From frustration-free parent Hamiltonians to off-diagonal long-range order: Moore-Read and related states in second quantization

Fanmao Zhang,[1] Matheus Schossler,[2] Alexander Seidel,[3,4,2] and Li Chen[1]
[1]*College of Physics and Electronic Science, Hubei Normal University, Huangshi 435002, China*
[2]*Department of Physics, Washington University, St. Louis, Missouri 63130, USA*
[3]*Physics Department, Technical University of Munich, TUM School of Natural Sciences, 85748 Garching, Germany*
[4]*Munich Center for Quantum Science and Technology (MCQST), Schellingstrasse 4, 80799 München, Germany*

(Received 8 June 2023; accepted 8 August 2023; published 18 August 2023)

We construct a recursive second-quantized formula for Moore-Read Pfaffian states. We demonstrate the utility of such second-quantized presentations by directly proving the existence of frustration-free parent Hamiltonians, without appealing to polynomial clustering properties. Furthermore, we show how this formalism is connected to the existence of a nonlocal order parameter for Moore-Read states, and we provide proof that the latter exhibit off-diagonal long-range order (ODLRO) in these quantities. We also develop a similar second-quantized presentation for the fermionic anti- and PH-Pfaffian states, as well as f- and higher wave paired composite fermion states, and we discuss ODLRO in most cases.

DOI: 10.1103/PhysRevB.108.075142

I. INTRODUCTION

The past few decades have witnessed tremendous efforts in the study of strongly correlated systems, including unconventional superconductors [1–3], quantum spin liquids [4–7], as well as fractional quantum Hall (FQH) systems [8–11]. In FQH systems, the kinetic energies of electrons are quenched as electrons occupy a certain Landau level, rendering Coulomb interactions as the major term in the Hamiltonian. The closed form for the ground state of the many-body Coulomb interaction is difficult to obtain; thus, theorists resort to model Hamiltonians for which the prototypical trial state of the closed-form wave function is the *exact* unique densest zero mode (zero-energy ground state with the minimum total angular momentum). These model Hamiltonians include a two-body pseudopotential [12] for the Laughlin state [13], a two-body parent Hamiltonian [14] for the *unprojected* Jain composite fermion state [10,15], a three-body parent Hamiltonian [16,17] for the Moore-Read Pfaffian state [18], and general multibody parent Hamiltonians for Read-Rezayi states [19–21]. Of all FQH states, much attention has been paid to those with non-Abelian anyonic excitation, a key necessary ingredient for topological quantum computation [22,23]. A typical example is the Moore-Read Pfaffian state, which is constructed from correlators in conformal field theory [24].

In the study of model FQH states and their corresponding parent Hamiltonian, it is common practice to focus on the *first-quantized* wave functions, whose algebraic clustering properties when two or more particles come together are traditionally utilized to construct closely related first-quantized parent Hamiltonians. More recently, a second-quantized approach was developed to yield alternative, second-quantized presentations of FQH model states, study their parent Hamiltonians, and establish new such Hamiltonians [14,25,26]. In particular, this approach has proven effective in constructing parent Hamiltonians [14] for unprojected Jain composite fermion states, which are, in general, not fully characterized by conventional clustering properties. It has also been used to explain the existence of a frustration-free parent Hamiltonian as a consequence of the matrix product structure of the Laughlin state [27]. Furthermore, it inspired a picture for particle fractionalization [28] that largely recovers a symmetry between quasiholes and *quasiparticles*, which is typically obscure in traditional treatments. A strength of the second-quantized approach is that it allows rigorous statements about the zero mode space of some frustration-free solvable models where traditional methods are inadequate. This is particularly so in the context of partonlike states (see Refs. [29–38] and references therein), where Landau-level mixing leads to wave functions that are no longer represented by holomorphic polynomials, barring established techniques from being used to prove uniqueness and/or completeness of zero-mode trial wave functions. Alternative methods to achieve such statements have recently been developed, emphasizing largely second-quantized methods over first-quantized ones. In some cases, one can develop the entire theory surrounding certain classes of trial wave functions, their parent Hamiltonians, and their associated zero-mode spaces using an exclusively second-quantized formalism that nowhere references the polynomials associated with first-quantized wave functions. This has been done, in particular, for Laughlin states [25] as well as all composite fermion states in the positive Jain sequence [14]. Here, the construction of traditional polynomial trial wave functions is replaced by certain recursion relations in particle number that allow the second-quantized trial states to be created from the vacuum via a corresponding operator product. The prototypical version of such products is Read's presentation [39] of the Laughlin state as a "condensate" involving a nonlocal order parameter (which was originally given in a mixed first-/second-quantized notation). Analogously, second-quantized constructions were recently discussed for composite fermion states [26].

In this paper, we put forth similar developments that yield a fully second-quantized construction of the Moore-Read sequence, and a concurrent discussion of its parent Hamiltonians. Our main result is a fully second-quantized expression of Moore-Read states as an operator product acting on the vacuum. As is well known, the parent Hamiltonians of Moore-Read states involve three-body terms [16,17]. While higher-body terms are quite common in the literature of quantum Hall parent Hamiltonians [40–45], the discussion is typically limited to the lowest Landau level utilizing first quantization. While we will not leave the lowest Landau level in this paper, one byproduct of our approach will be the extension of second-quantized methods so far exclusively applied to two-body interactions to solvable models involving higher-body terms. We thus make manifest how the "frustration-free property" of the Moore-Read state and its parent Hamiltonian arises in second quantization. We will further utilize these results to demonstrate the existence of off-diagonal long-range order in Moore-Read states. Finally, we will extend several of these results to the anti-Pfaffian and PH-Pfaffian states.

This paper is organized as follows. In Sec. II A, we set up the problem. In Sec. II B, we postulate a second-quantized recursive formula (2.16a) for fermionic (bosonic) $\nu = 1/M$ Pfaffian state, whose zero-mode property is proven in Secs. II C and II D. In Sec. II E, we perform a root analysis of the recursively defined state. In Sec. II F, we obtain its second-quantized nonlocal order parameter and prove the existence of off-diagonal long-range order. In Sec. II G, we generalize to Pfaffian states with higher angular momentum pairing. In Sec. III, we obtain the second-quantized recursive formulas (3.1) and (3.8) for fermionic anti- and PH-Pfaffian states, based on the recursive formula for the fermionic Pfaffian state. We present discussion and outlook in Sec. IV.

II. SECOND-QUANTIZED MOORE-READ PFAFFIAN STATE

A. Moore-Read Pfaffian state and its parent Hamiltonian

In this section, we review some defining properties of the Moore-Read state and its parent Hamiltonian, and we establish the second-quantized formulation of these properties.

The parent Hamiltonian for the $\nu = 1/M$ fermionic (bosonic) Moore-Read Pfaffian state [18], whose first-quantized wave function is given by

$$\mathrm{Pf}\left(\frac{1}{z_i - z_j}\right)\prod_{k<l}(z_k - z_l)^M \tag{2.1}$$

with even (odd) positive integer M for fermions (bosons), respectively, consists of two-body and three-body projection operators [46],

$$H = H^{(2\mathrm{bd})} + H^{(3\mathrm{bd})}. \tag{2.2}$$

The two-body projection operator $H^{(2\mathrm{bd})}$ in second quantization is of the following form [47]:

$$H^{(2\mathrm{bd})} = \sum_{\substack{0\leqslant m<M-1\\ (-1)^m=(-1)^{M-1}}} \sum_{J\in\mathbb{Z}^{0+}} T_J^{(2\mathrm{bd},m)\dagger} T_J^{(2\mathrm{bd},m)}, \tag{2.3}$$

where the positive-semidefinite two-body fermionic (bosonic) operator $T_J^{(2\mathrm{bd},m)\dagger} T_J^{(2\mathrm{bd},m)}$ is the second-quantized form of the Haldane V_m pseudopotential [12]. That is, it projects onto an antisymmetric (symmetric) two-body state of relative angular momentum $m\hbar$ and total angular momentum $J\hbar$ in the lowest Landau level (LLL). In disk geometry, it can be given a concrete form via

$$T_J^{(2\mathrm{bd},m)} = 2^{\frac{1-J}{2}} \sum_k p_{m,\frac{J}{2}}(k) \sqrt{\binom{J}{m}\binom{J}{\frac{J}{2}+k}}\, c_{\frac{J}{2}-k} c_{\frac{J}{2}+k}, \tag{2.4}$$

and similar expressions hold in other geometries [47]. Here, $\binom{J}{m} = J!/(J-m)!m!$ is the binomial coefficient, and c_i is a fermionic (bosonic) operator that annihilates a particle of angular momentum $i\hbar$ in the LLL. Throughout this paper, we are dealing with LLL orbitals on the disk, so only those c_i with non-negative i are of concern to us. We therefore let $c_i = 0$ whenever we formally encounter negative i in the calculation. $p_{m,\frac{J}{2}}(k)$ is a polynomial in k of degree m and parity $(-1)^m$, whose expression is given by

$$p_{m,\frac{J}{2}}(k) = (-1)^{m+\frac{J}{2}-k} \frac{\binom{m}{J/2-k}}{\binom{J}{J/2-k}} \times {}_2F_1\left(-\frac{J}{2}+k, -J+m, 1-\frac{J}{2}+k+m, -1\right) \tag{2.5}$$

with ${}_2F_1$ the hypergeometric function.

The zero mode, or ground space of $H^{(2\mathrm{bd})}$, is spanned by the $\nu = 1/(M-1)$ Laughlin state and its zero-energy excitations, which physically represent the edge and quasihole excitations of this state. The zero-mode condition associated with $H^{(2\mathrm{bd})}$ can be cast as

$$T_J^{(2\mathrm{bd},m)} |\psi_{\mathrm{zero}}\rangle = 0 \tag{2.6}$$

for all J and m in Eq. (2.3). This zero-mode condition is clearly invariant under the formation of new linearly independent linear combinations of the operators $T_J^{(2\mathrm{bd},m)}$, and thus it can be written as

$$Q_J^{(2\mathrm{bd},m)} |\psi_{\mathrm{zero}}\rangle = 0 \tag{2.7}$$

in terms of simpler operators

$$Q_J^{(2\mathrm{bd},m)} = \sum_{\substack{0\leqslant i_1,i_2\leqslant J\\ i_1+i_2=J}} \frac{(i_1-i_2)^m}{\sqrt{i_1!i_2!}} c_{i_2} c_{i_1}. \tag{2.8}$$

Here, J and m run over the same values as before. The simple monomial form of the last expression offers yet a more condensed version of the two-body zero-mode condition. Defining the operators

$$Q_J^{(2\mathrm{bd},\mathcal{P})} = \sum_{\substack{0\leqslant i_1,i_2\leqslant J\\ i_1+i_2=J}} \frac{\mathcal{P}(i_1,i_2)}{\sqrt{i_1!i_2!}} c_{i_2} c_{i_1}, \tag{2.9}$$

where $\mathcal{P}$ is any polynomial in two variables of the requisite symmetry, we may equivalently cast Eq. (2.7) as

$$Q_J^{(2\text{bd},\mathcal{P})}\,|\psi_{\text{zero}}\rangle = 0, \tag{2.10}$$

where J runs over all non-negative integers as before, and $\mathcal{P}$ can be *any* polynomial of degree less than $M-1$. To see the equivalence with Eq. (2.7), write $\mathcal{P}$ in terms of variables i_1+i_2 and i_1-i_2, and note that i_1+i_2 is a constant in the definition of Eq. (2.9).

We will now similarly cast the zero-mode condition associated with $H^{(3\text{bd})}$. $H^{(3\text{bd})}$, as given in the literature [21,46], is a three-body projection operator that projects onto states of relative angular momentum $3M-3$. To make the claim even stronger, we also include the three-body projection operator that projects onto states of relative angular momentum $3M-2$. The Moore-Read state will be the unique zero mode of the resulting Hamiltonian within its angular momentum sector with or without the addition of the $3M-2$ term. Note, however, that the latter must be taken to vanish identically if $M=2$ (fermionic case) or $M=1$ (bosonic case), since the corresponding three-body states do not exist [21]. The second-quantized form for $H^{(3\text{bd})}$ is thus given by

$$H^{(3\text{bd})} = \sum_{t=3M-3}^{3M-2}\sum_{J\in\mathbb{Z}^{0+}} T_J^{(3\text{bd},t)\dagger}T_J^{(3\text{bd},t)}, \tag{2.11}$$

with

$$T_J^{(3\text{bd},t)} = \sum_{\substack{0\leqslant i_1,i_2,i_3\leqslant J\\ i_1+i_2+i_3=J}} \frac{\mathcal{Q}_t(i_1,i_2,i_3)}{\sqrt{i_1!i_2!i_3!}}c_{i_3}c_{i_2}c_{i_1}. \tag{2.12}$$

Here, t runs over an index set that labels an orthonormal basis of three-particle states with total angular momentum J and relative angular momentum t (all in units of $\hbar$). Any such state can be expressed via Eq. (2.12) through an appropriately chosen polynomial $\mathcal{Q}_t$ in three variables, of the requisite symmetry for fermions/bosons. ($\mathcal{Q}_t$ will also depend on J and M; we will, however, leave this understood.) $\mathcal{Q}_t$ can be chosen to be of degree t (not necessarily homogeneous).

The zero-mode condition associated with $H^{(3\text{bd})}$ then reads, in complete analogy with the two-body case,

$$T_J^{(3\text{bd},t)}\,|\psi_{\text{zero}}\rangle = 0 \tag{2.13}$$

for all $J\geqslant 0$ and $t=3M-3, 3M-2$.

