Full counting statistics across the entanglement phase transition of non-Hermitian Hamiltonians with charge conservation

Tian-Gang Zhou [1] Yi-Neng Zhou [1] and Pengfei Zhang [2,3,*]
[1] *Institute for Advanced Study, Tsinghua University, Beijing 100084, China*
[2] *Department of Physics, Fudan University, Shanghai 200438, China*
[3] *Shanghai Qi Zhi Institute, AI Tower, Xuhui District, Shanghai 200232, China*

(Received 1 March 2023; accepted 12 September 2023; published 22 September 2023)

Performing quantum measurements produces not only the expectation value of a physical observable O but also the probability distribution $P(o)$ of all possible outcomes o. The full counting statistics (FCS) $Z(\phi, O) \equiv \sum_o e^{i\phi o} P(o)$, a Fourier transform of this distribution, contains the complete information of the measurement outcome. In this work, we study the FCS of Q_A, the charge operator in subsystem A, for one-dimensional systems described by non-Hermitian Sachdev-Ye-Kitaev-like models, which are solvable in the large-N limit. In both the volume-law entangled phase for interacting systems and the critical phase for noninteracting systems, the conformal symmetry emerges, which gives $F(\phi, Q_A) \equiv \ln Z(\phi, Q_A) \sim \phi^2 \ln |A|$. In short-range entangled phases, the FCS shows area-law behavior which can be approximated as $F(\phi, Q_A) \sim (1 - \cos\phi)|\partial A|$ for $\zeta \gg J$, regardless of the presence of interactions. Our results suggest the FCS is a universal probe of entanglement phase transitions in non-Hermitian systems with conserved charges, which does not require the introduction of multiple replicas. We also discuss the consequences of discrete symmetry, long-range hopping, and generalizations to higher dimensions.

DOI: 10.1103/PhysRevB.108.094308

I. INTRODUCTION

Unitary evolution and projective measurements are basic building blocks of quantum operations. Recent studies unveiled novel entanglement phase transitions driven by their competition in systems under repeated measurements [1–22]. For small (large) measurement rates, the steady state in a typical quantum trajectory is volume law (area law) entangled. Later, it was realized that entanglement transitions also exist in general nonunitary dynamics. In particular, Sachdev-Ye-Kitaev (SYK) large-N solvable models with non-Hermitian Hamiltonians have been proposed, in which the transition of the second Rényi entropy is mapped to a transition of classical spins [23–28]. The corresponding order parameter is the quantum correlation G^{ud} between forward and backward evolution branches on the (replicated) Keldysh contour: the Rényi entropy can be expressed as a correlator of the replica twist operator $\mathcal{T}_r$. For interacting systems, the spin model is Z_4 symmetric. When $G^{ud} \neq 0$, the spin model is in the ordered phase. The insertion of $\mathcal{T}_r$ excites a domain wall. This leads to a volume-law entangled phase. In noninteracting systems, the Z_4 symmetry is promoted to an $O(2)$ symmetry. Consequently, the domain wall is replaced by a half-vortex pair, which gives rise to logarithmic entanglement entropy. For $G^{ud} = 0$, the spin model is disordered regardless of the presence of interactions, which corresponds to an area-law entangled phase.

On the other hand, it is known that the existence of conserved charges plays an important role in the many-body dynamics of quantum information [29–38]. For example, the evolution of the out-of-time correlator shows a power-law tail due to the charge diffusion in systems with $U(1)$ symmetry [29,30]. In this work, we explore the signature of conserved charges across the entanglement phase transition in large-N non-Hermitian complex SYK chains. We compute the steady-state full counting statistics (FCS) [39–54]

$$Z(\phi, Q_A) = \lim_{t\to\infty} \mathrm{tr}[\rho(t) e^{i\phi Q_A}] \equiv e^{-F(\phi, Q_A)}. \qquad (1)$$

Here, $Q_A = \sum_{x\in A} Q_x$ is the total charge in subsystem A, which will be described more precisely later. We choose the convention that $\phi \in (-\pi, \pi]$. The FCS of charge operators is also known as the disorder parameter in [55–57]. We mainly focus on initial states described by the thermofield double (TFD) state [58,59], which has been widely studied in both high-energy and condensed matter physics. The FCS, which takes the form of a generating function, contains the complete information about charge fluctuations in subsystem A. A series of works observed that charge fluctuations and charge statistics are closely related to entanglement entropy [40–43]. Also, entanglement entropy and FCS work well to characterize bulk and edge spectrum problems [48,49]. Our work provides another perspective to check the profound relation between FCS and entanglement entropy.

We show that in our setup, FCS can also be viewed as a correlator of twist operators $\mathcal{T}_c$, which now generates a relative phase rotation in the charge $U(1)$ group between branches with forward and backward evolutions on the Keldysh

*pengfeizhang.physics@gmail.com

 ©2023 American Physical Society

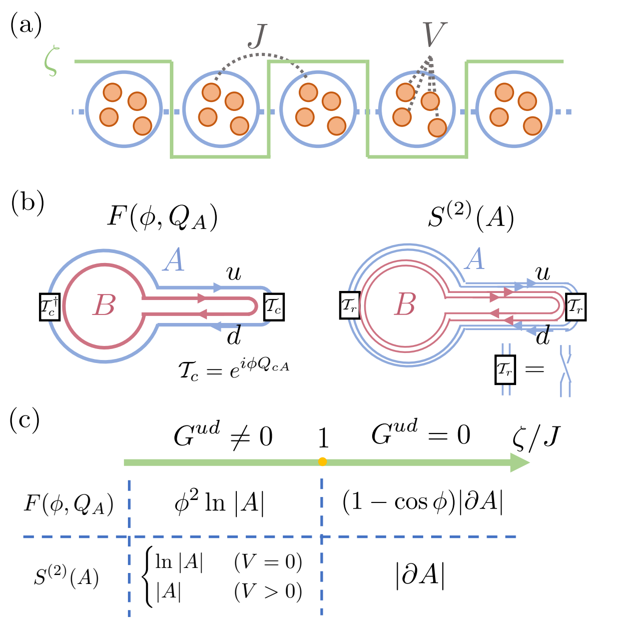

FIG. 1. (a) Schematics of the non-Hermitian complex SYK model. J and V are SYK random hopping and on-site interactions. ζ is a staggered imaginary potential. (b) A comparison between the path integral representations of $F(\phi, Q_A)$ and $S^{(2)}(A)$. Both quantities can be expressed as a correlator of twist operators. (c) The phase diagram of the non-Hermitian complex SYK model. The scaling of $F(\phi, Q_A)$ changes qualitatively when we tune ζ/J across the entanglement phase transition.

contour, as illustrated in Fig. 1(b). This leads to $F(\phi, Q_A) \sim \phi^2 \ln |A|$ when $G_{ud} \neq 0$, corresponding to both the volume-law entangled entropy phase for interacting systems and the critical phase for noninteracting systems. For $G_{ud} = 0$, $F(\phi, Q_A)$ satisfies an area law as the entanglement entropy. A pictorial illustration is presented in Fig. 1(c), which clearly shows the FCS can be used to probe the entanglement phase transition of non-Hermitian Hamiltonians. We also remark on generalizations to systems with discrete symmetry, in higher dimensions, or with long-range hopping.

II. MODEL AND SETUP

We consider the non-Hermitian complex SYK chains with Brownian couplings. The total Hamiltonian $H = H_R - iH_I$ reads

$$H_R = \sum_{ijx} [J_{ij}^x(t) c_{ix}^\dagger c_{jx+1} + \text{H.c.}] + \sum_{ijklx} \frac{V_{ij,kl}^x(t)}{4} c_{ix}^\dagger c_{jx}^\dagger c_{kx} c_{lx},$$
$$H_I = \zeta \sum_{ix} (-1)^{x-1} c_{ix}^\dagger c_{ix}. \tag{2}$$

Here, $i \in \{1, 2, \dots, N\}$ labels different fermion modes c_{ix} on each site $x \in \{1, 2, \dots, L\}$. The total charge $Q_c = \sum_{ix} c_{ix}^\dagger c_{ix}$ is conserved under the evolution. H_R contains random hopping J_{ij}^x between nearest-neighbor sites and random on-site interactions $V_{ij,kl}^x$, which are independent Brownian variables with

$$\overline{J_{ij}^x(t_1) J_{ij}^x(t_2)} = \frac{J\delta(t_{12})}{2N}, \quad \overline{V_{ij,kl}^x(t_1) V_{ij,kl}^x(t_2)} = \frac{2V\delta(t_{12})}{N^3}. \tag{3}$$

H_I describes a staggered imaginary potential with depth ζ. It can be realized by performing weak measurements and following quantum trajectories without quantum jumps [23,38]. As an example, let us consider weak measurements for operator O, which is described by Kraus operators:

$$K_O^0 = 1 - \gamma_O O^\dagger O + O(\gamma^2), \qquad K_O^1 = \sqrt{2\gamma_O} O, \tag{4}$$

where we have assumed $\gamma \ll 1$. We perform forced measurement by postselection of outcome 0. Introducing $\gamma_O = \zeta_O \delta t$, the evolution of ρ due to the measurement then takes the form of imaginary-time evolutions

$$\rho(t + \delta t) \propto e^{-h_I \delta t} \rho(t) e^{-h_I \delta t}, \tag{5}$$

with $h_I = \zeta_O O^\dagger O$. Adding contributions from measurements with different O and contributions from the unitary part, the total evolution is governed by the non-Hermitian Hamiltonian

$$H = H_R - iH_I, \qquad H_I = \sum_O \zeta_O O^\dagger O. \tag{6}$$

Choosing $O = c_{ix}^\dagger c_{ix}$ and $c_{ix} c_{ix}^\dagger$ for odd and even sites, respectively, with $\zeta_O = \zeta$, and following quantum trajectories without quantum jumps lead to the model introduced in Eq. (2).

We are interested in computing the FCS on the steady state of the non-Hermitian dynamics. In this work, we prepare the system in a TFD state, as in the study of entanglement phase transitions in SYK-like models. The definition of the TFD state requires introducing an auxiliary fermion system with annihilation operators η_{ix}. We first construct an Einstein-Podolsky-Rosen (EPR) state between c_{ix} and η_{ix} in the occupation basis as $|\text{EPR}\rangle = \otimes_{ix} \frac{1}{\sqrt{2}}(|00\rangle_{ix} + |11\rangle_{ix})$. The TFD state is obtained after adding imaginary-time evolutions to the EPR state $|\text{TFD}\rangle = \sqrt{Z^{-1}} e^{-\frac{\mu}{2} Q_c} |\text{EPR}\rangle$, where Z is a normalization factor.

