# Extension of the SYK model to 1+1 dimensions in the strong coupling limit

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We study a $1+1$ (space-time)-dimensional extension of the $0+1$-dimensional Sachdev-Ye-Kitaev model for $N$ Majorana fermions, with random all-to-all quartic interactions, averaged over disorder. At large interaction couplings and large $N$, the conformal symmetry of the effective action emerges, which is not broken spontaneously as in the original $0+1$ SYK model. Two-point correlators are obtained from a coupling expansion of the Schwinger-Dyson equations. For $N=4$, the model can be mapped onto complex fermions and solved exactly via the bosonization technique, featuring two branches of excitations: a gapped "pseudocharge" mode and a gapless "pseudospin" mode. We give an approximate analytic form of the two-point correlators at large distances and zero temperature for $N=4$, which is adopted heuristically to evaluate an approximation to the large-$N$ free energy in the zero temperature limit, numerically. The fact that this energy displays an absolute minimum at a finite value of the gap, though in a restricted range of parameters, suggests that a gapped branch of excitations is also present in our extension of the model at least in that range of parameters.

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## I. INTRODUCTION

The Sachdev-Ye-Kitaev (SYK) [1–3] model, which describes random all-to-all $J$ interaction between $N$ Majorana fermions in $0+1$ dimensions, has become highly popular as a holographic dual for gravity theories of black holes [4–7]. Disorder average, obtained by contracting the couplings of the interaction $\overline{J^a_{ijkl}J^b_{ijkl}} = 3!\,\delta_{ab}J^2/N^3$, where $a, b$ denote replica indices, makes the model exactly solvable in the limit of large $N$. A conformal symmetry emerges at strong coupling, which spontaneously breaks at low energies, down to the $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ group symmetry [8], giving rise to soft modes, finite zero-temperature entropy, and maximally chaotic behavior. As a tensor model [9,10], it is dominated by melon diagrams in the large-$N$ limit and fixed by the $N$-dependent parameter $J$. This melonic mean-field behavior has been found even in other nonrandom SYK-like tensor models [11–13], demonstrating that the random distribution of the model is not really important [14].

The short-range spectral correlations given by random matrix theory have shown the model to be quantum chaotic [15]. Indeed, the out-of-time-ordered-correlator grows exponentially on an inverse timescale which corresponds to a classical Lyapunov exponent $\lambda_L$ and saturates at times less than the "scrambling time" with [16,17] $\lambda_L \sim 2\pi k_B T/\hbar$.

Generalized SYK models and complex fermion versions of it [10,18–23] have been proposed with extension to higher-space dimensions [22,24–29], in particular in the context of condensed matter, having in mind dot arrays [30,31] with a hopping term or the embedding of 0+1 subsystems in a Fermi liquid environment [20,32]. These extensions to higher-space dimensionality appear to be a tractable benchmark for interacting quantum many-particle system with non-Fermi-liquid (NFL) behavior [33–35]. When dealing with hopping in a spatial lattice at lowest perturbative order, in the IR limit, the response of the fermionic excitations, in the conformal symmetry limit, to an external driving to be specified, gives rise to the celebrated linear temperature dependence of the transported current over a large range of temperatures and to the constancy in temperature of the thermal conductivity [22,36,37]. As this is a striking feature of the resistivity which is experimentally found in the normal phase of the high critical temperature ($\mathrm{HT}_c$) superconducting materials, these models, dubbed "strange metals," are extensively studied in that connection [32,38–46]. It is interesting that the addition of a kinetic term to the model carries a complex $U(1)$ phase with it, to be added to the real fields, which gives rise to bosonic collective gapped diffusive modes [20,47,48].

A different kind of generalization have been studied by extending the original $0+1$-dimensional ($0+1$ d) SYK model to $1+1$-dimensional ($1+1$ d) space-time dimensions in the continuum limit, as a field theoretical model. By adding an extra dimension, the canonical scale dimension of the fermions is changed to $1/2$. This makes the interaction term marginal at best. To avoid this problem, some authors have considered a topological kinetic term, to acquire zero scale dimension of the fields [49]. Of course, this term is nonlocal. However, a local higher-dimensional theory with scale invariance and Lorentz symmetry is found in the supersymmetry context [50]. In this model, it is necessary to define a superfield involving both bosons and fermions. Finally, a family of models have been proposed in the extension to $1+1$ d and $2+1$ d, described by a large number of bosons and fermions, via local random Yukawa coupling [42–44,46,51]. A quartic fermion interaction can be thought of as a two fermion fields interaction mediated by a bosonic field. These variants can exhibit critical saddle-point solution and maximal scrambling.

This paper is devoted to the study of the collective excitations of the extension of the $0+1$ d SYK model in the continuum limit, when a one-dimensional space dependence is added. We consider a generalized form of the nonchiral $1+1$ d SYK model involving two different sets of real couplings among $N$ Majorana fermions, $J_{ijkl}$ and $Q_{ijkl}$. The couplings are antisymmetric with respect to any two indices. When the $J$ couplings vanish, the model can be seen as the random Thirring model [52], while it becomes decoupled into left and right-mover chiral SYK systems [53], when the $Q$ couplings vanish.

In Sec. II, we present the $1+1$ d extended model by expressing the propagator, after disorder average, as a $2\times 2$ matrix in the chiral basis, with diagonal elements $g_\pm$ and off-diagonal nonchiral contributions $g_\cap, g_\cup$. The emergence of the conformal symmetry at large couplings $J, Q$, similar to the $0+1$ d case, is discussed. In this limit, when a UV cutoff $\Lambda$ is introduced in real time-space to regularize the singularity at small arguments of the correlators with a logarithmic factor, we show that the model indeed flows to strong coupling and that the Schwinger-Dyson equation in the $1/N \to 0$ limit can be solved within the conformal symmetry limit. When $Q = 0$ we prove that the model is still critical. Moving away from the strong coupling fixed point, the two-point correlation function displays a power-law dependence at large distance with a non-free-fermion-like exponent $\Gamma \neq -1$. This feature confirms the non-Fermi-liquid nature of the excitations.

We argue that the $Q \neq 0$ case is noncritical at the two-point level. Our goal is to try to infer the properties of the excitation spectrum of the $Q \neq 0$, large-$N$ model heuristically from the results of the $N = 4$ case, which are presented in Sec. III. We take advantage of the fact that the restriction of the model to $N = 4$ can be exactly solved, via mapping to complex fermions and bosonization. For $N = 4$ there are just two independent coupling parameters $J \equiv J_{1234}, Q_{1234}$ and we limit the analysis to the important case $J = Q$.

The action of the $N = 4$ version of the model, for a given realization of the interaction coupling $J$, can be expressed in terms of complex fermions with both chiralities labeled by $\pm$, $(c_{\sigma\pm}, c^\dagger_{\sigma\pm})$, where $\sigma =\uparrow,\downarrow$ is a pseudospin label. In turn, the complex fermion action can be mapped onto two uncoupled bosonic sine-Gordon [54] actions which we dubbed pseudospin and pseudocharge actions, characterized by the corresponding velocities $u_{s,c} = u_0\sqrt{1 \mp \frac{J}{\pi u_0}}$, where $u_0$ is a velocity scale. As it is well known, the sine-Gordon model is in the critical phase with a power-law decay of the two-point correlation function when the corresponding interaction coupling $\mathcal{K} > 1$, while it has a gapped spectrum with exponentially decaying correlators when $\mathcal{K} < 1$. The pseudospin velocity vanishes when $J \to \pi u_0$ and $\mathcal{K}_s = 1/\sqrt{1 - \frac{J}{\pi u_0}}$ diverges, thus marking the strong coupling limit of this $N = 4$ model. In conclusion, the $N = 4$ model, which is not Lorentz invariant, nor conformally symmetric, displays two excitation branches, the pseudospin gapless excitations and the pseudocharge gapped excitations. We expect that these general features are maintained in the large-$N$ limit.

Based on these findings, we surmise that the large-$N$ limit is characterized by the disorder averaged interaction coupling $J$ and by a gap $\Delta$, stemming from a gapped branch of the spectrum. In Sec. IV, we provide an approximate form for the two-point correlators, in analogy with the $N = 4$ case, which reduce to the free ones when $J, \Delta \to 0$ uniformly, and use these analytic expressions to evaluate the free energy at the lowest $1/N$ order, in the zero-temperature limit. The free energy is numerically minimized with respect to the parameter $\Delta$ for increasing $J$. We find an intermediate range of values of $J$ in which an absolute minimum of the energy is found, corroborating the idea that a gapped excitation branch may arise in the spectrum with increasing interaction among the fermions. However, the analogy with the $N = 4$ case cannot be pushed to large values of $J$ either, because of the finite value of $J$ at which the $N = 4$ reduced model breaks down.

A short summary and conclusions can be found in Sec. V. The Fourier transform of the correlators in the conformal limiting case are derived in Appendix A. Appendix B provides details about the mapping to complex fermions of the $N = 4$ case. Appendix C reports on the approximations used to extract an analytic form out of the correlators of the $N = 4$ reduced model in the pseudospin and pseudocharge representation.

## II. THE ACTION FOR A NONCHIRAL 1+1 SYK SYSTEM

We consider a system of fermionic degrees of freedom defined along the 1d line $x$ and labeled by a flavor index $i$ that can take $N$ (even) values. In the chiral representation, the real (Majorana) fermionic operators will be denoted by $\psi_i(x,\tau)$, where $\psi_i^T(x,\tau) \equiv (\psi_{i+}(x,\tau), \psi_{i-}(x,\tau))$ and $\pm$ labels the chirality. The $\psi_i(x,\tau)$'s of different flavor or site anticommute,

$$\psi_i(x,\tau)\,\psi_j(y,\tau) + \psi_j(y,\tau)\,\psi_i(x,\tau) = \delta_{i,j}\,\delta(x-y). \quad (1)$$

In Euclidean space, $\psi_{i+}(x,\tau)$ is only function of the complex coordinate $z = x + iu_0\tau$ while $\psi_{i-}(x,\tau)$ is only function of $\bar{z} = x - iu_0\tau$. In the following, $x$ in $\psi_{i\pm}(x)$ will denote both variables $(x,\tau)$ if no ambiguity arises.

The free Majorana spectrum is linearized around $k = 0$ with velocity $\pm u_0$ for right/left movers. The action for the free massless case is

$$S_0 = \frac{1}{2}\sum_{i=1}^{N}\int d^2x\,\psi_i^T(x)(-\partial_\tau + iu_0\sigma_z\partial_x)\psi_i(x). \quad (2)$$

Introducing the $\gamma$ matrices,

$$\gamma^0 = \sigma_x, \quad \gamma^1 = -i\sigma_y, \quad \gamma^5 = \sigma_z, \quad (3)$$

the interaction can be written as

$$S_I = \int d^2x\left[\frac{1}{2}\sum_{i<j<k<l} J_{ijkl}(\bar{\psi}_i\gamma^\mu\psi_j)(\bar{\psi}_k\gamma^\mu\psi_l) + \sum_{i<j<k<l} Q_{ijkl}\left(\frac{1}{2}(\bar{\psi}_i\gamma^\mu\psi_j)(\bar{\psi}_k\gamma_\mu\psi_l) + (\bar{\psi}_i\psi_j\bar{\psi}_k\psi_l)\right)\right], \quad (4)$$

where the couplings $J_{ijkl}$ and $Q_{ijkl}$ are real and antisymmetric with respect to any two indices. In the large-$N$ case, we assume that they obey the random Gaussian distribution $P$ with

$$\overline{(J_{ijkl})^2} = \frac{3!J^2}{N^3}, \qquad \overline{(Q_{ijkl})^2} = \frac{Q^2}{8N^3}. \tag{5}$$

If $J = 0$, then we have the random Thirring model [52]. If $Q = 0$, then we have two decoupled SYK models for right/left-moving fermions [53].

The Hubbard-Stratonovich procedure requires the introduction of the Green's functions $g_{aa'}(x,x') = \frac{1}{N}\langle\sum_j \psi_{ja}(x)\psi_{ja'}(x')\rangle$, where $a$ is the chirality label. The Green's function $\hat{G}$ and the self energy $\hat{\Sigma}$ are written as $2\times 2$ matrices in chirality space (where the label $\pm \equiv \pm\pm$, while $\cap \equiv +-$ and $\cup \equiv -+$):

$$\hat{G} = \begin{pmatrix} g_+ & g_\cap \\ -g_\cup & g_- \end{pmatrix}, \quad \hat{\Sigma} = \begin{pmatrix} \Sigma_+ & -\Sigma_\cap \\ \Sigma_\cup & \Sigma_- \end{pmatrix}, \tag{6}$$
$$\hat{G}^{-1} = \begin{pmatrix} -(\partial_\tau - iu_0\partial_x) - \Sigma_+ & \Sigma_\cap \\ -\Sigma_\cup & -(\partial_\tau + iu_0\partial_x) - \Sigma_- \end{pmatrix},$$

and we use the standard replica method to perform the ensemble average over random coupling constants, assuming that the replica symmetry is unbroken. In the chirality representation, the partition function

$$\overline{Z} = \int \mathcal{D}\psi \int \mathcal{D}\mathbf{J} \int \mathcal{D}\mathbf{Q}\, P(J_{ijkl})P(Q_{ijkl})e^{[-S_0+S_I]} \tag{7}$$

becomes, after disorder average [55],

$$\overline{Z} = \int \mathcal{D}\Sigma\mathcal{D}G\Bigg\{\int \mathcal{D}\psi\;\; e^{-\frac{1}{2}\sum_i^N \int d^2x(\bar{\psi}_i\gamma^\mu\partial_\mu\psi_i)} \times \exp\Bigg[-\int \frac{d^2xd^2x'}{2}\Bigg(\Sigma_+\psi_+(x)\psi_+(x') + \Sigma_\cap\psi_-(x)\psi_+(x') + \Sigma_\cup\psi_+(x)\psi_-(x')$$
$$+ \Sigma_-\psi_-(x)\psi_-(x') - \frac{J^2}{4}\sum_\alpha g_\alpha^4(x,x') - \frac{Q^2}{2}(g_+^2g_-^2 + g_\cap^2g_\cup^2 - 4\,g_+g_\cap g_-g_\cup)\Bigg)\Bigg]\Bigg\}^N, \tag{8}$$

where the label $\alpha$ runs over $\{+,-,\cap,\cup\}$. Integrating out the fermions, we obtain the effective action

$$-\overline{S}[\hat{\Sigma},\hat{G}] = N\Bigg[\ln \mathrm{Pf}[\hat{G}^{-1}] - \frac{1}{2}\int d^2xd^2x'\mathbf{Tr}[\hat{\Sigma}(x,x')\,\hat{G}(x',x)]$$
$$+\frac{1}{2}\int d^2xd^2x'\Bigg(\frac{J^2}{4}\sum_\alpha g_\alpha^4(x,x') + \frac{Q^2}{4}\{\mathbf{Tr}[(\mathcal{P}\hat{G}\mathcal{P}^\dagger\,\hat{G})^2] + 4\,g_+g_\cap g_-g_\cup\}\Bigg)\Bigg]. \tag{9}$$

Here Pf $[O]$ denotes the pfaffian of the operator $O$, and $\mathcal{P}\hat{G}\mathcal{P}^\dagger = (\begin{smallmatrix} g_- & g_\cap \\ -g_\cup & g_+ \end{smallmatrix})$ is the parity-transformed and transposed $\hat{G}$ function. Since the action is translationally invariant in both time and space, its two-point functions $g_\alpha, \Sigma_\alpha$ will depend on the difference of spacetime coordinates, e.g., $g_\alpha(\tau_1, ix_1; \tau_2, ix_2) = g_\alpha[(\tau_1-\tau_2), i(x_1-x_2)]$. In the rest of the paper, the pair $(\tau, x)$ will be often denoted simply by $r$.