For general M, the polynomials $\mathcal{Q}_t$ are rather complex, even more so than their two-body counterparts (2.5). Luckily, we will not need their precise form. For similar reasons, though perhaps less well known, the zero-mode condition (2.13) can be given an equivalent form analogous to Eq. (2.10). To this end, we define generic three-body destruction operators

$$Q_J^{(3\text{bd},\mathcal{Q})} = \sum_{\substack{0\leqslant i_1,i_2,i_3\leqslant J\\ i_1+i_2+i_3=J}} \frac{\mathcal{Q}(i_1,i_2,i_3)}{\sqrt{i_1!i_2!i_3!}}c_{i_3}c_{i_2}c_{i_1} \tag{2.14}$$

with $\mathcal{Q}$ a polynomial in three variables and of the desired (anti)symmetry. By definition, the zero modes we are interested in satisfy *both* the two-body and the three-body zero-mode conditions Eqs. (2.10) and (2.13). In this case, we may, however, replace the three-body zero-mode condition (2.13) with the seemingly stronger condition

$$Q_J^{(3\text{bd},\mathcal{Q})}\,|\psi_{\text{zero}}\rangle = 0 \tag{2.15}$$

for all integers $J\geqslant 0$ *and* all three-variable polynomials $\mathcal{Q}$ of degree less than or equal to $3M-2$. Clearly, ensuring Eq. (2.15) is sufficient to ensure that Eq. (2.13) is also satisfied. Below we will show that our second-quantized expression for the Moore-Read state satisfies both Eqs. (2.10) and (2.15). It is thus, in particular, a zero mode of the Hamiltonian Eq. (2.2). In turn, any state that is a zero mode of this Hamiltonian, and has the same angular momentum as the Moore-Read state, must be equal to the Moore-Read state (2.1) itself (up to a constant). This follows from known spectral properties of this Hamiltonian [21,46]. We will thus be able to establish, without referring to any explicit first-quantized polynomial construction, that the second-quantized expression, which will constitute the main result of this work below, is the Moore-Read state.

It may be instructive, however, to understand why fulfillment of the stronger equation (2.15) by the Moore-Read state is not coincidental, but indeed a zero mode satisfying both Eq. (2.10) (or any of its equivalents) and Eq. (2.13) also satisfies Eq. (2.15). This may be done as follows. One may convince oneself that any three-particle state generated from the vacuum $|0\rangle$ via $(Q_J^{3\text{bd},\mathcal{Q}})^\dagger\,|0\rangle$, with $\mathcal{Q}$ of degree L, lies in the subspace of relative angular momentum less than or equal to L. (Conversely, if a three-particle state of given total angular momentum J has relative angular momentum L, it can be written in this way by a polynomial of degree L.) Hence, for $L=3M-2$ and at given J, these three-particles states span the subspace spanned by the states associated with the $\mathcal{Q}_t$ defined after Eq. (2.12) *and* (all) additional states of relative angular momentum *less than* $3M-2$. However, it is well known that zero modes of $H^{(2\text{bd})}$ in Eq. (2.3) are automatically annihilated by three-particle projection operators onto states with relative angular momentum less than $3M-3$. It is for this reason that such three-particle projection operators are usually excluded from Eq. (2.2). Hence, in the presence of the two-body constraint (2.10), the three-body constraint (2.15) becomes truly equivalent to that of (2.13).

B. Recursive formula for the fermionic (bosonic) Pfaffian state

With its essential defining properties now in place, we postulate the following second-quantized recursive formula for the Moore-Read "Pfaffian" state, whose first-quantized wave function is Eq. (2.1):

$$|\text{Pf}_{N+2}\rangle = \frac{1}{N+2}\sum_{l=0}^{M-1}(-1)^l\binom{M-1}{l}\sum_{r,k=0}^{MN+M-1}\sqrt{r!k!}\times c_r^\dagger c_k^\dagger S_{MN+M-1-l-r}S_{MN+l-k}\,|\text{Pf}_N\rangle \tag{2.16a}$$

for even non-negative particle number N, where the particle-number-conserving operator S_ℓ is defined in Eq. (2.17) below. The beginning of the recursion is defined by $|\text{Pf}_0\rangle = |0\rangle$. As we comment below, the recursion (2.16a) can be viewed as a purely second-quantized version of a "mixed" first-/second-quantized presentation of the Pfaffian state that has already appeared in the original work by Moore and Read [18]. While Eq. (2.16a) can be derived directly from the Moore-Read wave function (2.1), we will emphasize here that one does not need to make contact with this first-quantized wave function, nor any other presentation given originally by Moore and Read, in order to show directly that (2.16a) defines the densest zero mode of a frustration-free parent Hamiltonian. Our approach is thus intrinsically second-quantized.

In addition, the recursion (2.16a) generalizes a similar second-quantized recursion for the Laughlin state [25] that, in turn, can be seen to be a (purely) second-quantized rendition of Read's presentation [39] of the Laughlin state as the "Bose condensate" of certain (nonlocal) "order parameter" operators that are off-diagonal in particle number. An important distinction between Eq. (2.16a) for the Moore-Read state and the earlier recursions for the Laughlin state is that we are increasing the particle number by 2, reflecting the paired nature of the state. However, the Moore-Read state with an odd particle number can also be accessed in this framework, simply via removal of one particle from $|\mathrm{Pf}_N\rangle$ with even N. We will comment in detail on particle removal further below. Wherever desired, we will notationally condense Eq. (2.16a) to

$$|\mathrm{Pf}_{N+2}\rangle = \mathcal{R}_N\,|\mathrm{Pf}_N\rangle\,, \tag{2.16b}$$

where $\mathcal{R}_N$ denotes the operator on the right-hand side of Eq. (2.16a).

To prove Eq. (2.16), we will utilize the strategy set up in the preceding section. That is, we will establish the state $|\mathrm{Pf}_N\rangle$ as defined in Eq. (2.16a) to be a zero mode of the parent Hamiltonian (2.2), which uniquely defines the state given that it has the proper total angular momentum. This serves the important additional purpose of exposing the inner workings that render complex (long-ranged) second-quantized positive-semidefinite Hamiltonians—like the one in question—frustration-free. It is also for this reason that we proceed without making any essential use of the first-quantized wave function (2.1). We will, however, comment on how Eq. (2.16a) *could* be derived in the first-quantized manner in Appendix A.

To proceed, we make contact with the operator formalism first established in Refs. [25,47,48], and then generalized to composite fermions in multiple Landau levels in Ref. [26]. The S operator [25,26] in Eq. (2.16a), which originates from $\prod_{i<j}(z_i - z_j)^M$ in the first quantization, is defined as

$$\begin{aligned} S_\ell &= (-1)^\ell \sum_{n_1+n_2+\cdots+n_M=\ell} e_{n_1} e_{n_2}\cdots e_{n_M} \quad \text{for} \quad \ell \geqslant 0,\\ S_\ell &= 0 \quad \text{for} \quad \ell < 0, \end{aligned} \tag{2.17}$$

where e_n, in turn, is the particle-number-conserving operator that, in first quantization, multiplies the wave function with the elementary symmetric polynomial $2^{-n/2}\sum_{1\leqslant i_1<i_2\cdots<i_n\leqslant N} z_{i_1}z_{i_2}\cdots z_{i_n}$. Second-quantized representations of these operators and other generators of symmetric polynomials have been discussed in detail in Ref. [48]. We have

$$\begin{aligned} e_n &= \frac{1}{n!}\sum_{l_1,\ldots,l_n=0}^{+\infty} \sqrt{l_1+1}\,c^\dagger_{l_1+1}\sqrt{l_2+1}\,c^\dagger_{l_2+1}\cdots\sqrt{l_n+1}\\ &\quad\times c^\dagger_{l_n+1}c_{l_n}\cdots c_{l_2}c_{l_1} \quad \text{for } n>0,\\ e_0 &= \mathbb{1},\\ e_n &= 0 \quad \text{for } n<0. \end{aligned} \tag{2.18}$$

This then allows for recursive generation of the second-quantized Moore-Read state via Eqs. (2.16) and (2.17).

e_n is related to the power-sum symmetric polynomial operator

$$P_d = \sum_{r=0}^{+\infty}\sqrt{\frac{(r+d)!}{r!}}\,c^\dagger_{r+d}c_r \tag{2.19}$$

for $d \geqslant 0$ by the Newton-Girard relation [26,48],

$$e_n = \frac{1}{n}\sum_{d=1}^{n}(-1)^{d-1}P_d\,e_{n-d}. \tag{2.20}$$

The action of P_d on an N-particle state is that of multiplying its first-quantized wave function with the power-sum symmetric polynomial $P_d \equiv 2^{-d/2}\sum_{i=1}^N z_i^d$.

P_d is a "zero-mode generator" in the sense that when acting on a zero mode $|\psi_{\text{zero}}\rangle$, as defined by Eqs. (2.10) and (2.15), it gives a new zero mode. The reason is that $Q_J^{(2\text{bd},\mathcal{P})}P_d\,|\psi_{\text{zero}}\rangle = 0$ since $[Q_J^{(2\text{bd},\mathcal{P})}, P_d]$ is of the form $Q_{J-d}^{(2\text{bd},\mathcal{P}')}$, with $\mathcal{P}'$ a polynomial of degree no larger than that of $\mathcal{P}$. Thus, $[Q_J^{(2\text{bd},\mathcal{P})}, P_d]$ vanishes on zero modes by Eq. (2.10). For analogous reasons, we also have $Q_J^{(3\text{bd},\mathcal{Q})}P_d\,|\psi_{\text{zero}}\rangle = 0$. By the Newton-Girard formula, every e_n can be expressed in terms of all P_d with $d = 1, 2, \ldots, n$. Therefore, e_n and S_ℓ are also zero-mode generators.

Another important property of S_ℓ is that different S_ℓ commute with each other. The commutative property of S_ℓ can likewise be established by first establishing the commutativity of the P_d among themselves, and then extending this property to the e_n via Newton-Girard relations.

A centerpiece of this work and the machinery to follow is the description of the effect of the removal of a single particle in state r from the state $|\mathrm{Pf}_{N+2}\rangle$ in terms of the addition of a particle to the state $|\mathrm{Pf}_N\rangle$, plus operators generating a "correlation hole" just big enough such that the net effect is the local charge depletion described by c_r:

$$\begin{aligned} c_r\,|\mathrm{Pf}_{N+2}\rangle &= \frac{\sqrt{r!}}{2}\sum_{l=0}^{M-1}(-1)^l\binom{M-1}{l}\sum_{k=0}^{MN+M-1}\sqrt{k!}\,c_k^\dagger\\ &\quad\times[S_{MN+M-1-l-r}S_{MN+l-k} + (-1)^{M-1}\\ &\quad\times S_{MN+M-1-l-k}S_{MN+l-r}]\,|\mathrm{Pf}_N\rangle\,. \end{aligned} \tag{2.21}$$

This equation can actually be derived as a pure consequence of Eq. (2.16a), that is, without resorting to the first-quantized wave function of the Moore-Read state. We show this in the Supplemental Material [49]. The derivation is lengthy, however. To the less patient reader, we thus offer an alternative proof of Eq. (2.21) [and by extension Eq. (2.16a)] that uses the first-quantized wave function. This proof is given in Appendix A.

We shall now proceed to show that the recursion Eq. (2.16a) defines the Moore-Read state at filling $1/M$ by showing that it is a zero mode of the appropriate parent Hamiltonian at the proper angular momentum. We begin with the two-body terms.

C. Proof that the recursively defined Pfaffian state is a zero mode of the two-body Hamiltonian (2.3)

We prove by the method of mathematical induction that the state as recursively defined in Eq. (2.16a) is a zero mode of all $Q_J^{(2\text{bd},\mathcal{P})}$ with degree of $\mathcal{P}$ less than $M-1$, thus a zero mode of the two-body Hamiltonian (2.3).

Proof. We begin the induction by proving the claimed property directly for $|\text{Pf}_0\rangle = |0\rangle$, $|\text{Pf}_2\rangle$, and $|\text{Pf}_4\rangle$. By using the recursive formula Eq. (2.16a), the second-quantized form of $|\text{Pf}_2\rangle$ is

$$
\begin{aligned}
|\text{Pf}_2\rangle &= \frac{1}{2}\sum_{l=0}^{M-1}(-1)^l\binom{M-1}{l}\sum_{r,k=0}^{M-1}\sqrt{r!k!}\,c_r^\dagger c_k^\dagger \\
&\quad\times S_{M-1-l-r}S_{l-k}\,|0\rangle \\
&= \frac{1}{2}\sum_{l=0}^{M-1}(-1)^l\binom{M-1}{l}\sqrt{(M-1-l)!l!} \\
&\quad\times c_{M-1-l}^\dagger c_l^\dagger\,|0\rangle\,. \qquad (2.22)
\end{aligned}
$$

In the above calculation of $|\text{Pf}_2\rangle$, we have used the fact that the S operator is the sum of products of e operators, which have annihilation operators on the right, thus $S_{M-1-l-r}S_{l-k}\,|0\rangle$ vanishes unless $M-1-l-r=0$ and $l-k=0$. The second-quantized form of $|\text{Pf}_4\rangle$ is given in Eq. (B1).

$|\text{Pf}_0\rangle$, $|\text{Pf}_2\rangle$, and $|\text{Pf}_4\rangle$ are annihilated by all $Q_J^{(2\text{bd},\mathcal{P})}$ with degree of $\mathcal{P}$ less than $M-1$, since $|\text{Pf}_0\rangle$ is vacuum and

$$
\begin{aligned}
Q_J^{(2\text{bd},\mathcal{P})}\,|\text{Pf}_2\rangle &= (-1)^{M-1}\delta_{J,M-1}\sum_{l=0}^{M-1}(-1)^l\binom{M-1}{l} \\
&\quad\times \mathcal{P}(l, M-1-l) = 0,
\end{aligned}
$$

due to Eq. (B4). The proof that $|\text{Pf}_4\rangle$ is annihilated by all $Q_J^{(2\text{bd},\mathcal{P})}$ is given in Appendix B.