For a realization of Brownian variables, the state at time T is given by

$$|\psi(T)\rangle = \frac{e^{-iHT}|\text{TFD}\rangle}{\sqrt{\langle \text{TFD}| e^{iH^\dagger T} e^{-iHT} |\text{TFD}\rangle}}. \tag{7}$$

For the FCS, we choose a bipartition of the total system into A and $\bar{A}$, where A contains fermion modes c_{ix} and η_{ix}, with $i \in \{1, 2, \dots, |A|\}$. We further take $Q_A \equiv Q_{cA} - Q_{\eta A} = \sum_x c_{ix}^\dagger c_{ix} - \eta_{ix}^\dagger \eta_{ix}$, which annihilates the initial state as $Q_A|\text{TFD}\rangle = 0$. The FCS (1) then reads

$$Z(\phi, Q_A) = \frac{\text{tr}_c[e^{iH^\dagger T} \mathcal{T}_c e^{-iHT} e^{-\mu Q_c/2} \mathcal{T}_c^\dagger e^{-\mu Q_c/2}]}{\text{tr}_c[e^{iH^\dagger T} e^{-iHT} e^{-\mu Q_c}]}. \tag{8}$$

Here, the trace is over the Hilbert space of fermion c_{ix}. The FCS takes the form of a correlator $\langle \mathcal{T}_c(T) \mathcal{T}_c^\dagger(0) \rangle_{\rho_c}$, with $\mathcal{T}_c = e^{i\phi Q_{cA}}$, on the ensemble of $\rho_c = Z^{-1} e^{-\mu Q_c}$. This is a close analog of the Rényi entropy calculation, as illustrated in Fig. 1(b). Moreover, for even L the FCS is symmetric across $|A| = L/2$. This is due to the invariance of (2) under a combination of the particle-hole transformation and the spatial reflection $c_{ix} \leftrightarrow$

$c^\dagger_{iL-x}$, which gives $Z(\phi, Q_A) = Z(-\phi, Q_{\bar{A}'}) = Z(\phi, Q_{A'})$, with $|A| = |\bar{A}'| = L - |A'|$.

After computing the FCS for given random couplings, we need to perform the disorder average, which requires introducing disorder replicas. In SYK-like models, it is known that the saddle-point solution is replica diagonal [60,61]. Consequently, we can make the approximation $Z(\phi, Q_A) = \mathcal{Z}(\phi, Q_A)/\mathcal{Z}(0, Q_A)$, with $\mathcal{Z}(\phi, Q_A) = \mathrm{tr}[e^{iH^\dagger T}\mathcal{T}_c e^{-iHT} e^{-\mu Q_c/2}\mathcal{T}_c^\dagger e^{-\mu Q_c/2}]$. In the following sections, we begin with an analysis of $\mathcal{Z}(0, Q_A)$ and then develop an effective theory for computing the response of twist operators for finite ϕ.

There are two main reasons for selecting the TFD state. First, measuring the relative charge $Q_{cA} - Q_{\eta A}$ in the doubled system, as shown in Fig. 1(b), is equivalent to measuring the charge Q_{cA} at two different times. This allows us to directly interpret the FCS as the statistics of the charge transfer across the boundary ∂A. Second, the TFD state can be easily represented by a continuous boundary condition in the path-integral approach, which makes it simpler to numerically verify our results. However, we want to emphasize that our analysis yields qualitative features that should hold for more general initial states from a symmetry perspective.

III. SADDLE-POINT SOLUTION

In the large-N limit, $\mathcal{Z}(\phi, Q_A)$ can be analyzed using the saddle-point approximation. We first focus on $\phi = 0$. The Green's functions are defined as $G^{ab}_x(t,t') = \langle c^a_{ix}(t)\bar{c}^b_{ix}(t')\rangle$, where $a, b \in \{u, d\}$ labels fermion fields on branches with forward and backward evolutions. In SYK-like models, the path integral of fermions can be transformed into a theory of collective fields $(G^{ab}_x, \Sigma^{ab}_x)$, in which the saddle-point equation is equivalent to the Schwinger-Dyson equation:

$$[-if^a\partial_t\delta^{ac} + \zeta(-1)^{x-1}\delta^{ac} - \Sigma^{ac}_x] \circ G^{cb}_x = \delta^{ab}\hat{I}, \tag{9}$$

where the summation over c is implicit. We have $f^{u/d} = \pm i$ and self-energy

$$\Sigma^{ac}_x = f^a f^c \hat{I}\left[\frac{J}{2}\left(G^{ac}_{x+1} + G^{ac}_{x-1}\right) - V\left(G^{ac}_x\right)^2\left(G^{ca}_x\right)^T\right]. \tag{10}$$

Here, we have viewed both G and Σ as matrices in the time domain and defined the identity matrix $\hat{I}(t,t') = \delta(t - t')$. The transpose is applied in the time domain, and the contour indexes are explicitly shown. Away from the boundary of branches at $t = T$ and $t = 0$, the Green's functions are time translational invariant $G^{ab}_x(t,t') = G^{ab}_x(t - t')$, which can be verified numerically. The solution can be obtained analytically, parametrized by $(\mathcal{P}, \mathcal{S}, z)$:

$$\tilde{G}_x(\omega) = \begin{pmatrix} -i\omega + (-1)^{x-1}\mathcal{P} & -z^{-1}\mathcal{S} \\ z\mathcal{S} & i\omega + (-1)^{x-1}\mathcal{P} \end{pmatrix}^{-1}. \tag{11}$$

Here, $\tilde{G}_x(\omega) = \int \frac{d\omega}{2\pi} e^{-i\omega t} G_x(t)$. $z = e^{\mu/2}$ is determined by the initial density matrix ρ. For $\zeta \geqslant J$, the solution is

$$\mathcal{P} = \zeta - J/2, \qquad \mathcal{S} = 0, \tag{12}$$

which gives a vanishing correlation between two branches $G^{ud} = 0$. Following the analysis in previous studies, this corresponds to the area-law entangled phase. For $\zeta < J$, we instead have $\mathcal{P} = \zeta/2$ and

$$1 = \frac{J}{2}\frac{1}{\sqrt{\mathcal{P}^2 + \mathcal{S}^2}} + \frac{V}{8}\frac{\mathcal{S}^2}{(\mathcal{P}^2 + \mathcal{S}^2)^{3/2}}. \tag{13}$$

Since $G_{ud} \neq 0$, this is a critical phase for $V = 0$ and a volume-law entangled phase for $V > 0$, as illustrated in Fig. 1(c). For $V = 0$, we find $\mathcal{S} = \frac{J}{2}\sqrt{1 - \zeta^2/J^2}$. For $V < 2J$, $\mathcal{S}$ increases from zero to $J/2 + V/8$ continuously when we tune ζ from J to zero. For $V > 2J$, the transition at $\zeta = J$ becomes first order.

IV. RELATIVE PHASE TWIST

We ask how the insertion of twist operators $\mathcal{T}_r$ changes the saddle-point solution. We begin with the simplest case where $\bar{A} = \emptyset$ and $Q_{cA} = Q_c$ is the conserved charge. The twist operator then commutes with the Hamiltonian H. As a result, it induces only a relative phase rotation between the u and d branches $(c^u_{ix}, c^d_{ix}) \to (c^u_{ix}e^{-i\phi}, c^d_{ix})$, which gives

$$G^{du}_x(t,t')_\phi = \left\langle c^d_{ix}(t)\mathcal{T}_c(T)\bar{c}^u_{ix}(t')\mathcal{T}^\dagger_c(0)\right\rangle_\rho = e^{i\phi}G^{du}_x(t,t'). \tag{14}$$

Here, $G^{du}_x(t,t')_\phi$ is the Green's function with twist operators. Similarly, we have $G^{ud}_x(t,t')_\phi = e^{-i\phi}G^{ud}_x(t,t')$. This reveals that the twist operator is coupled to the phase fluctuation in the off-diagonal components of the Green's functions.

We then consider a general subsystem size $|A|$. Due to the presence of twist operators, the system no longer exhibits translation symmetry, and an exact solution of the saddle-point equation is unavailable. However, for large $L > |A| \gg 1$, we expect only soft modes can be excited. For $G^{ud} = 0$, there is no symmetry reason for the existence of any soft mode, and the correlation in the system is generally short ranged. As a result, we expect $F(\phi, Q_A)$ to satisfy an area law. For $G^{ud} \neq 0$ the soft mode in the system is just the relative phase mode: The saddle-point equation (B4) is invariant under the relative phase rotation, while the solution (C1) breaks the symmetry when $G^{ud} \neq 0$. As a result, the relative phase rotation becomes a Goldstone mode. This indicates we can approximate

$$G_x(t,t)_\phi \approx \begin{pmatrix} G^{uu}_x(0) & e^{-i\varphi(x,t)}G^{ud}(0) \\ e^{i\varphi(x,t)}G^{du}(0) & G^{dd}_x(0) \end{pmatrix}. \tag{15}$$

The FCS is then determined by minimizing the effective action of $\varphi(x,t)$ with the boundary condition specified by the twist operator:

$$\varphi(x,T) = \begin{cases} 0 & x \in \bar{A}, \\ \phi & x \in A. \end{cases} \tag{16}$$

This can be derived by noticing $G^{du}_x(T,T)_\phi = G^{uu}_x(T,T)_\phi$ for $x \in \bar{A}$ and $G^{du}_x(T,T)_\phi = e^{i\phi}G^{uu}_x(T,T)_\phi$ for $x \in A$.

V. EFFECTIVE ACTION AND FCS

In this section, we explicitly derive the effective action governing the fluctuation around the saddle-point solution, which justifies our analysis above and gives closed-form expressions for $F(\phi, Q_A)$.

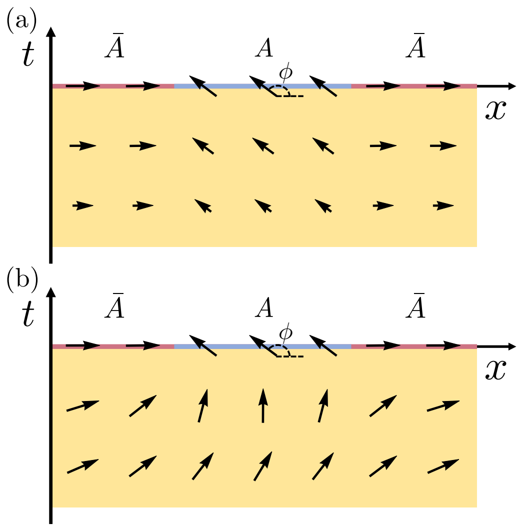

FIG. 2. An illustration of the typical configurations of $G_x^{ud}(t,t)=|G_x^{ud}(t,t)|e^{-i\varphi(x,t)}$ in the calculation of the FCS. Since $\varphi(x,t)$ is defined as a phase field, we represent it as the angle of a 2D vector (spin). Furthermore, the magnitudes of spins represent $|G_x^{ud}(t,t)|$. (a) For $\zeta>J$, $F(\phi,Q_A)$ can be estimated by a perturbative calculation of the hopping term J, which gives $F(\phi,Q_A)\sim(1-\cos\phi)|\partial A|$. (b) For $\zeta<J$, the twist operators create a ϕ-vortex pair, whose excitation energy gives $F(\phi,Q_A)\sim\phi^2\ln|A|$. There is also a similar contribution near $t=0$.