### A. The Schwinger-Dyson equations in the conformal limit

As in the $0+1$ SYK model, the Schwinger-Dyson equations derived from the action of Eq. (9) provide solutions which are invariant under reparametrizations in the limit of large $J, Q$. We prove this here first in the simpler setup which drops the off-diagonal terms in the matrices of Eq. (6). In this case, the effective action becomes

$$\overline{S}[\Sigma, G] \to N\sum_{a=\pm}\Bigg[-\ln \mathrm{Pf}\big[-(\partial_\tau - ai\partial_x) - \Sigma_a(x,x')\big]$$
$$+\frac{1}{2}\int d^2xd^2x'\Bigg(\Sigma_a(x,x')g_a(x,x') - \frac{J^2}{4}g_a^4(x,x') - \frac{Q^2}{2}g_+^2(x,x')g_-^2(x,x')\Bigg)\Bigg]. \tag{10}$$

In the large-$N$ limit, the saddle-point approximation for estimating $\overline{Z}$ gives the Schwinger-Dyson equations:

$$\Sigma_+(x,x') = J^2g_+^3(x,x') + Q^2g_+(x,x')g_-^2(x,x'), \tag{11}$$

$$\Sigma_-(x,x') = J^2g_-^3(x,x') + Q^2g_-(x,x')g_+^2(x,x'). \tag{12}$$

As the prefactor of $\Sigma_a$ given by Eqs. (11) and (12) includes positive powers of $J, Q$, the inverse free Green's function term appearing in Eqs. (6) and (10) can be dropped in the large $J, Q$ limit and the conformal symmetry emerges in this limit, as it occurs in the $0+1$ SYK model. Equations (11) and (12) become invariant under the conformal transformation $z \to f(z)$ and $\bar{z} \to \bar{f}(\bar{z})$, which reads

$$g_\pm(z, z'; \bar{z}, \bar{z}') \equiv [f^2]^{\Delta_\pm}[\bar{f}^2]^{\bar{\Delta}_\pm}\widetilde{g_\pm}, \tag{13}$$

where $[f^2]$ stands for $[f'(z)f'(z')]$, $[\bar{f}^2]$ stands for $[\bar{f}'(\bar{z})\bar{f}'(\bar{z}')]$, and $\widetilde{g_\pm} \equiv g_\pm(f(z), f(z'); \bar{f}(\bar{z}), \bar{f}(\bar{z}'))$. In a four-fields interacting model, the self-energy $\Sigma_a$, according to Eq. (12), transforms, with the same short-hand notation, as

$$\begin{aligned}\Sigma_\pm(z, z'; \bar{z}, \bar{z}') &= J^2[f^2]^{(4-1)\Delta_\pm}[\bar{f}^2]^{(4-1)\bar{\Delta}_\pm}\widetilde{g_\pm}^{4-1} \\ &\quad + Q^2[f^2]^{\frac{4}{2}(\Delta_\pm+\Delta_\mp)-\frac{4}{4}\Delta_\pm} \\ &\quad \times [\bar{f}^2]^{\frac{4}{2}(\bar{\Delta}_\pm+\bar{\Delta}_\mp)-\frac{4}{4}\bar{\Delta}_\pm}\widetilde{g_\pm}^{\frac{4}{4}}\widetilde{g_\mp}^{\frac{4}{2}}. \end{aligned} \tag{14}$$

Under the same approximations, the unitarity condition,

$$\begin{aligned}&\int d^2z'\, g_\pm(z, z'; \bar{z}, \bar{z}')\, \Sigma_\pm(z', z''; \bar{z}', \bar{z}'') \\ &\quad = -\delta(z - z'')\,\delta(\bar{z} - \bar{z}''),\end{aligned} \tag{15}$$

arises from minimization of the action with respect to $\Sigma_\pm$. The unitarity condition of Eq. (15) implies that $\frac{4}{2}(\Delta_\pm + \Delta_\mp) = \frac{4}{2}(\bar{\Delta}_\pm + \bar{\Delta}_\mp) = 1$. Unbroken parity implies that $g_+(z, \bar{z}) = g_-(\bar{z}, z) \equiv g(z, \bar{z})$. Under these assumptions, $\Delta_+ = 0$ and $\bar{\Delta}_- = 0$, so that we can just redefine $\bar{\Delta}_+ \to \Delta_-$ and $\bar{\Delta}_- \to \Delta_+$, and we can conclude that the saddle point and unitarity equations are invariant under conformal transformation $z \to f(z), \bar{z} \to f(\bar{z})$ with $\Delta \equiv \Delta_- = \frac{1}{2}$. Reparametrization invariance suggests the following solutions for Eqs. (11), (12), and (15):

$$\begin{aligned}g(z, \bar{z}) &= \frac{C}{z}\ln^\alpha(z\bar{z}\Lambda^2), \\ \Sigma(z, \bar{z}) &= C^3\left(\frac{J^2}{z^3} + \frac{Q^2}{z\bar{z}^2}\right)\ln^{3\alpha}(z\bar{z}\Lambda^2),\end{aligned} \tag{16}$$

where $\alpha$ and $C$ are constants to be fixed by Eq. (15). Here the Lorentz invariance is explicitly broken, starting from the action where a UV regularization has to be introduced all the way down to the IR limit [52] with an UV cutoff $\Lambda$. The Fourier transforms are derived in Appendix A. In the large $J, Q$ limit they are

$$g(p, \bar{p}) = i\pi\frac{C}{\bar{p}}\ln^\alpha\left(\frac{\Lambda^2}{|p|^2}\right), \tag{17}$$

$$\Sigma(p, \bar{p}) \approx i\bar{p}C^3(J^2 + Q^2)\frac{\pi}{3\alpha + 1}\ln^{3\alpha+1}\left(\frac{\Lambda^2}{|p|^2}\right), \tag{18}$$

where $|p|^2 = p\bar{p}$. The log-term softens the RG flow and regularizes the Fourier transformation of the self-energy given by Eqs. (17) and (18) [3,52].

From the unitary condition Eq. (15), transformed to momentum space, we get $\alpha = -\frac{1}{4}$ and $4\pi^2C^4(J^2 + Q^2) = 1$.

Here symmetry breaking was produced on purpose. This is at variance with the $0+1$ SYK model, in which the solution of the Schwinger-Dyson equations spontaneously breaks the conformal symmetry.

In the small $J, Q$ limit, we can obtain solutions by carrying out perturbation theory [52], considering the Fourier transform $\mathcal{FT}[\frac{1}{z}F_\alpha(\ln(z\bar{z}\Lambda^2))] = \frac{i\pi}{\bar{p}}F_\alpha(\ln(\frac{\Lambda^2}{|p|^2}))$, and $F_\alpha$ being the solution of the Schwinger-Dyson equation for the Green's function, once one factorizes the free-fermion-like $1/z$ part. Rewriting Eq. (15) as

$$\frac{1}{F_\alpha} = \pi^2(J^2 + Q^2)\int^{\ln(\Lambda^2/|p|^2)} dy\, F_\alpha^3[y], \tag{19}$$

one can get the differential equation $F_\alpha' = -\pi^2(J^2 + Q^2)F_\alpha^5$ (the prime means derivative with respect to the argument), which provides the solution

$$F_\alpha = \left[1 + 4\pi^2(J^2 + Q^2)\ln\left(\frac{\Lambda^2}{|p|^2}\right)\right]^{-1/4}.$$

Correlation functions involving $F_\alpha$ can be plugged into the Callan-Symanzik equation (with $\tilde{J}^2 = J^2 + Q^2$):

$$\left[\Lambda\frac{\partial}{\partial\Lambda} + \beta(\tilde{J})\frac{\partial}{\partial\tilde{J}} + 2\,\gamma(\tilde{J})\right]\frac{\bar{p}}{\left[1 + 4\pi^2\tilde{J}^2\ln\left(\frac{\Lambda^2}{|p|^2}\right)\right]^{1/4}} = 0. \tag{20}$$

The $\beta$ function satisfied by the fermion propagator $g(p, \bar{p})$ can be obtained as

$$\beta(\tilde{J}) = 4\pi^2\tilde{J}^3, \quad \gamma(\tilde{J}) = \pi^2\tilde{J}^2. \tag{21}$$

As the $\beta$ function is positive, the coupling increases with increasing energy scale and the model becomes strongly coupled at high energy.

We must notice that the RG flow has been obtained by considering the average coupling $\tilde{J}$, which is coming from random coupling for realizations $i, j, k, l$. One can think that some realizations make also the coupling scale invariant or decreasing with increasing energy scale. Then, there are relevant and irrelevant operators that will grow or decrease as we flow into the IR and these can also change as the couplings themselves evolve. This point is discussed in Ref. [52], where the authors consider the flow of ensemble of couplings for the random Thirring model.

When the off-diagonal terms are included from Eq. (9), the full action becomes

$$\begin{aligned}-\overline{S}[\hat{\Sigma}, \hat{G}] = N\Bigg[&\ln \mathrm{Pf}[\hat{G}^{-1}] - \frac{1}{2}\int d^2x d^2x'\Bigg(\Sigma_+ g_+ + \Sigma_\cap g_\cup + \Sigma_\cup g_\cap + \Sigma_- g_- \\ &- \frac{J^2}{4}\sum_\alpha g_\alpha^4(x, x') - \frac{Q^2}{2}(g_+^2 g_-^2 + g_\cap^2 g_\cup^2 - 4\, g_+ g_\cap g_- g_\cup)\Bigg)\Bigg],\end{aligned} \tag{22}$$

where the label $\alpha$ runs over $\{+, -, \cap, \cup\}$. Most of the arguments developed for the diagonal SD solution can be extended and applied to the off-diagonal case. Maximization with respect to $g_\alpha$ gives

$$\Sigma_\pm(z,\bar z) = J^2 g_\pm^3 + Q^2[g_\mp^2 g_\pm - 2\, g_\cap g_\mp g_\cup],$$
$$\Sigma_\cap(z,\bar z) = J^2 g_\cup^3 + Q^2[g_\cap^2 g_\cup - 2\, g_- g_\cap g_+],$$
$$\Sigma_\cup(z,\bar z) = J^2 g_\cap^3 + Q^2[g_\cup^2 g_\cap - 2\, g_+ g_\cup g_-].$$

The equations can be solved in the conformal limit. Thus, in real space we assume the Ansatz

$$g_+(z,\bar z) = \overline{g_-(z,\bar z)} = \frac{a}{z}\ln^\alpha(|z|^2\Lambda^2),$$
$$g_\cap(z,\bar z) = g_\cup(z,\bar z) = \frac{b}{|z|}\ln^\alpha(|z|^2\Lambda^2), \tag{23}$$

with $a$, $b$ real, obtaining

$$\Sigma_+(z,\bar z) = \left\{J^2\frac{a^3}{z^3} + Q^2\left[\frac{a^3}{\bar z|z|^2} - 2\,\frac{ab^2}{\bar z|z|^2}\right]\right\}\ln^{3\alpha}(|z|^2\Lambda^2),$$
$$\Sigma_-(z,\bar z) = \overline{\Sigma_+(z,\bar z)},$$
$$\Sigma_\cap(z,\bar z) = \left\{J^2\frac{b^3}{|z|^3} + Q^2\left[\frac{b^3}{|z|^3} - 2\,\frac{a^2 b}{|z|^3}\right]\right\}\ln^{3\alpha}(|z|^2\Lambda^2),$$
$$\Sigma_\cup(z,\bar z) = \Sigma_\cap(z,\bar z). \tag{24}$$

With these definitions, we show that the Schwinger-Dyson equations can be easily solved in the strong coupling, conformal limit.

The Fourier transforms of the given functions are derived in Appendix A, Eqs. (A3), (A11), (A12), and (A13), giving

$$g_+(p,\bar p) = i\pi\frac{a}{\bar p}\ln^\alpha\left(\frac{\Lambda^2}{|p|^2}\right),$$
$$g_\cap(p,\bar p) = i\pi\frac{b}{|p|}\ln^\alpha\left(\frac{\Lambda^2}{|p|^2}\right), \tag{25}$$
$$\Sigma_+(p,\bar p) \approx i\pi\ \bar p\ \frac{(a^3J^2 + (a^3 - 2ab^2)Q^2)}{3\alpha+1}\ \ln^{3\alpha+1}\left(\frac{\Lambda^2}{|p|^2}\right),$$
$$\Sigma_\cap(p,\bar p) \approx i\pi\ |p|\frac{(b^3J^2 + (b^3 - 2a^2b)Q^2)}{3\alpha+1}\ln^{3\alpha+1}\left(\frac{\Lambda^2}{|p|^2}\right). \tag{26}$$

The unitarity condition in the conformal limit requires the following equality in Fourier space:

$$\hat G(p,\bar p) = \begin{pmatrix} g_+ & g_\cap \\ -g_\cup & g_- \end{pmatrix}(p,\bar p),$$
$$\hat G^{-1}(p,\bar p) = \frac{1}{g_+g_- + g_\cap g_\cup}\begin{pmatrix} g_- & -g_\cap \\ g_\cup & g_+ \end{pmatrix}(p,\bar p)$$
$$= \begin{pmatrix} -\Sigma_+ & \Sigma_\cap \\ -\Sigma_\cup & -\Sigma_- \end{pmatrix}(p,\bar p). \tag{27}$$

By plugging the Ansatz, Eqs. (25) and (26), into Eq. (27), it is straightforward to obtain two independent equations for the constants $a$ and $b$:

$$\frac{a}{a^2+b^2} = \frac{\pi^2}{3\alpha+1}[a^3J^2 + (a^3 - 2ab^2)Q^2]\ln^{4\alpha+1}\left(\frac{\Lambda^2}{|p|^2}\right),$$
$$\frac{b}{a^2+b^2} = \frac{\pi^2}{3\alpha+1}[b^3J^2 + (b^3 - 2a^2b)Q^2]\ln^{4\alpha+1}\left(\frac{\Lambda^2}{|p|^2}\right),$$

which have solution $\alpha = -1/4$ and $b = \pm a$, so that

$$a^4 = b^4 = \frac{1}{8\pi^2(J^2 - Q^2)}. \tag{28}$$

In Secs. III and IV, we will consider the case $Q = J$. Our derivation shows that, for $a$ and $b$ to be finite in this limiting case, the given solution only holds for $Q^2 \to J^2 \to \infty$, provided $\lim_{\substack{a^2\to b^2 \\ J^2\to\infty}} 8\pi^2J^2(a^4 - b^4) = 1$. We have shown that our Ansatz by which $g_+(z)$ decays as a free-fermion-like power law $\sim 1/z$ in the infinity limit, holds in the conformal symmetry limit. This can be checked directly. Choosing a different power, $g_+(z) = (r_0/z)^{1/2K}$ which, in Fourier space implies, for dimensional reasons only,

$$g_+ \sim i\bar p^{\frac{1}{2K}-2}, \qquad g_\cap \sim |p|^{\frac{1}{2K}-2},$$
$$\Sigma_+ \sim -iJ^2\bar p^{\frac{3}{2K}-2}, \quad \Sigma_\cap \sim J^2|p|^{\frac{3}{2K}-2}.$$

Comparing the exponents of the $p$, $\bar p$ powers in the unitarity relation $G^{-1}(\omega) \leftrightarrow \Sigma(\omega)$:

$$\frac{g_\cap}{g_+g_- + g_\cap g_\cup} \sim \frac{|p|^{\frac{1}{2K}-2}}{(a^2+b^2)|p|^{\frac{2}{2K}-4}} = \frac{|p|^{2-\frac{1}{2K}}}{(a^2+b^2)}$$

to be compared to $J^2|p|^{\frac{3}{2K}-2} \sim \Sigma_\cap$, and

$$\frac{g_-}{g_+g_- + g_\cap g_\cup} \sim \frac{ip^{\frac{1}{2K}-2}}{(a^2+b^2)|p|^{\frac{2}{2K}-4}} = \frac{i\bar p^{2-\frac{1}{2K}}}{(a^2+b^2)}$$

to be compared to $-iJ^2\bar p^{\frac{3}{2K}-2} \sim \Sigma_+$, we find that the exponents coincide if and only if $2K = 1$, as expected.

In the next subsection we set $Q = 0$, so that the chiralities become decoupled and we show that, by moving $J$ away from infinity to large but finite values, $g_+(z)$ remains a power law at large distances, but it acquires an exponent $\Gamma \neq -1$. By contrast, the case $N = 4$ discussed in Sec. III shows that our Ansatz of Eq. (23) may not be justified at strong but finite coupling, as the two-point correlator is found to be noncritical and exponentially decaying at finite $J = Q$.

### B. Critical correlator at large distances away from the strong coupling conformal symmetry limit

The previous case shows that, except for a very soft breaking obtained by the envelope function $\ln(\frac{|p|^2}{\Lambda^2})^{\frac{1}{4}}$, the conformal symmetry forces the correlation function $g_+(z,\bar z) \propto 1/z$ as in the free $1+1$ d case. Here we show that a critical power-law decay of the correlators at large distance with non-free-fermion-like exponent $\Gamma \neq -1$ can also be obtained from the conformal symmetry limit, close to the infinitely strong coupling fixed point.