Now we establish the induction step, assuming that

$$
Q_J^{(2\text{bd},\mathcal{P})}\,|\text{Pf}_N\rangle = 0 \qquad (2.23)
$$

holds for all J, with some $N \geqslant 4$. Then we have

$$
\begin{aligned}
(N+2)Q_J^{(2\text{bd},\mathcal{P})}\,|\text{Pf}_{N+2}\rangle &= Q_J^{(2\text{bd},\mathcal{P})}\sum_{l=0}^{M-1}(-1)^l\binom{M-1}{l}\sum_{r,k=0}^{MN+M-1}\sqrt{r!k!}\,c_r^\dagger c_k^\dagger S_{MN+M-1-l-r}S_{MN+l-k}\,|\text{Pf}_N\rangle \\
&= 2\sum_{\substack{0\leqslant i_1,i_2\leqslant J\\ i_1+i_2=J}}\frac{\mathcal{P}(i_1,i_2)}{\sqrt{i_2!}}c_{i_2}\sum_{l=0}^{M-1}(-1)^l\binom{M-1}{l}\sum_{k=0}^{MN+M-1}\sqrt{k!}\,c_k^\dagger \\
&\quad\times\left[S_{MN+M-1-l-i_1}S_{MN+l-k} + (-1)^{M-1}S_{MN+M-1-l-k}S_{MN+l-i_1}\right]|\text{Pf}_N\rangle \\
&\quad+ 2(-1)^M\sum_{\substack{0\leqslant i_1,i_2\leqslant J\\ i_1+i_2=J}}\mathcal{P}(i_1,i_2)\sum_{l=0}^{M-1}(-1)^l\binom{M-1}{l}S_{MN+M-1-l-i_2}S_{MN+l-i_1}\,|\text{Pf}_N\rangle \\
&\quad+ \sum_{l=0}^{M-1}(-1)^l\binom{M-1}{l}\sum_{r,k=0}^{MN+M-1}\sqrt{r!k!}\,c_r^\dagger c_k^\dagger Q_J^{(2\text{bd},\mathcal{P})}S_{MN+M-1-l-r}S_{MN+l-k}\,|\text{Pf}_N\rangle \\
&= 4Q_J^{(2\text{bd},\mathcal{P})}\,|\text{Pf}_{N+2}\rangle + 2(-1)^M\sum_{\substack{0\leqslant i_1,i_2\leqslant J\\ i_1+i_2=J}}\mathcal{P}(i_1,i_2)\sum_{l=0}^{M-1}(-1)^l\binom{M-1}{l}S_{MN+M-1-l-i_2}S_{MN+l-i_1}\,|\text{Pf}_N\rangle\,,
\end{aligned}
\qquad (2.24)
$$

where we have used Eq. (2.16a) in the first step, $c_{i_2}c_{i_1}c_r^\dagger c_k^\dagger = \delta_{r,i_1}c_{i_2}c_k^\dagger + (-1)^{M-1}\delta_{r,i_2}c_{i_1}c_k^\dagger + \delta_{k,i_2}c_{i_1}c_r^\dagger + (-1)^{M-1}\delta_{k,i_1}c_{i_2}c_r^\dagger - \delta_{r,i_1}\delta_{k,i_2} + (-1)^M\delta_{r,i_2}\delta_{k,i_1} + c_r^\dagger c_k^\dagger c_{i_2}c_{i_1}$ in the second step, and Eq. (2.21) in the last step so as to reassemble the first expression after the second step into the first expression on the last line. We have also used the identity $Q_J^{(2\text{bd},\mathcal{P})}S_{MN+M-1-l-r}S_{MN+l-k}\,|\text{Pf}_N\rangle = 0$ since $S_{MN+M-1-l-r}$ and S_{MN+l-k} are zero-mode generators, and $|\text{Pf}_N\rangle$ is assumed to be a zero mode.

Now we need to simplify the last term

$$
\begin{aligned}
&\sum_{\substack{0\leqslant i_1,i_2\leqslant J\\ i_1+i_2=J}}\mathcal{P}(i_1,i_2)\sum_{l=0}^{M-1}(-1)^l\binom{M-1}{l} \\
&\quad\times S_{MN+M-1-l-i_2}S_{MN+l-i_1}\,|\text{Pf}_N\rangle\,. \qquad (2.25)
\end{aligned}
$$

Under a change of variables, $i_1 - l = i_1'$ and $i_2 + l = i_2'$, the above term becomes

$$\sum_{l=0}^{M-1}(-1)^l \binom{M-1}{l} \sum_{\substack{-l\leqslant i_1'\leqslant J-l \\ l\leqslant i_2'\leqslant J+l \\ i_1'+i_2'=J}} \mathcal{P}(i_1'+l, i_2'-l) \times S_{MN+M-1-i_2'} S_{MN-i_1'} |\mathrm{Pf}_N\rangle . \tag{2.26}$$

Now we shall use an important identity,

$$S_\ell |\mathrm{Pf}_N\rangle = 0 \quad \text{for } \ell > MN. \tag{2.27}$$

The reason for its validity is the following: the state $|\mathrm{Pf}_N\rangle$ has N particles, while nonzero S_ℓ is defined as $(-1)^\ell \sum_{n_1+n_2+\cdots+n_M=\ell} e_{n_1} e_{n_2} \cdots e_{n_M}$, in which e_{n_i} will move the orbitals of n_i particles for $i = 1, 2, \ldots, M$. For $\ell > MN$, there must be an n_i larger than the particle number N, thus S_ℓ annihilates $|\mathrm{Pf}_N\rangle$ in this case.

As a result of the above identity, the lower limit of both i_1' and i_2' can be changed to 0, which does not affect the summation. Therefore, the upper limit of both i_1' and i_2' can be changed to J on account of $i_1' + i_2' = J$. After the change of limits of summations, Eq. (2.25) can be finally simplified to

$$\sum_{\substack{0\leqslant i_1', i_2'\leqslant J \\ i_1'+i_2'=J}} \left[\sum_{l=0}^{M-1}(-1)^l \binom{M-1}{l} \mathcal{P}(i_1'+l, i_2'-l)\right] \times S_{MN+M-1-i_2'} S_{MN-i_1'} |\mathrm{Pf}_N\rangle , \tag{2.28}$$

which vanishes since the summation in the square brackets is exactly 0 as a result of Eq. (B4), considering that the degree of $\mathcal{P}$ is less than $M-1$.

After this lengthy simplification, we obtain $(N+2)Q_J^{(2\mathrm{bd},\mathcal{P})} |\mathrm{Pf}_{N+2}\rangle = 4Q_J^{(2\mathrm{bd},\mathcal{P})} |\mathrm{Pf}_{N+2}\rangle$. Therefore, if $|\mathrm{Pf}_N\rangle$ is a zero mode of all $Q_J^{(2\mathrm{bd},\mathcal{P})}$ for some $N \geqslant 4$, so will be $|\mathrm{Pf}_{N+2}\rangle$. By mathematical induction, the fermionic (bosonic) Pfaffian state, as recursively defined in Eq. (2.16a), is thus a zero mode of the two-body Hamiltonian (2.3). ■

D. Proof that a recursively defined Pfaffian state is a zero mode of the three-body Hamiltonian (2.11)

Next, we prove by the method of mathematical induction that the fermionic (bosonic) Pfaffian state, as recursively defined in Eq. (2.16a), is a zero mode of all $Q_J^{(3\mathrm{bd},\mathcal{Q})}$ with degree of $\mathcal{Q}$ less than $3M-1$, thus a zero mode of the three-body Hamiltonian (2.11).

Proof. To begin the induction, we prove the claimed property directly for $|\mathrm{Pf}_0\rangle = |0\rangle$, $|\mathrm{Pf}_2\rangle$, $|\mathrm{Pf}_4\rangle$, and $|\mathrm{Pf}_6\rangle$.

It is easy to see that $|\mathrm{Pf}_0\rangle$ and $|\mathrm{Pf}_2\rangle$ are annihilated by all $Q_J^{(3\mathrm{bd},\mathcal{Q})}$, since in these cases the particle numbers are less than 3. We prove that $|\mathrm{Pf}_4\rangle$ and $|\mathrm{Pf}_6\rangle$ are annihilated by all $Q_J^{(3\mathrm{bd},\mathcal{Q})}$ in Appendix C.

Now prove the induction step and assume that

$$Q_J^{(3\mathrm{bd},\mathcal{Q})} |\mathrm{Pf}_N\rangle = 0 \quad \text{for all } J \text{ and some } N \geqslant 6. \tag{2.29}$$

Similar to Eq. (2.24), we obtain

$$\begin{aligned}
(N+2)Q_J^{(3\mathrm{bd},\mathcal{Q})} |\mathrm{Pf}_{N+2}\rangle &= Q_J^{(3\mathrm{bd},\mathcal{Q})} \sum_{l=0}^{M-1}(-1)^l \binom{M-1}{l} \sum_{r,k=0}^{MN+M-1} \sqrt{r!k!}\, c_r^\dagger c_k^\dagger S_{MN+M-1-l-r} S_{MN+l-k} |\mathrm{Pf}_N\rangle \\
&= 3 \sum_{\substack{0\leqslant i_1,i_2,i_3\leqslant J \\ i_1+i_2+i_3=J}} \frac{\mathcal{Q}(i_1,i_2,i_3)}{\sqrt{i_1!i_2!}} c_{i_2} c_{i_1} \sum_{l=0}^{M-1}(-1)^l \binom{M-1}{l} \sum_{k=0}^{MN+M-1} \sqrt{k!}\, c_k^\dagger \\
&\quad \times [S_{MN+M-1-l-i_3} S_{MN+l-k} + (-1)^{M-1} S_{MN+M-1-l-k} S_{MN+l-i_3}] |\mathrm{Pf}_N\rangle \\
&\quad - 6 \sum_{\substack{0\leqslant k_1,k_2,i\leqslant J \\ k_1+k_2+i=J}} \mathcal{Q}(k_1,k_2,i) \frac{c_i}{\sqrt{i!}} \sum_{l=0}^{M-1}(-1)^l \binom{M-1}{l} S_{MN+M-1-l-k_1} S_{MN+l-k_2} |\mathrm{Pf}_N\rangle \\
&\quad + \sum_{l=0}^{M-1}(-1)^l \binom{M-1}{l} \sum_{r,k=0}^{MN+M-1} \sqrt{r!k!}\, c_r^\dagger c_k^\dagger Q_J^{(3\mathrm{bd},\mathcal{Q})} S_{MN+M-1-l-r} S_{MN+l-k} |\mathrm{Pf}_N\rangle \\
&= 6Q_J^{(3\mathrm{bd},\mathcal{Q})} |\mathrm{Pf}_{N+2}\rangle - 6 \sum_{\substack{0\leqslant k_1,k_2,i\leqslant J \\ k_1+k_2+i=J}} \frac{\mathcal{Q}(k_1,k_2,i)}{\sqrt{i!}} c_i \sum_{l=0}^{M-1}(-1)^l \binom{M-1}{l} S_{MN+M-1-l-k_1} S_{MN+l-k_2} |\mathrm{Pf}_N\rangle ,
\end{aligned} \tag{2.30}$$

where we have used Eq. (2.16a) in the first step, $c_{i_3} c_{i_2} c_{i_1} c_r^\dagger = \delta_{r,i_1} c_{i_3} c_{i_2} + (-1)^{M-1} \delta_{r,i_2} c_{i_3} c_{i_1} + \delta_{r,i_3} c_{i_2} c_{i_1} + (-1)^{M-1} c_r^\dagger c_{i_3} c_{i_2} c_{i_1}$ twice, and $c_r^\dagger c_{i_2} c_{i_1} = c_{i_2} c_{i_1} c_r^\dagger - \delta_{r,i_1} c_{i_2} + (-1)^M \delta_{r,i_2} c_{i_1}$ in the second step, and again Eq. (2.21) in the third step in order to condense terms into the first term on the last line. We have also used the identity $Q_J^{(3\mathrm{bd},\mathcal{Q})} S_{MN+M-1-l-r} S_{MN+l-k} |\mathrm{Pf}_N\rangle = 0$ since $S_{MN+M-1-l-r}$ and S_{MN+l-k} are zero-mode generators, and $|\mathrm{Pf}_N\rangle$ is assumed to be a zero mode.