We first consider the short-range entangled phase with $\zeta>J$. The effective action is given by expanding the G-Σ action around the saddle point (C1) and (C2) [62]. Leaving details for Appendix C, to the quadratic order we find

$$\frac{S_{\rm eff}}{N}=\int_{\Omega,k}\mathfrak{g}_{-k}^{du}\begin{pmatrix}-2\zeta+2J-\frac{Jk^2}{2} & i\Omega\\ i\Omega & -2\zeta\end{pmatrix}\mathfrak{g}_k^{ud},\qquad(17)$$

where we have introduced $\mathfrak{g}_k^{ud}=(\delta G_k^{ud},\delta G_{k+\pi}^{ud})^T$ and $\mathfrak{g}_k^{du}=(\delta G_k^{du},\delta G_{k+\pi}^{du})$. To estimate $F(\phi,Q_A)$, we take a perturbation approach in terms of small J. Although $G^{ud}=0$ for $0\ll t\ll T$, it becomes finite near the boundaries due to the boundary condition of the contour. Without the hopping term J, different sites decouple, and G^{ud} can be computed as in (14), which gives $\varphi(x,t)=\phi$ for $x\in A$ and $\varphi(x,t)=0$ for $x\in\bar{A}$. This is illustrated in Fig. 2(a). To further estimate the decay of $|G^{ud}|$, we integrate out $\mathfrak{g}_k^{du}$ in (17). For small J, this imposes the constraint

$$\left(-\frac{1}{2\zeta}\partial_t^2+2\zeta-2J\right)\delta G_x^{ud}(t,t)=0,\qquad(18)$$

which gives $|G_x^{ud}(t,t)|\sim e^{-2(T-t)\sqrt{\zeta(\zeta-J)}}$. A similar calculation works for $|G_x^{du}(t,t)|$. Then we can compute contributions from small hopping terms in $\mathcal{Z}(\phi,Q_A)$ as

$$JN\,\mathrm{Re}\int dt\;G_A^{ud}G_{\bar{A}}^{du}\sim-\frac{J}{\sqrt{\zeta(\zeta-J)}}N\cos\phi.\qquad(19)$$

Subtracting the contribution from $\mathcal{Z}(0,Q_A)$, we finally obtain

$$F(\phi,Q_A)\sim\frac{J}{\sqrt{\zeta(\zeta-J)}}N(1-\cos\phi).\qquad(20)$$

Now we consider the long-range correlated phase with $\zeta<J$. As explained in the last section, we need to derive the effective theory for the relative phase mode $\varphi(x,t)$. Similar to the derivation of (17), we now expand the G-Σ action around the saddle point (C1) and (C3) to quadratic order, with the identification that $\delta G_x^{ud}(t,t)=-i\varphi(x,t)G^{ud}(0)$ and $\delta G_x^{du}(t,t)=i\varphi(x,t)G^{du}(0)$. As derived in Appendix C, the result reads

$$S_{\rm eff}=\frac{N\mathcal{S}^2}{4(\zeta^2+4\mathcal{S}^2)}\int_{x,t}\left(\frac{(\partial_t\varphi)^2}{2\sqrt{\zeta^2+4\mathcal{S}^2}-J}+J(\partial_x\varphi)^2\right).\qquad(21)$$

This is a large-N XY model. Consequently, the system exhibits an emergent conformal symmetry. Unlike the emergent replica conformal symmetry in non-Hermitian free-fermion systems, here, the conformal symmetry is a consequence of charge $U(1)$ symmetry, which is stable against adding interactions. The boundary condition in (16) then excites a ϕ-vortex pair, as sketched in Fig. 2(b). The FCS is equal to the excitation of the vortex pair. In the limit of $L\to\infty$, the result reads [24,63]

$$F(\phi,Q_A)\sim\frac{\mathcal{S}^2\phi^2N}{\zeta^2+4\mathcal{S}^2}\left(\frac{J}{2\sqrt{\zeta^2+4\mathcal{S}^2}-J}\right)^{1/2}\ln|A|.\qquad(22)$$

In particular, it shows nonanalyticity near $\phi=\pi$. For finite L, $\ln|A|$ should be replaced by $\ln[L\sin(\pi|A|/L)/\pi]$ due to the conformal invariance. Comparing (20) and (22), we find $F(\phi,Q_A)$ shows qualitatively different scalings of both $|A|$ and ϕ for both interacting and noninteracting systems and thus serves as a universal probe of the entanglement phase transition in non-Hermitian Hamiltonian dynamics.

To justify our theoretical predictions (20) and (22), we numerically study the FCS in the large-N limit by solving the saddle-point equation with twist operators $\mathcal{T}_c$ and computing the on-shell action. Similar approaches have been widely adopted to simulate dynamics of Rényi entropies in SYK-like models [25–27,64]. In Fig. 3, we present results for $L=20$ with $\mu=0.5$. For $\zeta<J$, we check that $F(\phi,Q_A)$ is a linear function of $\ln[L\sin(\pi|A|/L)/\pi]$ and is a quadratic function of ϕ. Near $\phi=\pm\pi$, we have two different saddle-point solutions, which leads to a nonanalyticity. For $\zeta>J$, we check that $F(\phi,Q_A)$ shows area-law behavior for large $|A|$ and is proportional to $(1-\cos\phi)$. All results are tested for both the interacting case ($V=J$) and the noninteracting case ($V=0$).

VI. DISCUSSION

In this work, we studied the full counting statistics of the steady state of non-Hermitian complex SYK models. We found $F(\phi,Q_A)$ show different scaling for both $|A|$ and ϕ in phases with different entanglement properties. Using an effective spin model, we showed the following: For $\zeta>J$, the system is short range correlated. We can approximate $F(\phi,Q_A)\sim(1-\cos\phi)|\partial A|$ for $\zeta\gg J$. For $\zeta<J$, the system is long range correlated with $F(\phi,Q_A)\sim\phi^2\ln|A|$, which exhibits nonanalyticity near $\phi=\pi$. We further validated our theoretical predictions by numerically solving the saddle-point equation.

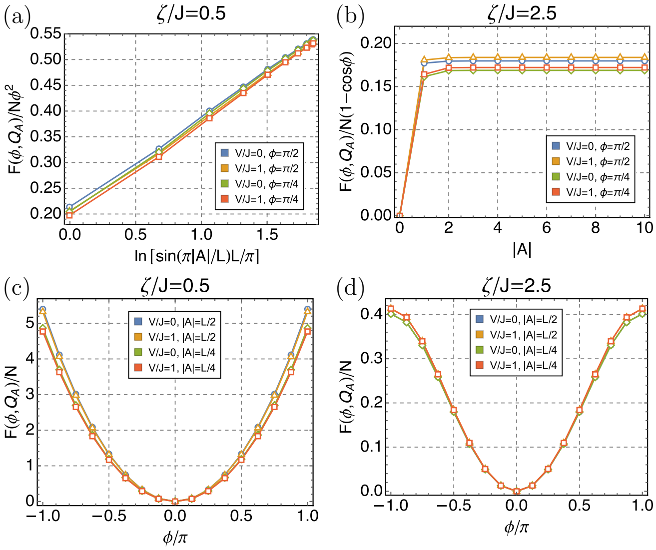

FIG. 3. The numerical calculation of the FCS in the large-N limit by solving the Schwinger-Dyson equation with $\mu = 0.5$. (a) and (b) $F(\phi, Q_A)$ for different subsystem sizes $|A|$. The results show $F(\phi, Q_A) \propto \ln |A|$ for $\zeta/J = 0.5$ and $F(\phi, Q_A) \propto |\partial A|$ for $\zeta/J = 2.5$. (c) and (d) $F(\phi, Q_A)$ for different twist strengths ϕ. The results show $F(\phi, Q_A) \propto \phi^2$ for $\zeta/J = 0.5$ and $F(\phi, Q_A) \propto (1 - \cos\phi)$ for $\zeta/J = 2.5$.

We point out the identification of the relative phase twist is not restricted to the TFD states and is valid for general initial states that are eigenstates of local charge operators. Without loss of generality, we assume $Q_A|\psi\rangle = 0$. This gives $Z(\phi, Q_A) \propto \langle\psi|e^{iH^\dagger T}e^{i\phi Q_A}e^{-iHT}|\psi\rangle = \langle\psi|e^{iH^\dagger T}e^{i\phi Q_A}e^{-iHT}e^{-i\phi Q_A}|\psi\rangle$. Like in (8), a pair of twist operators appears, which leads to a relative phase twist between forward evolution and backward evolution branches. Moreover, there is no difference between the TFD state and more general initial states from the symmetry perspective: In the absence of boundary conditions at $t = 0$ and $t = T$, fermion fields on distinct branches of the system can undergo independent transformations. As a result, the system exhibits a $U(1) \otimes U(1)$ symmetry. However, in the volume-law phase where $G_{ud} \neq 0$, this symmetry is broken down to a single $U(1)$ symmetry, leading to the emergence of a Goldstone mode. As a result, we believe our theoretical predictions reveal universal features of the FCS across the entanglement transition of non-Hermitian Hamiltonians.

Several additional remarks are in order: First, instead of systems with $U(1)$ symmetry, we can consider models with discrete symmetries. It is then natural to expect that the phase with small non-Hermitian strength is still a symmetry-breaking phase, except with no Goldstone mode. The dominant contribution now becomes a domain wall, instead of a vortex pair. This is consistent with recent numerics in [65]. Second, our results can be generalized to higher dimensions straightforwardly. As an example, let us consider a non-Hermitian complex SYK chain in two dimensions. The short-range correlated phase with $\zeta > J$ is still area law entangled with $F(\phi, Q_A) \sim (1 - \cos\phi)R$ for a subsystem A with radius R. For the phase with $\zeta < J$, the effective theory becomes a three-dimensional XY model. The vortex pair is then replaced by a vortex ring. Consequently, we have $F(\phi, Q_A) \sim \phi^2 R \ln R$. Finally, in experimental systems, long-range interactions may present. If we consider a long-range hopping term that decays as $1/r^\alpha$ as in [24], a fractal phase with $F(\phi, Q_A) \sim \phi^2 L_A^{1-z}$ appears, with $z = \frac{2\alpha-1}{2}$ for $\alpha \in (0.5, 1.5)$.

ACKNOWLEDGMENTS

We thank H. Zhai for bringing our attention to the full counting statistics. We thank M. Cheng, Y. Gu, and S.-Q. Ning for helpful discussions. P.Z. would like to thank X. Chen, S.-K. Jian, and C. Liu for valuable discussions during the collaboration on related topics. This project is supported by NSFC under Grant No. 12374477.

APPENDIX A: THE DERIVATION OF THE FULL COUNTING STATISTICS FOR THE TFD STATE

The FCS is defined as

$$\mathcal{Z}(\phi, Q_A) \equiv \mathrm{tr}_{c,\eta}[\rho(T)e^{-i\phi Q_A}], \tag{A1}$$

with $Q_A \equiv Q_{cA} - Q_{\eta A} = \sum_x c_{ix}^\dagger c_{ix} - \eta_{ix}^\dagger \eta_{ix}$. Here, the trace is over the c fermion system and η fermion system, and $\rho(T) = \frac{1}{Z(T)}e^{-iHT}\rho(0)e^{iH^\dagger T}$ is the density matrix at time T. $Z(T) = \mathrm{tr}[e^{-iHT}e^{-i\mu Q_c}e^{iH^\dagger T}]$ is the normalization factor. The initial state is the $|\mathrm{TFD}\rangle = \sqrt{Z^{-1}}e^{-\frac{\mu}{2}Q_c}|\mathrm{EPR}\rangle$. Putting all the definitions into the FCS, we obtain

$$\begin{aligned}\mathcal{Z}(\phi, Q_A) &= \frac{1}{Z(T)}\mathrm{tr}_{c,\eta}[e^{-iHT}|\mathrm{TFD}\rangle\langle\mathrm{TFD}|e^{iH^\dagger T}e^{-i\phi Q_A}]\\ &= \frac{1}{Z(T)}\langle\mathrm{TFD}|e^{iH^\dagger T}e^{-i\phi Q_A}e^{-iHT}|\mathrm{TFD}\rangle\\ &= \frac{1}{Z(T)}\langle\mathrm{TFD}|e^{iH^\dagger T}e^{-i\phi Q_{cA}}e^{i\phi Q_{\eta A}}e^{-iHT}|\mathrm{TFD}\rangle.\end{aligned} \tag{A2}$$

Since the Hamiltonian H is defined on the c fermion system and $Q_{\eta A}$ is defined on the auxiliary η fermion system, the operator H and $Q_{\eta A}$ commute with each other. Thus, we have

$$\mathcal{Z}(\phi, Q_A) = \frac{1}{Z(T)}\langle\mathrm{TFD}|e^{iH^\dagger T}e^{-i\phi Q_{cA}}e^{-iHT}e^{i\phi Q_{\eta A}}|\mathrm{TFD}\rangle. \tag{A3}$$