Let us put $Q = 0$, so that chiralities are decoupled and consider just the $z$-chiral contribution in the saddle point equations for $\Sigma_+$, given in Eq. (24). We keep just the lowest orders of the expansion in inverse powers of $z$, for large $z$. We also drop the regularizing log factors, so that the chiral Green's function takes the form

$$g(z,\bar{z}) \approx \frac{i}{C_\Gamma}\frac{r_0}{z}\left[1+\frac{\lambda}{r_0}\,\eta\left(\frac{z}{\lambda}\right)\right]. \tag{29}$$

Here $\eta(\frac{z}{\lambda})$ is assumed to be an expansion in powers of $1/z$ (unit of $r_0$ are assumed): $\eta(\frac{z}{\lambda}) = (\frac{\lambda}{z})^\Gamma + ...$, with $\Gamma$ and $\lambda$ to be determined and the dots refer to higher powers. The saddle-point equation for $\Sigma$, given in Eq. (24), takes the form

$$\Sigma_J(z,\bar{z}) = -i\left(\frac{J\lambda}{\pi u_0}\right)^2\frac{1}{C_\Gamma^3}\left(\frac{1}{z}\right)^3\left[1+\eta\left(\frac{z}{\lambda}\right)\right]^3. \tag{30}$$

$C_\Gamma$ appearing in Eq. (29) is fixed by the unitarity condition. The cubic power on the right-hand side of Eq. (30) can undergo a rearrangement of powers. For large $z$, by keeping just the lowest order in the $\lambda/z$ expansion of $\eta_J(\frac{z}{\lambda})$, the function $\eta(\frac{z}{\lambda})$ itself is approximately reproduced, giving rise to a sort of linearization:

$$\Sigma_J(z,\bar{z}) \approx -i\left(\frac{J\lambda}{\pi u_0}\right)^2\frac{1}{C_\Gamma^3}\left(\frac{1}{z}\right)^3\left[1+3\,\eta\left(\frac{z}{\lambda}\right)\right].$$

We invert the Fourier transform of Eq. (29) and use the Dyson equation $g^{-1}(q) = g_0^{-1}(q) - \Sigma(q)$, obtaining [16]

$$\begin{aligned} C_\Gamma \times g^{-1}(q) &\to \frac{1}{\frac{1}{i\bar{q}} - i\mathcal{FT}[(1/z)\times\eta](q,\bar{q})} \\ &\approx i\bar{q} - i\bar{q}^2\mathcal{FT}[(1/z)\times\eta](q,\bar{q}) \\ &= i\bar{q} - C_\Gamma\Sigma(q,\bar{q}), \end{aligned} \tag{31}$$

where $\mathcal{FT}$ stands for Fourier transformation. The last equality, which is valid for $q \to 0$, allows us to write down a differential equation for $\eta(\frac{z}{\lambda})$,

$$\partial_z^2\left[\frac{\lambda}{z}\eta\left(\frac{z}{\lambda}\right)\right] \approx \left(\frac{J}{\pi u_0 C_\Gamma}\right)^2\left(\frac{\lambda}{z}\right)^3\left[1+3\,\eta\left(\frac{z}{\lambda}\right)\right]. \tag{32}$$

Introducing $h_J(\frac{z}{\lambda}) = \frac{\lambda}{z}\eta(\frac{z}{\lambda})$ and defining $b_J = (\frac{J}{\pi u_0 C_\Gamma})^2$, we get the simple differential equation

$$\partial_z^2\, h_J\left(\frac{z}{\lambda}\right) - \frac{3b_J\lambda^2}{z^2}\,h_J\left(\frac{z}{\lambda}\right) = \frac{b_J\lambda^3}{z^3}, \tag{33}$$

whose solution is

$$\begin{aligned} h_J\left(\frac{z}{\lambda}\right) &= -\frac{\lambda}{z}\frac{1}{\left[1+2\left(1-\frac{1}{b_J\lambda^2}\right)\right]} \\ &\quad + \left(\frac{z}{\lambda}\right)^{\frac{1}{2}}\left(c_1\left(\frac{z}{\lambda}\right)^{s/2} + c_2\left(\frac{z}{\lambda}\right)^{-s/2}\right), \end{aligned}$$

with $s^2 = 1 + 12b_J\lambda^2$. Putting $c_1 = 0$, we get from Eq. (29)

$$\frac{\tilde{g}(z)}{C_\Gamma} = \frac{i}{z} - \frac{i\lambda}{z}\frac{1}{\left[1+2\left(1-\frac{1}{b_J\lambda^2}\right)\right]} + c_2\frac{i}{\left(\frac{z}{\lambda}\right)^{\frac{s}{2}-\frac{1}{2}}}. \tag{34}$$

If we want that the free-fermion-like $1/z$ dependence disappears in favour of $1/(\frac{z}{\lambda})^{\frac{s}{2}-\frac{1}{2}}$, then we have to choose $\lambda$ such that $3\lambda^2 - \lambda^3 = 2/b_J$. For $b_J \to \infty$ both $\lambda = 0$ and $\lambda \to 0$ give the reparametrization invariant solution. In fact, the $\lambda \to 0$ solution, implying $b_J\lambda^2 \to 2/3$, gives an exponent $\Gamma = \frac{s}{2} - \frac{1}{2} = \frac{1}{2}(\sqrt{[1+12\,b_J\lambda^2]} - 1) \to 1$. This confirms that the conformal symmetry emerges at strong coupling even in the $1+1$ d, $Q = 0$, SYK model.

With increasing $\lambda$, $\Gamma$ increases slightly. The inverse power $z^{-\Gamma} = 1/z^{\frac{s}{2}-\frac{1}{2}}$ can be Fourier transformed with respect to time, yielding a power-law $\sim\omega^{\Gamma-1}$ behavior of $g(q \to 0, \omega)$. Away from the infinitely strong coupling fixed point, for $0 < \lambda \ll 3$, we get

$$\Gamma - 1 = \frac{1}{2}\sqrt{\left[1+\frac{8}{1-\frac{\lambda}{3}}\right]} - \frac{3}{2} < 1. \tag{35}$$

This derivation proves that the conformal symmetry can be broken also by moving away from the infinitely strong coupling fixed point. On the other side, $\Gamma < 2$ points out that the critical state at strong coupling has NFL nature.

The nonchiral $Q$-dependent term in Eq. (24) can be dealt with additively in a similar way. However, the differential equation corresponding to Eq. (32) cannot be linearized.

## III. COMPLEX FERMION MAPPING AND BOSONIZATION FOR $N = 4$

The approximate conformal symmetry emerging at large $J, Q$ and large $N$ when dropping the contribution due to the free Green function in the Dyson equation provides the close form given in Eq. (23) for the correlators $\frac{1}{N}\langle\sum_j \psi_{ja}(x)\psi_{ja'}(x')\rangle$ of the model. In this section, we evaluate the two-point function $g(z,\bar{z})$ directly for the case of $N = 4$, which allows for mapping onto complex fermions and their bosonization.

For $N = 4$ and $Q = J$ there is just one interaction parameter $J$ and the Lagrangian density is

$$\begin{aligned} \mathcal{L} &= \mathcal{L}_0 + \mathcal{L}_I \\ &\equiv \frac{i}{2}\sum_{i=1}^{4}[\psi_{i+}(\partial_0 + u_0\partial_x)\psi_{i+} + \psi_{i-}(\partial_0 - u_0\partial_x)\psi_{i-}] \\ &\quad - J[\psi_{1+}\psi_{2+}\psi_{3+}\psi_{4+} + \psi_{1+}\psi_{2+}\psi_{3-}\psi_{4-} \\ &\quad + \psi_{1+}\psi_{2-}\psi_{3-}\psi_{4+} + \psi_{1+}\psi_{2-}\psi_{3+}\psi_{4-} + (+\leftrightarrow -)]. \end{aligned} \tag{36}$$

It is convenient to define the complex fermion fields [53]

$$\begin{aligned} c_{\uparrow\pm} &= \frac{1}{\sqrt{2}}(\psi_{1\pm} + i\psi_{2\pm}), \quad c_{\downarrow\pm} = \frac{1}{\sqrt{2}}(\psi_{3\pm} + i\psi_{4\pm}), \\ c^\dagger_{\uparrow\pm} &= \frac{1}{\sqrt{2}}(\psi_{1\pm} - i\psi_{2\pm}), \quad c^\dagger_{\downarrow\pm} = \frac{1}{\sqrt{2}}(\psi_{3\pm} - i\psi_{4\pm}), \end{aligned} \tag{37}$$

distinguished by a pseudospin label $\sigma \equiv \uparrow, \downarrow$. For any pair of indices $\sigma\pm$ and $x \neq x'$, $\{c(x), c^\dagger(x')\} = \{c(x), c(x')\} = \{c^\dagger(x), c^\dagger(x')\} = 0$. In the free action for $N = 4$, right and left movers are decoupled

$$\begin{aligned} S_0 = \int d^2x \sum_{\sigma=\uparrow,\downarrow}[&c^\dagger_{\sigma+}(-\partial_\tau + iu_0\partial_x)c_{\sigma+} \\ &+ c^\dagger_{\sigma-}(-\partial_\tau - iu_0\partial_x)c_{\sigma-}]. \end{aligned} \tag{38}$$

The interaction terms in Eq. (36) are $c^\dagger_{\uparrow\pm}c_{\uparrow\pm}c^\dagger_{\downarrow\pm}c_{\downarrow\pm}$, $c^\dagger_{\uparrow\pm}c_{\uparrow\pm}c^\dagger_{\downarrow\mp}c_{\downarrow\mp}$, $c^\dagger_{\uparrow\pm}c_{\uparrow\mp}c^\dagger_{\downarrow\mp}c_{\downarrow\pm}$, $c^\dagger_{\uparrow\pm}c_{\uparrow\mp}c^\dagger_{\downarrow\pm}c_{\downarrow\mp}$, so that the interacting IR action $S_I$ is

$$S_I = J\int d^2x\, c^\dagger_\uparrow(x)\, c_\uparrow(x)\, c^\dagger_\downarrow(x)\, c_\downarrow(x), \tag{39}$$

where $c_\sigma = c_{\sigma-} + c_{\sigma+}$. In the present form the problem is similar to the Tomonaga-Luttinger model solved by Dzyaloshinski and Larkin [56,57] but in the absence of a Fermi sea. The interaction term in Eq. (39) is Hubbard-like, but, because of particle-hole symmetry, it is unbounded from below. Nevertheless, the Hamiltonian can be bosonized in terms of the chiral bosonic fields $\phi_{\sigma+}, \phi_{\sigma-}$, obeying the commutation relations

$$[\phi_{\sigma a}(x), \phi_{\sigma' a}(x')] = \frac{i}{4}\delta_{\sigma\sigma'}\,\mathrm{sgn}(x - x'), \tag{40}$$

$$[\phi_{\sigma+}(x), \phi_{\sigma'-}(x')] = \frac{i}{4}\delta_{\sigma\sigma'}, \tag{41}$$

and of the dual fields $\phi_\sigma = \phi_{\sigma+} + \phi_{\sigma-}$ and $\theta_\sigma = \phi_{\sigma-} - \phi_{\sigma+}$, with

$$[\phi_\sigma(x_1), \theta_{\sigma'}(x_2)] = \frac{i}{2}\,\delta_{\sigma\sigma'}\,\mathrm{sgn}(x_2 - x_1).$$

Furthermore, the pseudocharge ($c$) and pseudospin ($s$) operators can be defined from the combinations $\phi_{c/s} = \frac{1}{\sqrt{2}}(\phi_\uparrow \pm \phi_\downarrow)$ (the same for $\theta_{c/s}$), so that we represent the complex fermion $c_{\sigma\pm}(x)$ as

$$c_{\sigma\pm}(x) = \frac{1}{\sqrt{2\pi\alpha}} e^{i\sqrt{\frac{\pi}{2}}[\pm\phi_c(x) - \theta_c(x) + \sigma(\pm\phi_s(x) - \theta_s(x))]}, \tag{42}$$

where $\alpha$ is a spatial short-distance cutoff [58].

The Hamiltonian density separates in this representation so that the pseudocharge and pseudospin sectors have a separate spectrum:

$$\begin{aligned}
\mathcal{H} &= \mathcal{H}_c + \mathcal{H}_s,\\
\mathcal{H}_c &= \frac{u_0}{2}\left[\Pi_c^2 + \left(1 + \frac{J}{\pi u_0}\right)(\partial_x\phi_c)^2 + \frac{J}{u_0\pi^2\alpha^2}\cos\left(\sqrt{8\pi}\phi_c\right)\right],\\
\mathcal{H}_s &= \frac{u_0}{2}\left[\Pi_s^2 + \left(1 - \frac{J}{\pi u_0}\right)(\partial_x\phi_s)^2 + \frac{J}{u_0\pi^2\alpha^2}\cos\left(\sqrt{8\pi}\phi_s\right)\right],
\end{aligned} \tag{43}$$

where the canonical momentum field conjugate to $\phi_\sigma(x)$ is

$$\Pi_\rho(x) = \partial_x\theta_\rho(x), \qquad \rho = c, s. \tag{44}$$

In the following, we will denote as "critical," quantities derived from Hamiltonians including just the first two quadratic terms appearing in the Hamiltonians of Eq. (43), i.e., when cosine term's effects are neglected. In the usual approach to the Hubbard model, the cosine term does not appear in the pseudocharge sector if Umklapp processes are neglected. Here the absence of an underlying Fermi sea puts the interaction of both sectors on an equal footing. In Appendix B [see Eq. (B5)] we show that the addition of an extra interaction term $(\bar{\psi}_i\gamma^5\psi_j)(\bar{\psi}_k\gamma^5\psi_l)$ can cancel one of the two cosine terms as in the random Gross-Neveu-like interaction. Note that the pseudospin Hamiltonian $\mathcal{H}_s$ signals an instability as $\mathcal{H}_s$ is unbounded from below when $J/\pi u_0 > 1$. A similar situation can happen when electron-phonon interaction is introduced in a low-dimensional electronic system [57,59]. Therefore, the mapping is only meaningful for $0 < J/\pi u_0 < 1$.

By a Legendre transformation we obtain the Lagrangian density for the two separate sectors,

$$\mathcal{L} = \frac{1}{2}\sum_{\rho=c,s}\frac{1}{\mathcal{K}_\rho}\left[\frac{1}{u_\rho}(\partial_t\phi_\rho)^2 - u_\rho(\partial_x\phi_\rho)^2 - \frac{J\mathcal{K}_\rho}{\pi^2\alpha^2}\cos\sqrt{8\pi}\phi_\rho\right], \tag{45}$$

where $u_\rho$ is the velocity of the mode and we have introduced $u_\rho\mathcal{K}_\rho = u_0$, with $\mathcal{K}_\rho = 1/\sqrt{1 \pm \frac{J}{\pi u_0}}$. In the expression for $\mathcal{K}_\rho$, the upper sign is for $\rho = c$ while the lower sign is for $\rho = s$.

The energy-momentum tensor, defined as

$$T^{\mu\nu} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_\rho)}\partial^\nu\phi_\rho - g^{\mu\nu}\mathcal{L}, \tag{46}$$

gives information about the energy density $T^{00}$, the energy current $T^{x0}$, the momentum density $T^{0x}$ and the pressure $T^{xx}$. Lowering the $\nu$ index using the Minkowski metric, we have, from Eq. (46), $T^0{}_0 = \mathcal{H}$, $T^0{}_x = \sum_{\rho=c,s}\frac{1}{u_0}\partial_t\phi_\rho\partial_x\phi_\rho$, $T^x{}_0 = -\sum_{\rho=c,s}\frac{u_\rho}{\mathcal{K}_\rho}\partial_t\phi_\rho\partial_x\phi_\rho$ and

$$T^x{}_x = -\frac{1}{2}\sum_{\rho=c,s}\left[\frac{1}{u_0}(\partial_t\phi_\rho)^2 + \frac{u_\rho}{\mathcal{K}_\rho}(\partial_x\phi_\rho)^2 - \frac{\mathcal{K}_\rho J}{u_0\pi^2\alpha^2}\cos\left(\sqrt{8\pi\mathcal{K}_\rho}\phi_\rho\right)\right] \neq -\mathcal{H}. \tag{47}$$

The fact that $T^\mu{}_\nu$ is not symmetric and not traceless confirms that the $N = 4$ model is not Lorentz invariant nor conformally invariant [60]. However, even if traceless energy-momentum implies conformal symmetry, the opposite cannot be assured. We have shown in Sec. II B that in the case of the 1 + 1 d, large-$N$ model, an approximate conformal symmetry emerges, in the strong coupling limit, as in the 0 + 1 d case. Here, at $N = 4$, this is excluded, due to the fact that $N$ is finite and the limitation $0 < J/\pi u_0 < 1$, as well. The bosonized form of the diagonal element of correlator $\hat{G}$ in the chiral basis, $G_\pm(z)$, with $G(z,\bar{z}) \equiv G_+(z,\bar{z}) = \overline{G_-(z,\bar{z})}$ ($r$ stands for $z - z'$), is

$$G(r) = \frac{i}{2\pi\alpha}\left(e^{\frac{\pi}{2}\langle\phi_c(r)\phi_c(0) - \phi_c^2(0)\rangle} e^{\frac{\pi}{2}\langle\theta_c(r)\theta_c(0) - \theta_c^2(0)\rangle} \times e^{\frac{\pi}{2}\langle\phi_s(r)\phi_s(0) - \phi_s^2(0)\rangle} e^{\frac{\pi}{2}\langle\theta_s(r)\theta_s(0) - \theta_s^2(0)\rangle}\right). \tag{48}$$

Here and in the following, we denote with upper case letter $G$ the fermionic Green's functions of the $N = 4$ model, while the Green's function appearing in $\hat{G}$ for the large-$N$ model will be denoted with lower case. All the correlators, which we are going to derive, are in the zero-temperature limit.