Now we need to simplify the last term,

$$\sum_{\substack{0\leqslant k_1,k_2,i\leqslant J\\ k_1+k_2+i=J}} \mathcal{Q}(k_1,k_2,i)\frac{c_i}{\sqrt{i!}}\sum_{l=0}^{M-1}(-1)^l\binom{M-1}{l}S_{MN+M-1-l-k_1}S_{MN+l-k_2}\left|\mathrm{Pf}_N\right\rangle . \tag{2.31}$$

By using the commutator

$$[c_i,S_l]=\sum_{k=1}^{M}(-1)^k\binom{M}{k}\sqrt{\frac{i!}{(i-k)!}}\,S_{l-k}c_{i-k}, \tag{2.32}$$

this term can be rewritten as

$$\sum_{m_1,m_2=0}^{M}(-1)^{m_1+m_2}\binom{M}{m_1}\binom{M}{m_2}\sum_{\substack{0\leqslant k_1,k_2,i\leqslant J\\ k_1+k_2+i=J}}\sum_{l=0}^{M-1}(-1)^l\binom{M-1}{l}\mathcal{Q}(k_1,k_2,i)S_{MN+M-1-l-k_1-m_1}S_{MN+l-k_2-m_2}$$
$$\times\frac{c_{i-m_1-m_2}}{\sqrt{(i-m_1-m_2)!}}\left|\mathrm{Pf}_N\right\rangle . \tag{2.33}$$

Under a change of variables, $k_1+m_1+l=k_1'$, $k_2+m_2-l=k_2'$, and $i-m_1-m_2=i'$, the above term will be

$$\sum_{m_1,m_2=0}^{M}(-1)^{m_1+m_2}\binom{M}{m_1}\binom{M}{m_2}\sum_{l=0}^{M-1}(-1)^l\binom{M-1}{l}\sum_{\substack{m_1+l\leqslant k_1'\leqslant J+m_1+l\\ m_2-l\leqslant k_2'\leqslant J+m_2-l\\ -m_1-m_2\leqslant i'\leqslant J-m_1-m_2\\ k_1'+k_2'+i'=J}}\mathcal{Q}(k_1'-m_1-l,k_2'-m_2+l,i'+m_1+m_2)$$
$$\times S_{MN+M-1-k_1'}S_{MN-k_2'}\frac{c_{i'}}{\sqrt{i'!}}\left|\mathrm{Pf}_N\right\rangle . \tag{2.34}$$

Similar to Eq. (2.27), we shall use a constraint

$$S_\ell c_{i'}\left|\mathrm{Pf}_N\right\rangle=0\quad\text{for }\ell>M(N-1). \tag{2.35}$$

As a result of this constraint, the lower limit of k_1' can be raised to $2M-1$, the lower limit of k_2' can be raised to M, and the upper limit of i' can be lowered to $J-(3M-1)$ on account of $k_1'+k_2'+i'=J$. Also, observe that i' should be non-negative; therefore, the upper limit of both k_1' and k_2' can be changed to J.

After these changes of limits of summations, Eq. (2.31) can be finally simplified to

$$\sum_{\substack{2M-1\leqslant k_1'\leqslant J\\ M\leqslant k_2'\leqslant J\\ 0\leqslant i'\leqslant J-(3M-1)\\ k_1'+k_2'+i'=J}}\left[\sum_{m_1,m_2=0}^{M}(-1)^{m_1+m_2}\binom{M}{m_1}\binom{M}{m_2}\sum_{l=0}^{M-1}(-1)^l\binom{M-1}{l}\mathcal{Q}(k_1'-m_1-l,k_2'-m_2+l,i'+m_1+m_2)\right]$$
$$\times S_{MN+M-1-k_1'}S_{MN-k_2'}\frac{c_{i'}}{\sqrt{i'!}}\left|\mathrm{Pf}_N\right\rangle . \tag{2.36}$$

As a result of Eq. (B4), for the summations in the square brackets not to vanish, there should exist at least one term in $\mathcal{Q}$ in which the power of l, m_1, and m_2 should be greater than or equal to $M-1$, M, and M, respectively. However, the degree of $\mathcal{Q}$ is less than $3M-1$. Therefore, the term in the square brackets vanishes, rendering Eq. (2.31) zero.

After this lengthy simplification, we obtain $(N+2)Q_J^{(3\mathrm{bd},\mathcal{Q})}\left|\mathrm{Pf}_{N+2}\right\rangle=6Q_J^{(3\mathrm{bd},\mathcal{Q})}\left|\mathrm{Pf}_{N+2}\right\rangle$. Therefore, if $\left|\mathrm{Pf}_N\right\rangle$ is a zero mode of all $Q_J^{(3\mathrm{bd},\mathcal{Q})}$ for some $N\geqslant 6$, so will be $\left|\mathrm{Pf}_{N+2}\right\rangle$. By mathematical induction, the fermionic (bosonic) Pfaffian state, as recursively defined in Eq. (2.16a), is thus a zero mode of the three-body Hamiltonian (2.11). ■

E. Root state and filling factor of the fermionic (bosonic) Pfaffian state $\left|\mathrm{Pf}_N\right\rangle$

The Moore-Read FQH state belongs to a large class of trial wave functions that follow a "root state + squeezing" paradigm. This holds true for all Jack polynomial FQH trial states [50–54] and their fermionic counterparts, of which Moore-Read states are examples, and it has recently been generalized to a considerable number of mixed Landau level FQH states [26,55–58]. Consider those occupation number eigenstates $|\{n_i\}\rangle$ in the angular momentum LLL eigenbasis that appear with a nonzero coefficient in an N-particle state $|\psi\rangle$. $|\{n_i\}\rangle$ is a Slater determinant for fermions and a symmetrized monomial (permanent) for bosons, but we will prefer the neutral term *configuration* to refer to both cases. The case of interest will be where $|\psi\rangle$ is a zero mode of the parent Hamiltonian. Then we write

$$|\psi\rangle = |\psi\rangle_{\rm root} + \sum_{|\{n_i\}\rangle \neq |\psi\rangle_{\rm root}} C_{\{n_i\}} \, |\{n_i\}\rangle \, , \qquad (2.37)$$

where $|\psi\rangle_{\rm root}$ is comprised of those configurations in the expansion that cannot be obtained from any other configuration, appearing with nonzero coefficient in $|\psi\rangle$, through so-called inward-squeezing processes [51]. These inward-squeezing processes are generated by the operations

$$c_j^\dagger c_i^\dagger c_{i-m} c_{j+m}, \qquad (2.38)$$

where $i \leqslant j$ and $m > 0$. Usually, $|\psi\rangle_{\rm root}$ is proportional to a single configuration such that all the other configurations in the expansion in Eq. (2.37) can be obtained from it via inward squeezing. However, by our definition, $|\psi\rangle_{\rm root}$ can also be a linear combination of such configurations, as it may happen that the zero mode $|\psi\rangle$ is a linear combination of simpler zero modes. We refer the reader to the referenced literature [26,47,50–58] for details. Also, we note that Jack states and their fermionic counterparts have been associated with certain types of recursion relations [54,59,60]. These are recursion relations for the coefficients of the mode expansion, where the particle number is fixed and the recursion proceeds along increasing "squeezing level" of the associated modes. This is to be distinguished from the present case, where we defined states recursively in particle number.

The root states satisfy Pauli-like principles. In the case of a single-component state in a single Landau level, these are known as generalized Pauli principles [25,50,51]. For example, there is no more than one particle in any three consecutive orbitals in the root state of the $\nu = 1/3$ Laughlin state, which corresponds to the familiar 100100100... configuration. The same generalized Pauli principle does, however, apply to other zero modes (not necessarily of the highest density) of the state's parent Hamiltonian. For multicomponent and/or multi-Landau-level states, our definition of a root state will generally lead to more than one configuration entering $|\psi\rangle_{\rm root}$, and especially will lead to root level entanglement. In this case, we speak of "entangled" Pauli principles [56]. The unprojected $\nu = 2/5$ Jain state may serve as an example of this, where this entangled Pauli principle requires next-nearest neighbors to be singlets of an SU(2)-algebra related to the Landau level degrees of freedom, in addition to ruling out double occupancies (with the same angular momentum but different Landau level indices) [55]. Effectively, this leads to a situation in which there can be no more than two particles in any five consecutive orbitals, in the root state. By contrast, basis states inward-squeezed from root states do not satisfy these Pauli-like principles.

As root states contain much information about the universal properties of the underlying state, including statistics [61], their uses are manifold. In an obvious way, they encode the filling fractions of the underlying state, commonly defined as the ratio of the particle number to the highest angular momentum of any orbital occupied in the state (in the thermodynamic limit).

In this subsection, we will now prove that $|{\rm Pf}_N\rangle$ has a root state

$$c_0^\dagger c_{M-1}^\dagger c_{2M}^\dagger c_{3M-1}^\dagger \dots c_{(N-2)M}^\dagger c_{(N-1)M-1}^\dagger \, |0\rangle \qquad (2.39)$$

for even particle number N. This will reaffirm that it has the correct highest occupied orbital [angular momentum $(N-1)M-1$], rendering it the unique densest zero mode of its parent Hamiltonian, thus identical (up to normalization) to the Moore-Read state at the respective filling factor. This will also serve to close one loop-hole in the reasoning so far. As for that shown above, it might be possible that the state $|{\rm Pf}_N\rangle$ as defined in Eq. (2.16a) vanishes, at least for some sufficiently high particle number N. We can rule this out below, as we show in particular that the state $|{\rm Pf}_N\rangle$ has a nonzero overlap with the root state (2.39).

Again, we prove this by mathematical induction. For $N = 2$, the above statement is true, as seen from Eq. (2.22). Now we assume

$$|{\rm Pf}_N\rangle_{\rm root} \propto c_0^\dagger c_{M-1}^\dagger c_{2M}^\dagger c_{3M-1}^\dagger \dots c_{(N-2)M}^\dagger c_{(N-1)M-1}^\dagger \, |0\rangle \qquad (2.40)$$

for $N \geqslant 2$, and its coefficient $C_{N_{\rm root}}$ in the expansion of $|{\rm Pf}_N\rangle$ in terms of occupation number basis states is nonzero.

We plug $|{\rm Pf}_N\rangle_{\rm root}$ into Eq. (2.16) to obtain

$$\begin{aligned}
\mathcal{R}_N \, |{\rm Pf}_N\rangle_{\rm root} &= \frac{1}{N+2} \sum_{l=0}^{M-1} (-1)^l \binom{M-1}{l} \sum_{r_1,r_2=0}^{MN+M-1} \sqrt{r_1! r_2!} \, c_{r_1}^\dagger c_{r_2}^\dagger S_{MN+M-1-l-r_1} S_{MN+l-r_2} \, |{\rm Pf}_N\rangle_{\rm root} \\
&= \frac{1}{N+2} \sum_{l=0}^{M-1} (-1)^l \binom{M-1}{l} \sum_{r_1,r_2=0}^{MN+M-1} \sum_{p_1,\dots,p_N,q_1,\dots,q_N=0}^{M} (-1)^{\sum_{i=1}^N (p_i+q_i)} \prod_{i=1}^N \binom{M}{p_i} \binom{M}{q_i} \\
&\quad \times \sqrt{\frac{(q_1+p_1)!}{0!} \frac{(M-1+q_2+p_2)!}{(M-1)!} \frac{(2M+q_3+p_3)!}{(2M)!} \frac{(3M-1+q_4+p_4)!}{(3M-1)!} \dots \frac{[(N-2)M+q_{N-1}+p_{N-1}]!}{[(N-2)M]!}} \\
&\quad \times \sqrt{\frac{[(N-1)M-1+q_N+p_N]! r_1! r_2!}{[(N-1)M-1]!}} \, c_{r_1}^\dagger c_{r_2}^\dagger c_{q_1+p_1}^\dagger c_{M-1+q_2+p_2}^\dagger c_{2M+q_3+p_3}^\dagger c_{3M-1+q_4+p_4}^\dagger \dots c_{(N-2)M+q_{N-1}+p_{N-1}}^\dagger \\
&\quad \times c_{(N-1)M-1+q_N+p_N}^\dagger S_{MN+M-1-l-r_1-\sum_{i=1}^N p_i} S_{MN+l-r_2-\sum_{i=1}^N q_i} \, |0\rangle
\end{aligned}$$

$$= \frac{1}{N+2}\sum_{l=0}^{M-1}(-1)^l\binom{M-1}{l}\sum_{p_1,\dots,p_N,q_1,\dots,q_N=0}^{M}(-1)^{\sum_{i=1}^N(p_i+q_i)}\prod_{i=1}^{N}\binom{M}{p_i}\binom{M}{q_i}$$
$$\times\sqrt{\frac{(q_1+p_1)!}{0!}\frac{(M-1+q_2+p_2)!}{(M-1)!}\frac{(2M+q_3+p_3)!}{(2M)!}\frac{(3M-1+q_4+p_4)!}{(3M-1)!}\cdots\frac{[(N-2)M+q_{N-1}+p_{N-1}]!}{[(N-2)M]!}}$$
$$\times\sqrt{\frac{[(N-1)M-1+q_N+p_N]!}{[(N-1)M-1]!}(MN+M-1-l-\textstyle\sum_{i=1}^N p_i)!(MN+l-\sum_{i=1}^N q_i)!}$$
$$\times c^\dagger_{MN+M-1-l-\sum_{i=1}^N p_i}c^\dagger_{MN+l-\sum_{i=1}^N q_i}c^\dagger_{q_1+p_1}c^\dagger_{M-1+q_2+p_2}c^\dagger_{2M+q_3+p_3}c^\dagger_{3M-1+q_4+p_4}\cdots c^\dagger_{(N-2)M+q_{N-1}+p_{N-1}}$$
$$\times c^\dagger_{(N-1)M-1+q_N+p_N}|0\rangle,$$

where we have used Eq. (B2) to move S to the right of $c^\dagger$. We have also used the fact that both indices of S operators, $MN+M-1-l-r_1-\sum_{i=1}^N p_i$ and $MN+l-r_2-\sum_{i=1}^N q_i$, have to be 0, following the same logic used in the derivation of Eq. (2.22).

