(q) &= -ge|\Delta_q|^2 \frac{d}{dq} \int_k \frac{\tanh\left(\frac{\varepsilon_{k+q}}{2T}\right) + \tanh\left(\frac{\varepsilon_{-k}}{2T}\right)}{2(\varepsilon_{k+q} + \varepsilon_{-k})} \\ &\equiv -ge|\Delta_q|^2 F'(q). \end{aligned} \quad (8)$$

Note that $F(q)$ depends purely on the normal state band structure. It peaks when q is such that k and $k + q$ are distinct Fermi points. Consequently, $F'(q)$ changes signs when q connects a pair of Fermi momenta. For instance, in a minimal 1D dispersion with a single left (right) mover with Fermi momentum, $-K_L$ (K_R), $F'(q^*) = 0$, where $q^* = K_R - K_L$. We demonstrate this property in Figs. 2(c) and 2(d) for a lattice dispersion of the form $\varepsilon_k = -2t \cos k - 2t' \sin(2k + \theta) - \mu$, which corresponds to ordinary nearest-neighbor and complex second-neighbor hopping. $F(q)$ is analyzed more closely in the Appendix B.

The implication of this behavior for the SDE, illustrated in Figs. 2(e) and 2(f), is the following: If $q_c < |q^*|$, a current I such that $0 < I < I^{\text{eq}}$ (we choose the convention $I^{\text{eq}} > 0$) will be nondissipative and be carried by Cooper pairs in the BAM with the appropriate q , whereas no value of q can accommodate a negative supercurrent. Thus, the critical currents are $I_c^+ = I^{\text{eq}}$ and $I_c^- = 0$, and the diode coefficient $\eta = \frac{I_c^+ - I_c^-}{I_c^+ + I_c^-} = 1$.

We emphasize that this reasoning for the perfect SDE is immune to the specific form of Δ_q as long it vanishes beyond a critical value of $|q|$. In fact, the perfect SDE will persist even if the parent SC inherits a slight asymmetry due to the BAM and acquires unequal critical momenta, $q_c^+ \neq q_c^-$, provided $|q^*| > \max q_c^\pm$. Then, $I(q)$ has the same sign $\forall q$ for which $\Delta_q \neq 0$.

On the other hand, if $q_c > |q^*|$, then Cooper pairs with momentum q such that $|q^*| < |q| < q_c$ will enable negative supercurrents in the BAM and yield a diode coefficient $0 < \eta < 1$. Naively, if a supercurrent vanishes at a certain q and has a negative slope at that point—as happens for $q = q^*$ in Fig. 2(f)—the superconducting phase is rendered unstable as the the corresponding free energy reaches a local maximum or saddle point [73]. However, this is only true in intrinsic SCs; for proximity-induced superconductivity in the BASC, the parent SC effectively provides a training field in Nambu pseudospin space that creates a nonzero $\Delta(q)$ proportional to the pairing amplitude in the parent SC. As a result, the BASC