As is well known, the excitation spectrum of each of the Hamiltonians of Eq. (43) is gapped if $\mathcal{K}_\rho < 1$, while it is gapless in the case $\mathcal{K}_\rho > 1$. The pseudospin sector is gapless. However, the pseudocharge sector, having $\mathcal{K}_c < 1$, has a gapped spectrum.

The small velocity of the pseudospin sector, $u_s \ll u_0$, gives rise to two effects. On the one hand, fluctuations of the phase $\langle\phi_s^2\rangle_0$ grow enormously. On the other hand, they renormalize the cosine interaction term which, by normal ordering is strongly suppressed as $\frac{J}{u_\rho}\cos(\sqrt{8\pi}\phi_\rho) \to \frac{J}{u_\rho}e^{-\langle\phi_s^2\rangle_0} : \cos(\sqrt{8\pi}\phi_\rho) :$ by making it irrelevant in the strong coupling limit. This implies that criticality shows up in the pseudospin sector. The fixed-point Hamiltonian of the pseudospin sector is quadratic and critical. The two-point bosonic correlator for $\phi_s$ can be readily found as [54]

$$\mathcal{I}_{\phi_s}(r) = \langle\phi_s(r)\phi_s(0) - \phi_s^2(0)\rangle = \frac{\mathcal{K}_s}{2\pi}\ln\left(\frac{\alpha}{\sqrt{x^2 + (u_s\tau + \alpha)^2}}\right). \tag{49}$$

Similarly, for its dual field $\theta_s$, it is found that $\mathcal{I}_{\theta_s}(r) = \langle\theta_s(r)\theta_s(0) - \theta_s^2(0)\rangle = \frac{1}{\mathcal{K}_s^2}\mathcal{I}_{\phi_s}(r)$. The fermionic two-point correlators can be related with the bosonic ones through Eq. (42) as

$$\begin{aligned} G_\pm(r) &= \frac{1}{4}\sum_i\langle\psi_{i\pm}(r)\psi_{i\pm}(0)\rangle \\ &= \frac{1}{4}\sum_{\sigma=\uparrow,\downarrow}\langle(c_{\sigma\pm}(r)c^\dagger_{\sigma\pm}(0) + \text{H.c.})\rangle \\ &= \pm\frac{i}{2\pi\alpha}[e^{\frac{\pi}{2}\mathcal{I}_{\phi_c}}e^{\frac{\pi}{2}\mathcal{I}_{\theta_c}}e^{\frac{\pi}{2}\mathcal{I}_{\phi_s}}e^{\frac{\pi}{2}\mathcal{I}_{\theta_s}}], \end{aligned} \tag{50}$$

appearing in a factorized form. We can define for the pseudospin sector:

$$\begin{aligned} G_s(r) &\equiv G_{\phi_s\phi_s}(r) \times G_{\theta_s\theta_s}(r) = e^{\frac{\pi}{2}\mathcal{I}_{\phi_s}} \times e^{\frac{\pi}{2}\mathcal{I}_{\theta_s}} \\ &= \left(\frac{\alpha}{\sqrt{x^2 + (u_s\tau + \alpha)^2}}\right)^{\frac{1}{4}(\mathcal{K}_s + \frac{1}{\mathcal{K}_s})}, \end{aligned} \tag{51}$$

which appears also as factorized in two chiral terms $G_s(r) = \prod_\pm G_{\pm,s}(r)$. In the pseudocharge sector, the variational method provides the self-consistent equation for the gap $\Delta$,

$$\frac{\Delta^2}{L^2} = \frac{4\,u_c J}{\pi\alpha^2}\left(\frac{\Delta}{u_c\Lambda}\right)^{2K_c}, \tag{52}$$

where $\Lambda$ is a large momentum cutoff [54]. In the large gap limit, the gapped pseudocharge degrees of freedom can be approximately described by the Hamiltonian density

$$\mathcal{H}_c = \frac{1}{2}\left[u_0(\Pi_c)^2 + \frac{u_c}{\mathcal{K}_c}(\partial_x\phi_c)^2 + \frac{4J}{\pi\alpha^2}\phi_c^2\right], \tag{53}$$

where the conjugate momentum $\Pi_c = \frac{1}{u_0}\partial_t\phi_c = \partial_x\theta_c$. In Fourier space, the action corresponding to the quadratic Hamiltonian (53), in terms of the dual fields $\theta_c, \phi_c$, is

$$S_c = \frac{1}{2\beta\Omega}\sum_{\mathbf{q}}\begin{pmatrix}\theta_c(-\mathbf{q}) & \phi_c(-\mathbf{q})\end{pmatrix} \times \begin{pmatrix} u_0k^2 & -ik\omega_n \\ -ik\omega_n & \frac{u_c}{\mathcal{K}_c}k^2 + \frac{4J}{\pi\alpha^2}\end{pmatrix}\begin{pmatrix}\theta_c(\mathbf{q}) \\ \phi_c(\mathbf{q})\end{pmatrix}, \tag{54}$$

with $\mathbf{q} = (k, \omega_n)$. The field correlators required in $G_{\phi_c\phi_c}(r) \times G_{\theta_c\theta_c}(r)$ corresponding to Eq. (51) for the pseudocharge sector are obtained by inverting the matrix of Eq. (54) and Fourier transforming back:

$$\begin{aligned} \mathcal{I}_{\phi_c} &= \langle\phi_c(r)\phi_c(0) - \phi_c^2(0)\rangle = \mathcal{K}_c\int_{-\infty}^{+\infty}\frac{d\omega'}{2\pi}\int_0^{+\infty}\frac{dk}{\pi}\frac{1}{k^2 + \frac{\Delta^2}{L^2} + \omega'^2}(e^{iu_c\omega'\tau}\cos kx - 1), \\ \mathcal{I}_{\theta_c} &= \langle\theta_c(r)\theta_c(0) - \theta_c^2(0)\rangle = \frac{1}{\mathcal{K}_c}\int_{-\infty}^{+\infty}\frac{d\omega'}{2\pi}\int_0^{+\infty}\frac{dk}{\pi}\frac{\left(k^2 + \frac{\Delta^2}{L^2}\right)}{k^2\left(k^2 + \frac{\Delta^2}{L^2} + \omega'^2\right)}(e^{iu_c\omega'\tau}\cos kx - 1). \end{aligned} \tag{55}$$

The mass $\Delta$ scales with $L$, the size of the sample, and its specific form is $\propto \frac{4J}{\pi\alpha^2}$:

$$\frac{\Delta}{L} = \frac{2}{\alpha}\frac{\sqrt{\mathcal{K}_c^{-2} - 1}}{\mathcal{K}_c}, \tag{56}$$

with $L/\alpha = N$. The two correlators of Eq. (55) are derived in Appendix C [see Eq. (C9)]. They are nonchiral and are approximated in such a way that they reproduce the critical result of the same form as Eq. (51) in the limit $\Delta \to 0$. In the limit of large distances $(\tau, x)$ they read:

$$\begin{aligned} \mathcal{I}_{\phi_c}(\tau, x) &\approx -\frac{\mathcal{K}_c}{4\pi}\left[\int_0^{ix+u_c\tau}\frac{\Delta}{L}\sqrt{\frac{-2i\,x}{z+\alpha}}K_1\left(\frac{\Delta}{L}\sqrt{-2i\,x}\sqrt{z+\alpha}\right)dz + \ln(\alpha^2) + \text{c.c.}\right], \\ \mathcal{I}_{\theta_c}(\tau, x) &\approx \frac{1}{4\pi\mathcal{K}_c}\ln\left\{e^{\frac{4\pi}{\mathcal{K}_c}\mathcal{I}_{\phi_c}(\tau,x)}\prod_\pm e^{\Delta\left[e^{-\frac{\Delta u_c\tau \pm i\,x}{L}}\right]}\left[\sqrt{\frac{\pm i\,x}{2\pi L}}\,\Gamma\left(\frac{\pm i\,x}{L}\right)e^{\mp i\,\frac{x}{L}(\gamma+1)}\right]^{\Delta e^{-\frac{\Delta u_c\tau}{L}}}\right\}, \end{aligned} \tag{57}$$

where terms proportional to $\mathcal{O}(\Delta^4)$ have been neglected. $\Gamma(z)$ is the $\Gamma$ function and $\gamma$ is the Euler's constant. $K_1(z)$ is the modified Bessel function of integer order [61].

The final form of the chiral diagonal correlator $G(r)$ of Eq. (48), which includes both $G_s(r)$ given in Eq. (51) for the pseudospin sector and the corresponding contribution for the pseudocharge sector (see Appendix C), is

$$G(r) = \frac{i}{2\pi\alpha}\left(\frac{\alpha}{\sqrt{(u_s\tau+\alpha)^2+x^2}}\right)^{\frac{1}{4}\left(\mathcal{K}_s+\frac{1}{\mathcal{K}_s}\right)} e^{-\frac{1}{8}\left(\mathcal{K}_c+\frac{1}{\mathcal{K}_c}\right)\left[\int_0^{ix+u_c\tau}\frac{\Delta}{L}\sqrt{\frac{-2ix}{z+\alpha}}K_1\left(\frac{\Delta}{L}\sqrt{-2ix}\sqrt{z+\alpha}\right)dz+\text{c.c.}\right]}$$
$$\times\prod_{\pm} e^{\frac{\Delta}{8\mathcal{K}_c}\left[e^{-\frac{\Delta u_c\tau\pm ix}{L}}\right]}\left[\sqrt{\frac{\pm ix}{2\pi L}}\,\Gamma\left(\frac{\pm ix}{L}\right)e^{\mp i\frac{x}{L}(\gamma+1)}\right]^{\frac{\Delta}{8\mathcal{K}_c}e^{-\frac{\Delta u_c\tau}{L}}}. \tag{58}$$

This is nonchiral, due to the gap-dependent part. Again, by expanding the Bessel function $K_1(z)$ in the limit $\Delta\to 0$, the critical result, factorized in the two chiral terms, is recovered.

The off-diagonal elements of the correlator $\hat{G}$ in the chiral basis, $G_{\cap/\cup}(r)$, vanish in the $N=4$ model:

$$\begin{aligned} G_{\cap/\cup}(r) &\equiv \frac{1}{4}\sum_{i=1}^{4}\langle\psi_{i\pm}(r)\psi_{i\mp}(0)\rangle \\ &= \frac{1}{4}\sum_{\sigma=\uparrow,\downarrow}\langle c_{\sigma\pm}(r)c^\dagger_{\sigma\mp}(0)+c^\dagger_{\sigma\pm}(r)c_{\sigma\mp}(0)\rangle = 0. \end{aligned} \tag{59}$$

Other combinations can be envisaged, which do not conserve chirality nor fermion number:

$$\begin{aligned} \mathcal{O}^z_{\text{TS}} &= [\psi_{1+}\psi_{1-}-\psi_{2+}\psi_{2-}] - i[\psi_{1+}\psi_{2-}+\psi_{2+}\psi_{1-}] \\ &= 2c^\dagger_{\uparrow+}c^\dagger_{\uparrow-} \\ &= \frac{1}{\pi\alpha}e^{i\sqrt{2\pi}\theta_c}e^{i\sqrt{2\pi}\theta_s}, \end{aligned} \tag{60}$$

$$\begin{aligned} \mathcal{O}^{\bar{z}}_{\text{TS}} &= [\psi_{3+}\psi_{3-}-\psi_{4+}\psi_{4-}] - i[\psi_{3+}\psi_{4-}+\psi_{4+}\psi_{3-}] \\ &= 2c^\dagger_{\downarrow+}c^\dagger_{\downarrow-} \\ &= \frac{1}{\pi\alpha}e^{i\sqrt{2\pi}\theta_c}e^{-i\sqrt{2\pi}\theta_s}, \end{aligned} \tag{61}$$

where $\mathcal{O}^z_{\text{TS}}$ and $\mathcal{O}^{\bar{z}}_{\text{TS}}$ are the operators which describe superconducting zero-momentum triplet pairing in Luttinger liquid models [54]. The number and chirality conserving correlator corresponding to $\sum_{i,j=1}^4\langle\psi_{i+}(r)\psi_{i-}(0)\psi_{j-}(r)\psi_{j+}(0)\rangle$ does not vanish in the $N=4$ model:

$$\begin{aligned} \sum_{i,j=1}^{4}\langle\psi_{i+}(r)\psi_{i-}(0)\psi_{j-}(r)\psi_{j+}(0)\rangle &\sim \sum_{\sigma,\sigma'=\uparrow,\downarrow}[\langle c_{\sigma+}(r)c^\dagger_{\sigma-}(0)c_{\sigma'-}(r)c^\dagger_{\sigma'+}(0)+c_{\sigma+}(r)c^\dagger_{\sigma-}(0)c^\dagger_{\sigma'-}(r)c_{\sigma'+}(0) \\ &\quad + c^\dagger_{\sigma+}(r)c_{\sigma-}(0)c_{\sigma'-}(r)c^\dagger_{\sigma'+}(0)+c^\dagger_{\sigma+}(r)c_{\sigma-}(0)c^\dagger_{\sigma'-}(r)c_{\sigma'+}(0)\rangle] \\ &\sim \sum_{\sigma=\uparrow,\downarrow}[\langle c_{\sigma+}(r)c^\dagger_{\sigma-}(0)c_{\sigma-}(r)c^\dagger_{\sigma+}(0)\rangle+\text{H.c.}]\neq 0. \end{aligned} \tag{62}$$

Each of the terms appearing in the sum of Eq. (62) can be expressed in terms of the operators $\mathcal{O}^{z/\bar{z}}_{\text{TS}}(r)$ of Eqs. (60) and (61). As an example, we give explicitly the expectation value of $\mathcal{O}^{z\dagger}_{\text{TS}}(r)\mathcal{O}^z_{\text{TS}}(0) = 4c_{\uparrow+}(r)c^\dagger_{\uparrow-}(0)c_{\uparrow-}(r)c^\dagger_{\uparrow+}(0)$. Its approximate expression, derived from Eqs. (60) and (61), can be extracted from Appendix C, Eq. (C22) and is reported here:

$$\begin{aligned} \langle\mathcal{O}^{z\dagger}_{\text{TS}}(r)\mathcal{O}^z_{\text{TS}}(0)\rangle &\approx \frac{1}{\pi^2\alpha^2}e^{2\pi\langle\theta_c(r)\theta_c(0)-\theta_c^2(0)\rangle}e^{2\pi\langle\theta_s(r)\theta_s(0)-\theta_s^2(0)\rangle} \\ &\approx \frac{1}{\pi^2\alpha^2}\left(\frac{\alpha}{\sqrt{(u_s\tau+\alpha)^2+x^2}}\right)^{\frac{1}{\mathcal{K}_s}} e^{-\frac{1}{2\mathcal{K}_c}\left[\int_0^{ix+u_c\tau}\frac{\Delta}{L}\sqrt{\frac{-2ix}{z+\alpha}}K_1\left(\frac{\Delta}{L}\sqrt{-2ix}\sqrt{z+\alpha}\right)dz+\ln(\alpha^2)+\text{c.c.}\right]} \\ &\quad\times\prod_{\pm}e^{\frac{\Delta}{2\mathcal{K}_c}\left[e^{-\frac{\Delta u_c\tau\pm ix}{L}}\right]}\left[\sqrt{\frac{\pm ix}{2\pi L}}\,\Gamma\left(\frac{\pm ix}{L}\right)e^{\mp i\frac{x}{L}(\gamma+1)}\right]^{\frac{\Delta}{2\mathcal{K}_c}e^{-\frac{\Delta u_c\tau}{L}}}. \end{aligned} \tag{63}$$

The same result is obtained for $\langle \mathcal{O}_{\rm TS}^{\bar z \dagger}(r)\mathcal{O}_{\rm TS}^{\bar z}(0)\rangle$ in these approximations.