The only solutions for

$$c^\dagger_{MN+M-1-l-\sum_{i=1}^N p_i}c^\dagger_{MN+l-\sum_{i=1}^N q_i}c^\dagger_{q_1+p_1}c^\dagger_{M-1+q_2+p_2}$$
$$\times c^\dagger_{2M+q_3+p_3}c^\dagger_{3M-1+q_4+p_4}\cdots c^\dagger_{(N-2)M+q_{N-1}+p_{N-1}}$$
$$\times c^\dagger_{(N-1)M-1+q_N+p_N}|0\rangle \tag{2.41}$$

in the above expression to be proportional to $|\text{Pf}_{N+2}\rangle_\text{root}\propto c_0^\dagger c_{M-1}^\dagger c_{2M}^\dagger c_{3M-1}^\dagger\cdots c_{NM}^\dagger c_{(N+1)M-1}^\dagger|0\rangle$ are parametrized by a choice of $j=0,1,2,\dots,N/2$, where $q_1=p_1=q_2=p_2=\dots=q_{2j}=p_{2j}=0$ and $q_{2j+1}=p_{2j+1}=q_{2j+2}=p_{2j+2}=\dots=q_N=p_N=M$, and furthermore a choice of $l=0,M-1$. One checks that all these solutions enter with the same sign, and thus $|\text{Pf}_{N+2}\rangle_\text{root}$ will be generated from $|\text{Pf}_N\rangle_\text{root}$ via Eq. (2.16). On the other hand, by acting with $\mathcal{R}_N$ on any $|\{n_i\}\rangle$ that can be obtained from $|\text{Pf}_N\rangle_\text{root}$ via inward squeezing, similar considerations show that $|\text{Pf}_{N+2}\rangle_\text{root}$ cannot be generated, and the only configurations that can be generated are obtainable from $|\text{Pf}_{N+2}\rangle_\text{root}$ via inward squeezing. Together, these results show that $|\text{Pf}_{N+2}\rangle_\text{root}$ is the root state of $|\text{Pf}_{N+2}\rangle$ not only in name, but according to the definition given at the beginning of this section.

In summary, the fermionic (bosonic) Pfaffian state $|\text{Pf}_N\rangle$, as recursively defined in Eq. (2.16) for even particle number N, has a root state proportional to

$$c_0^\dagger c_{M-1}^\dagger c_{2M}^\dagger c_{3M-1}^\dagger\cdots c_{(N-2)M}^\dagger c_{(N-1)M-1}^\dagger|0\rangle, \tag{2.42}$$

thus possessing the filling factor $1/M$.

F. Off-diagonal long-range-order operator of the Pfaffian state in second quantization

In this subsection, we establish the connection between the foregoing results and the existence of off-diagonal long-range order (ODLRO) in a nonlocal order parameter for the Moore-Read state. Such a connection is natural, as the second-quantized recursion (2.16) we use to define the Moore-Read state in this paper is a generalization of a similar recursion for the Laughlin state that, in its original form [39], emerged as the interpretation of the Laughlin state as a condensate of a nonlocal order parameter. This is quite manifest also in Eq. (2.16), and it can be further emphasized by its trivial formal "integration" via

$$|\text{Pf}_N\rangle=(\mathcal{R})^{N/2}|0\rangle \tag{2.43}$$

for N even, where

$$\mathcal{R}=\sum_{N\text{even}}\mathcal{R}_N P_N, \tag{2.44}$$

and P_N is the projection onto N-particle subspace of the Fock space. In this form, one may see this equation as equivalent to Eq. (5.8) by Moore and Read [18], with the important difference that the latter is presented in mixed first-/second-quantized notations.

Fully second-quantized forms similar to ours have been given before for the Laughlin state [25], concurrent with second-quantized expressions for the associated nonlocal order parameter [48]. Both have been successfully generalized to composite fermion states [26], which became instrumental in constructing parent Hamiltonians for these states [14]. To complete our second-quantized picture for the Moore-Read state, it is thus prudent to construct the nonlocal order parameter directly and demonstrate its display of ODLRO. Similar to previously studied examples, the key ingredient is the action of an electron destruction operator on the incompressible ground state, as facilitated in the present case by Eq. (2.21). While Refs. [26,48] demonstrated the ODLRO in the orbital basis, a formulation in real space is equally possible. We will aim for the demonstration of real-space ODLRO here, and to this end, utilize some notation developed in Ref. [28].

We thus introduce the field operator annihilating a particle (we again treat fermions and bosons on an equal footing) at $z=x+iy$, projected onto the lowest Landau level, via its mode expansion $\Lambda(z)=\sum_{r\geqslant 0}\phi_r(z)c_r$, where the single-particle wave function on the disk is

$$\phi_r(z)=N_r^{-1}z^r e^{-|z|^2/4} \tag{2.45}$$

with the normalization factor $N_r=\sqrt{2\pi 2^r r!}$. By introducing pseudofermionic (bosonic) operators [26] $\bar{c}_r:=c_r/N_r$ and $\bar{c}_r^\dagger:=N_r c_r^\dagger$ for compactness, Eq. (2.21) can be recast in the form

$$\Lambda(z)\,|\mathrm{Pf}_{N+2}\rangle = \frac{e^{-|z|^2/4}}{4\pi}\sum_{l=0}^{M-1}(-1)^l\binom{M-1}{l}\sum_{r,k\geqslant 0}\frac{z^r\bar{c}_k^\dagger}{\sqrt{2^{r+k}}}\times[S_{MN+M-1-l-r}S_{MN+l-k}+(-1)^{M-1}\times S_{MN+M-1-l-k}S_{MN+l-r}]\,|\mathrm{Pf}_N\rangle\,. \tag{2.46}$$

This may be simplified by introducing the second-quantized N-body quasihole operator $\widehat{U}_N(z)=\sum_{d=0}^N(-z)^{N-d}2^{\frac{d}{2}}e_d$, which creates a Laughlin-quasihole at z. Its Mth power is given by $\widehat{U}_N^M(z)=(-1)^{MN}\sum_{r\geqslant 0}z^r2^{\frac{MN-r}{2}}S_{MN-r}$ (see the supplementary notes of Ref. [28]). Note that Read's order parameter for the $1/M$-Laughlin state is precisely $\Lambda^\dagger(z)\widehat{U}_N^M(z)$, albeit with the role of fermions and bosons reversed compared to the present case. Using the commutativity of the S-operators among themselves, and the fact that

$$S_m\,|\mathrm{Pf}_N\rangle = 0 \quad \text{for } m > MN, \tag{2.47}$$

we now obtain

$$\Lambda(z)\,|\mathrm{Pf}_{N+2}\rangle = \mathcal{F}_{M,N}(z)\,|\mathrm{Pf}_N\rangle\,, \tag{2.48}$$

where

$$\mathcal{F}_{M,N}(z) = \frac{(-1)^{MN}e^{-|z|^2/4}}{2\pi\sqrt{2^{MN+M-1}}}\sum_{l=0}^{M-1}(-1)^l\binom{M-1}{l}\times z^{M-1-l}\sum_{k\geqslant 0}\frac{\bar{c}_{k+l}^\dagger}{\sqrt{2^k}}S_{MN-k}\widehat{U}_N^M(z). \tag{2.49}$$

In line with Read's original reasoning for the Laughlin state [39], we can argue that

$$\begin{aligned}&\langle\mathrm{Pf}_N|\,\mathcal{F}_{M,N}^\dagger(z)\Lambda(z)\Lambda^\dagger(z')\mathcal{F}_{M,N}(z')\,|\mathrm{Pf}_N\rangle\\ &= \langle\mathrm{Pf}_{N+2}|\,\rho(z)\rho(z')\,|\mathrm{Pf}_{N+2}\rangle\\ &\to \langle\rho\rangle^2,\end{aligned} \tag{2.50}$$

where we use the Landau-level projected fields $\Lambda(z)$ to define local densities $\rho(z)=\Lambda^\dagger(z)\Lambda(z)$, such that $\hat{N}=\int d^2z\,\rho(z)$ is the Landau-level projected particle number operator. We also assumed the exponential decay of correlations as $|z-z'|\to\infty$, such that the expression approaches the square of the particle density $\langle\rho\rangle$ of the homogeneous fluid, which is determined by the filling factor ν.

We thus infer the existence of ODLRO of the $\nu=1/M$ Moore-Read Pfaffian state in the nonlocal operator given by

$$O(z) = \Lambda^\dagger(z)\mathcal{F}_{M,N}(z). \tag{2.51}$$

It is worth noting that, in spite of deliberately writing (2.51) in a form similar to the Laughlin-state order parameter $\Lambda^\dagger(z)\widehat{U}_N^M(z)$, there are important differences. The most crucial difference lies in the fact that Eq. (2.51) changes particle number by 2, as a change by 1 is also "hidden" in the field operator $\mathcal{F}_{M,N}(z)$. The fact that the order parameter changes the particle number by 2 is, of course, a direct signature of the paired nature of the Moore-Read state. We emphasize once more that the presentation of the Moore-Read state in the form (2.43) is by itself not sufficient to demonstrate ODLRO. For this, we crucially needed Eq. (2.21).

Given the above, following again Read's construction [39], we could alternatively use Eq. (2.43) [together with Eq. (2.21)] to construct a condensate of a well-defined phase conjugate to particle number, for which the order parameter (2.51) itself assumes an expectation value. The only difference with the Laughlin-state case would be that such a condensate would have well-defined particle number parity, i.e., it would be a coherent superposition of states (2.43) with even N only. We leave the (simple) details to the reader.

G. Higher angular momentum paired Pfaffian states

We generalize the results of Sec. II to a Pfaffian state of the form

$$\Psi_N^m \sim \mathrm{Pf}\left[\frac{1}{(z_i-z_j)^m}\right]\prod_{1\leqslant i<j\leqslant N}(z_i-z_j)^M, \tag{2.52}$$

where odd $m\leqslant M$. $\mathrm{Pf}[\frac{1}{(z_i-z_j)^m}]$ signifies paired composite fermions beyond p-wave pairing, where in particular the case $m=3$ has recently been studied [33].

For this state, the recursion relation Eq. (2.16) generalizes straightforwardly via the modification $\mathcal{R}_N\to\mathcal{R}_N^m$, where

$$\mathcal{R}_N^m = \frac{1}{N+2}\sum_{l=0}^{M-m}(-1)^l\binom{M-m}{l}\sum_{r,k=0}^{MN+M-m}\sqrt{r!k!}\,c_r^\dagger c_k^\dagger\times S_{MN+M-m-l-r}S_{MN+l-k}, \tag{2.53a}$$

such that

$$\left|\mathrm{Pf}_{N+2}^m\right\rangle = \mathcal{R}_N^m\left|\mathrm{Pf}_N^m\right\rangle, \tag{2.53b}$$

where we also introduced a ket $|\mathrm{Pf}_N^m\rangle$ associated with the wave function (2.52).

For the state (2.52), we do not know an appropriate parent Hamiltonian at this point, so the proof of Eq. (2.53) necessarily proceeds by making contact with the first-quantized form given in Eq. (2.52). This is done in Appendix A, where we also specify pertinent normalization conventions. Equally importantly, one can generalize the effect of particle removal, Eq. (2.21), as follows:

$$c_r\left|\mathrm{Pf}_{N+2}^m\right\rangle = \frac{\sqrt{r!}}{2}\sum_{l=0}^{M-m}(-1)^l\binom{M-m}{l}\sum_{k=0}^{MN+M-m}\sqrt{k!}\,c_k^\dagger\times[S_{MN+M-m-l-r}S_{MN+l-k}+(-1)^{M-m}\times S_{MN+M-m-l-k}S_{MN+l-r}]\,|\mathrm{Pf}_N^m\rangle\,. \tag{2.54}$$

A derivation of Eq. (2.54) from the first-quantized Eq. (2.52) is again given in Appendix A, or, from the second-quantized Eq. (2.53), in the Supplemental Material [49]. The benefit of Eq. (2.54) is, among other things, a straightforward generalization of the derivation of ODLRO given in the preceding section to the case of Eq. (2.52). This leads to ODLRO in the following nonlocal operator:

$$O(z) = \Lambda^\dagger(z)\mathcal{F}_{M,m,N}(z), \tag{2.55}$$

where

$$\mathcal{F}_{M,m,N}(z) = \frac{(-1)^{MN} e^{-|z|^2/4}}{2\pi\sqrt{2^{MN+M-m}}} \sum_{l=0}^{M-m} (-1)^l \binom{M-m}{l} \times z^{M-m-l} \sum_{k\geqslant 0} \frac{\bar{c}^{\dagger}_{k+l}}{\sqrt{2^k}} S_{MN-k} \widehat{U}^M_N(z). \tag{2.56}$$

We leave other possible uses of Eq. (2.54), such as in the construction of possible parent Hamiltonians for Eq. (2.52), to future work.

III. RECURSIVE FORMULA FOR FERMIONIC $\nu = 1/2$ ANTI- AND PH-PFAFFIAN STATES

At Landau level filling factor $\nu = 1/2$, several inequivalent topological phases featuring Majorana fermions are possible. Among possible competitors, the anti-Pfaffian state has been proposed as the particle-hole conjugate of the $\nu = 1/2$ Pfaffian state [62,63]. Generally, a particle-hole conjugate of a state can be obtained by replacing $c \to h^{\dagger}$, $c^{\dagger} \to h$, and $|0\rangle_e \to \prod_{i=0}^{l_{\max}(N)} h_i^{\dagger} |0\rangle_h$, where $l_{\max}(N)$ is the highest occupied orbital in the $\nu = 1/2$ Pfaffian state, and $l_{\max}(N) = 2N - 3$ for N even. As long as we restrict ourselves to the Fock space associated with the orbitals $0, \dots, l_{\max}(N)$, these relations merely facilitate a reinterpretation of the Pfaffian state. A new state is obtained when the "holes" created by the operators $h^{\dagger}$ are again reinterpreted as the particles (i.e., once more replaced by $c^{\dagger}$'s). We leave this understood. On the half-infinite lattice, however, the replacement $|0\rangle_e \to \prod_{i=0}^{l_{\max}(N)} h_i^{\dagger} |0\rangle_h$ does change the vacuum. It replaces the "particle vacuum" for orbitals with angular momenta $l > l_{\max}(N)$ with the "hole vacuum," i.e., a $\nu = 1$ integer quantum Hall state. The result is that once the $h^{\dagger}$-operators are reinterpreted as particles, we obtain the $(N-2)$-particle anti-Pfaffian state $|\mathrm{aPf}_{N-2}\rangle$ from the N-particle $\nu = 1/2$ Pfaffian state $|\mathrm{Pf}_N\rangle$, where $|\mathrm{aPf}_{N-2}\rangle$ has the same highest occupied orbital $l_{\max}(N) = 2N - 3$, and it has an edge with vacuum. The following example illustrates this: The four-particle Pfaffian state on the disk is $(c_0^{\dagger}c_1^{\dagger}c_4^{\dagger}c_5^{\dagger} - \sqrt{2}c_0^{\dagger}c_2^{\dagger}c_3^{\dagger}c_5^{\dagger} + \sqrt{10}c_1^{\dagger}c_2^{\dagger}c_3^{\dagger}c_4^{\dagger})\,|0\rangle_e$. After replacing $c \to h^{\dagger}$, $c^{\dagger} \to h$, and $|0\rangle_e \to \prod_{i=0}^{5} h_i^{\dagger} |0\rangle_h$, we obtain a two-particle anti-Pfaffian state on the disk $(h_2^{\dagger}h_3^{\dagger} - \sqrt{2}\,h_1^{\dagger}h_4^{\dagger} + \sqrt{10}\,h_0^{\dagger}h_5^{\dagger})\,|0\rangle_h$. We note that $l_{\max}(N)$ agrees with the number of flux quanta on the sphere that the respective state would require to represent a rotationally invariant state.