Given that Q_A annihilates the initial TFD state $Q_A|\mathrm{TFD}\rangle = 0$, we obtain $e^{i\phi Q_A}|\mathrm{TFD}\rangle = |\mathrm{TFD}\rangle$. Thus, we further have $e^{i\phi Q_{cA}}|\mathrm{TFD}\rangle = e^{i\phi Q_{\eta A}}|\mathrm{TFD}\rangle$. Therefore, FCS can be further written as

$$\begin{aligned}\mathcal{Z}(\phi, Q_A) &= \frac{1}{Z(T)}\langle\mathrm{TFD}|e^{iH^\dagger T}e^{-i\phi Q_{cA}}e^{-iHT}e^{i\phi Q_{cA}}|\mathrm{TFD}\rangle\\ &= \frac{1}{Z(T)}\langle\mathrm{TFD}|\mathcal{T}_c(T)\mathcal{T}_c^\dagger(0)|\mathrm{TFD}\rangle,\end{aligned} \tag{A4}$$

with $\mathcal{T}_c = e^{-i\phi Q_{cA}}$. Therefore, we find that the FCS takes the form of a correlator. We also have

$$\begin{aligned}
&\mathcal{Z}(\phi, Q_A)\\
&= \frac{1}{Z(T)}\langle \mathrm{TFD}|e^{iH^\dagger T}e^{-i\phi Q_{cA}}e^{-iHT}e^{i\phi Q_{cA}}|\mathrm{TFD}\rangle\\
&= \frac{1}{Z(T)}\langle \mathrm{EPR}|e^{-\frac{\mu}{2}Q_c}e^{iH^\dagger T}e^{-i\phi Q_{cA}}e^{-iHT}e^{i\phi Q_{cA}}e^{-\frac{\mu}{2}Q_c}|\mathrm{EPR}\rangle\\
&= \frac{1}{Z(T)}\mathrm{tr}_c\big[e^{-\frac{\mu}{2}Q_c}e^{iH^\dagger T}e^{-i\phi Q_{cA}}e^{-iHT}e^{i\phi Q_{cA}}e^{-\frac{\mu}{2}Q_c}\big]\\
&= \frac{1}{Z(T)}\mathrm{tr}_c\big[e^{iH^\dagger T}e^{-i\phi Q_{cA}}e^{-iHT}e^{i\phi Q_{cA}}e^{-\mu Q_c}\big]\\
&= \frac{1}{Z(T)}\mathrm{tr}_c\big[e^{iH^\dagger T}\mathcal{T}_c e^{-iHT}\mathcal{T}_c^\dagger e^{-\mu Q_c}\big]\\
&= \frac{1}{Z(T)}\mathrm{tr}_c\big[e^{iH^\dagger T}\mathcal{T}_c e^{-iHT}e^{-\frac{\mu}{2}Q_c}\mathcal{T}_c^\dagger e^{-\frac{\mu}{2}Q_c}\big]. \qquad \text{(A5)}
\end{aligned}$$

We have used the fact that $\mathcal{T}_c^\dagger$ and Q_c commute with each other to obtain the last equality. This is Eq. (5) in the main text. Notice that here, the trace is over the c fermion system.

APPENDIX B: THE PATH-INTEGRAL REPRESENTATION OF THE FULL COUNTING STATISTICS

As explained in the main text, we focus on the path-integral representation of the FCS. The time evolution of a FCS $\mathcal{Z}(\phi, Q_A)$ is

$$\mathcal{Z}(\phi, Q_A) = \overline{\mathrm{tr}[e^{iH^\dagger T}\mathcal{T}_c e^{-iHT}e^{-\mu Q_c/2}\mathcal{T}_c^\dagger e^{-\mu Q_c/2}]}, \qquad \text{(B1)}$$

and it does not preserve normalization. To evaluate $\mathcal{Z}(\phi, Q_A)$ using the path integral, one needs two contours similar to the Keldysh contour. Denoted by u and d for forward and backward evolution, the action on these two contours schematically is

$$-I = \int dt \sum_{x,a}\left(-if^a\bar{c}_x^a\partial_t c_x^a + f^a\big[J^x(t)\bar{c}_x^a c_{x+1}^a + J^x(t)\bar{c}_{x+1}^a c_x^a\big] + f^a\frac{V^x(t)}{4}\bar{c}_x^a\bar{c}_x^a c_x^a c_x^a + \zeta(-1)^{x-1}\bar{c}_x^a c_x^a\right), \qquad \text{(B2)}$$

where $a = u, d$ denotes two contours and the superscript for the fermion species on each site is suppressed. Here, $f^{u/d} = \pm i$. The last term in the equation does not have a contour-dependent prefactor because of the nonunitary H_I and the definition of Eq. (B1). The effective action on these two contours after integrating out disorder and introducing $1 = \int dt dt' \Sigma_x^{ab}(t,t')[G_x^{ba}(t',t) - c_x^b(t')\bar{c}_x^a(t)]$ is

$$\begin{aligned}
-\frac{I}{N} = &\sum_x \mathrm{Tr}\ln\big[-if^a\partial_t\delta^{ac} + \zeta(-1)^{x-1}\delta^{ac} - \Sigma_x^{ac}\big] + \int dt dt'\Bigg[\Sigma_x^{ac}\big(G_x^{ca}\big)^T\\
&+ f^a f^c\hat{I}\bigg(\frac{V}{4}\big(G_x^{ac}\big)^2\big[\big(-G_x^{ca}\big)^2\big]^T + \frac{J}{4}\big[G_{x+1}^{ac}\big(-G_x^{ca}\big)^T + G_{x-1}^{ac}\big(-G_x^{ca}\big)^T\big]\bigg)\Bigg], \qquad \text{(B3)}
\end{aligned}$$

where G^{ab} and Σ^{ab} are the bilocal fields with time arguments t and t' omitted, which characterize the two-point function of Majorana fermions or the corresponding self-energy at the a and b contours. Here, $\hat{I} = \delta(t - t')$ means the Brownian condition. As a result, the saddle-point equation is

$$\begin{aligned}
&\sum_c\big[-if^a\partial_t\delta^{ac} + \zeta(-1)^{x-1}\delta^{ac} - \Sigma_x^{ac}\big]\circ G_x^{cb} = \delta^{ab}\hat{I},\\
&\Sigma_x^{ac} = f^a f^c\hat{I}\Big[V\big(G_x^{ac}\big)^2\big(-G_x^{ca}\big)^T + J\big(G_{x+1}^{ac} + G_{x-1}^{ac}\big)/2\Big]. \qquad \text{(B4)}
\end{aligned}$$

APPENDIX C: THE DERIVATION OF THE EFFECTIVE ACTION

The effective action is given by expanding the $G-\Sigma$ action (B3) around the saddle-point solution. In this section, we give a detailed derivation for effective actions.

The solution can be obtained analytically, parametrized by $(\mathcal{P}, \mathcal{S}, z)$:

$$\tilde{G}_x(\omega) = \begin{pmatrix} -i\omega + (-1)^{x-1}\mathcal{P} & -z^{-1}\mathcal{S}\\ z\mathcal{S} & i\omega + (-1)^{x-1}\mathcal{P}\end{pmatrix}^{-1}. \qquad \text{(C1)}$$

Here, $\tilde{G}_x(\omega) = \int \frac{d\omega}{2\pi}e^{-i\omega t}G_x(t)$. $z = e^{\mu/2}$ is determined by the initial density matrix ρ. For $\zeta \geqslant J$, the solution is

$$\mathcal{P} = \zeta - J/2, \qquad \mathcal{S} = 0, \qquad \text{(C2)}$$

which gives a vanishing correlation between two branches $G^{ud} = 0$. Following the analysis in previous studies, this corresponds to the area-law entangled phase. For $\zeta < J$, we instead have

$$\mathcal{P} = \zeta/2, \qquad 1 = \frac{J}{2}\frac{1}{\sqrt{\mathcal{P}^2 + \mathcal{S}^2}} + \frac{V}{8}\frac{\mathcal{S}^2}{(\mathcal{P}^2 + \mathcal{S}^2)^{3/2}}. \qquad \text{(C3)}$$

Now we consider saddle-point fluctuations,

$$\begin{aligned}
\Sigma(t_1, t_2) &= \Sigma_s(t_1, t_2) + \delta\Sigma(t_1)\delta(t_{12}),\\
G(t_1, t_2) &= G_s(t_1, t_2) + \delta G(t_1, t_2). \qquad \text{(C4)}
\end{aligned}$$

We will separately evaluate the fluctuation in Eq. (B3), including the Tr ln term, linear coupling ΣG term, and interaction term related to J and V. Now we discuss both the $\zeta > J$ and $\zeta < J$ cases.

1. $\zeta > J$

a. Tr ln *term.* Expanding the Tr ln term leads to the following equation:

$$\begin{aligned}
&\mathrm{Tr}\ln\left[-if^a\partial_t\delta^{ac}+\zeta(-1)^{x-1}\delta^{ac}-\Sigma_x^{ac}\right]\\
&\quad=\mathrm{Tr}\ln\left[\left(G_s^{-1}\right)_x^{ac}\right]+\mathrm{Tr}\ln\left[\hat{I}^{ab}-(G_s)_x^{ca}\circ(\delta\Sigma_s)_x^{ab}\right]\\
&\quad=\mathrm{Tr}\ln\left[\left(G_s^{-1}\right)_x^{ac}\right]-\int d\tau_1d\tau_2\left[(G_s)_x^{ca}(\tau_1,\tau_2)(\delta\Sigma_s)_x^{ac}(\tau_2)\delta(\tau_2-\tau_1)\right]\\
&\qquad-\int d\tau_1d\tau_2d\tau_3d\tau_4\frac{1}{2}\left[(G_s)_x^{ab}(\tau_1,\tau_2)(\delta\Sigma_s)_x^{bc}(\tau_2)\delta(\tau_2-\tau_3)(G_s)_x^{cd}(\tau_3,\tau_4)(\delta\Sigma_s)_x^{da}(\tau_4)\delta(\tau_4-\tau_1)\right].
\end{aligned}\tag{C5}$$