We will adopt $\langle \mathcal{O}_{\rm TS}^{z \dagger}(r)\mathcal{O}_{\rm TS}^{z}(0)\rangle$ and $\langle \mathcal{O}_{\rm TS}^{\bar z \dagger}(r)\mathcal{O}_{\rm TS}^{\bar z}(0)\rangle$ from Eq. (63), to give an heuristic analytical approximation for the product $g_\cap(z,\bar z)\, g_\cup(z,\bar z)$ in the large-$N$ extension of the model, when evaluating the approximate free energy of the model in the next section. Two representative plots for the dependence on the space coordinate at equal times of these quantities are presented in Fig. 1 and are used in the next section.

## IV. APPROXIMATE ENERGY IN THE NONCHIRAL 1+1-DIMENSIONAL EXTENDED SYK MODEL

In this section, we go back to the partition function of the original model for large $N$, after disorder average, Eq. (8), in the limit $Q \to J$, and discuss the physical properties arising from the effective action $\overline{S}[\hat\Sigma, \hat G]$ of the model, Eq. (9). Our goal is to give an approximate derivation of the energy, or zero-temperature limit, $\beta \to \infty$, of the free energy with $-\beta F \sim -\overline{S}$ to be derived from the Eq. (9). We assume that the features of the model restricted to $N = 4$, which were derived in the previous section, can be appropriately extended to the large-$N$ model, at least in some range of parameters to be discussed.

Our starting assumption is that the features of the spectrum derived in the $N = 4$ case still hold when $N \gg 4$, i.e., that the spectrum of the $N \gg 4$ case still includes two branches as in the $N = 4$ case: a gapless one corresponding to the $N = 4$ pseudospin one, with renormalized velocity $u_s = u_0\sqrt{1 - J/\pi u_0}$ and a gapped one, with gap $\Delta$, corresponding to the pseudocharge branch with velocity $u_c = u_0\sqrt{1 + J/\pi u_0}$. The gapless one is remnant of the Goldstone boson branch which comes together with the spontaneous conformal symmetry breaking at large coupling $J$ in the 0 + 1 d SYK model. The addition of the space dimension and of the interchiral scattering determines an extra excitation branch in the spectrum, which, we assume, corresponds to the $N = 4$ pseudocharge one. While the pseudocharge excitations allow for whatever large value of $J$, the velocity of the pseudospin branch vanishes at $J/\pi u_0 = 1$, so that we consider $J/\pi u_0 \lesssim 1$ as the strong coupling limit of this approach. It has been shown that in the 1 + 1 d chiral, large-$N$ SYK model, the leading Lyapunov exponent reaches the maximal chaos bound at vanishing $u_s$ [53].

In this section, we use the Green's function of Eq. (6) to plot the energy as a function of the parameters $u_s/u_0$, $u_c/u_0$, and $\Delta$, in search for an absolute minimum of the free energy. To evaluate the Green's function given by Eq. (6), the functions $g_\pm(z,\bar z)$ and $g_\cap(z,\bar z)$, $g_\cup(z,\bar z)$, in the zero-temperature limit, are required. We adopt heuristically the functional form of $g_\pm(z,\bar z)$ obtained in the previous section, Eq. (58), for the $N = 4$ case. These functions tend to the "critical" limit when $J/\pi u_0$ is small and $\Delta \to 0$, a limiting form that has been discussed in the previous section. We take the diagonal chirality Green's function, $g_\pm(z,\bar z)$, in the strong coupling limit, as $g(z,\bar z) \sim G(r)$, where $G(r)$ is given by Eq. (58) (with $g(z,\bar z) \equiv g_+(z,\bar z) = \overline{g_-(z,\bar z)}$, as usual). As for the off-chirality functions, $g_{\cap/\cup}(z,\bar z)$, they individually vanish when $N = 4$, as explained already in Sec. III. A direct evaluation of these correlators would require a precise knowledge of the excitation spectrum and this is out of the present possibilities [62]. They do not vanish in the large-$N$ limit, because the interaction in the disorder average does not conserve chirality. However, we will see in Eq. (69) that, in the limit $Q \to J$, only the product $g_\cap(z,\bar z)\cdot g_\cup(z,\bar z)$ appears in the free energy. We use this fact to proceed with our extension to the large-$N$ case and we trade the four-point correlators $\langle \mathcal{O}_{\rm TS}^{\bar z \dagger}(r)\mathcal{O}_{\rm TS}^{\bar z}(0)\rangle$ and $\langle \mathcal{O}_{\rm TS}^{z \dagger}(r)\mathcal{O}_{\rm TS}^{z}(0)\rangle$ from Eq. (63), for the product $g_\cap(z,\bar z)\cdot g_\cup(z,\bar z)$. They are nonvanishing, because they conserve both number and chirality. According to Eq. (62),

$$\left\langle \mathcal{O}_{\rm TS}^{\bar z\dagger}(r)\mathcal{O}_{\rm TS}^{\bar z}(0)\right\rangle + \left\langle \mathcal{O}_{\rm TS}^{z\dagger}(r)\mathcal{O}_{\rm TS}^{z}(0)\right\rangle \to g_\cap(z,\bar z)\cdot g_\cup(z,\bar z). \tag{64}$$

In the limit $\Delta \to 0$, this product tends to the critical result

$$\lim_{J\to 0} g_\cap(z,\bar z)\cdot g_\cup(z,\bar z) = \lim_{J\to 0}\frac{1}{\pi^2\alpha^2}\left(\frac{\alpha^2}{(u_s\tau+\alpha)^2+x^2}\right)^{\frac{1}{\mathcal{K}_s}+\frac{1}{\mathcal{K}_c}}. \tag{65}$$

However, as explained already in Sec. III, the limiting $G$ expressions for the $N = 4$ model, Eqs. (58) and (64) were derived in the limit of sizable $\Delta$ and $J/\pi u_0$. It follows that our results, which are in any case just qualitative, cannot reproduce the real features of the large-$N$ model in the two opposite limits of small and large coupling $J/\pi u_0$. This will be apparent in our results because the free energy which we derive in this way is not at a minimum in these two limits. The plots of $\beta F$ as a function of $\Delta$, at given $J/\pi u_0$, are

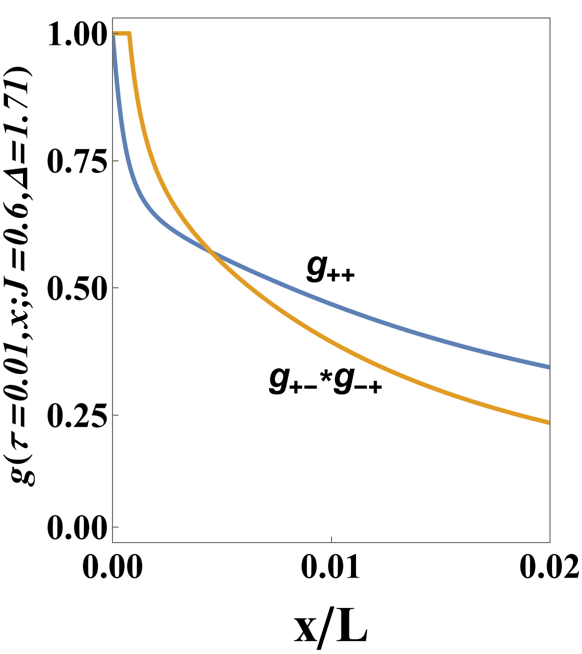


FIG. 1. Decay in space of Green's functions $g_{++}(z,\bar z)$ and $g_{+-}(z,\bar z) * g_{-+}(z,\bar z) \equiv g_\cap(z,\bar z) g_\cup(z,\bar z)$, used to compute the free energy ($\cap \equiv +-$, while $\cup \equiv -+$), at almost equal time $\tau - \tau' = 0.01$. The figure shows them for the specific fixed values $\Delta = 1.71$, $\tau \equiv u_0(\tau-\tau')/L = 0.01$, and $J \equiv J/\pi u_0 = 0.6$, in the gapped regime. $g_{+-}(z,\bar z) * g_{-+}(z,\bar z)$ has been chopped close to $x - x' \sim 0$.

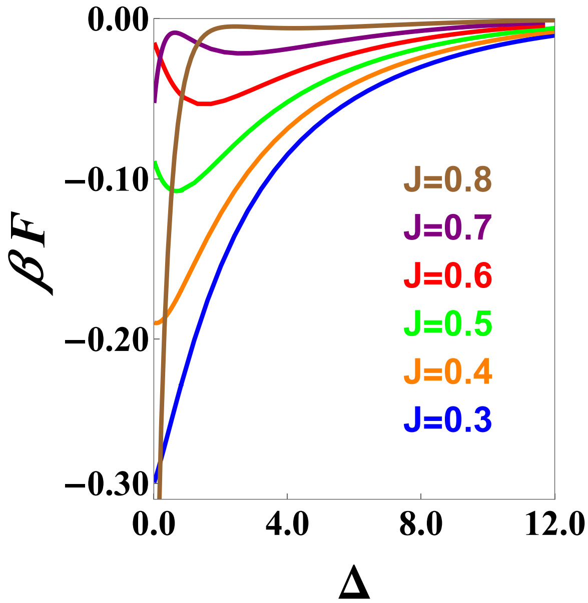


FIG. 2. Free energy $\beta F$ vs the gap $\Delta$ for different values of the dimensionless coupling $J \equiv J/(\pi u_0)$. The energy does not develop any minimum when $J = 0.3$ or lower. At $J = 0.4$ a minimum appears at low $\Delta$ which is the absolute minimum of the energy till $J = 0.6$ when it becomes metastable (see Fig. 4).

presented in Figs. 2–4. They indeed show that there is a range of intermediate values for $J/\pi u_0$ in which the free energy of our model has an absolute minimum at finite $\Delta$, in the sense that the minimum is indeed lower in energy than the reference energy at $\Delta = 0$.

Having anticipated the final result, we now turn to the derivation of the free energy which follows the same lines as for the $0 + 1$ d SYK model [16]. In the $1/N$ limit, $\beta F$

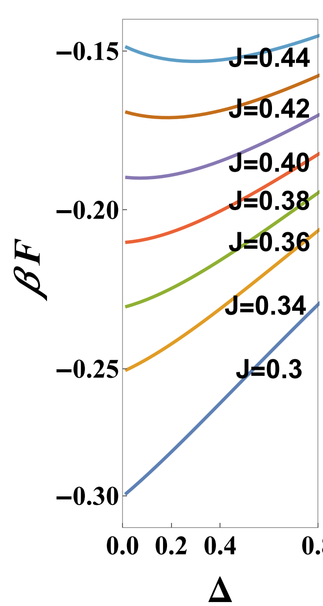


FIG. 3. Blowup of the appearance of the absolute minimum of the free energy $\beta F$ at $\Delta \geqslant 0$, which develops continuously at $\Delta > 0$ for increasing coupling $J \equiv J/(\pi u_0) > 0.4$.

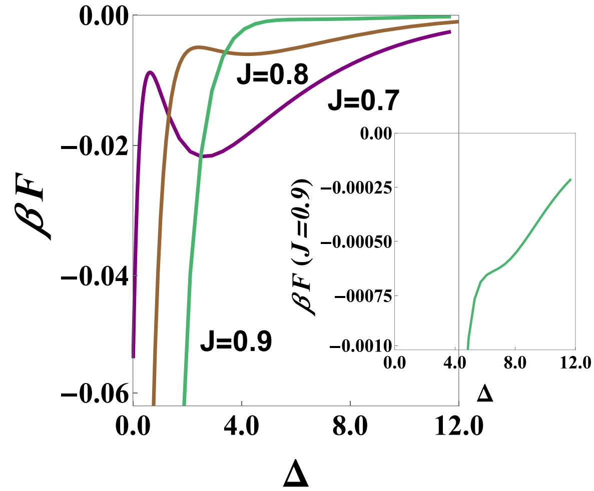


FIG. 4. The free energy $\beta F$ vs the gap $\Delta$ appearing in Fig. 2, restricted to the values of the dimensionless coupling $J \equiv J/(\pi u_0) = 0.7, 0.8, 0.9$. For $J$ beyond 0.6, the absolute minimum is lost and a metastable minimum arises at finite $\Delta$. *In the inset*: Blowup of the free energy $\beta F$ vs the gap $\Delta$ for $J \equiv J/(\pi u_0) = 0.9$, around $\Delta_{\min} \approx 7.0$, where the minimum disappears.

is derived from the effective action in which the Green's function and the self-energy solving the Schwinger-Dyson equations are inserted. For the time being, the parameters for the interaction couplings $J$ and $Q$ will be kept as separate in the next derivation, to make results more transparent. Eventually, we perform the limit $Q \to J$. Due to the fact that $\hat{G}$ and $\hat{\Sigma}$ obey the SD equations and explicitly depend on the coupling parameters in the conformal symmetry limit (see Sec. II A), it is enough to take the derivative of the free energy with respect to the coupling parameters: $J\,\partial_J(-\beta F/N) + Q\,\partial_Q(-\beta F/N)$. Since the partition function only depends on the combinations $\beta J$, $\beta Q$, the derivative $J\partial_J + Q\partial_Q$ produces the derivative $\beta\partial_\beta$. At the energy minimum, $E$, the equality holds, $\partial_\beta(-\beta F/N) = -E/N$. We will still talk of free energy for simplicity, in the following, no matter that we consider the zero-temperature limit only. Using the identity for a given matrix $A$

$$\frac{1}{\mathrm{Pf}(A)}\frac{\partial \mathrm{Pf}(A)}{\partial x} = \frac{1}{2}\,\mathrm{Tr}\left(A^{-1}\frac{\partial A}{\partial x}\right),$$

we have

$$\partial_J \ln \mathrm{Pf}[-(\partial_\tau - ai\partial_x) - \Sigma_a(x, x')] = \frac{1}{2}\,\mathrm{Tr}\left(G\,\frac{\partial \Sigma}{\partial J}\right) \quad (66)$$

and similarly for $\partial_Q$. From the action (22) some cancellation of terms occurs and the final result is (with $\Omega = u_0\hbar\alpha N$)

$$\begin{aligned} -\frac{E}{N} &= \frac{1}{\beta}[J\,\partial_J(-\beta F/N) + Q\,\partial_Q(-\beta F/N)] \\ &= \frac{J^2}{4\Omega}\int d^2z \sum_a g_a^4(z,\bar{z}) + \frac{Q^2}{2\Omega}\int d^2z \\ &\quad \times [g_+^2 g_-^2 + g_\cap^2 g_\cup^2 - 4\,g_+ g_\cap g_- g_\cup]\,(z,\bar{z}). \end{aligned} \quad (67)$$

The first term contains the single $g_{\cap/\cup}(z,\bar{z})$ in the large-$N$ limit but, restricting the model to the case $J^2 = Q^2$ and rearranging the related terms,

$$g_\cap^4 + g_\cup^4 + 2g_\cap^2 g_\cup^2 = (g_\cap^2 + g_\cup^2)^2 \equiv (g_\cap^2 + \overline{g_\cap}^2)^2$$
$$= 4[(\Re e g_\cap)^2 - (\Im m g_\cap)^2]^2$$
$$= 4[\Re e(g_\cap g_\cup) + \Im m(g_\cap g_\cup)]^2$$
$$= 4|g_\cap g_\cup|^2, \tag{68}$$

we obtain an explicit dependence only on the product $g_\cap g_\cup$, which we approximate from Eqs. (63) and (64). The energy takes the form

$$-\frac{E}{N} = \frac{J^2}{\Omega}\int d^2z\,\{(g_+g_-)^2 + (g_\cap g_\cup)^2 - 2\,g_+g_\cap g_-g_\cup\}$$
$$= \frac{J^2}{2\Omega}\int d^2z\,(\mathbf{Tr}\,\hat{G}^2)^2, \tag{69}$$

with $\hat{G}$ given in Eq. (6). This is the extension of the result for the $0+1$ d Sachdev-Ye-Kitaev model [16] to the $1+1$ d nonchiral case for $J^2 = Q^2$.