Using the above, by particle-hole conjugating the second-quantized recursive formula Eq. (2.16a) from the $(N+2)$-particle fermionic $\nu = 1/2$ Pfaffian state to the $(N+4)$-particle state with $M = 2$, we can arrive at the second-quantized recursive formula for the fermionic $\nu = 1/2$ anti-Pfaffian (aPf) state,

$$|\mathrm{aPf}_{N+2}\rangle = \frac{2}{N+4} \sum_{r,k=0}^{2N+5} \sqrt{r!k!}\, h_r h_k R_{2N+5-r} R_{2N+4-k} \times h^{\dagger}_{2N+2} h^{\dagger}_{2N+3} h^{\dagger}_{2N+4} h^{\dagger}_{2N+5}\, |\mathrm{aPf}_N\rangle\,, \tag{3.1}$$

for even non-negative N. The beginning of recursion is $|\mathrm{aPf}_0\rangle = |0\rangle_h$, the vacuum for holes. Four hole creation operators appear in the recursive formula, since each time we increase the particle number by 2, the "edge" between vacuum and $\nu = 1$ phase in the vacuum replacement $|0\rangle_e \to \prod_{i=0}^{l_{\max}(N)} h_i^{\dagger} |0\rangle_h$ shifts by four orbitals. The R operator in the above recursive formula for the anti-Pfaffian state is obtained from S operator in Eq. (2.17) with $M = 2$ by particle-hole conjugation. Explicitly,

$$R_\ell = (-1)^\ell \sum_{n_1+n_2=\ell} f_{n_1} f_{n_2} \quad \text{for} \quad \ell \geqslant 0, \qquad R_\ell = 0 \quad \text{for} \quad \ell < 0. \tag{3.2}$$

Here, f_n is the particle-hole conjugate of e_n in Eq. (2.18),

$$\begin{aligned} f_n &= \frac{1}{n!} \sum_{l_1,\dots,l_n=0}^{+\infty} \sqrt{l_1+1}\, h_{l_1+1} \sqrt{l_2+1}\, h_{l_2+1} \cdots \times \sqrt{l_n+1} h_{l_n+1} h^{\dagger}_{l_n} \cdots h^{\dagger}_{l_2} h^{\dagger}_{l_1} \quad \text{for } n > 0,\\ f_0 &= \mathbb{1},\\ f_n &= 0 \quad \text{for } n < 0. \end{aligned} \tag{3.3}$$

Note that different R_ℓ still commute with each other. For $\ell > 0$, S_ℓ increases the total angular momentum of an electronic state by ℓ, whereas its particle-hole conjugate R_ℓ decreases the total angular momentum, as measured by occupied $h^{\dagger}$-states, by the same amount.

The parent Hamiltonian for the N-particle anti-Pfaffian state is the particle-hole conjugate of the three-body parent Hamiltonian for the $\nu = 1/2$ Pfaffian state in Eq. (2.11) with $M = 2$,

$$H_{\mathrm{aPf}_N} = \sum_J U^{\dagger}_{J,N} U_{J,N}, \tag{3.4}$$

with

$$U_{J,N} = \sum_{i_1+i_2+i_3=J\in[3,6N]} \frac{\sqrt{6(J-3)!}}{3^{\frac{J}{2}} 4\sqrt{i_1!i_2!i_3!}} (i_1-i_2)(i_1-i_3) \times (i_2-i_3) h^{\dagger}_{i_3} h^{\dagger}_{i_2} h^{\dagger}_{i_1}. \tag{3.5}$$

Note that, however, the above Hamiltonian has an N-particle anti-Pfaffian state as the unique incompressible zero mode only if orbital indices in the above sum are restricted by the additional constraint $0 \leqslant i_1, i_2, i_3 \leqslant 2N+1$, *or* if the edge with (h-) vacuum is instead replaced with an edge with a $\nu = 1$ state. This is the reason why the edge of the anti-Pfaffian with vacuum is more complicated than that of the original Moore-Read state [62,63].

We remark that although the case $M = 2$ is of greatest interest, one may generalize the above straightforwardly to obtain recursions for the particle-hole conjugates of $\nu = 1/M$ Moore-Read states, although these would then not live at the same filling factor in the thermodynamic limit, but instead would have filling factor $1 - 1/M$.

Note, moreover, that by straightforwardly taking the particle-hole conjugate of Eq. (2.51), we may define nonlocal order parameters for these particle-hole conjugates of Moore-Read states, as arguments leading to Eq. (2.50) will, *mutatis mutandis*, hold. In particular, by particle-hole conjugation of Eq. (2.21), one obtains a similar equation for particle addition into the particle-hole conjugate of Moore-Read states.

While so far we have mostly focused on states at even particle number N, we can easily obtain the incompressible Moore-Read state at odd particle number N via

$$|\mathrm{Pf}_N\rangle = c_{l_{\max}(N+1)}\,|\mathrm{Pf}_{N+1}\rangle\,. \tag{3.6}$$

Note that for general M, $l_{\max}(N) = M(N-1) - 1$ for N even, and $l_{\max}(N) = M(N-1)$ for N odd [see Eq. (2.42)]. For odd N, the particle-hole conjugate of $|\mathrm{Pf}_N\rangle$ has $l_{\max}(N) + 1 - N = MN - M - N + 1$ particles within the orbitals $0, 1, 2, \dots, l_{\max}(N)$, which is also even. (Note that we are dealing with fermionic states in this section, so M is even.) It is thus more natural to define the $\nu = 1/2$ anti-Pfaffian ($M = 2$) for odd N in analogy with Eq. (3.6) via

$$|\mathrm{aPf}_N\rangle = c_{l_{\max}(N+3)}\,|\mathrm{aPf}_{N+1}\rangle\,, \tag{3.7}$$

since the $(N+1)$-particle $\nu = 1/2$ anti-Pfaffian state is obtained from the $(N+3)$-particle $\nu = 1/2$ Pfaffian state by particle-hole conjugation.

Lastly, the PH-Pfaffian phase, which recently attracted much interest [64–66], is the universality class of a particle-hole symmetric state at $\nu = 1/2$. Inspired by the latter and with the help of the above developments, we may easily construct a *particle-hole symmetric* state defined by a straightforward modification and amalgamation of the recursions for the $\nu = 1/2$ Pfaffian and anti-Pfaffian states,

$$\begin{aligned}|\mathrm{PH}_{N+2}\rangle = \sum_{r,k=0}^{2N+3} \sqrt{r!k!}\,(c_r^\dagger c_k^\dagger S_{2N+3-r} S_{2N-k} + c_r c_k \\ \times R_{2N+3-r} R_{2N-k} c_{2N}^\dagger c_{2N+1}^\dagger c_{2N+2}^\dagger c_{2N+3}^\dagger) \\ \times |\mathrm{PH}_N\rangle\,,\end{aligned} \tag{3.8}$$

for even non-negative N. The beginning of recursion is given by $|\mathrm{PH}_0\rangle = |0\rangle$, the vacuum for electrons. The state $|\mathrm{PH}_N\rangle$ so constructed is manifestly particle-hole symmetric on the orbital lattices given by the orbitals with indices $0, \dots, 2N+3$. In particular, $|\mathrm{PH}_{N+2}\rangle$ would thus suitably fit onto a sphere with the correct number of flux quanta $2(N+2) - 1$. In the above, the R operator is still defined as in Eq. (3.2), but with all h-operators in f_n replaced by c-operators, as they must be creating the same particles as those in the S-operator part of the recursion.

We defer further analysis of the state defined in Eq. (3.8) and its relation to the first-quantized particle-hole symmetric Pfaffian state defined in the literature [65,67–69], or possibly a gapless particle-hole symmetric state at half-filling [64], to future work.

IV. DISCUSSION AND OUTLOOK

In this paper, we developed a second-quantized presentation for the Moore-Read state at filling factor $\nu = 1/M$. In practice, this presentation is realized as a recursive definition of Moore-Read states in second quantization. Such recursions are of interest in connection with the recent body of literature about the construction of frustration-free parent Hamiltonians for FQH states in second quantization, which can, in principle, lead to new Hamiltonians that are difficult to construct following the established first-quantized principles. The prime example for such a development is given by the recently constructed Hamiltonians for the (positive) Jain sequence [14]. Two types of presentations for fractional quantum Hall trial wave functions can be distinguished that are both far removed from traditional first-quantized constructions and lend themselves to the scheme for the discussion of parent Hamiltonians that is the subject of this paper. One is the MPS-presentation of fractional quantum Hall trial wave functions, which also exists for Moore-Read states, but not, to our knowledge, for composite fermion states or the anti-Pfaffian state. The other consists in recursion relations that are closely related to an understanding of the state in question as a condensate of a nonlocal order parameter. The latter kind of presentation is what we utilized and further developed in this work for the Moore-Read states. A closely related mixed first-/second-quantized presentation of this kind has been known for some time [18]. While we give a fully second-quantized version of this presentation, this, by itself, was not sufficient for the second-quantized discussion of parent Hamiltonians we have given in this work. Instead, a key ingredient developed in this paper is the second-quantized description of particle removal from this state in the form of Eq. (2.21). On the one hand, this allows us to develop a fully second-quantized understanding of the parent Hamiltonian of Moore-Read states. As the example of the composite fermion states shows, such an understanding furnishes a promising foundation on which to base the construction of new parent Hamiltonians that are not based on simple clustering properties manifest in first quantization. Moreover, Eq. (2.21) also makes possible our derivation of off-diagonal long-range order in these states, in terms of nonlocal order parameters. We have also shown how both the second-quantized presentation as well as the definition of the nonlocal order parameter extend to particle-hole conjugates of Moore-Read states. Some of our findings are complementary to similar developments utilizing MPS presentation of Moore-Read states [70]. We are hopeful that these findings will continue to facilitate developments of trial fractional quantum Hall states and accompanying parent Hamiltonians that are not conveniently available in the traditional first-quantized approach. Moreover, the distinction between various similar non-Abelian phases at half-filling has inspired several proposals in the past, guiding both physical [71–73] and numerical experiment [31,65,67–69,74–77]. We hope that the formulas we developed here for nonlocal order parameters can provide additional tools to distinguish the underlying states at least in numerical experiments.

Note added. While preparing this manuscript, we became aware of a work in parallel by A. Bochniak and G. Ortiz [78], which contains a second-quantized presentation of the Moore-Read states equivalent to ours, but otherwise focuses on different aspects of the physics of these states.

ACKNOWLEDGMENTS

L.C. is supported by NSFC Grant No. 12004105. A.S. acknowledges support by the National Science Foundation under Grant No. DMR-2029401. We gratefully acknowledge insightful discussions with G. Ortiz, A. Bochniak, and A. Balram.

APPENDIX A: THE DERIVATION OF EQS. (2.16a) AND (2.21) IN FIRST QUANTIZATION

We can write Moore-Read's (unnormalized) first-quantized Pfaffian wave function as

$$\Psi_N = \mathcal{N}_N \, \mathrm{Pf}\left(\frac{1}{z_i - z_j}\right) \prod_{1\leqslant i<j\leqslant N} (z_i - z_j)^M, \tag{A1}$$

with even (odd) M for fermions (bosons) and even particle number N, and an as yet arbitrary normalization constant $\mathcal{N}_N$. We will fix the normalization convention below.

Moore-Read's original Pfaffian state has also been generalized to an f-wave paired state of first-quantized wave function [33]

$$\mathrm{Pf}\left[\frac{1}{(z_i - z_j)^3}\right] \prod_{1\leqslant i<j\leqslant N} (z_i - z_j)^M, \tag{A2}$$

which inspires us to consider a generalized Pfaffian state

$$\Psi_N^m = \mathcal{N}_N^m \, \mathrm{Pf}\left[\frac{1}{(z_i - z_j)^m}\right] \prod_{1\leqslant i<j\leqslant N} (z_i - z_j)^M, \tag{A3}$$

with an odd positive integer m as the pairing parameter, which must obey $1 \leqslant m \leqslant M$ and on which the normalization $\mathcal{N}_N^m$ may depend.