The contour index a, b, c, d implicitly uses the Einstein summation. The second-order term can be transformed:

$$\begin{aligned}
&-\int d\tau_1d\tau_2d\tau_3d\tau_4\frac{1}{2}\left[(G_s)_x^{ab}(\tau_1,\tau_2)(\delta\Sigma_s)_x^{bc}(\tau_2)\delta(\tau_2-\tau_3)(G_s)_x^{cd}(\tau_3,\tau_4)(\delta\Sigma_s)_x^{da}(\tau_4)\delta(\tau_4-\tau_1)\right]\\
&\quad=-\int d\tau_1d\tau_2\frac{1}{2}\left[(G_s)_x^{ab}(\tau_1,\tau_2)(\delta\Sigma_s)_x^{bc}(\tau_2)(G_s)_x^{cd}(\tau_2,\tau_1)(\delta\Sigma_s)_x^{da}(\tau_1)\right]\\
&\quad=-\frac{1}{2}\int_{\omega,\Omega}\mathrm{Tr}[G_s(\omega+\Omega)\delta\Sigma(\Omega)G_s(\omega)\delta\Sigma(-\Omega)],
\end{aligned}\tag{C6}$$

where $\int_\Omega\equiv\int_{-\infty}^{\infty}\frac{d\Omega}{2\pi}$ and we have used the Fourier transform $\delta\Sigma(\Omega)=\int dt\,\delta\Sigma(t)e^{i\Omega t}$. In the last step of Eq. (C6), $\int_\omega$ can be integrated out using the residue theorem, and G_s can be inserted into Eqs. (C1) and (C2). Finally, we arrive at

$$-\delta I_1^{(1)}/N=\sum_x\int_\Omega\sigma_x^T(\Omega)\mathcal{M}_1^{(1)}\sigma_x(-\Omega)\tag{C7}$$

for the trace log term. There are two independent diagonal fields $\delta\Sigma^{uu}$ and $\delta\Sigma^{dd}$ and two independent off-diagonal fields $\delta\Sigma^{ud}$ and $\delta\Sigma^{du}$. The full kernel implies that (a) off-diagonal fields decouple from the diagonal field and (b) two independent off-diagonal fields have nontrivial interactions. We denote these two nontrivial off-diagonal fields as $\sigma_x=(\delta\Sigma_x^{ud},\delta\Sigma_x^{du})^T$, and the corresponding kernel reads

$$\mathcal{M}_1^{(1)}=\frac{1}{2(\Omega^2+(J-2\zeta)^2)}\begin{pmatrix}0 & J-2\zeta+i\Omega(-1)^x\\ J-2\zeta-i\Omega(-1)^x & 0\end{pmatrix}.\tag{C8}$$

b. ΣG *term.* For the ΣG term,

$$\begin{aligned}
\int dtdt'\,\Sigma_x^{ac}\left(G_x^{ca}\right)^T&=-I_2^{(1)}/N+\text{Diag. term}+\sum_x\frac{1}{2}\int_\Omega\left[g_x^T(\Omega)\begin{pmatrix}0&1\\1&0\end{pmatrix}\sigma_x(-\Omega)+\sigma_x^T(\Omega)\begin{pmatrix}0&1\\1&0\end{pmatrix}g_x(-\Omega)\right]\\
&\equiv-I_2^{(1)}/N-\delta I_2^{(1)}/N,
\end{aligned}\tag{C9}$$

where $g_x=(\delta G_x^{ud},\delta G_x^{du})^T$ and σ was defined before.

c. V *term.*

$$\int dtdt'\sum_x f^af^c\hat{I}\frac{V}{4}\left(G_x^{ac}\right)^2\left[\left(-G_x^{ca}\right)^2\right]^T=\text{Diag. term}+\frac{1}{4}\sum_x\int dt\left\{\left[G_x^{12}(t,t)\right]^2\left[-G_x^{21}(t,t)\right]^2+\left[G_x^{21}(t,t)\right]^2\left[-G_x^{12}(t,t)\right]^2\right\}.\tag{C10}$$

After applying the saddle-point fluctuation (C4) and keeping to second order of δG, we find all fluctuation terms are proportional to $G_s^{12}(\Omega)$ or $G_s^{21}(\Omega)$. According to condition (C2), this means V term does not contribute to action in the second order.

d. J *term.*

$$\begin{aligned}
&\int dtdt'\sum_x\frac{J}{4}\left[G_{x+1}^{ac}\left(-G_x^{ca}\right)^T+G_{x-1}^{ac}\left(-G_x^{ca}\right)^T\right]\\
&\quad=-I_4^{(1)}/N+\text{Diag. term}-\int_k\int dt\frac{J}{2}\left[\delta G_k^{12}(t,t)\delta G_{-k}^{21}(t,t)\cos(k)+\delta G_k^{21}(t,t)\delta G_{-k}^{12}(t,t)\cos(k)\right]\\
&\quad=-I_4^{(1)}/N+\text{Diag. term}+\int_k\int dt\frac{-J}{2}g_k^T(\Omega)\begin{pmatrix}0&\cos k\\ \cos k&0\end{pmatrix}g_k(-\Omega)\\
&\quad\equiv-I_4^{(1)}/N-\delta I_4^{(1)}/N.
\end{aligned}\tag{C11}$$

In the last step, we apply the saddle fluctuation solution and perform the Fourier transition on k, where $\delta G_x^{ab}(t,t)=\int_k G_x^{ab}(t,t)e^{ikx}$. Here, $\int_k\equiv\int\frac{dk}{2\pi}$.

e. Effective action. Summing all contributions (C7), (C9), and (C11) together and keeping the second-order off-diagonal terms, we arrive at

$$-\delta I/N = \int_\Omega \left[\sum_x \left\{ \sigma^T(\Omega)\mathcal{M}_1\sigma(-\Omega) + \frac{1}{2}\left[g_x^T(\Omega)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\sigma_x(-\Omega) + \sigma_x^T(\Omega)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} g_x(-\Omega) \right] \right\} + \sum_k \frac{-J}{2} g_k^T(\Omega)\begin{pmatrix} 0 & \cos k \\ \cos k & 0 \end{pmatrix} g_k(-\Omega) \right]. \quad \text{(C12)}$$

Integrating out σ leads to

$$-\delta I^{(1)}/N = \int_\Omega \left\{ \sum_x g_x^T(\Omega)\begin{pmatrix} 0 & \frac{1}{2}[2\zeta - J + i(-1)^x\Omega] \\ \frac{1}{2}[2\zeta - J - i(-1)^x\Omega] & 0 \end{pmatrix} g_x(-\Omega) + \sum_k \frac{-J}{2} g_k^T(\Omega)\begin{pmatrix} 0 & \cos k \\ \cos k & 0 \end{pmatrix} g_k(-\Omega) \right\}. \quad \text{(C13)}$$

To deal with site-dependent phase $(-1)^x$, we can define new enlarged bases $\mathfrak{g}_k^{ud} = (\delta G_k^{ud}, \delta G_{k+\pi}^{ud})$ and $\mathfrak{g}_k^{du} = (\delta G_k^{du}, \delta G_{k+\pi}^{du})^T$. Focusing on $k \ll 1$, the action becomes

$$-\delta I^{(1)}/N = \int_{\Omega, k\ll 1} \mathfrak{g}_k^{ud}(\Omega)\begin{pmatrix} 2\zeta - J\cos(k) - J & -i\Omega \\ -i\Omega & 2\zeta + J\cos(k) - J \end{pmatrix}\mathfrak{g}_{-k}^{du}(-\Omega). \quad \text{(C14)}$$

An expansion of (C14) leads to the result cited in the main text. We also compute the smaller eigenvalue of the matrix as

$$\frac{1}{2}(4\zeta - \sqrt{2}\sqrt{J^2\cos(2k) + J^2 - 2\Omega^2} - 2J) = \frac{\Omega^2}{2J} + \frac{Jk^2}{2} + (2\zeta - 2J). \quad \text{(C15)}$$

After integrating out $\mathfrak{g}_k^{du}$, the resulting δ function leads to two coupled constraint equations,

$$\begin{pmatrix} -2\zeta + 2J - \frac{Jk^2}{2} & i\Omega \\ i\Omega & -2\zeta + \frac{Jk^2}{2} \end{pmatrix}\begin{pmatrix} \delta G_k^{ud} \\ \delta G_{k+\pi}^{ud} \end{pmatrix} = 0, \quad \text{(C16)}$$

which can be simplified as

$$\left(-2\zeta + 2J - \frac{Jk^2}{2} + \frac{\Omega^2}{-2\zeta + Jk^2/2}\right)\delta G_k^{ud} = 0$$
$$\Rightarrow -\left(2\zeta - 2J + \frac{1}{2\zeta}\Omega^2 + \frac{Jk^2}{2} + O(\Omega^2 k^2)\right)\delta G_k^{ud} = 0. \quad \text{(C17)}$$

This leads to the differential equation in the main text:

$$\left(-\frac{1}{2\zeta}\partial_t^2 + 2\zeta - 2J\right)\delta G_x^{ud}(t,t) = 0. \quad \text{(C18)}$$

According to the translationally invariant solution, the equal-time Green's function in this case is

$$G_x(0^+) = \begin{pmatrix} -\frac{1}{2}[(-1)^x - 1] & 0 \\ 0 & -\frac{1}{2}[(-1)^x + 1] \end{pmatrix},$$
$$G_x(0^-) = \begin{pmatrix} -\frac{1}{2}((-1)^x + 1) & 0 \\ 0 & -\frac{1}{2}[(-1)^x - 1] \end{pmatrix}. \quad \text{(C19)}$$

Several observations help us to determine the detailed form of δG_k^{ud}. First, $\delta G^{ud}(t,t)$ is bounded by $G^{ud}(0^\pm) = 0$ when the fermion operator is away from the boundary, i.e., $t \ll T$. Therefore, $|\delta G_x^{ud}(t,t)| \sim e^{2t\sqrt{\zeta(\zeta-J)}}$ corresponds to the correct decaying direction.

Second, in the main text, we noticed the boundary condition $G_x^{du}(T,T)_\phi = G_x^{uu}(T,T)_\phi$ for $x \in \bar{A}$ and $G_x^{du}(T,T)_\phi = e^{i\phi}G_x^{uu}(T,T)_\phi$ for $x \in A$. Without loss of generality, we consider a special case where x is at the boundary of A, namely, $x \in A$ and $x+1 \in \bar{A}$. We can have $|\delta G_x^{ud}(T+0^-,T)| \sim |G_x^{uu}(0^-)| = \frac{1}{2}|(-1)^x + 1|$ and $|\delta G_{x+1}^{du}(T, T+0^-)| \sim |G_{x+1}^{uu}(0^+)| = \frac{1}{2}|(-1)^{x+1} - 1|$. If x is an even site, then we find both δG^{ud} and δG^{du} have a $O(1)$ value. Then we can write $|\delta G_x^{ud}(t,t)| \sim e^{-2(t-T)\sqrt{\zeta(\zeta-J)}}$ and $|\delta G_x^{ud}(t,t)| \sim e^{-2(t-T)\sqrt{\zeta(\zeta-J)}}$.

Third, we consider the phase difference between $\delta G_A^{ud}(t,t)$ and $\delta G_{\bar{A}}^{du}(t,t)$. As in our previous discussion, the differences are a minus sign [referring to Eq. (C1)] and an $e^{i\phi}$ phase [referring to Eqs. (12) and (13) in the main text:

$$JN\ \mathrm{Re}\int dt\ G_A^{ud}(t,t)G_{\bar{A}}^{du}(t,t)\sim JN\ \mathrm{Re}\int dt\ e^{-4(t-T)\sqrt{\zeta(\zeta-J)}}(-e^{i\phi})\sim-\frac{J}{\sqrt{\zeta(\zeta-J)}}N\cos\phi. \tag{C20}$$