We now comment on the numerical results. In the following and in the figures, we denote as $J$ the dimensionless coupling $J/\pi u_0$ and the time parameter $\tau \sim u_0(\tau-\tau')/L \to 0$. In Fig. 1 we plot the space dependence of the Green's functions $g(z,\bar{z}) \equiv g_+(z,\bar{z}) = \overline{g_-(\bar{z},z)}$ and $g_\cap(z,\bar{z})g_\cup(z,\bar{z})$, used to compute the free energy (omitting the prefactor $i$). All of them only depend on the difference of the space/time arguments because of translational invariance. The figure shows them for the specific fixed values $\Delta = 1.71$, $\tau = 0.01$, and $J = 0.6$ versus the dimensionless space coordinate $x/L$. These parameter values belong to the interval in which the energy is at an absolute minimum. Both $g_\pm$ and $g_\cap g_\cup$ are taken from the $N = 4$ case, in such a way that $g_+ \sim G_{++}$ and $g_\cap g_\cup \sim G_{+-}G_{-+}$, by assuming that the functional form of the correlators is the same for $N \gg 4$. While $g_+ = \overline{g_-}$ tends to unity by construction when $x \to 0$ and shows a crossover from power-law decay to exponential decay $\sim e^{-\frac{\pi x}{8}\frac{\Delta}{\mathcal{K}_c}}$ at large $x$, the product $g_\cap(z,\bar{z})g_\cup(z,\bar{z})$ has been chopped to unity at small distances and has an exponential decay $\sim e^{-\frac{\pi x}{2}\frac{\Delta}{\mathcal{K}_c}}$ at large distances.

Figure 2 shows the free energy $\beta F$ versus the gap $\Delta$ for different values of $J$, in the limit of zero temperature. The free energy increases with $\Delta$ when $J$ increases from zero. In the range $J \lesssim 0.3$, the model is unable to produce a minimum of the free energy not only at finite gap, but also at zero gap. The fact that the zero gap limit at low $J$ is not accurate happens because, both the diagonal and the off-diagonal terms of the Green's function of Eq. (6), adopted from the $N = 4$ case, acquire factorized contributions from the two excitation branches in this limit, as seen from Eqs. (58) and (63). While this is expected to be so for terms which conserve the chirality, it is highly unphysical for terms which do not conserve the chirality, because the gapless and gapped branch merge in energy when $\Delta \to 0$ and their contributions to the propagator cannot come out to be factorized. The situation improves when $J$ increases, because, with increasing $J$, also the gap between the two branches in the spectrum increases (see the inset of Fig. 5). It is remarkable that the absolute minimum appears at zero gap for $J \leqslant 0.4$ (orange curve). Figure 3 highlights the formation of the minimum for nonzero $\Delta$ around $J \sim 0.4$. The minimum emerges with continuity and becomes more pronounced by increasing $J$ beyond $J \sim 0.4$, as it happens in Landau's second order phase transitions. For increasing values of $J$, the energy develops an absolute minimum which confirms, at least within our approximations, the presence of gapped excitations in the spectrum of the model, in the range $0.4 \leqslant J \leqslant 0.6$. At higher values of $J$, let us say $0.7 \leqslant J \leqslant 0.8$, the minimum is still present but becomes metastable, meanwhile a stable minimum of the energy is lost and the assumptions on which the model rests appear to break down. A very shallow minimum appears at $\Delta \sim 0.5$ for $J \leqslant 0.8$ (brown curve in Fig. 2) and more in detail in Fig. 4. Close to the physical bound, $u_s \to 0$, the minimum fully disappears, as it is shown in the inset of Fig. 4 for $J = 0.9$.

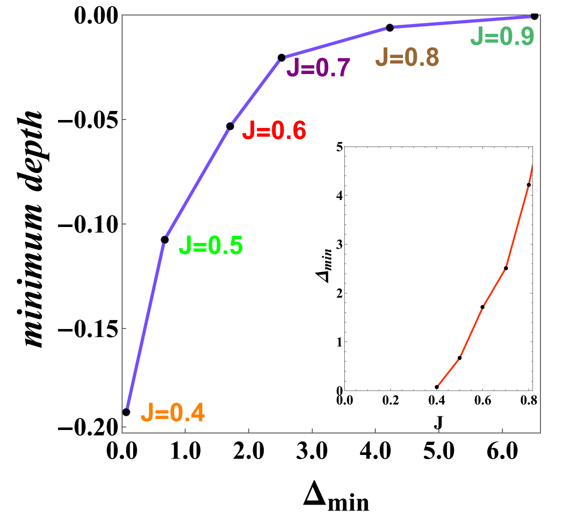


FIG. 5. The minimum of $\beta F$ becomes shallower when the dimensionless coupling $J \equiv J/(\pi u_0)$ and $\Delta_{\min}$ increase and the depth at the minimum decreases down to zero. *In the inset:* Value of the gap at the minimum, $\Delta_{\min}$, vs the dimensionless coupling $J \equiv J/(\pi u_0)$. $\Delta_{\min}$ increases with increasing $J$.

As stressed already, as $J$ increases, so does $\Delta_{\min}$, the $\Delta$ value corresponding to the minimum of the free energy (see *inset* of Fig. 5), while the minimum itself becomes shallower and shallower as can be seen in Fig. 5.

From the numerical results we conclude that our basic assumption that the extension to $1+1$ d of the SYK model, extrapolated from the exactly soluble $N = 4$ case to large $N$ and relatively strong coupling $J$, entails two excitation branches, the gapless one originating from the emergent conformal symmetry at strong coupling and the gapped one originating from nonconservation of the number and chirality, may hold at least for a limited range of values of the coupling constant $J$ inside the physical bounds $0 < J/\pi u_0 < 1$. The excitation modes have different velocities and the limit of $u_s \to 0$ which occurs at finite $J/\pi u_0$ is clearly intrinsic of the finite $N$ model, but not of its extension to large $N$.

## V. CONCLUSIONS

The $0+1$ d SYK model of $N$ Majorana fermions, with disorder average of the random interaction appears extremely rich and exactly solvable in the $N \to \infty$ limit. The challenge of applying the model to real condensed matter systems demands an extension of the model to higher-space dimensionality. Such an extension would be extremely fruitful because in the $N \to \infty$ limit, an approximate conformal symmetry emerges, which is spontaneously broken in the IR limit. The two-point function plays the role of an order parameter and is nonlocal in imaginary time $\tau$. Its Fourier transformed dependence, $\sim\omega^{1/2}$, points to NFL excitations, which are induced by the soft modes arising in connection with the spontaneous symmetry breaking [3,16]. Extension of the model could give an hint for interpreting as NFL behavior the unusual properties of the "strange" metals among which the high temperature superconductors are classified.

Most of the attempts to introduce higher-space dimensionality in the model start from parent complex fermion Hamiltonians which include hopping between sites. The disorder average of the interaction is performed at the partition function level, expressed in terms of Grassman variables. To our view this approach overlooks the fact that the original Hamiltonian is written in terms of Majorana fields, i.e., real fermions, which do not conserve the charge, nor the momentum, in the added space dimensions. By contrast, we believe that nonconservation is the crucial interesting feature of the model, which the parent Hamiltonians loose in the IR and $T \to 0$ limit. By adding an hopping term, made up of Majorana's [63]: $\sum_{i<j} K_{ij}\psi_i\psi_j$, a closer Hamiltonian to the original one is obtained instead, but the flavor is changed into a site label, a procedure that obscures what is electron transport in the lattice.

From the field theory point of view, by adding an extra dimension, the canonical scale dimension of the fermions is changed to 1/2. This makes the interaction term marginal at best. Since the $0+1$ d SYK model has relevant interactions, the above is an important obstacle in the generalization of the model to higher dimensions, especially in the nonchiral case. To avoid this problem, some authors have considered a theory with a nonlocal topological kinetic term, to acquire zero scale dimension of the fields [49], or models described by a large number of bosons and fermions, via local random Yukawa coupling [42–44,46,51], among other options.

Our choice was to consider a generalized form of the $1+1$ d SYK model in the continuum limit, involving two different sets of real couplings, $J_{ijkl}$ and $Q_{ijkl}$ among $N$ Majorana fermions, which mediate the interactions between fermions of the same and of different chirality branches, respectively. In the large-$N$ case, when the $J$ couplings vanish, the model reduces to the random Thirring model [52], while left- and right-movers decouple when the $Q$ couplings vanish. We have shown that there is an approximate conformal symmetry emerging in the disorder averaged action when $N, J, Q \to \infty$, as in the $0+1$ d case, and the model is still critical, with a strong coupling fixed point. However, this symmetry is not spontaneously broken in the $1+1$ d case, and the system can be considered as a marginal Fermi liquid, in which the Schwinger-Dyson equation only admits free-fermion-like two-point correlators as solutions. Nevertheless, by moving away from the fixed point in the chiral case $Q = 0$, this symmetry is indeed broken by hand and we obtain NFL power-law decaying propagators. The case $Q \neq 0$ requires the solution of a nonlinear Dyson equation and does not exclude a noncritical solution, with the appearance of an exponential decay.

In the limiting case $N = 4$, the model can be exactly solved by bosonization. The Hamiltonian is not conformally invariant and the need to introduce a UV cutoff in the space dependence of the correlators also breaks the Lorenz invariance. The $N = 4$ action can be mapped onto two uncoupled bosonic sine-Gordon [54] actions. This fact shows that there are two branches of excitations with velocities $u_{c,s} = u_0\sqrt{1 \pm \frac{J}{\pi u_0}}$, which we denote as "pseudocharge" ($c$) and "pseudospin" ($s$), in analogy with the well known sine-Gordon case. Since the pseudospin velocity $u_s$ vanishes when $J \to \pi u_0$, we found that there is a physical window $0 < J/u_0\pi < 1$ within which the bosonization picture is meaningful. A similar case of instability [57,59,64] was found in the charge channel of the Luttinger liquid, when the attractive interaction arising from the electron-phonon interaction is introduced. As the corresponding interaction parameter $\mathcal{K}_s = 1/\sqrt{1 - \frac{J}{\pi u_0}}$ diverges when $J \to \pi u_0$, we interpret the $J$ values close to this limit as strong coupling, also for the large-$N$ case.

It has been shown in the chiral case [53] $Q = 0$ that the functional form of the two-point correlator for $N = 4$ and large $N$ are equal (upon average over random couplings) and the model remains critical with gapless excitations. However, in our nonchiral $N = 4$ case, it is found that the pseudocharge branch is gapped, while the pseudospin branch is gapless, and it is not difficult to accept that cross-chiral scattering ($Q \neq 0$) can add a gapped branch to the spectrum also in the large-$N$ case. It is tempting to assume that this feature is maintained in the $N \to \infty$ limiting case. To check this Ansatz, we have derived the expression for the free energy $\beta F$ in the limit of zero-temperature and $Q \to J$ at the lowest order in $1/N$, in terms of the diagonal and off-diagonal chirality correlators, $g_\pm$ and $g_\cap$, $g_\cup$. To proceed, we have assumed heuristically that the analytic correlator $G_\pm$, evaluated in the $N = 4$ case, can be considered as an indication of the functional form of the corresponding correlators $g_\pm$ in the large-$N$, large-$J$ case.

There is no correspondent expression of $g_\cap$, $g_\cup$ in the $N = 4$ case. This is because the ground state conserves number and chirality at finite $N$, and two-point off-diagonal chirality correlators vanish. However, the energy in the $Q \to J$ limit only depends on the product $g_\cap \cdot g_\cup$, which we assume to be given by a corresponding four-point correlator, Eq. (63) factorized at lowest order. We have obtained the limiting form of the free energy $\beta F$ at zero-temperature numerically, as a function of the parameter $J$, which determines the velocities $u_c$, $u_s$ of the excitations, and of the gap $\Delta$. By plotting the free energy versus the gap $\Delta$, for increasing values of $J$, we have found an intermediate range of values of $J$ in which an absolute minimum of the energy is found, corroborating the idea that a gapped excitation branch may arise in the spectrum with increasing interaction among the fermions. However, our approach seem to be justified only at intermediate $J$ couplings and fails at $J \to 0$ and $\frac{J}{\pi u_0} \lesssim 1$ because the $N = 4$ analogy breaks down at these extrema.

We have used the dual sine-Gordon version of the model, which can be solved exactly by bosonization in the limit $N = 4$, as an approximation to infer the features of the system in the large-$N$ limit. Because in our approach we are neither forcing a zero canonical scale dimension of the fermions nor changing the nature of the quartic fermionic interaction, our model keeps similar in form to the original one. However, the inclusion of the spatial dimension generally makes 1 + 1 d nonchiral SYK models statistically marginal irrelevant [52], in the sense that after averaging over disorder and using conformal perturbation theory, the $\beta$ function is positive. However, there are relevant and irrelevant operators that will grow or decrease as we flow into the IR. Since all these contributions are screened by the net effect of the average over disorder, added to the fact that our model is not truly conformal symmetric, we found that the theory can be studied as an effective model in a range of couplings $J$ determined by the physical bound $0 < J/u_0\pi < 1$, coming from the gapless pseudospin sector, where the theory can be studied safely.

Two excitation branches were found, the gapless one, originating from the emergent conformal symmetry at strong coupling, and the gapped one, originating from nonconservation of the number and chirality. The excitation modes have different velocities and the limit of $u_s \to 0$ which occurs at finite $J/\pi u_0$ is clearly intrinsic of the finite $N$ model, but not of its extension to large $N$. Since our approximation cannot reproduce the large-$N$ results in the entire physical bound, the model being restricted to an intermediate range of values of coupling constant $J$, a deeper analysis of the large-$J$ limit, which does not rely on the assumption made, is required.

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## APPENDIX A: FOURIER TRANSFORMS OF THE GREEN'S FUNCTION IN THE CONFORMAL SYMMETRY LIMIT

In this Appendix we provide the Fourier transforms of the two-point function $g(z,\bar z)$, Eqs. (17) and (23), and of the self-energy $\Sigma(z,\bar z)$, Eqs. (18) and (24), for the solution of the SD equations in the large-$N$ case.

The diagonal terms in the matrix $\hat G$, $\hat\Sigma$ are

$$g_+(p,\bar p) = a\int \frac{d^2z}{z}\ln(z\bar z\Lambda^2)^{\alpha} e^{ipz+i\bar p\bar z}, \tag{A1}$$

$$\begin{aligned}\Sigma_+(p,\bar p) &= a^3\left(J^2\int \frac{d^2z}{z^3}\ln(z\bar z\Lambda^2)^{3\alpha}e^{ipz+i\bar p\,\bar z}\right.\\ &\quad\left.+Q^2\int\frac{d^2z}{z\bar z^2}\ln(z\bar z\Lambda^2)^{3\alpha}e^{ipz+i\bar p\,\bar z}\right)\\ &= \Sigma_J+\Sigma_Q.\end{aligned} \tag{A2}$$

For the Fourier transforms of the Green's function $g_+(z,\bar z)$ and for the $Q$ part of the self-energy, $\Sigma_Q$, we define [52]

$$\begin{aligned}F_\beta(|p|/\Lambda) &\equiv \int\frac{d^2z}{z\bar z}\ln(z\bar z\Lambda^2)^{\beta}e^{ipz+i\bar p\bar z}\\ &= 2^{\beta+1}\pi\int_0^\infty\frac{dr}{r}\ln^{\beta}(\Lambda r)J_0(|p|r)\\ &= \frac{\pi}{\beta+1}\ln\left(\frac{\Lambda^2}{|p|^2}\right)^{\beta+1},\end{aligned}$$

where $|p|^2 = p\bar p$. It follows that

$$g_+(p,\bar p) = -ia\partial_{\bar p}F_\alpha(|p|/\Lambda) = i\pi\frac{a}{\bar p}\ln\left(\frac{\Lambda^2}{|p|^2}\right)^{\alpha}. \tag{A3}$$

Similarly,

$$\partial_{\bar p}\Sigma_Q = i\,a^3Q^2F_{3\alpha}(|p|/\Lambda) = i\,a^3Q^2\frac{\pi}{3\alpha+1}\ln\left(\frac{\Lambda^2}{|p|^2}\right)^{3\alpha+1},$$

so that, neglecting $\mathcal{O}(\ln^{3\alpha}\frac{\Lambda^2}{|p|^2})$ terms,

$$\Sigma_Q(p,\bar p)\approx i\bar p\,a^3Q^2\frac{\pi}{3\alpha+1}\ln\left(\frac{\Lambda^2}{|p|^2}\right)^{3\alpha+1}. \tag{A4}$$