In all of the above, Pf is the Pfaffian of an antisymmetric matrix with element $1/(z_i - z_j)^m$,

$$\mathrm{Pf}\left[\frac{1}{(z_i - z_j)^m}\right] = \frac{1}{2^{\frac{N}{2}}(\frac{N}{2})!} \sum_{\sigma\in S_N} (-1)^\sigma \prod_{k=1}^{\frac{N}{2}} \frac{1}{(z_{\sigma_{2k-1}} - z_{\sigma_{2k}})^m}. \tag{A4}$$

The permutation σ can be viewed as encoding a way of pairing indices into pairs $(\sigma_{2k-1}, \sigma_{2k})$. There is then, however, much overcounting, as both the order within pairs and between pairs does not matter. This is compensated by a factor $\frac{1}{2^{N/2}(N/2)!}$. As the order of pairs plays no role, we can, in particular, still generate all pairings if we fix $\sigma_N = N$. We write such σ as $\sigma \in S_{N-1}$. Thus, adjusting the combinatorial overcounting factor,

$$\begin{aligned}\mathrm{Pf}\left[\frac{1}{(z_i - z_j)^m}\right] &= \frac{1}{2^{\frac{N-2}{2}}(\frac{N-2}{2})!} \sum_{\sigma\in S_{N-1}} (-1)^\sigma \\ &\quad\times \prod_{k=1}^{\frac{N}{2}} \frac{1}{(z_{\sigma_{2k-1}} - z_{\sigma_{2k}})^m} \\ &= \frac{(N-1)!}{2^{\frac{N-2}{2}}(\frac{N-2}{2})!} \mathcal{A}_{N-1} \prod_{k=1}^{\frac{N}{2}} \frac{1}{(z_{2k-1} - z_{2k})^m},\end{aligned} \tag{A5}$$

where $\mathcal{A}_{N-1}$ denotes the antisymmetrization in just $z_1, \ldots, z_{N-1}$. Thus,

$$\Psi_N^m = \mathcal{N}_N^{m\prime} \prod_{1\leqslant i<j\leqslant N} (z_i - z_j)^M \mathcal{A}_{N-1} \prod_{k=1}^{\frac{N}{2}} \frac{1}{(z_{2k-1} - z_{2k})^m}, \tag{A6}$$

where we have absorbed all combinatorial factors into a new normalization constant $\mathcal{N}_N^{m\prime}$.

For even M, the Laughlin-Jastrow factor is totally symmetric, and we can pull it into the antisymmetrization. For odd M, the Laughlin-Jastrow factor is totally antisymmetric, and we can change the antisymmetrization into a symmetrization after pulling the Laughlin-Jastrow factor inside. We thus define $\mathcal{S}_N^{(M)}$ to be the (anti)symmetrization operator in $z_1, \ldots, z_N$ for (even) odd M. Changing from N to $N+2$:

$$\begin{aligned}\Psi_{N+2}^m &= \mathcal{N}_{N+2}^{m}{}' \, \mathcal{S}_{N+1}^{(M)} \prod_{1\leqslant i<j\leqslant N+2} (z_i - z_j)^M \prod_{k=1}^{\frac{N+2}{2}} \frac{1}{(z_{2k-1} - z_{2k})^m} \\ &= \mathcal{N}_{N+2}^{m}{}' \, \mathcal{S}_{N+1}^{(M)} (z_{N+1} - z_{N+2})^{M-m} \prod_{1\leqslant i\leqslant N} (z_{N+2} - z_i)^M \prod_{1\leqslant i\leqslant N} (z_{N+1} - z_i)^M \mathcal{S}_{N-1}^{(M)} \prod_{1\leqslant i<j\leqslant N} (z_i - z_j)^M \prod_{k=1}^{\frac{N}{2}} \frac{1}{(z_{2k-1} - z_{2k})^m}.\end{aligned} \tag{A7}$$

In the above, it might be beneficial to insert an additional (anti)symmetrization operator $\mathcal{S}_{N-1}^{(M)}$ in front of the last line as shown, because the products in the first line are already symmetric in the variables z_i for $i = 1, \ldots, N$, whereas the second line depends on only these variables; we were thus able to write $\mathcal{S}_{N+1}^{(M)} = \mathcal{S}_{N+1}^{(M)}\mathcal{S}_{N-1}^{(M)}$ and permute the $\mathcal{S}_{N-1}^{(M)}$ to the position shown. This gives

$$\Psi_{N+2}^m = \frac{\mathcal{N}_{N+2}^{m}{}'}{\mathcal{N}_N^{m\prime}} \mathcal{S}_{N+1}^{(M)} (z_{N+1} - z_{N+2})^{M-m} \prod_{1\leqslant i\leqslant N} (z_{N+2} - z_i)^M \prod_{1\leqslant i\leqslant N} (z_{N+1} - z_i)^M \Psi_N^m. \tag{A8}$$

Now we need to expand $\prod_{1\leqslant i\leqslant N} (z_{N+2} - z_i)^M$. To do so, we first expand

$$\begin{aligned}\prod_{1\leqslant i\leqslant N} (z_{N+2} - z_i) &= \sum_{k=0}^{N} z_{N+2}^k (-1)^{N-k} \sum_{1\leqslant i_1<i_2\cdots<i_{N-k}\leqslant N} z_{i_1} z_{i_2} \cdots z_{i_{N-k}} \\ &= \sum_{k=0}^{N} z_{N+2}^k (-1)^{N-k} \, 2^{\frac{N-k}{2}} e_{N-k},\end{aligned} \tag{A9}$$

where we have identified $\sum_{1\leqslant i_1<i_2\cdots<i_{N-k}\leqslant N} z_{i_1}z_{i_2}\cdots z_{i_{N-k}}$ as $2^{\frac{N-k}{2}}e_{N-k}$. Then we have

$$\prod_{1\leqslant i\leqslant N}(z_{N+2}-z_i)^M=\sum_{k=0}^{MN} z_{N+2}^k\, 2^{\frac{MN-k}{2}}S_{MN-k}, \tag{A10}$$

where S is related to e by Eq. (2.17). $\prod_{1\leqslant i\leqslant N}(z_{N+1}-z_i)^M$ is expanded in the same way. $(z_{N+1}-z_{N+2})^{M-m}$ can be expanded via binomial expansion.

With these expansions, we obtain

$$\Psi_{N+2}^m=\frac{1}{2\pi}\sqrt{\frac{N+1}{N+2}}\sum_{l=0}^{M-m}(-1)^l\binom{M-m}{l}\sum_{k,r}2^{\frac{-k-r}{2}}\mathcal{S}_{N+1}^{(M)}z_{N+2}^r z_{N+1}^k S_{MN+M-m-l-r}S_{MN+l-k}\Psi_N^m, \tag{A11}$$

where we finally fix the arbitrary normalization constants via

$$\frac{\mathcal{N}_{N+2}^{m\,\prime}}{\mathcal{N}_N^{m\prime}}(-1)^{M-m}2^{\frac{2MN+M-m}{2}}2\pi\sqrt{\frac{N+2}{N+1}}=1. \tag{A12}$$

Equation (A11) is equivalent to

$$\Psi_{N+2}^m=\frac{1}{\sqrt{N+2}}\sum_{l=0}^{M-m}(-1)^l\binom{M-m}{l}\sum_{k,r}\sqrt{r!k!}\frac{z_{N+2}^r}{\sqrt{2\pi\,2^r r!}}\sqrt{N+1}\,\mathcal{S}_{N+1}^{(M)}\frac{z_{N+1}^k}{\sqrt{2\pi\,2^k k!}}S_{MN+M-m-l-r}S_{MN+l-k}\Psi_N^m. \tag{A13}$$

Since we rigorously derived the above to yield the manifestly (anti)symmetric wave function (A3), we may *optionally* act on it with the (anti)symmetrizer $\mathcal{S}_{N+2}^{(M)}$, giving

$$\Psi_{N+2}^m=\frac{1}{N+2}\sum_{l=0}^{M-m}(-1)^l\binom{M-m}{l}\sum_{k,r}\sqrt{r!k!}\sqrt{N+2}\,\mathcal{S}_{N+2}^{(M)}\frac{z_{N+2}^r}{\sqrt{2\pi\,2^r r!}}\sqrt{N+1}\,\mathcal{S}_{N+1}^{(M)}\frac{z_{N+1}^k}{\sqrt{2\pi\,2^k k!}}S_{MN+M-m-l-r}S_{MN+l-k}\Psi_N^m. \tag{A14}$$

Upon second quantization by using Eq. (1.13) of Ref. [79], with Eq. (2.45) in mind, the above formula leads to

$$|\mathrm{Pf}_{N+2}^m\rangle=\frac{1}{N+2}\sum_{l=0}^{M-m}(-1)^l\binom{M-m}{l}\sum_{r,k=0}^{MN+M-m}\sqrt{r!k!}\,c_r^\dagger c_k^\dagger S_{MN+M-m-l-r}S_{MN+l-k}\left|\mathrm{Pf}_N^m\right\rangle, \tag{A15}$$

of which Eq. (2.16a) is a special case with $m=1$.

Now we derive the expression for the general Pfaffian state with one particle removed by using Eq. (1.12) of Ref. [79] (Gaussians are included in the integration measure):

$$c_r\Psi_{N+2}^m=\sqrt{N+2}\int d^2z_{N+2}\frac{\bar{z}_{N+2}^r}{\sqrt{2\pi\,2^r r!}}\Psi_{N+2}^m. \tag{A16}$$

We now see why we went through the effort not only to derive Eq. (A14), which could have been arrived at more directly, but also to derive Eq. (A13). This equation has the much needed advantage to expose the dependence on z_{N+2} by having this variable appear outside of the symmetrization. Via the change of variable $l\to M-m-l$, we rewrite Eq. (A13) as

$$\begin{aligned}\Psi_{N+2}^m=&\frac{1}{2\sqrt{N+2}}\sum_{l=0}^{M-m}(-1)^l\binom{M-m}{l}\sum_{k,r}\sqrt{r!k!}\frac{z_{N+2}^r}{\sqrt{2\pi\,2^r r!}}\sqrt{N+1}\,\mathcal{S}_{N+1}^{(M)}\frac{z_{N+1}^k}{\sqrt{2\pi\,2^k k!}}\\&\times[S_{MN+M-m-l-r}S_{MN+l-k}+(-1)^{M-m}S_{MN+M-m-l-k}S_{MN+l-r}]\Psi_N^m.\end{aligned} \tag{A17}$$

Then, Eq. (A16) leads to

$$\begin{aligned}c_r\left|\mathrm{Pf}_{N+2}^m\right\rangle=&\frac{\sqrt{r!}}{2}\sum_{l=0}^{M-m}(-1)^l\binom{M-m}{l}\sum_{k=0}^{MN+M-m}\sqrt{k!}\,c_k^\dagger\\&\times[S_{MN+M-m-l-r}S_{MN+l-k}+(-1)^{M-m}S_{MN+M-m-l-k}S_{MN+l-r}]\left|\mathrm{Pf}_N^m\right\rangle,\end{aligned} \tag{A18}$$

of which Eq. (2.21) is a special case with $m=1$.

APPENDIX B: THE ANNIHILATION OF $|\mathrm{Pf}_4\rangle$ BY ALL $Q_J^{(2\mathrm{bd},\mathcal{P})}$

By using the recursive formula Eq. (2.16a), the second-quantized form of $|\mathrm{Pf}_4\rangle$ is

$$
\begin{aligned}
|\mathrm{Pf}_4\rangle = \frac{1}{8}\sum_{p_1,p_2,q_1,q_2=0}^{M}(-1)^{\sum_{i=1}^{2}(p_i+q_i)}\prod_{i=1}^{2}\binom{M}{p_i}\binom{M}{q_i}\sum_{l_1,l_2=0}^{M-1}(-1)^{l_1+l_2}\prod_{i=1}^{2}\binom{M-1}{l_i}\\
\times\sqrt{(3M-1-l_2-q_1-q_2)!(2M+l_2-p_1-p_2)!(M-1-l_1+p_1+q_1)!(l_1+p_2+q_2)!}\\
\times c^\dagger_{3M-1-l_2-q_1-q_2}c^\dagger_{2M+l_2-p_1-p_2}c^\dagger_{M-1-l_1+p_1+q_1}c^\dagger_{l_1+p_2+q_2}|0\rangle, \qquad \text{(B1)}
\end{aligned}
$$

where we have used the commutator

$$
[S_l, c_r^\dagger] = \sum_{k=1}^{M}(-1)^k\binom{M}{k}\sqrt{\frac{(r+k)!}{r!}}c^\dagger_{r+k}S_{l-k} \qquad \text{(B2)}
$$

to move S to the right of $c^\dagger$. We act with $Q_J^{(2\mathrm{bd},\mathcal{P})}$ on $|\mathrm{Pf}_4\rangle$ to obtain