2. $\zeta < J$

a. Tr ln *term.* Taking saddle points (C1) and (C3) and expanding the Tr ln term leads to

$$\begin{aligned}-\delta I_1^{(2)}/N &= -\frac{1}{2}\int_{\omega,\Omega}\mathrm{Tr}[G_s(\omega+\Omega)\delta\Sigma(\Omega)G_s(\omega)\delta\Sigma(-\Omega)]\\ &=\sum_x\int_\Omega\tilde{\sigma}_x^T(\Omega)\mathcal{M}_1^{(2)}\tilde{\sigma}_x(-\Omega),\end{aligned} \tag{C21}$$

where $\tilde{\sigma}_x=(\delta\Sigma_x^{uu},\delta\Sigma_x^{dd},\delta\Sigma_x^{ud},\delta\Sigma_x^{du})^T$. Since the off-diagonal couplings G_s^{ud} and G_s^{du} are nonzero, the diagonal and off-diagonal parts of the self-energy are coupled. Therefore, we consider all fluctuation components in $\tilde{\sigma}_x$. The kernel reads

$$\begin{aligned}&\mathcal{M}_1^{(2)}\sqrt{\zeta^2+4\mathcal{S}^2}(\zeta^2+4\mathcal{S}^2+\Omega^2)\\ &=\begin{pmatrix}\mathcal{S}^2 & \mathcal{S}^2 & -\frac{1}{2}\mathcal{S}z[\zeta(-1)^x+i\Omega] & \frac{\mathcal{S}[\zeta(-1)^x-i\Omega]}{2z}\\ \mathcal{S}^2 & \mathcal{S}^2 & -\frac{1}{2}\mathcal{S}z[\zeta(-1)^x+i\Omega] & \frac{\mathcal{S}(\zeta(-1)^x-i\Omega)}{2z}\\ -\frac{1}{2}\mathcal{S}z[\zeta(-1)^x-i\Omega] & -\frac{1}{2}\mathcal{S}z[\zeta(-1)^x-i\Omega] & -\mathcal{S}^2z^2 & \frac{1}{2}\{-2\mathcal{S}^2-\zeta[\zeta-i(-1)^x\Omega]\}\\ \frac{\mathcal{S}[\zeta(-1)^x+i\Omega]}{2z} & \frac{\mathcal{S}[\zeta(-1)^x+i\Omega]}{2z} & \frac{1}{2}\{-2\mathcal{S}^2-\zeta[\zeta+i(-1)^x\Omega]\} & -\frac{\mathcal{S}^2}{z^2}\end{pmatrix}.\end{aligned} \tag{C22}$$

b. ΣG *term.* For the ΣG term, we also need to consider all components of the fluctuating Green's function. Similarly, we define $\tilde{g}_x=(\delta G_x^{uu},\delta G_x^{dd},\delta G_x^{ud},\delta G_x^{du})^T$. The linear coupling term reads

$$\begin{aligned}\int dtdt'\ \Sigma_x^{ac}\big(G_x^{ca}\big)^T &=-I_2^{(2)}/N+\sum_x\frac{1}{2}\int_\Omega\left[\tilde{g}_x^T(\Omega)\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}\tilde{\sigma}_x(-\Omega)+\tilde{\sigma}_x^T(\Omega)\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}\tilde{g}_x(-\Omega)\right]\\ &\equiv -I_2^{(2)}/N-\delta I_2^{(2)}/N.\end{aligned} \tag{C23}$$

c. V *term.* In contrast to the $\zeta > J$ case, the second-order off-diagonal term contributed by the V term is nonzero. Together with the diagonal term, we calculate the second-order fluctuation action

$$\begin{aligned}&\int dtdt'\sum_x f^af^c\hat{I}\frac{V}{4}\big(G_x^{ac}\big)^2\big[\big(-G_x^{ca}\big)^2\big]^T\\ &=-I_3^{(2)}/N+\frac{1}{4}\big\{-6\delta G_x^{uu}(-\Omega)\delta G_x^{uu}(\Omega)G_{s,x}^{uu}(t=0)^2-6\delta G_x^{dd}(-\Omega)\delta G_x^{dd}(\Omega)G_{s,x}^{dd}(t=0)^2\\ &\quad+4G_{s,x}^{ud}(t=0)G_{s,x}^{du}(t=0)\big[\delta G_x^{ud}(-\Omega)\delta G_x^{du}(\Omega)+\delta G_x^{du}(-\Omega)\delta G_x^{ud}(\Omega)\big]\\ &\quad+2\delta G_x^{ud}(-\Omega)\delta G_x^{ud}(\Omega)G_{s,x}^{du}(t=0)^2+2\delta G_x^{du}(-\Omega)\delta G_x^{du}(\Omega)G_{s,x}^{ud}(t=0)^2\big\}\\ &=-I_3^{(2)}/N+\sum_x\int_\Omega\tilde{g}_x^T(\Omega)\mathcal{M}_3^{(2)}\tilde{g}_x(-\Omega)\\ &\equiv -I_3^{(2)}/N-\delta I_3^{(2)}/N,\end{aligned} \tag{C24}$$

where the kernel reads

$$\mathcal{M}_3^{(2)}=\frac{1}{\zeta^2+4\mathcal{S}^2}\begin{pmatrix}\frac{-3\zeta(-1)^xz^2\sqrt{\zeta^2+4\mathcal{S}^2}-6\mathcal{S}^2z^2-3\zeta^2z^2}{4z^2} & 0 & 0 & 0\\ 0 & \frac{3\zeta(-1)^xz^2\sqrt{\zeta^2+4\mathcal{S}^2}-6\mathcal{S}^2z^2-3\zeta^2z^2}{4z^2} & 0 & 0\\ 0 & 0 & \frac{\mathcal{S}^2z^2}{2} & -\mathcal{S}^2\\ 0 & 0 & -\mathcal{S}^2 & \frac{\mathcal{S}^2}{2z^2}\end{pmatrix}. \tag{C25}$$

d. J term.

$$\int dtdt' \sum_x \frac{J}{4}\Big[G^{ac}_{x+1}\big(-G^{ca}_x\big)^T + G^{ac}_{x-1}\big(-G^{ca}_x\big)^T\Big] = -I_4^{(1)}/N + \int_{k,\Omega} \frac{J}{2}\tilde{g}_k^T(\Omega)\begin{pmatrix} \cos k & 0 & 0 & 0 \\ 0 & \cos k & 0 & 0 \\ 0 & 0 & 0 & -\cos k \\ 0 & 0 & -\cos k & 0 \end{pmatrix}\tilde{g}_{-k}(-\Omega)$$
$$\equiv -I_4^{(1)}/N - \delta I_4^{(1)}/N. \tag{C26}$$

e. Effective action. To obtain the effective action, we need to integrate out the self-energy $\tilde{\sigma}$. However, we find there are two zero eigenvalues in the kernel of the fluctuating self-energy (C22). Since the self-energy $\delta\Sigma$ and Green's function δG are linearly coupled, these two zero modes lead to two constraints on δG. After applying these constraints, we can safely integrate out the self-energy in the reduced subspace. Finally, we expect to obtain effective action with $4-2=2$ fluctuation fields in the x space.

We perform the calculations in detail following the arguments above. First, the four eigenvalues of $\mathcal{M}_1^{(2)}$ are

$$\begin{aligned} \lambda_{1,2} &= 0, \\ \lambda_3 &= \frac{\sqrt{\Omega^2 z^2[2\mathcal{S}^2(z^4+1)+\zeta^2 z^2]+[\mathcal{S}^2(z^2+1)^2+\zeta^2 z^2]^2} - \mathcal{S}^2(z^2-1)^2}{2z^2\sqrt{\zeta^2+4\mathcal{S}^2}(\zeta^2+4\mathcal{S}^2+\Omega^2)}, \\ \lambda_4 &= \frac{-\sqrt{\Omega^2 z^2[2\mathcal{S}^2(z^4+1)+\zeta^2 z^2]+[\mathcal{S}^2(z^2+1)^2+\zeta^2 z^2]^2} - \mathcal{S}^2(z^2-1)^2}{2z^2\sqrt{\zeta^2+4\mathcal{S}^2}(\zeta^2+4\mathcal{S}^2+\Omega^2)}. \end{aligned} \tag{C27}$$

The corresponding eigenvectors without normalization are

$$\begin{aligned} u_1 &= (1,-1,0,0)^T, \\ u_2 &= \left(0, \frac{\zeta(-1)^{x+1}}{Sz}, -\frac{1}{z^2}, 1\right)^T, \\ u_3 &= \begin{pmatrix} \frac{S\{\sqrt{\Omega^2z^2[2S^2(z^4+1)+\zeta^2z^2]+[S^2(z^2+1)^2+\zeta^2z^2]^2}+S^2(z^2+1)^2+\zeta z^4[\zeta+i(-1)^x\Omega]\}}{2S^2(\zeta(-1)^x(z^3+z)+i\Omega z)+\zeta^2z^3(\zeta(-1)^x+i\Omega)} \\ \frac{S\{\sqrt{\Omega^2z^2[2S^2(z^4+1)+\zeta^2z^2]+[S^2(z^2+1)^2+\zeta^2z^2]^2}+S^2(z^2+1)^2+\zeta z^4[\zeta+i(-1)^x\Omega]\}}{2S^2(\zeta(-1)^x(z^3+z)+i\Omega z)+\zeta^2z^3(\zeta(-1)^x+i\Omega)} \\ \frac{\zeta(-1)^{x+1}\sqrt{\Omega^2z^2[2S^2(z^4+1)+\zeta^2z^2]+[S^2(z^2+1)^2+\zeta^2z^2]^2}+S^2[\zeta(-1)^x(z^4-1)+2i\Omega z^2]}{2S^2(\zeta(-1)^x(z^2+1)+i\Omega)+\zeta^2z^2[\zeta(-1)^x+i\Omega]} \\ 1 \end{pmatrix}, \\ u_4 &= \begin{pmatrix} \frac{S\{-\sqrt{\Omega^2z^2[2S^2(z^4+1)+\zeta^2z^2]+[S^2(z^2+1)^2+\zeta^2z^2]^2}+S^2(z^2+1)^2+\zeta z^4[\zeta+i(-1)^x\Omega]\}}{2S^2(\zeta(-1)^x(z^3+z)+i\Omega z)+\zeta^2z^3(\zeta(-1)^x+i\Omega)} \\ \frac{S\{-\sqrt{\Omega^2z^2[2S^2(z^4+1)+\zeta^2z^2]+[S^2(z^2+1)^2+\zeta^2z^2]^2}+S^2(z^2+1)^2+\zeta z^4[\zeta+i(-1)^x\Omega]\}}{2S^2(\zeta(-1)^x(z^3+z)+i\Omega z)+\zeta^2z^3(\zeta(-1)^x+i\Omega)} \\ \frac{\zeta(-1)^{x}\sqrt{\Omega^2z^2[2S^2(z^4+1)+\zeta^2z^2]+[S^2(z^2+1)^2+\zeta^2z^2]^2}+S^2[\zeta(-1)^x(z^4-1)+2i\Omega z^2]}{2S^2(\zeta(-1)^x(z^2+1)+i\Omega)+\zeta^2z^2(\zeta(-1)^x+i\Omega)} \\ 1 \end{pmatrix}. \end{aligned} \tag{C28}$$

The eigenvalues and eigenvectors satisfy $\mathcal{M}_1 U = U\,\mathbf{Diag}\{\lambda_1,\lambda_2,\lambda_3,\lambda_4\}$, where $U = (\frac{u_1}{\|u_1\|}, \frac{u_2}{\|u_2\|}, \frac{u_3}{\|u_3\|}, \frac{u_4}{\|u_4\|})$ and **Diag** means constructing diagonal matrix using the elements. The transformation matrix U is normalized and therefore satisfies the unitary condition $UU^\dagger = \hat{1}$. We consider ΣG and the Tr ln term,

$$\begin{aligned} &\sum_x \int_\Omega \tilde{\sigma}_x^T(\Omega) U\,\mathbf{Diag}\{\lambda_1,\lambda_2,\lambda_3,\lambda_4\}U^\dagger \tilde{\sigma}_x(-\Omega) \\ &\quad + \sum_x \frac{1}{2}\int_\Omega \left[\tilde{g}_x^T(\Omega)\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix} UU^\dagger\tilde{\sigma}_x(-\Omega) + \tilde{\sigma}_x^T(\Omega)UU^\dagger\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}\tilde{g}_x(-\Omega)\right] \\ &= \sum_x \int_\Omega \tilde{\sigma}^\dagger_{\mathrm{eig},x}(\Omega)\mathbf{Diag}\{\lambda_1,\lambda_2,\lambda_3,\lambda_4\}\tilde{\sigma}_{\mathrm{eig},x}(-\Omega) + \sum_x \frac{1}{2}\int_\Omega [\tilde{g}^\dagger_{\mathrm{eig},x}(\Omega)\tilde{\sigma}_{\mathrm{eig},x}(-\Omega) + \tilde{\sigma}^\dagger_{\mathrm{eig},x}(\Omega)\tilde{g}_{\mathrm{eig},x}(-\Omega)]. \end{aligned} \tag{C29}$$