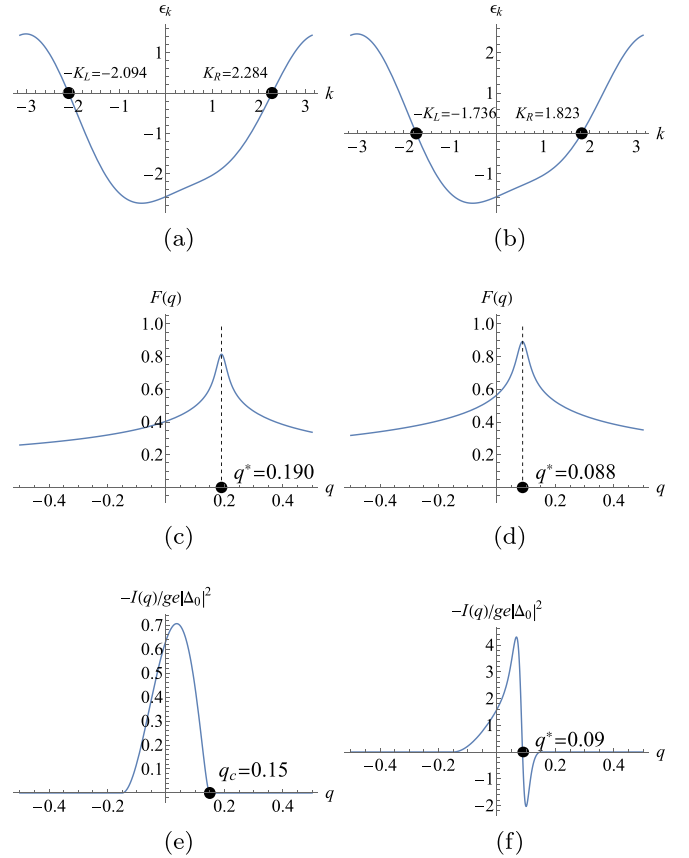


FIG. 2. An asymmetric dispersion (a), (b), the corresponding $F(q)$ (c), (d) and the resulting $I(q)$ following Eq. (8). We use $\varepsilon_k = -2 \cos k + 0.5 \sin(2k + \pi/3) - \mu$ with $\mu = 1$ for (a), (c), (e) and $\mu = 0$ for (b), (d), (f), set $T = 0.01$, and phenomenologically choose $\Delta_q = \Delta_0(1 - q^2/q_c^2)$ with $q_c = 0.15$. Note that $q^* = K_R - K_L$ to very good accuracy in both columns. When $|q^*| > q_c$ (left column), $I(q)$ is always positive and vanishes at $q = q_c$, resulting in a perfect SDE. If $|q^*| < q_c$, $I(q) < 0$ for $|q| \in (|q^*|, q_c)$ and the SDE is imperfect.

does not extremize the free energy of an isolated BAM and can remain stable as long as the parent SC is well-behaved.

IV. TOPOLOGICAL DEFECTS

In d dimensions, a BAM would naturally be described by a vector order parameter with symmetries of velocity or momentum. For instance, at low energies compared to the bandwidth, an intuitive choice for an order parameter is the average Fermi momentum. Alternately, a real-space quantity with the same symmetries as \mathbf{Q} that has gained recent interest is the toroidal moment, $\sim \mathbf{r} \times \mathbf{m}$, where \mathbf{r} is a position vector and \mathbf{m} is a magnetic moment [74–80]. Yet another choice is a spin chiral order along a preferred direction [81,82]. On purely symmetry grounds, \mathbf{Q} can couple to electromagnetic fields as $\mathbf{Q} \cdot (\mathbf{E} \times \mathbf{B})$ and, hence, can be trained by mutually perpendicular electric and magnetic fields. We now investigate the effects of topological defects in \mathbf{Q} on the equilibrium current density \mathbf{J}^{eq} and demonstrate an anomalous nonconservation of electric charge.

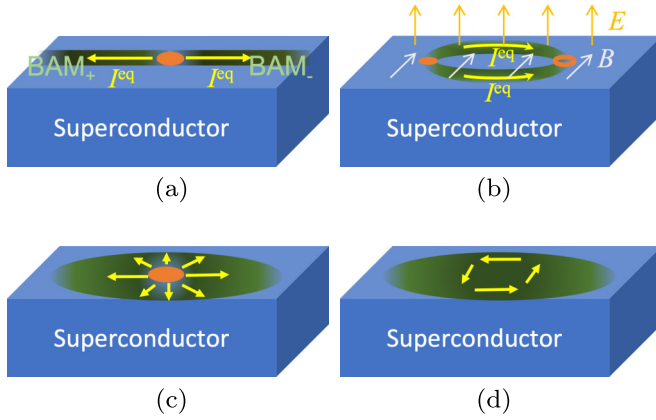


FIG. 3. (a) Domain wall (orange dot) in the BAM leads to charge nonconservation in the BASC due to opposite I^{eq} on either side of the domain wall. Here, BAM_{\pm} schematically denotes 1D BAMs with opposite Q_x . (b) Applying a spatially uniform $\mathbf{E} \times \mathbf{B}$ on a ring geometry drives currents in the same average direction in two halves of the ring (left to right in the figure), which results in a pair of domain walls that emit (orange dot) and absorb (orange circle) charge. (c) An electric vortex with $\nabla \cdot \mathbf{Q} \neq 0$ also acts as a charge source or sink based on the sign of $\nabla \cdot \mathbf{Q}$ while (d) a magnetic vortex with $\nabla \cdot \mathbf{Q} = 0$ does not exhibit charge nonconservation.