However,

$$\begin{aligned}\partial_p\Sigma_J &= a^3J^2 i\int\frac{d^2z}{z^2}\ln(z\bar z\Lambda^2)^{3\alpha}e^{i(pz+\bar p\bar z)}\\ &= -a^3J^2 i\int_0^\infty r\,dr\int_{-\pi}^{\pi}d\theta\,\frac{\cos 2\theta - i\sin 2\theta}{r^2}\ln(r^2\Lambda^2)^{3\alpha}e^{i|p|r\cos\eta},\end{aligned}$$

where we put $e^{i\,pz+\bar p\bar z} = e^{i|p|r[\cos\gamma\cos\theta+\sin\gamma\sin\theta]} = e^{i|p|r\cos\eta}$ with $\eta=\theta-\gamma$. As $\cos 2\theta - i\sin 2\theta = (\cos 2\eta - i\sin 2\eta)(\cos 2\gamma - i\sin 2\gamma)$, integrating over $\eta$ and dropping $\sin 2\eta$ which is odd in the integration, we obtain

$$\begin{aligned}\partial_p\Sigma_J &= -a^3J^2 i(\cos 2\gamma - i\sin 2\gamma)\int_0^\infty r\,dr\int_{-\pi}^{\pi}d\eta\,\frac{\cos 2\eta}{r^2}\ln(r^2\Lambda^2)^{3\alpha}e^{i|p|r\cos\eta}\\ &= 2\pi\,a^3J^2 i\frac{\bar p}{p}\int_0^\infty\frac{dr}{r}\ln(r^2\Lambda^2)^{3\alpha}J_2(|p|r) = a^3J^2 i\frac{\bar p}{p}F^{(J)}_{3\alpha}(|p|/\Lambda).\end{aligned} \tag{A5}$$

The last equality defines the function $F_\alpha^{(J)}(|p|/\Lambda)$ which can be approximated by use of the following integral:

$$G_\epsilon^{(J)}\left(\frac{|p|}{\Lambda}\right) = \int_0^\infty \frac{dr}{r}\,(\Lambda r)^\epsilon \ln(\Lambda r)\, J_2(|p|r) = \frac{1}{2^{2-\epsilon}}\frac{\Gamma\left[1+\frac{\epsilon}{2}\right]}{\Gamma\left[2-\frac{\epsilon}{2}\right]}\left(\frac{\Lambda}{|p|}\right)^\epsilon\left[\psi\left(1+\frac{\epsilon}{2}\right)+\psi\left(2-\frac{\epsilon}{2}\right)+2\ln\frac{2\Lambda}{|p|}\right]. \tag{A6}$$

We analytically continue the relation

$$F_{n+1}^{(J)}(|p|/\Lambda) = 2^{n+2}\pi \lim_{\epsilon\to 0}\partial_\epsilon^n G_\epsilon^{(J)}\left(\frac{|p|}{\Lambda}\right),$$

to noninteger index $n+1 \to 3\alpha$, obtaining

$$F_{3\alpha+1}^{(J)}(|p|/\Lambda) = 2^{3\alpha+2}\pi \lim_{\epsilon\to 0}\partial_\epsilon^{3\alpha} G_\epsilon^{(J)}\left(\frac{|p|}{\Lambda}\right).$$

Expanding $G_\epsilon^{(J)}$ for small $\epsilon$,

$$\begin{aligned} G_\epsilon^{(J)}\left(\frac{|p|}{\Lambda}\right) &\approx \left(\frac{\Lambda}{|p|}\right)^\epsilon \frac{1}{4}\left[\gamma(\gamma-1)+2\ln\frac{2\Lambda}{|p|}\right] \\ &= \frac{1}{2}\left(\frac{\Lambda}{|p|}\right)^\epsilon \ln\frac{\Lambda}{|p|}\left[1+\mathcal{O}\left(1/\ln\frac{\Lambda}{|p|}\right)\right] \\ &= \frac{1}{2}\sum_{m=0}^\infty \frac{\epsilon^m \ln^{m+1}(\frac{\Lambda}{|p|})}{m!}\left[1+\mathcal{O}\left(1/\ln\frac{\Lambda}{|p|}\right)\right], \end{aligned} \tag{A7}$$

Eq. (A5) becomes

$$\partial_p \Sigma_J = a^3 J^2 i\frac{\bar p}{p}\,\pi\,\ln^{3\alpha}(\Lambda^2/|p|^2). \tag{A8}$$

Finally, the primitive function with respect to the variable $p$, provides the desired result:

$$\begin{aligned} \Sigma_J(p,\bar p) &= -a^3J^2 i\pi\,\bar p\int^p \frac{dp}{p}\,\ln\left(\frac{\bar p}{\Lambda^2}p\right)^{3\alpha} \\ &= \frac{a^3J^2 i\pi\,\bar p}{3\alpha+1}\,\ln\left(\frac{\Lambda^2}{|p|^2}\right)^{3\alpha+1}. \end{aligned} \tag{A9}$$

In conclusion, the Fourier transforms in the large $J$, $Q$ limit are given by

$$g_+(p,\bar p) = i\pi\frac{a}{\bar p}\ln^\alpha\left(\frac{\Lambda^2}{|p|^2}\right), \tag{A10}$$

$$\Sigma_+(p,\bar p) \approx i\pi\,\bar p\,a^3\,\frac{(J^2+Q^2)}{3\alpha+1}\,\ln^{3\alpha+1}\left(\frac{\Lambda^2}{|p|^2}\right). \tag{A11}$$

The off-diagonal terms are, according to Eqs. (23) and (24),

$$g_\cap(p,\bar p) \propto \int \frac{d^2z}{|z|}\ln(|z|^2\Lambda^2)^\alpha e^{i|p||z|},$$

$$\Sigma_\cap(p,\bar p) \propto \int \frac{d^2z}{|z|^3}\ln(|z|^2\Lambda^2)^{3\alpha} e^{i|p||z|}.$$

$g_\cap(p,\bar p)$ can be obtained directly:

$$\begin{aligned} g_\cap(p,\bar p) &\sim \int |z|d|z|d\theta\,\frac{\ln(|z|^2\Lambda^2)^\alpha}{|z|}e^{i|p||z|\cos\theta} \\ &\sim \frac{2\alpha}{\Lambda}\int d|z|\ln(|z|)J_0\left(\frac{|p||z|}{\Lambda}\right) \\ &\sim -\frac{2\alpha}{\Lambda}\frac{\Lambda}{|p|}\ln\left(\frac{|p|}{\Lambda}\right) \\ &\sim \frac{1}{|p|}\ln\left(\frac{\Lambda^2}{|p|^2}\right)^\alpha, \end{aligned} \tag{A12}$$

or, with a procedure similar to the one of Eq. (A3), we define

$$\begin{aligned} F'_\beta(|p|/\Lambda) &\equiv \frac{1}{2}\int\frac{d^2z}{|z|^2}\ln(|z|^2\Lambda^2)^\beta e^{i|p||z|} \\ &= \frac{1}{2}\frac{\pi}{\beta+1}\ln\left(\frac{\Lambda^2}{|p|^2}\right)^{\beta+1}, \end{aligned}$$

and obtain

$$g_\cap(p,\bar p) \propto -i\,\partial_{|p|}F'_\alpha(|p|/\Lambda) = i\pi\frac{1}{|p|}\ln\left(\frac{\Lambda^2}{|p|^2}\right)^\alpha,$$

as above. Similarly,

$$\partial_{|p|}\Sigma_\cap(p,\bar p) \propto i\,F'_{3\alpha}(|p|/\Lambda) = i\,\frac{\pi}{3\alpha+1}\ln\left(\frac{\Lambda^2}{|p|^2}\right)^{3\alpha+1},$$

where, neglecting $\mathcal{O}(\ln^{3\alpha}\frac{\Lambda^2}{|p|^2})$ terms, we obtain

$$\Sigma_\cap(p,\bar p) \propto i\,\frac{\pi}{3\alpha+1}|p|\ln\left(\frac{\Lambda^2}{|p|^2}\right)^{3\alpha+1}. \tag{A13}$$

## APPENDIX B: BOSONIZATION OF THE INTERACTION FOR $N = 4$

Here we derive the bosonized form of the action of the model given in Eq. (4) for the case $N = 4$. Putting $\{J\} = \{Q\}$ from the outset, Eq. (4) has the following structure:

$$\sum_{i<j<k<l}\left[\frac{1}{2}((\bar\psi_i\gamma^\mu\psi_j)(\bar\psi_k\gamma^\mu\psi_l)+(\bar\psi_i\gamma^\mu\psi_j)(\bar\psi_k\gamma_\mu\psi_l))+\bar\psi_i\psi_j\bar\psi_k\psi_l\right],$$

or, in an expanded way,

$$\sum_{i<j<k<l}\sum_{a=\pm}[\psi_{ia}\psi_{ja}\psi_{ka}\psi_{la}+\psi_{ia}\psi_{ja}\psi_{k\bar{a}}\psi_{l\bar{a}}+\psi_{ia}\psi_{j\bar{a}}\psi_{k\bar{a}}\psi_{la}+\psi_{ia}\psi_{j\bar{a}}\psi_{ka}\psi_{l\bar{a}}]. \tag{B1}$$

In the $N=4$ case, there is only one possibility of ordering $i, j, k, l$ fermions

$$\sum_{a=\pm}(\psi_{1a}\psi_{2a}\psi_{3a}\psi_{4a}+\psi_{1a}\psi_{2a}\psi_{3\bar{a}}\psi_{4\bar{a}}+\psi_{1a}\psi_{2\bar{a}}\psi_{3\bar{a}}\psi_{4a}+\psi_{1a}\psi_{2\bar{a}}\psi_{3a}\psi_{4\bar{a}}),$$

and, using the definitions of the complex fermions $c_{\sigma\pm}$ given in Eq. (37), the interaction becomes

$$\sum_{a=\pm}(c^\dagger_{\uparrow a}c_{\uparrow a}c^\dagger_{\downarrow a}c_{\downarrow a}+c^\dagger_{\uparrow a}c_{\uparrow a}c^\dagger_{\downarrow\bar{a}}c_{\downarrow\bar{a}}+c^\dagger_{\uparrow a}c_{\uparrow\bar{a}}c^\dagger_{\downarrow\bar{a}}c_{\downarrow a}+c^\dagger_{\uparrow a}c_{\uparrow\bar{a}}c^\dagger_{\downarrow a}c_{\downarrow\bar{a}})=-c^\dagger_{\uparrow}c_{\uparrow}c^\dagger_{\downarrow}c_{\downarrow}, \tag{B2}$$

where $c_\sigma=c_{\sigma-}+c_{\sigma+}$. Under bosonization, the theory can be separated into pseudospin and pseudocharge sectors, by defining the bosonic phases $\phi_{\uparrow,\downarrow}=\frac{1}{\sqrt{2}}(\phi_c\pm\phi_s)$. We obtain

$$\sum_{a=\pm}(c^\dagger_{\uparrow a}c_{\uparrow a}c^\dagger_{\downarrow a}c_{\downarrow a}+c^\dagger_{\uparrow a}c_{\uparrow a}c^\dagger_{\downarrow\bar{a}}c_{\downarrow\bar{a}})=\frac{1}{2\pi}((\partial_x\phi_c)^2-(\partial_x\phi_s)^2),$$

$$\sum_{a=\pm}(c^\dagger_{\uparrow a}c_{\uparrow\bar{a}}c^\dagger_{\downarrow\bar{a}}c_{\downarrow a})=\frac{1}{2\pi^2\alpha^2}\cos(\sqrt{8\pi}\phi_s),$$

$$\sum_{a=\pm}(c^\dagger_{\uparrow a}c_{\uparrow\bar{a}}c^\dagger_{\downarrow a}c_{\downarrow\bar{a}})=\frac{1}{2\pi^2\alpha^2}\cos(\sqrt{8\pi}\phi_c). \tag{B3}$$

The terms in Eq. (B3) contribute to rescaling velocities in the action, giving rise to pseudocharge/pseudospin separation, while the last two terms are the cosine interactions. Therefore, the interaction part

$$\frac{1}{2}((\bar{\psi}_i\gamma^\mu\psi_j)(\bar{\psi}_k\gamma^\mu\psi_l)+(\bar{\psi}_i\gamma^\mu\psi_j)(\bar{\psi}_k\gamma_\mu\psi_l)) \longleftrightarrow \sum_{a=\pm}(c^\dagger_{\uparrow a}c_{\uparrow a}c^\dagger_{\downarrow a}c_{\downarrow a}+c^\dagger_{\uparrow a}c_{\uparrow a}c^\dagger_{\downarrow\bar{a}}c_{\downarrow\bar{a}}) \tag{B4}$$

just rescales the velocity, while the interaction part

$$\bar{\psi}_i\psi_j\bar{\psi}_k\psi_l \longleftrightarrow \sum_{a=\pm}(c^\dagger_{\uparrow a}c_{\uparrow\bar{a}}c^\dagger_{\downarrow\bar{a}}c_{\downarrow a}+c^\dagger_{\uparrow a}c_{\uparrow\bar{a}}c^\dagger_{\downarrow a}c_{\downarrow\bar{a}})$$

introduces cosine-like interaction terms in both sectors, in the $N=4$ case.

As a last comment, if we add to the interaction the term

$$(\bar{\psi}_i\gamma^5\psi_j)(\bar{\psi}_k\gamma^5\psi_l) \longleftrightarrow \sum_{a=\pm}(c^\dagger_{\uparrow a}c_{\uparrow\bar{a}}c^\dagger_{\downarrow a}c_{\downarrow\bar{a}}-c^\dagger_{\uparrow a}c_{\uparrow\bar{a}}c^\dagger_{\downarrow\bar{a}}c_{\downarrow a}),$$

it is possible to obtain a model having the cosine interaction term in just one of the spin/charge sectors. For instance, in random Gross-Neveu-like interaction

$$\frac{1}{2}[(\bar{\psi}_i\psi_j\bar{\psi}_k\psi_l)-(\bar{\psi}_i\gamma^5\psi_j)(\bar{\psi}_k\gamma^5\psi_l)] \longleftrightarrow \sum_{\alpha=\pm}(c^\dagger_{\uparrow a}c_{\uparrow\bar{a}}c^\dagger_{\downarrow\bar{a}}c_{\downarrow a})=\frac{1}{2\pi^2\alpha^2}\cos(\sqrt{8\pi}\phi_s), \tag{B5}$$

the cosine in the charge sector disappears.

## APPENDIX C: CORRELATORS IN THE N = 4 CASE

In this Appendix, we compute the different correlators for the case $N=4$ introduced in Sec. III. In the limit of strong coupling, the critical action holds for the pseudospin degree of freedom:

$$S=\int dt dx \frac{1}{2}\left[\frac{1}{u_s\mathcal{K}_s}(\partial_t\phi_s)^2-\frac{u_s}{\mathcal{K}_s}(\partial_x\phi_s)^2\right]. \tag{C1}$$

As the critical Hamiltonian is invariant under the duality transformation $\phi\leftrightarrow\theta, \mathcal{K}\leftrightarrow\frac{1}{\mathcal{K}}$, we can use this correspondence to get $\mathcal{I}_{\theta_s}(x,\tau)$ straightforwardly, once the phase-phase correlator $\langle\phi_s(r_1)\phi_s(r_2)\rangle$ in known. This is defined as [54]

$$\langle\phi_s(r_1)\phi_s(r_2)\rangle=\frac{\int\mathcal{D}\phi\mathcal{D}\theta\, e^{-S[\phi,\theta]}\phi_s(r_1)\phi_s(r_2)}{\int\mathcal{D}\phi\mathcal{D}\theta\, e^{-S[\phi,\theta]}}. \tag{C2}$$

In Fourier space, $\phi_s(r)=\frac{1}{\beta\Omega}\sum_{k,\omega_n}e^{i(kx-\omega_n\tau)}\phi_s(k,\omega_n)$. Real fields satisfy $\phi^*(q)=\phi(-q)$. The action can be represented as

$$S[\phi_s]=\frac{1}{\beta\Omega}\sum_{k,\omega_n}\frac{1}{2u_s\mathcal{K}_s}\left[\omega_n^2+u_s^2k^2\right]\phi_s(k,\omega_n)\phi_s^*(k,\omega_n). \tag{C3}$$

The correlator in momentum space is

$$\langle\phi_s(q_1)\phi_s(q_2)\rangle=u_s\mathcal{K}_s\frac{\beta\Omega}{\omega_n^2+u_s^2k^2}\delta_{q_1,-q_2}.$$

Back to real space, we obtain the correlator $\mathcal{I}_{\phi_s}(x,\tau)$ appearing in Eq. (49):

$$\mathcal{I}_{\phi_s}(x,\tau)=\langle\phi_s(x,\tau)\phi_s(0,0)-\phi_s^2(0,0)\rangle =\frac{\mathcal{K}_s}{2\pi}\ln\left(\frac{\alpha}{\sqrt{x^2+(u_s\tau+\alpha)^2}}\right). \tag{C4}$$

We now turn to the pseudocharge mode. Due to the presence of the gap $m=\frac{\Delta}{L}$, the approximate action is given by Eq. (54) and the correlators $\mathcal{I}_{\phi_c}(x,\tau)$ and $\mathcal{I}_{\theta_c}(x,\tau)$ are given by Eqs. (55) ($\omega=\omega' u_c$):

$$\left\langle\phi_c(r)\phi_c(0)-\phi_c^2(0)\right\rangle =2\mathcal{K}_c\int_{-\infty}^{+\infty}\frac{d\omega'}{2\pi}\int_0^{+\infty}\frac{dk}{2\pi}\frac{(e^{iu_c\omega'\tau}\cos kx-1)}{k^2+m^2+\omega'^2},$$

$$\langle\theta_c(r)\theta_c(0)-\theta_c^2(0)\rangle = \frac{2}{\mathcal{K}_c}\int_{-\infty}^{+\infty}\frac{d\omega'}{2\pi}\int_0^{+\infty}\frac{dk}{2\pi}\frac{(k^2+m^2)(e^{iu_c\omega'\tau}\cos kx-1)}{k^2(k^2+m^2+\omega'^2)}.$$

The Hamiltonian is not invariant under the transformation $\phi_c \leftrightarrow \theta_c$, $\mathcal{K}_c \leftrightarrow \frac{1}{\mathcal{K}_c}$ as before, due to the presence of the gap. To stick to analytic approximate expressions for the correlators, we are going to assume small $m$ and large distances $\zeta \equiv (u_c\tau \mp i\,x)$ in the following, by keeping just the leading contributions.