Here, we define the field in the eigenbasis

$$\tilde{\sigma}_{\mathrm{eig},x} = U^\dagger \tilde{\sigma}_x,\ \tilde{\sigma}^\dagger_{\mathrm{eig},x} = \tilde{\sigma}_x^T U,$$
$$\tilde{g}^\dagger_{\mathrm{eig},x}(\Omega) = \tilde{g}_x^T(\Omega)\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}U,$$
$$\tilde{g}_{\mathrm{eig},x}(-\Omega) = U^\dagger\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}\tilde{g}_x(-\Omega). \tag{C30}$$

Since $\lambda_{1,2}=0$, the absence of quadratic $(\tilde{\sigma}_{\mathrm{eig},x})_1$ and $(\tilde{\sigma}_{\mathrm{eig},x})_2$ leads to two delta functions, $\delta((\tilde{g}_{\mathrm{eig},x})_1)$ and $\delta((\tilde{g}_{\mathrm{eig},x})_2)$, when performing the integration on the self-energy, which leads to two constraints in the Green's function. In detail,

$$(\tilde{g}^\dagger_{\mathrm{eig},x})_1 = \left(\delta G_x^{uu}, \delta G_x^{dd}, \delta G_x^{du}, \delta G_x^{ud}\right)u_1 \propto -\delta G_x^{uu} + \delta G_x^{dd} = 0,$$
$$(\tilde{g}^\dagger_{\mathrm{eig},x})_2 = \left(\delta G_x^{uu}, \delta G_x^{dd}, \delta G_x^{du}, \delta G_x^{ud}\right)u_2 \propto \frac{\zeta(-1)^{x+1}}{\mathcal{S}z}\delta G_x^{dd} - \frac{1}{z^2}\delta G_x^{du} + \delta G_x^{ud} = 0. \tag{C31}$$

These two constraints reduce to $\delta G_x^{uu} = \delta G_x^{dd} = 0$ when we set $\mathcal{S}=0$. This is consistent with the $\zeta > J$ case where the diagonal terms are decoupled from the off-diagonal terms and therefore should be ignored. Returning to Eq. (C29), we can integrate out the residual self-energy

$$\sum_x \int_\Omega \tilde{\sigma}^\dagger_{\mathrm{eig},x}(\Omega)\mathbf{Diag}\{\lambda_1,\lambda_2,\lambda_3,\lambda_4\}\tilde{\sigma}_{\mathrm{eig},x}(-\Omega) + \sum_x \frac{1}{2}\int_\Omega [\tilde{g}^\dagger_{\mathrm{eig},x}(\Omega)\tilde{\sigma}_{\mathrm{eig},x}(-\Omega) + \tilde{\sigma}^\dagger_{\mathrm{eig},x}(\Omega)\tilde{g}_{\mathrm{eig},x}(-\Omega)]$$
$$\Rightarrow \sum_x \int_\Omega \tilde{\sigma}^\dagger_{\mathrm{eig,R},x}(\Omega)\mathbf{Diag}\{\lambda_3,\lambda_4\}\tilde{\sigma}_{\mathrm{eig,R},x}(-\Omega) + \sum_x \frac{1}{2}\int_\Omega \left[\tilde{g}^T_{\mathrm{eig,R},x}(\Omega)\tilde{\sigma}_{\mathrm{eig,R},x}(-\Omega) + \tilde{\sigma}^\dagger_{\mathrm{eig,R},x}(\Omega)\tilde{g}_{\mathrm{eig,R},x}(-\Omega)\right]$$
$$\Rightarrow -\frac{1}{4}\sum_x \int_\Omega \tilde{g}^\dagger_{\mathrm{eig,R},x}(\Omega)\mathbf{Diag}\left\{\lambda_3^{-1},\lambda_4^{-1}\right\}\tilde{g}_{\mathrm{eig,R},x}(-\Omega). \tag{C32}$$

Here, the R subscript means to take 3, 4 components in the eigenbasis, according to the definition in Eq. (C30). In addition to ΣG and the Tr ln term, we consider the V term and the J term. We apply constraint (C31) in Eq. (C24) and (C26), respectively. Then we sum all contributions together and simplify the result, which leads to

$$-\delta I^{(2)}/N = \sum_x \int_\Omega g_x^T(\Omega)\begin{pmatrix} -\frac{\mathcal{S}^2z^2\sqrt{\zeta^2+4\mathcal{S}^2}}{\zeta^2} & \frac{\sqrt{\zeta^2+4\mathcal{S}^2}(2\mathcal{S}^2+\zeta(\zeta+(-1)^x i\Omega))}{2\zeta^2} \\ \frac{\sqrt{\zeta^2+4\mathcal{S}^2}\{2\mathcal{S}^2+\zeta[\zeta-(-1)^x i\Omega]\}}{2\zeta^2} & -\frac{\mathcal{S}^2\sqrt{\zeta^2+4\mathcal{S}^2}}{\zeta^2z^2}\end{pmatrix}g_x(-\Omega) + V\sum_x\int_\Omega g_x^T(\Omega)$$
$$\times\begin{pmatrix} -\frac{\mathcal{S}^2z^2(\zeta^2+3\mathcal{S}^2)}{\zeta^4+4\zeta^2\mathcal{S}^2} & \frac{\zeta^2\mathcal{S}^2+6\mathcal{S}^4}{2\zeta^4+8\zeta^2\mathcal{S}^2} \\ \frac{\zeta^2\mathcal{S}^2+6\mathcal{S}^4}{2\zeta^4+8\zeta^2\mathcal{S}^2} & -\frac{\mathcal{S}^2(\zeta^2+3\mathcal{S}^2)}{\zeta^2z^2(\zeta^2+4\mathcal{S}^2)}\end{pmatrix}g_x(-\Omega) + \frac{J}{2}\sum_k\int_\Omega g_k^T(\Omega)\begin{pmatrix}\frac{2\mathcal{S}^2z^2\cos(k)}{\zeta^2} & \cos(k)\left(-\frac{2\mathcal{S}^2}{\zeta^2}-1\right)\\ \cos(k)\left(-\frac{2\mathcal{S}^2}{\zeta^2}-1\right) & \frac{2\mathcal{S}^2\cos(k)}{\zeta^2z^2}\end{pmatrix}g_{-k}(-\Omega). \tag{C33}$$

Here, the effective action is still in the bilinear form of vector $g_x = (\delta G_x^{ud}, \delta G_x^{du})^T$ since we have applied two constraints in the derivation. As we pointed out in the main text, we need two identifications,

$$(\delta G_x^{ud}(t,t), \delta G_x^{du}(t,t)) = \varphi_1(x,t)(-iG^{ud}(0), iG^{du}(0)),$$
$$(\delta G_x^{ud}(t,t), \delta G_x^{du}(t,t)) = \varphi_2(x,t)(-iG^{du}(0), -iG^{ud}(0)). \tag{C34}$$

With these two identifications, the action is projected onto two new subspaces, where

$$-\frac{\delta I^{(2)}}{N} = \int_{\Omega,k}(\varphi_1,\varphi_2)\begin{pmatrix}\frac{J\mathcal{S}^2(\cos(k)-1)}{\zeta^2+4\mathcal{S}^2} & -\frac{J\mathcal{S}^2(z^4-1)[\cos(k)-1]}{2z^2(\zeta^2+4\mathcal{S}^2)}\\ -\frac{J\mathcal{S}^2(z^4-1)[\cos(k)-1]}{2z^2(\zeta^2+4\mathcal{S}^2)} & K^\varphi_{2,2}\end{pmatrix}\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}$$
$$+\int_\Omega\sum_x(\varphi_1,\varphi_2)(x,\Omega)\begin{pmatrix}0 & -\frac{i\mathcal{S}^2(-1)^x\Omega(z^4+1)}{2\zeta z^2\sqrt{\zeta^2+4\mathcal{S}^2}}\\ \frac{i\mathcal{S}^2(-1)^x\Omega(z^4+1)}{2\zeta z^2\sqrt{\zeta^2+4\mathcal{S}^2}} & 0\end{pmatrix}\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}(x,-\Omega), \tag{C35}$$

with

$$K^{\varphi}_{2,2} = \frac{\mathcal{S}^2\{-J\mathcal{S}^2(z^4+1)^2[\cos(k)+3] - \zeta^2 J(z^4[\cos(k)+1]+z^8+1) + (z^4+1)^2(\zeta^2+4\mathcal{S}^2)^{3/2}\}}{\zeta^2 z^4(\zeta^2+4\mathcal{S}^2)}. \tag{C36}$$

Here, we have used the identity of parameters (C3). Finally, like in the $\zeta > J$ section, we rewrite the action in the enlarged basis $\boldsymbol{\varphi}(k,\Omega) = (\varphi_1(k,\Omega), \varphi_1(k+\pi,\Omega), \varphi_2(k,\Omega), \varphi_2(k+\pi,\Omega))^T$, which reads

$$-\frac{\delta I^{(2)}}{N} = \int_{\Omega,k} \boldsymbol{\varphi}(k,\Omega)\mathcal{M}^{(2)}\boldsymbol{\varphi}(-k,-\Omega), \tag{C37}$$

where the kernel is

$$\mathcal{M}^{(2)} = \begin{pmatrix} \frac{J\mathcal{S}^2(\cos(k)-1)}{2(\zeta^2+4\mathcal{S}^2)} & \boldsymbol{v} \\ \boldsymbol{v}^\dagger & \mathcal{M}^{(2)}_{\text{gap}} \end{pmatrix}, \tag{C38}$$

where

$$\boldsymbol{v} = \begin{pmatrix} 0 & -\frac{J\mathcal{S}^2(z^4-1)[\cos(k)-1]}{4z^2(\zeta^2+4\mathcal{S}^2)} & -\frac{i\mathcal{S}^2\Omega(z^4+1)}{4\zeta z^2\sqrt{\zeta^2+4\mathcal{S}^2}} \end{pmatrix}, \tag{C39}$$

$$\mathcal{M}^{(2)}_{\text{gap}} = \begin{pmatrix} -\frac{J\mathcal{S}^2[\cos(k)+1]}{2(\zeta^2+4\mathcal{S}^2)} & -\frac{i\mathcal{S}^2\Omega(z^4+1)}{4\zeta z^2\sqrt{\zeta^2+4\mathcal{S}^2}} & \frac{J\mathcal{S}^2(z^4-1)[\cos(k)+1]}{4z^2(\zeta^2+4\mathcal{S}^2)} \\ \frac{i\mathcal{S}^2\Omega(z^4+1)}{4\zeta z^2\sqrt{\zeta^2+4\mathcal{S}^2}} & \left(\mathcal{M}^{(2)}_{\text{gap}}\right)_{2,2} & 0 \\ \frac{J\mathcal{S}^2(z^4-1)[\cos(k)+1]}{4z^2(\zeta^2+4\mathcal{S}^2)} & 0 & (\mathcal{M}^{(2)}_{\text{gap}})_{3,3} \end{pmatrix}, \tag{C40}$$

with

$$\left(\mathcal{M}^{(2)}_{\text{gap}}\right)_{2,2} = \frac{\mathcal{S}^2\{-J\cos(k)[\mathcal{S}^2(z^4+1)^2+\zeta^2 z^4] - 3J\mathcal{S}^2(z^4+1)^2 - \zeta^2 J(z^8+z^4+1) + (z^4+1)^2(\zeta^2+4\mathcal{S}^2)^{3/2}\}}{2\zeta^2 z^4(\zeta^2+4\mathcal{S}^2)},$$

$$\left(\mathcal{M}^{(2)}_{\text{gap}}\right)_{3,3} = \frac{\mathcal{S}^2\{J\cos(k)[\mathcal{S}^2(z^4+1)^2+\zeta^2 z^4] - 3J\mathcal{S}^2(z^4+1)^2 - \zeta^2 J(z^8+z^4+1) + (z^4+1)^2(\zeta^2+4\mathcal{S}^2)^{3/2}\}}{2\zeta^2 z^4(\zeta^2+4\mathcal{S}^2)}. \tag{C41}$$