Suppose \mathbf{Q} has a domain wall, $\mathbf{Q}(\mathbf{r}) = Q_0 \Theta(x) \hat{\mathbf{x}}$. Then the $x > 0$ and $x < 0$ regions will carry opposite I^{eq} [Fig. 3(a)], so $x = 0$ will be a source or sink of electric charge depending on the directions of I^{eq} in the two regions. Interestingly, a uniform $\mathbf{E} \times \mathbf{B}$ field will create a pair of domain walls which will result in a charge source and a sink in the BASC [Fig. 3(b)]. Similarly, an electric vortex $\mathbf{Q}(\mathbf{r}) = Q_0 \frac{x\hat{\mathbf{x}} + y\hat{\mathbf{y}}}{\sqrt{x^2 + y^2}}$ also acts as a source/sink of charge [Fig. 3(c)], but a magnetic vortex $\mathbf{Q}(\mathbf{r}) = Q_0 \frac{y\hat{\mathbf{x}} - x\hat{\mathbf{y}}}{\sqrt{x^2 + y^2}}$ does not [Fig. 3(d)]. In general, time independence of physical quantities at equilibrium means the continuity equation for charge conservation is violated when

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J}^{\text{eq}} \propto \nabla \cdot \mathbf{Q} \neq 0. \quad (9)$$

This is an anomalous charge nonconservation that resembles the chiral anomaly in 1D quantum Hall edges and 3D Weyl semimetals [83–91]. Like the chiral anomaly, charge depleted from one region appears in a different region that could be macroscopically far away. Also, the violation is enabled by a charge reservoir in both cases—the parent SC for the BASC and a bulk insulator for the chiral anomaly. On the other hand, the charge nonconservation here differs from that in the chiral anomaly in crucial ways. First, it can occur in any number of dimensions in principle—including 2D, where a chiral anomaly is absent. Moreover, it depends on material details whereas the chiral anomaly is determined by universal constants e and \hbar .

Such anomalous nonconservation of charge does not occur in isolated SCs. Even though their mean-field condensates violate charge conservation, the nature of the violation is very different. In particular, they break gauge symmetry spontaneously, which leads to charge nonconserving microscopic processes such as Andreev reflection at an interface with a

nonsuperconducting material. However, the actual bulk material still conserves particle number and obeys the continuity equation. Inhomogeneities do create local currents in the equilibrium state, for example, around SC vortices. However, the continuity equation and time independence of equilibrium ensure that such currents necessarily form loops and are divergence-free. Thus, they do not contain local sources or sinks or charge, let alone a mechanism for pumping charge nonlocally over macroscopic distances. The latter is a unique property of BASCs.

V. EXPERIMENTAL PLATFORMS

BAMs are the generic low-energy limit of any metal that breaks \mathcal{T} and \mathcal{I} . This encompasses well-studied systems where \mathcal{I} is broken by spin-orbit coupling or ferroelectricity and \mathcal{T} is broken by a Zeeman field or magnetic order. It also includes an emerging family of metals with toroidal order such as UNi_4B , where \mathcal{T} and \mathcal{I} are broken by a vector order parameter but $\mathcal{T}\mathcal{I}$ is preserved [75–81]. Finally, itinerant electrons in the background of certain unidirectional spin chiral orders are BAMs too [81,82]. In principle, any of these systems proximity coupled to a conventional SC should exhibit the phenomena discussed in this paper, as the essential ingredient is an order parameter with the symmetries of $\mathbf{E} \times \mathbf{B}$. However, determining the ideal platform with a large effect will require a more sophisticated study of the proximity effect from a parent SC that is left for future work.

VI. SUMMARY

We have studied proximity-induced conventional superconductivity in metals with asymmetric dispersions, which can be viewed as the low-energy limit of generic metals that break \mathcal{T} and \mathcal{I} symmetries. We showed that the resulting SC carries a persistent equilibrium supercurrent that causes topological defects in the band asymmetry to act as sources and sinks of charge, both of which are absent in isolated SCs. For large enough band asymmetry, the SC also exhibits a perfect SDE while smaller asymmetry still gives a nonzero SDE. This work reveals strange behaviors in systems that would normally be considered ordinary, as they are at equilibrium, lack band Berry phases, and acquire conventional, s -wave superconductivity.

ACKNOWLEDGMENTS

We acknowledge useful discussions with L. Hao, J. Wei, K. Chen, and B. Karki. This work was supported by the Department of Energy, Office of Basic Energy Sciences under Grant No. DE-SC0022264.