We define the following integrals:

$$I_{1\pm} \equiv \int_{-\infty}^{+\infty}\frac{d\omega'}{2\pi}e^{iu_c\omega'\tau}\int_0^{+\infty}\frac{dk}{2\pi}e^{\pm ikx}\frac{1}{k^2+m^2+\omega'^2}, \tag{C5}$$

$$I_{2\pm} \equiv \int_{-\infty}^{+\infty}\frac{d\omega'}{2\pi}e^{iu_c\omega'\tau}\int_0^{+\infty}\frac{dk}{2\pi}e^{\pm ikx} \times \frac{m^2}{(\omega'^2+m^2)(k^2+m^2+\omega'^2)}, \tag{C6}$$

$$I_{3\pm} \equiv m^2\int_{-\infty}^{+\infty}\frac{d\omega'}{2\pi}\frac{e^{iu_c\omega'\tau}}{(m^2+\omega'^2)}\int_0^{+\infty}\frac{dk}{2\pi}\frac{e^{\pm ikx}}{k^2} \equiv I_{3a}I_{3b\pm}, \tag{C7}$$

which allow to rewrite the correlators according to

$$\mathcal{I}_{\phi_c}(r) = \langle\phi_c(r)\phi_c(0)-\phi_c^2(0)\rangle = \mathcal{K}_c[I_1(r)-I_1(0)], \tag{C8}$$

$$\mathcal{I}_{\theta_c}(r) = \langle\theta_c(r)\theta_c(0)-\theta_c^2(0)\rangle = \frac{1}{\mathcal{K}_c}[I_1(r)-I_2(r)+I_3(r)-(I_1(0)-I_2(0)+I_3(0))], \tag{C9}$$

where $I_n = I_{n+}+I_{n-}$. The integral $I_{1\pm}$ is computed as follows:

$$I_{1\pm} = \int_{-\infty}^{+\infty}\frac{d\omega'}{2\pi}\int_0^{+\infty}\frac{dk}{2\pi}\frac{e^{\pm ikx}e^{i\omega' u_c\tau}}{2i\sqrt{k^2+m^2}}\left(\frac{1}{\omega'-i\sqrt{k^2+m^2}}-\frac{1}{\omega'+i\sqrt{k^2+m^2}}\right) = \int_0^{+\infty}\frac{dk}{4\pi}\frac{e^{\pm ikx}e^{-u_c\tau\sqrt{k^2+m^2}}}{\sqrt{k^2+m^2}}, \tag{C10}$$

where the circuit has been closed in the upper complex half plane. In the limit $m\to 0$ we want to obtain back the critical result. In the case $m=0$, $I_{1\pm}$ satisfies the relation

$$\left.\frac{\partial}{\partial(-u_c\tau\pm i\,x)}I_{1\pm}\right|_{m=0} = \int_0^{+\infty}\frac{dk'}{4\pi}\,e^{-(u_c\tau\mp i\,x)k'}e^{-\alpha k'} = -\frac{1}{4\pi(\mp i\,x+u_c\tau+\alpha)}, \tag{C11}$$

so that $I_{1\pm} = -\frac{1}{4\pi}\ln(\mp i\,x+u_c\tau+\alpha)$. However, in the limit $0<m\ll 1$, the derivative $\frac{\partial}{\partial(-u_c\tau\pm i\,x)}I_{1\pm}$ takes the form

$$\frac{\partial}{\partial(-u_c\tau\pm i\,x)}I_{1\pm} \approx \pm i\int_m^{+\infty}\frac{dz}{4\pi}\,e^{\pm i\,x\sqrt{z^2-m^2}}e^{-u_c\tau z}, \tag{C12}$$

with $\sqrt{z^2-m^2}\to z(1-\frac{m^2}{2z^2})$ and, considering again the convergent factor $\alpha$, Eq. (C11) becomes [65]

$$\frac{\partial I_{1\pm}}{\partial(-u_c\tau\pm i\,x)} \approx \int_0^{+\infty}\frac{dz}{4\pi}\,e^{-(u_c\tau\mp i\,x)z}e^{-(\pm 2\,i\,xm^2)\frac{1}{4z}} = -\frac{1}{4\pi}\sqrt{\frac{\pm 2i\,m^2x}{\mp i\,x+u_c\tau+\alpha}} \times K_1(\sqrt{\pm 2i\,m^2x}\sqrt{\mp i\,x+u_c\tau+\alpha}). \tag{C13}$$

Equation (C11) is recovered in the limit of vanishing $m$, because, at first order, the Bessel function $K_1(z)\to\frac{1}{z}$. We have added a spurious contribution in Eq. (C13),

$$\left|\int_0^m\frac{dz}{4\pi}\,e^{-(u_c\tau\mp i\,x)z}e^{-(\pm 2\,i\,xm^2)\frac{1}{4z}}\right| < \left|\int_0^m\frac{dz}{4\pi}\,e^{-\zeta z}\,e^{\mp i\,x\frac{m}{2}}\right| < \frac{m}{4\pi}\,\frac{1}{\zeta}, \tag{C14}$$

where $\zeta = (u_c\tau\mp i\,x)$. We disregard it, because $\zeta$ is assumed to be large. Hence, for small $m$, the nonchiral result is

$$I_{1\pm} = -\int_0^{\mp i\,x+u_c\tau}\frac{dz}{4\pi}\frac{\Delta}{L}\sqrt{\frac{\pm 2i\,x}{z+\alpha}}\,K_1\!\left(\frac{\Delta}{L}\sqrt{\pm 2i\,x}\sqrt{z+\alpha}\right).$$

As for the the second pair of integrals $I_{2\pm}$, we use the identity $\frac{1}{a^2+b^2} = \int_0^\infty e^{-s(a^2+b^2)}ds$. In the generalized Gaussian's integral, both integrals give the same result and we can sum them, thus obtaining

$$I_2 = m^2\int_{-\infty}^{+\infty}\frac{d\omega'}{2\pi}e^{iu_c\omega'\tau}\frac{e^{-x\sqrt{\omega'^2+m^2}}}{(\omega'^2+m^2)^{3/2}}. \tag{C15}$$

In the limit of small $m$, this integral can be approximated by reducing it to the chiral form as $e^{iu_c\omega'\tau}e^{-x\sqrt{\omega'^2+m^2}} \approx e^{-(x-iu_c\tau)\sqrt{\omega'^2+m^2}}$. In doing this, we can view the integral as

$$\frac{\partial^2}{\partial(x-i\,u_c\tau)^2}I_{2\pm} \approx m^2\int_0^\infty\frac{d\omega'}{\pi}\frac{e^{-(x-iu_c\tau)\sqrt{\omega'^2+m^2}}}{\sqrt{\omega'^2+m^2}} = \frac{m^2}{\pi}K_0[m(x-iu_c\tau)], \tag{C16}$$

which shows that this contribution can be neglected in the limit of small $m$ and large distances. In this form, it appears that $I_2$ can be estimated to give rise to a contribution $< m^2 e^{-(ix+u_c\tau)^2 v[m(x-iu_c\tau)]}$ with $v[\zeta]$ a positive function of $\zeta$.

In the last pair of integrals $I_{3\pm}$, the $\omega'$ and $k$ integrals can be separated. The $k$ integral contains a nonsingular contribution $I_{3a}$ and a singular contribution $I_{3b\pm}$. The nonsingular integral for $I_{3a}$ gives

$$I_{3a} = m^2 \int_{-\infty}^{+\infty} \frac{d\omega'}{2\pi} \frac{e^{iu_c\omega'\tau}}{(\omega'^2+m^2)} = m^2 \frac{i}{4\pi m}[e^{mu_c\tau}\mathrm{Ei}(iu_c\tau\omega - u_c\tau m) - e^{-mu_c\tau}\mathrm{Ei}(iu_c\tau\omega + u_c\tau m)]\Big|_{-\infty}^{+\infty} \approx \frac{m}{2}e^{-mu_c\tau}$$

(where $\mathrm{Ei}(z)$ is the exponential integral, with the following properties $\mathrm{Ei}(i\infty) = i\pi$ and $\mathrm{Ei}(-i\infty) = -i\pi$, and where just the leading decaying exponential part was considered). For $I_{3b-}$ we have

$$\begin{aligned} I_{3b-} &\approx \int_{1/L}^{+\infty} dk \lim_{\epsilon\to 0} \frac{e^{-ikx}}{\epsilon^2+k^2} = x \lim_{\epsilon\to 0}\int dy \frac{e^{-iy}}{\epsilon^2+y^2} \\ &= x \lim_{\epsilon\to 0} \frac{i}{2\epsilon}[e^{\epsilon}E_1(\epsilon+iy) - e^{-\epsilon}E_1(-\epsilon+iy)] \approx L\, e^{-i\frac{x}{L}} + i\,x\Big[\gamma + \ln i\,\frac{x}{L}\Big]. \end{aligned} \tag{C17}$$

It follows that the singular part $I_{3\pm}$ becomes

$$m^2 \int \frac{dk}{2\pi} \int \frac{d\omega}{2\pi} \frac{e^{i\,\omega\tau \pm i\,kx}}{k^2(\omega^2+m^2)} \approx \frac{mL}{4\pi}\left[e^{-(m\tau \mp i\,\frac{x}{L})} \mp i\,e^{-m\tau}\,\frac{x}{L}\left(\gamma + \ln\frac{\mp i\,x}{L}\right)\right]. \tag{C18}$$

However, $\Gamma(z)$ is the $\Gamma$ function, which can be approximated as $\ln\Gamma(z) \approx (z-\frac{1}{2})\ln z - z + \frac{1}{2}\ln 2\pi + \mathcal{O}(\frac{1}{z})$, so that the final expression for $I_{3\pm}$ is

$$I_{3\pm} \approx \frac{mL}{4\pi}\,e^{-m\tau}\left[e^{\pm i\,\frac{x}{L}} + \ln\left[\sqrt{\frac{\mp i\,x}{2\pi L}}\right]\Gamma\left(\frac{\mp i\,x}{L}\right) + \frac{\mp i\,x}{L}(\gamma+1)\right]. \tag{C19}$$

Putting these results in Eqs. (C9), we obtain Eqs. (57) of the main text. Together with $\mathcal{I}_{\phi_s}(x,\tau)$ given by Eq. (C4) and $\mathcal{I}_{\theta_s}(x,\tau)$, the bosonized version of this two-point Green's function propagator can be constructed,

$$G_{\pm}(r) = \pm\frac{i}{2\pi\alpha}\left(e^{\frac{\pi}{2}\langle\phi_c(r)\phi_c(0)-\phi_c^2(0)\rangle}e^{\frac{\pi}{2}\langle\theta_c(r)\theta_c(0)-\theta_c^2(0)\rangle}e^{\frac{\pi}{2}\langle\phi_s(r)\phi_s(0)-\phi_s^2(0)\rangle}e^{\frac{\pi}{2}\langle\theta_s(r)\theta_s(0)-\theta_s^2(0)\rangle}\right), \tag{C20}$$

which gives Eq. (58) of the main text.

It can be proved that the off-diagonal correlators vanish identically in the bosonized $N=4$ model because they correspond to nonnumber conserving correlators which are absent in the $N=4$ case. In fact, recalling the initial definition of complex fermions, it is possible to write [see Eq. (60)]

$$\begin{aligned} G_{\cap/\cup}(r) &= \frac{1}{4}\sum_i \langle\psi_{i\pm}(r)\psi_{i\mp}(0)\rangle \\ &= \frac{1}{4}\sum_{\sigma=\uparrow,\downarrow}\langle c_{\sigma\pm}(r)c^{\dagger}_{\sigma\mp}(0) + c^{\dagger}_{\sigma\pm}(r)c_{\sigma\mp}(0)\rangle. \end{aligned} \tag{C21}$$

Using the bosonization dictionary and the fact that pseudocharge and pseudospin sectors can be completely separated, the correlators become

$$\begin{aligned} G_{\cap/\cup}(r) = \pm\frac{i}{8\pi\alpha}[&\langle e^{\pm i\sqrt{\frac{\pi}{2}}\phi_c(r)}e^{\pm i\sqrt{\frac{\pi}{2}}\phi_c(0)}\rangle\langle e^{-i\sqrt{\frac{\pi}{2}}\theta_c(r)}e^{i\sqrt{\frac{\pi}{2}}\theta_c(0)}\rangle(\phi_c\to\phi_s,\quad \theta_c\to\theta_s) \\ &+\langle e^{\mp i\sqrt{\frac{\pi}{2}}\phi_c(r)}e^{\mp i\sqrt{\frac{\pi}{2}}\phi_c(0)}\rangle\langle e^{i\sqrt{\frac{\pi}{2}}\theta_c(r)}e^{-i\sqrt{\frac{\pi}{2}}\theta_c(0)}\rangle(\phi_c\to\phi_s,\quad \theta_c\to\theta_s) \\ &+\langle e^{\pm i\sqrt{\frac{\pi}{2}}\phi_c(r)}e^{\pm i\sqrt{\frac{\pi}{2}}\phi_c(0)}\rangle\langle e^{-i\sqrt{\frac{\pi}{2}}\theta_c(r)}e^{i\sqrt{\frac{\pi}{2}}\theta_c(0)}\rangle(\phi_c\to-\phi_s,\quad \theta_c\to-\theta_s) \\ &+\langle e^{\mp i\sqrt{\frac{\pi}{2}}\phi_c(r)}e^{\mp i\sqrt{\frac{\pi}{2}}\phi_c(0)}\rangle\langle e^{i\sqrt{\frac{\pi}{2}}\theta_c(r)}e^{-i\sqrt{\frac{\pi}{2}}\theta_c(0)}\rangle(\phi_c\to-\phi_s,\quad \theta_c\to-\theta_s)]. \end{aligned}$$

For the correlator to be nonzero, the sum of the factors multiplying the fields in the exponentials has to vanish. In the $\phi$ sector this does not happen and off-diagonal correlators vanish. On the contrary, in the $\theta$ sector opposite signs occur in the exponents, so that correlators involving cross-chirality fermions, like

$$\left\langle\mathcal{O}^{z\dagger}_{\mathrm{TS}}(r)\mathcal{O}^{z}_{\mathrm{TS}}(0)\right\rangle = \frac{1}{\pi^2\alpha^2}e^{2\pi\langle\theta_c(r)\theta_c(0)-\theta_c^2(0)\rangle}e^{2\pi\langle\theta_s(r)\theta_s(0)-\theta_s^2(0)\rangle} \tag{C22}$$

of Eq. (63), provide a nonzero result. They can be obtained from Eqs. (49) and (57).

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