We derive the final action in the $k \to 0, \Omega \to 0$ limit. We find $\varphi_1(k)$ corresponds to a gapless mode and the other fields correspond to gapped modes. Therefore, we can integrate out the gapped modes to obtain effective action. In detail, we keep $\boldsymbol{v}$ to leading order,

$$\tilde{\boldsymbol{v}} = \begin{pmatrix} 0 & \frac{Jk^2\mathcal{S}^2(z^4-1)}{8z^2(\zeta^2+4\mathcal{S}^2)} & -\frac{i\mathcal{S}^2\Omega(z^4+1)}{4\zeta z^2\sqrt{\zeta^2+4\mathcal{S}^2}} \end{pmatrix}, \tag{C42}$$

and take $\lim_{k\to 0,\Omega\to 0}\mathcal{M}^{(2)}_{\text{gap}}$. The final gapless mode can be obtained as

$$\begin{aligned} -\delta I^{(2)}/N &= \int_{\Omega,k}\varphi_1(k,\Omega)\left[-\frac{Jk^2\mathcal{S}^2}{4(\zeta^2+4\mathcal{S}^2)} - \tilde{\boldsymbol{v}}\Big(\lim_{k\to 0,\Omega\to 0}\mathcal{M}^{(2)}_{\text{gap}}\Big)^{-1}\tilde{\boldsymbol{v}}^\dagger\right]\varphi_1(-k,-\Omega) \\ &\approx \int_{\Omega,k}\varphi_1(k,\Omega)\left[-\frac{Jk^2\mathcal{S}^2}{4(\zeta^2+4\mathcal{S}^2)} - \frac{\mathcal{S}^2\Omega^2}{4(\zeta^2+4\mathcal{S}^2)(2\sqrt{\zeta^2+4\mathcal{S}^2}-J)}\right]\varphi_1(-k,-\Omega) \\ &\approx \int_{\Omega,k}\varphi_1(x,\Omega)\left[-\frac{Jk^2\mathcal{S}^2}{4(\zeta^2+4\mathcal{S}^2)} - \frac{\mathcal{S}^2\Omega^2}{4(\zeta^2+4\mathcal{S}^2)(2\sqrt{\zeta^2+4\mathcal{S}^2}-J)}\right]\varphi_1(-k,-\Omega). \end{aligned} \tag{C43}$$

In leading order, we drop the high-order k^4 term. Finally, the effective action reads

$$S_{\text{eff}} = \frac{N\mathcal{S}^2}{4(\zeta^2+4\mathcal{S}^2)}\int_{x,t}\left(\frac{1}{2\sqrt{\zeta^2+4\mathcal{S}^2}-J}(\partial_t\varphi)^2 + J(\partial_x\varphi)^2\right). \tag{C44}$$

According to Eq. (C3), we have $\sqrt{\zeta^2+4S^2} \geqslant J$, and therefore, the coefficients of the gapless mode are always positive.

[1] G. Mazzucchi, W. Kozlowski, S. F. Caballero-Benitez, T. J. Elliott, and I. B. Mekhov, Phys. Rev. A **93**, 023632 (2016).

[2] Y. Li, X. Chen, and M. P. A. Fisher, Phys. Rev. B **98**, 205136 (2018).

[3] B. Skinner, J. Ruhman, and A. Nahum, Phys. Rev. X **9**, 031009 (2019).
[4] Y. Li, X. Chen, and M. P. A. Fisher, Phys. Rev. B **100**, 134306 (2019).
[5] M. Szyniszewski, A. Romito, and H. Schomerus, Phys. Rev. B **100**, 064204 (2019).
[6] A. Chan, R. M. Nandkishore, M. Pretko, and G. Smith, Phys. Rev. B **99**, 224307 (2019).
[7] R. Vasseur, A. C. Potter, Y.-Z. You, and A. W. W. Ludwig, Phys. Rev. B **100**, 134203 (2019).
[8] T. Zhou and A. Nahum, Phys. Rev. B **99**, 174205 (2019).
[9] M. J. Gullans and D. A. Huse, Phys. Rev. Lett. **125**, 070606 (2020).
[10] C.-M. Jian, Y.-Z. You, R. Vasseur, and A. W. W. Ludwig, Phys. Rev. B **101**, 104302 (2020).
[11] Y. Fuji and Y. Ashida, Phys. Rev. B **102**, 054302 (2020).
[12] A. Zabalo, M. J. Gullans, J. H. Wilson, S. Gopalakrishnan, D. A. Huse, J. H. Pixley, Phys. Rev. B **101**, 060301(R) (2020).
[13] M. J. Gullans and D. A. Huse, Phys. Rev. X **10**, 041020 (2020).
[14] S. Choi, Y. Bao, X.-L. Qi, and E. Altman, Phys. Rev. Lett. **125**, 030505 (2020).
[15] Y. Bao, S. Choi, and E. Altman, Phys. Rev. B **101**, 104301 (2020).
[16] A. Nahum, S. Roy, B. Skinner, and J. Ruhman, PRX Quantum **2**, 010352 (2021).
[17] R. Fan, S. Vijay, A. Vishwanath, and Y.-Z. You, Phys. Rev. B **103**, 174309 (2021).
[18] S. Sang and T. H. Hsieh, Phys. Rev. Res. **3**, 023200 (2021).
[19] O. Alberton, M. Buchhold, and S. Diehl, Phys. Rev. Lett. **126**, 170602 (2021).
[20] A. Lavasani, Y. Alavirad, and M. Barkeshli, Nat. Phys. **17**, 342 (2021).
[21] X. Turkeshi, A. Biella, R. Fazio, M. Dalmonte, and M. Schiró, Phys. Rev. B **103**, 224210 (2021).
[22] Y. Le Gal, X. Turkeshi, and M. Schirò, SciPost Phys. **14**, 138 (2023).
[23] S.-K. Jian, C. Liu, X. Chen, B. Swingle, and P. Zhang, Phys. Rev. Lett. **127**, 140601 (2021).
[24] P. Zhang, C. Liu, S.-K. Jian, and X. Chen, Quantum **6**, 723 (2022).
[25] C. Liu, P. Zhang, and X. Chen, SciPost Phys. **10**, 048 (2021).
[26] P. Zhang, S.-K. Jian, C. Liu, and X. Chen, Quantum **5**, 579 (2021).
[27] P. Zhang, Front. Phys. **17**, 43201 (2022).
[28] S. Sahu, S.-K. Jian, G. Bentsen, and B. Swingle, Phys. Rev. B **106**, 224305 (2022).
[29] T. Rakovszky, F. Pollmann, and C. W. von Keyserlingk, Phys. Rev. X **8**, 031058 (2018).
[30] V. Khemani, A. Vishwanath, and D. A. Huse, Phys. Rev. X **8**, 031057 (2018).
[31] H. Guo, Y. Gu, and S. Sachdev, Phys. Rev. B **100**, 045140 (2019).
[32] A. J. Friedman, A. Chan, A. De Luca, and J. T. Chalker, Phys. Rev. Lett. **123**, 210603 (2019).
[33] T. Rakovszky, F. Pollmann, and C. W. von Keyserlingk, Phys. Rev. Lett. **122**, 250602 (2019).
[34] T. Zhou and A. W. W. Ludwig, Phys. Rev. Res. **2**, 033020 (2020).
[35] Y. Huang, IOP SciNotes **1**, 035205 (2020).
[36] J. Kudler-Flam, R. Sohal, and L. Nie, SciPost Phys. **12**, 117 (2022).
[37] L. Agarwal and S. Xu, J. High Energy Phys. 02 (2022) 045.
[38] P. Zhang, J. High Energ. Phys. 04 (2023) 105.
[39] L. S. Levitov, H. Lee, and G. B. Lesovik, J. Math. Phys. **37**, 4845 (1996).
[40] I. Klich and L. Levitov, Phys. Rev. Lett. **102**, 100502 (2009).
[41] H. F. Song, C. Flindt, S. Rachel, I. Klich, and K. Le Hur, Phys. Rev. B **83**, 161408(R) (2011).
[42] P. Calabrese, M. Mintchev, and E. Vicari, Europhys. Lett. **98**, 20003 (2012).
[43] R. Süsstrunk and D. A. Ivanov, Europhys. Lett. **100**, 60009 (2012).
[44] H. F. Song, S. Rachel, and K. Le Hur, Phys. Rev. B **82**, 012405 (2010).
[45] G. C. Levine, M. J. Bantegui, and J. A. Burg, Phys. Rev. B **86**, 174202 (2012).
[46] H. F. Song, S. Rachel, C. Flindt, I. Klich, N. Laflorencie, and K. Le Hur, Phys. Rev. B **85**, 035409 (2012).
[47] D. A. Ivanov and A. G. Abanov, Phys. Rev. E **87**, 022114 (2013).
[48] V. Eisler, Phys. Rev. Lett. **111**, 080402 (2013).
[49] V. Eisler and Z. Rácz, Phys. Rev. Lett. **110**, 060602 (2013).
[50] I. Klich, J. Stat. Mech. (2014) P11006.
[51] B. Bertini, P. Calabrese, M. Collura, K. Klobas, and C. Rylands, arXiv:2212.06188.
[52] F. Barratt, U. Agrawal, S. Gopalakrishnan, D. A. Huse, R. Vasseur, and A. C. Potter, Phys. Rev. Lett. **129**, 120604 (2022).
[53] E. McCulloch, J. De Nardis, S. Gopalakrishnan, and R. Vasseur, arXiv:2302.01355.
[54] H. Oshima and Y. Fuji, Phys. Rev. B **107**, 014308 (2023).
[55] B.-B. Chen, H.-H. Tu, Z. Y. Meng, and M. Cheng, Phys. Rev. B **106**, 094415 (2022).
[56] Y.-C. Wang, M. Cheng, and Z. Y. Meng, Phys. Rev. B **104**, L081109 (2021).
[57] Y.-C. Wang, N. Ma, M. Cheng, and Z. Y. Meng, SciPost Phys. **13**, 123 (2022).
[58] W. Israel, Phys. Lett. A **57**, 107 (1976).
[59] J. Maldacena, J. High Energy Phys. 04 (2003) 021.
[60] A. Kitaev and S. J. Suh, J. High Energy Phys. 05 (2018) 183.
[61] Y. Gu, A. Kitaev, S. Sachdev, and G. Tarnopolsky, J. High Energy Phys. 02 (2020) 157.
[62] J. Maldacena and D. Stanford, Phys. Rev. D **94**, 106002 (2016).
[63] H. Zhai, *Ultracold Atomic Physics* (Cambridge University Press, Cambridge, 2021).
[64] P. Zhang, J. High Energy Phys. 06 (2020) 143.
[65] E. Tirrito, A. Santini, R. Fazio, and M. Collura, SciPost Phys. **15**, 096 (2023).