APPENDIX A: DERIVATION OF I^{eq} AND $I(q)$

1. I^{eq}

We begin with

$$I^{\text{eq}} = \int_k \text{Tr} \{ j_k [f(H_k^\Delta) - f(H_k^0)] \}, \quad (\text{A1})$$

where $H_k^\Delta = \begin{pmatrix} \varepsilon_k & \Delta_0^* \\ \Delta_0 & -\varepsilon_{-k} \end{pmatrix}$, $j_k = \frac{e}{2} \begin{pmatrix} v_k & 0 \\ 0 & -v_{-k} \end{pmatrix} \otimes \mathbb{I}_g$ is the current operator, $f_k \equiv \int \frac{dk}{2\pi}$, $f(X) = [e^{X/T} + 1]^{-1}$. It is easiest to evaluate the trace in the eigenbasis of H_k^Δ . This gives

$$I^{\text{eq}} = \frac{ge}{4} \int_k \sum_{n=\pm} f(E_k^n) \langle \psi_k^n | (v_k - v_{-k}) + (v_k + v_{-k}) \tau_z | \psi_k^n \rangle - (\Delta = 0 \text{ contribution}) \quad (\text{A2})$$

$$= \frac{ge}{4} \int_k \sum_{n=\pm} f(E_k^n) \left[(v_k - v_{-k}) + \frac{n(v_k + v_{-k})}{\sqrt{1 + \left| \frac{2\Delta_0}{\varepsilon_k + \varepsilon_{-k}} \right|^2}} \right] - (\Delta = 0 \text{ contribution}), \quad (\text{A3})$$

where τ_z is a Nambu Pauli matrix and $H_k^\Delta | \psi_k^\pm \rangle = E_k^\pm | \psi_k^\pm \rangle$ with $E_k^\pm = \frac{\varepsilon_k - \varepsilon_{-k}}{2} \pm \text{sgn}(\varepsilon_k + \varepsilon_{-k}) \sqrt{\left(\frac{\varepsilon_k + \varepsilon_{-k}}{2} \right)^2 + |\Delta_0|^2}$. Thanks to particle-hole symmetry, $E_k^\pm = -E_{-k}^\mp$, changing $k \rightarrow -k$ in the $n = -$ term simplifies this to

$$I^{\text{eq}} = -\frac{ge}{4} \int_k \tanh\left(\frac{E_k^+}{2T}\right) \left[(v_k - v_{-k}) + \frac{v_k + v_{-k}}{\sqrt{1 + \left| \frac{2\Delta_0}{\varepsilon_k + \varepsilon_{-k}} \right|^2}} \right] - (\Delta = 0 \text{ contribution}). \quad (\text{A4})$$

To leading order in Δ_0 ,

$$I^{\text{eq}} = -\frac{ge|\Delta_0|^2}{2} \int_k \left[\frac{v_k}{2T(\varepsilon_k + \varepsilon_{-k})} \text{sech}^2\left(\frac{\varepsilon_k}{2T}\right) - \frac{v_k + v_{-k}}{(\varepsilon_k + \varepsilon_{-k})^2} \tanh\left(\frac{\varepsilon_k}{2T}\right) \right]. \quad (\text{A5})$$

Integrating the first term by parts further reduces this to

$$I^{\text{eq}} = ge|\Delta_0|^2 \int_k \frac{v_{-k}}{(\varepsilon_k + \varepsilon_{-k})^2} \tanh\left(\frac{\varepsilon_k}{2T}\right), \quad (\text{A6})$$

which is the expression in the main paper.

2. $I(q)$

The starting point now is

$$H_k^\Delta(q) = \begin{pmatrix} \varepsilon_{k+q} & \Delta_q^* \\ \Delta_q & -\varepsilon_{-k} \end{pmatrix}, \quad (\text{A7})$$

$$I(q) = \int_k \text{Tr} \{ j_k(q) (f[H_k^\Delta(q)] - f[H_k^0(q)]) \}, \quad (\text{A8})$$

where $j_k(q) = \frac{e}{2} \begin{pmatrix} v_{k+q} & 0 \\ 0 & -v_{-k} \end{pmatrix} \otimes \mathbb{I}_g$. Particle-hole symmetry now reads $E_{k+q/2}^\pm = -E_{-k+q/2}^\mp$. Thus, it is convenient to shift $k \rightarrow k - q/2$ and change $k \rightarrow -k$ in the $n = -$ term. Paralleling the steps used for I^{eq} yields

$$I(q) = ge|\Delta_q|^2 \int_k \frac{v_{-k+q/2} \tanh\left(\frac{\varepsilon_{k+q/2}}{2T}\right)}{(\varepsilon_{k+q/2} + \varepsilon_{-k+q/2})^2} \quad (\text{A9})$$

to leading order in Δ_q , which clearly equals I^{eq} at $q = 0$. To shed light on the q dependence, we use the freedom in redefining the integration variable k to rewrite $I(q)$ as

$$I(q) = ge|\Delta_q|^2 \int_k \frac{v_{-k+q} \tanh\left(\frac{\varepsilon_k}{2T}\right)}{(\varepsilon_k + \varepsilon_{-k+q})^2} \quad (\text{A10})$$

$$= -ge|\Delta_q|^2 \frac{d}{dq} \int_k \frac{\tanh\left(\frac{\varepsilon_k}{2T}\right)}{\varepsilon_k + \varepsilon_{-k+q}} \quad (\text{A11})$$

$$= -\frac{ge|\Delta_q|^2}{2} \frac{d}{dq} \int_k \frac{\tanh\left(\frac{\varepsilon_{k+q}}{2T}\right) + \tanh\left(\frac{\varepsilon_{-k}}{2T}\right)}{\varepsilon_{k+q} + \varepsilon_{-k}} \quad (\text{A12})$$

$$= -ge|\Delta_q|^2 F'(q), \quad (\text{A13})$$

as given in the main paper.

APPENDIX B: EVALUATION OF $F(q)$

We have

$$F(q) = \frac{1}{4T} \int_k \frac{\tanh\left(\frac{\varepsilon_{k+q/2}}{2T}\right) + \tanh\left(\frac{\varepsilon_{-k+q/2}}{2T}\right)}{\frac{\varepsilon_{k+q/2} + \varepsilon_{-k+q/2}}{2T}} \quad (\text{B1})$$

$$= T \int_k \frac{\sinh\left(\frac{\varepsilon_{k+q/2} + \varepsilon_{-k+q/2}}{2T}\right)}{(\varepsilon_{k+q/2} + \varepsilon_{-k+q/2}) [\cosh\left(\frac{\varepsilon_{k+q/2} + \varepsilon_{-k+q/2}}{2T}\right) + \cosh\left(\frac{\varepsilon_{k+q/2} - \varepsilon_{-k+q/2}}{2T}\right)]}. \quad (\text{B2})$$

This integral is sharply peaked at $\varepsilon_{k+q/2} = \varepsilon_{-k+q/2}$. Let us refer to the points that satisfy $\varepsilon_{k+q/2} = \varepsilon_{-k+q/2}$ as K_i and the corresponding energies as ε_i . Note that K_i and ε_i depend on q . We can evaluate $F(q)$ using Laplace's method by defining

$$G_q(k) = \ln \left\{ \frac{\tanh\left(\frac{\varepsilon_{k+q/2}}{2T}\right) + \tanh\left(\frac{\varepsilon_{-k+q/2}}{2T}\right)}{\frac{\varepsilon_{k+q/2} + \varepsilon_{-k+q/2}}{2T}} \right\}, \quad (\text{B3})$$

so

$$F(q) \approx \sum_i \sqrt{\frac{2\pi}{|G_q''(K_i)|}} \frac{\tanh\left(\frac{\varepsilon_i}{2T}\right)}{2\varepsilon_i}. \quad (\text{B4})$$

Straightforward algebra yields

$$G'_q(K_i) = -\frac{v_{K_i+q/2}^2 + v_{-K_i+q/2}^2}{4T^2 \cosh^2\left(\frac{\varepsilon_i}{2T}\right)} + \frac{v'_{K_i+q/2} + v'_{-K_i+q/2}}{2T} \left[\frac{1}{\sinh\left(\frac{\varepsilon_i}{T}\right)} - \frac{1}{\left(\frac{\varepsilon_i}{T}\right)} \right] + \left(\frac{v_{K_i+q/2} - v_{-K_i+q/2}}{2T} \right)^2 \left[\frac{1}{\left(\frac{\varepsilon_i}{T}\right)^2} - \frac{1}{\sinh^2\left(\frac{\varepsilon_i}{T}\right)} \right] \quad (\text{B5})$$

$$\approx -\frac{v'_{K_i+q/2} + v'_{-K_i+q/2}}{2\varepsilon_i} \text{ for } |\varepsilon_i| \gg T, |q(v_{K_i} - v_{-K_i})|. \quad (\text{B6})$$

Thus,

$$F(q) \approx \sum_i \sqrt{\frac{\pi}{|\varepsilon_i(v'_{K_i+q/2} + v'_{-K_i+q/2})|}} \text{ for } |\varepsilon_i| \gg T, |q(v_{K_i} - v_{-K_i})|. \quad (\text{B7})$$

Clearly, $F(q)$ is maximum when ε_i , i.e., the momenta K_i are Fermi momenta. At larger ε_i , it decays as $1/\sqrt{\varepsilon_i}$.

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