$$
\begin{aligned}
Q_J^{(2\mathrm{bd},\mathcal{P})}|\mathrm{Pf}_4\rangle = \frac{1}{4}\sum_{p_1,p_2,q_1,q_2=0}^{M}(-1)^{p_1+p_2+q_1+q_2}\binom{M}{p_1}\binom{M}{p_2}\binom{M}{q_1}\binom{M}{q_2}\sum_{l_2=0}^{M-1}(-1)^{l_2}\binom{M-1}{l_2}\\
\times\left[\sum_{l_1=0}^{M-1}(-1)^{l_1}\binom{M-1}{l_1}\mathcal{P}(M-1-l_1+p_1+q_1, l_1+p_2+q_2)\right]\delta_{J,M+p_1+p_2+q_1+q_2-1}\\
\times\sqrt{(3M-1-l_2-q_1-q_2)!(2M+l_2-p_1-p_2)!}\,c^\dagger_{3M-1-l_2-q_1-q_2}c^\dagger_{2M+l_2-p_1-p_2}|0\rangle\\
+\frac{1}{4}\sum_{p_1,q_1,q_2=0}^{M}(-1)^{p_1+q_1+q_2}\binom{M}{p_1}\binom{M}{q_1}\binom{M}{q_2}\sum_{l_1,l_2=0}^{M-1}(-1)^{l_1+l_2}\binom{M-1}{l_1}\binom{M-1}{l_2}\\
\times\left[\sum_{p_2=0}^{M}(-1)^{p_2}\binom{M}{p_2}\mathcal{P}(l_1+p_2+q_2, 2M+l_2-p_1-p_2)\right]\delta_{J,l_1+l_2+2M-p_1+q_2}\\
\times\sqrt{(3M-1-l_2-q_1-q_2)!(M-1-l_1+p_1+q_1)!}\,c^\dagger_{3M-1-l_2-q_1-q_2}c^\dagger_{M-1-l_1+p_1+q_1}|0\rangle\\
+\frac{1}{4}\sum_{p_2,q_1,q_2=0}^{M}(-1)^{p_2+q_1+q_2}\binom{M}{p_2}\binom{M}{q_1}\binom{M}{q_2}\sum_{l_1,l_2=0}^{M-1}(-1)^{l_1+l_2}\binom{M-1}{l_1}\binom{M-1}{l_2}\\
\times\left[\sum_{p_1=0}^{M}(-1)^{p_1}\binom{M}{p_1}\mathcal{P}(2M+l_2-p_1-p_2, M-1-l_1+p_1+q_1)\right]\delta_{J,-l_1+l_2+3M-p_2+q_1-1}\\
\times\sqrt{(3M-1-l_2-q_1-q_2)!(l_1+p_2+q_2)!}\,c^\dagger_{3M-1-l_2-q_1-q_2}c^\dagger_{l_1+p_2+q_2}|0\rangle\\
+\frac{1}{4}\sum_{p_1,p_2,q_1=0}^{M}(-1)^{p_1+p_2+q_1}\binom{M}{p_1}\binom{M}{p_2}\binom{M}{q_1}\sum_{l_1,l_2=0}^{M-1}(-1)^{l_1+l_2}\binom{M-1}{l_1}\binom{M-1}{l_2}\\
\times\left[\sum_{q_2=0}^{M}(-1)^{q_2}\binom{M}{q_2}\mathcal{P}(3M-1-l_2-q_1-q_2, l_1+p_2+q_2)\right]\delta_{J,l_1-l_2+3M+p_2-q_1-1}\\
\times\sqrt{(2M+l_2-p_1-p_2)!(M-1-l_1+p_1+q_1)!}\,c^\dagger_{2M+l_2-p_1-p_2}c^\dagger_{M-1-l_1+p_1+q_1}|0\rangle\\
+\frac{1}{4}\sum_{p_1,p_2,q_2=0}^{M}(-1)^{p_1+p_2+q_2}\binom{M}{p_1}\binom{M}{p_2}\binom{M}{q_2}\sum_{l_1,l_2=0}^{M-1}(-1)^{l_1+l_2}\binom{M-1}{l_1}\binom{M-1}{l_2}\\
\times\left[\sum_{q_1=0}^{M}(-1)^{q_1}\binom{M}{q_1}\mathcal{P}(M-1-l_1+p_1+q_1, 3M-1-l_2-q_1-q_2)\right]\delta_{J,-l_1-l_2+4M+p_1-q_2-2}\\
\times\sqrt{(2M+l_2-p_1-p_2)!(l_1+p_2+q_2)!}\,c^\dagger_{2M+l_2-p_1-p_2}c^\dagger_{l_1+p_2+q_2}|0\rangle
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{4}\sum_{p_1,p_2,q_1,q_2=0}^{M}(-1)^{p_1+p_2+q_1+q_2}\binom{M}{p_1}\binom{M}{p_2}\binom{M}{q_1}\binom{M}{q_2}\sum_{l_1=0}^{M-1}(-1)^{l_1}\binom{M-1}{l_1}\\
&\times\left[\sum_{l_2=0}^{M-1}(-1)^{l_2}\binom{M-1}{l_2}\mathcal{P}(3M-1-l_2-q_1-q_2,2M+l_2-p_1-p_2)\right]\delta_{J,5M-p_1-p_2-q_1-q_2-1}\\
&\times\sqrt{(M-1-l_1+p_1+q_1)!(l_1+p_2+q_2)!}\,c^\dagger_{M-1-l_1+p_1+q_1}c^\dagger_{l_1+p_2+q_2}\left|0\right\rangle\\
=&\ 0,
\end{aligned}\tag{B3}
$$

where the summation in each of $\binom{4}{2}$ square brackets is zero by using a combinatorial identity [80],

$$
\sum_{i=0}^{n}(-1)^i\binom{n}{i}i^p=0\quad\text{for any integer } p\in[0,n-1],\tag{B4}
$$

considering that the degree of $\mathcal{P}$ is less than $M-1$.

APPENDIX C: THE ANNIHILATION OF $|\mathrm{Pf}_4\rangle$ AND $|\mathrm{Pf}_6\rangle$ BY ALL $Q_J^{(\mathrm{3bd},\mathcal{Q})}$

The second-quantized form of $|\mathrm{Pf}_4\rangle$ has been given in Eq. (B1), and the second-quantized form of $|\mathrm{Pf}_6\rangle$ is

$$
\begin{aligned}
|\mathrm{Pf}_6\rangle=&\frac{1}{48}\sum_{l_1,l_2,l_3=0}^{M-1}(-1)^{\sum_{i=1}^{3}l_i}\prod_{i=1}^{3}\binom{M-1}{l_i}\sum_{p_1,\dots,p_6,q_1,\dots,q_6=0}^{M}(-1)^{\sum_{i=1}^{6}(p_i+q_i)}\prod_{i=1}^{6}\binom{M}{p_i}\binom{M}{q_i}\\
&\times\sqrt{(5M-1-l_3-q_3-q_4-q_5-q_6)!(4M+l_3-p_3-p_4-p_5-p_6)!(3M-1-l_2-q_1-q_2+p_3+q_3)!}\\
&\times\sqrt{(2M+l_2-p_1-p_2+p_4+q_4)!(M-1-l_1+p_1+q_1+p_5+q_5)!(l_1+p_2+q_2+p_6+q_6)!}\\
&\times c^\dagger_{5M-1-l_3-q_3-q_4-q_5-q_6}c^\dagger_{4M+l_3-p_3-p_4-p_5-p_6}c^\dagger_{3M-1-l_2-q_1-q_2+p_3+q_3}c^\dagger_{2M+l_2-p_1-p_2+p_4+q_4}c^\dagger_{M-1-l_1+p_1+q_1+p_5+q_5}\\
&\times c^\dagger_{l_1+p_2+q_2+p_6+q_6}\left|0\right\rangle.
\end{aligned}\tag{C1}
$$

We act $Q_J^{(\mathrm{3bd},\mathcal{Q})}$ on $|\mathrm{Pf}_4\rangle$ to obtain

$$
\begin{aligned}
Q_J^{(\mathrm{3bd},\mathcal{Q})}|\mathrm{Pf}_4\rangle=&\frac{3(-1)^{M-1}}{4}\sum_{q_1,q_2=0}^{M}(-1)^{q_1+q_2}\binom{M}{q_1}\binom{M}{q_2}\sum_{l_2=0}^{M-1}(-1)^{l_2}\binom{M-1}{l_2}\\
&\times\left[\sum_{p_1,p_2=0}^{M}\sum_{l_1=0}^{M-1}(-1)^{l_1}\binom{M-1}{l_1}(-1)^{p_1+p_2}\binom{M}{p_1}\binom{M}{p_2}\right.\\
&\left.\times\mathcal{Q}(2M+l_2-p_1-p_2,M-1-l_1+p_1+q_1,l_1+p_2+q_2)\right]\\
&\times\delta_{J,l_2+3M+q_1+q_2-1}\sqrt{(3M-1-l_2-q_1-q_2)!}\,c^\dagger_{3M-1-l_2-q_1-q_2}\left|0\right\rangle\\
&+\frac{3}{4}\sum_{p_1,p_2=0}^{M}(-1)^{p_1+p_2}\binom{M}{p_1}\binom{M}{p_2}\sum_{l_2=0}^{M-1}(-1)^{l_2}\binom{M-1}{l_2}\\
&\times\left[\sum_{q_1,q_2=0}^{M}\sum_{l_1=0}^{M-1}(-1)^{l_1}\binom{M-1}{l_1}(-1)^{q_1+q_2}\binom{M}{q_1}\binom{M}{q_2}\right.\\
&\left.\times\mathcal{Q}(3M-1-l_2-q_1-q_2,M-1-l_1+p_1+q_1,l_1+p_2+q_2)\right]\\
&\times\delta_{J,-l_2+4M+p_1+p_2-2}\sqrt{(2M+l_2-p_1-p_2)!}\,c^\dagger_{2M+l_2-p_1-p_2}\left|0\right\rangle\\
&+\frac{3(-1)^{M-1}}{4}\sum_{p_1,q_1=0}^{M}(-1)^{p_1+q_1}\binom{M}{p_1}\binom{M}{q_1}\sum_{l_1=0}^{M-1}(-1)^{l_1}\binom{M-1}{l_1}
\end{aligned}
$$

$$
\begin{aligned}
&\times \left[\sum_{p_2,q_2=0}^{M} \sum_{l_2=0}^{M-1} (-1)^{l_2} \binom{M-1}{l_2} (-1)^{p_2+q_2} \binom{M}{p_2}\binom{M}{q_2} \right.\\
&\left. \times\, \mathcal{Q}(3M-1-l_2-q_1-q_2, 2M+l_2-p_1-p_2, l_1+p_2+q_2) \right]\\
&\times\, \delta_{J,l_1+5M-p_1-q_1-1} \sqrt{(M-1-l_1+p_1+q_1)!}\, c^\dagger_{M-1-l_1+p_1+q_1} |0\rangle\\
&+ \frac{3}{4} \sum_{p_2,q_2=0}^{M} (-1)^{p_2+q_2} \binom{M}{p_2}\binom{M}{q_2} \sum_{l_1=0}^{M-1} (-1)^{l_1} \binom{M-1}{l_1}\\
&\times \left[\sum_{p_1,q_1=0}^{M} \sum_{l_2=0}^{M-1} (-1)^{l_2} \binom{M-1}{l_2} (-1)^{p_1+q_1} \binom{M}{p_1}\binom{M}{q_1} \right.\\
&\left. \times\, \mathcal{Q}(3M-1-l_2-q_1-q_2, 2M+l_2-p_1-p_2, M-1-l_1+p_1+q_1) \right]\\
&\times\, \delta_{J,-l_1+6M-p_2-q_2-2} \sqrt{(l_1+p_2+q_2)!}\, c^\dagger_{l_1+p_2+q_2} |0\rangle\\
&= 0, \qquad \text{(C2)}
\end{aligned}
$$

where the term in each of $\binom{4}{3}$ square brackets is zero. Take the first square bracket as an example: on account of Eq. (B4), for the summations inside the first square bracket not to vanish, there should exist at least one term in $\mathcal{Q}$ in which the power of l_1, p_1, and p_2 should be greater than or equal to $M-1$, M, and M, respectively. However, the degree of $\mathcal{Q}$ is less than $3M-1$. Therefore, the term in the first square bracket vanishes. Likewise, summations in all other square brackets are zero.

Using the same line of reasoning, it is easy to verify $Q_J^{(3\text{bd},\mathcal{Q})} |\text{Pf}_6\rangle = 0$.

[1] A. Chubukov, Annu. Rev. Condens. Matter Phys. **3**, 57 (2012).
[2] A. Gurevich, Annu. Rev. Condens. Matter Phys. **5**, 35 (2014).
[3] P. Dai, Rev. Mod. Phys. **87**, 855 (2015).
[4] L. Savary and L. Balents, Rep. Prog. Phys. **80**, 016502 (2017).
[5] M. R. Norman, Rev. Mod. Phys. **88**, 041002 (2016).
[6] Y. Zhou, K. Kanoda, and T.-K. Ng, Rev. Mod. Phys. **89**, 025003 (2017).
[7] C. Broholm, R. J. Cava, S. A. Kivelson, D. G. Nocera, M. R. Norman, and T. Senthil, Science **367**, eaay0668 (2020).
[8] H. L. Stormer, D. C. Tsui, and A. C. Gossard, Rev. Mod. Phys. **71**, S298 (1999).
[9] G. Murthy and R. Shankar, Rev. Mod. Phys. **75**, 1101 (2003).
[10] J. K. Jain, Annu. Rev. Condens. Matter Phys. **6**, 39 (2015).
[11] T. H. Hansson, M. Hermanns, S. H. Simon, and S. F. Viefers, Rev. Mod. Phys. **89**, 025005 (2017).
[12] F. D. M. Haldane, Phys. Rev. Lett. **51**, 605 (1983).
[13] R. B. Laughlin, Phys. Rev. Lett. **50**, 1395 (1983).
[14] S. Bandyopadhyay, G. Ortiz, Z. Nussinov, and A. Seidel, Phys. Rev. Lett. **124**, 196803 (2020).
[15] J. K. Jain, *Composite Fermions* (Cambridge University Press, Cambridge, 2007).
[16] M. Greiter and F. Wilczek, Nucl. Phys. B **370**, 577 (1992).
[17] M. Greiter, X. G. Wen, and F. Wilczek, Nucl. Phys. B **374**, 567 (1992).
[18] G. Moore and N. Read, Nucl. Phys. B **360**, 362 (1991).
[19] N. Read and E. Rezayi, Phys. Rev. B **59**, 8084 (1999).
[20] S. H. Simon, E. H. Rezayi, and N. R. Cooper, Phys. Rev. B **75**, 075318 (2007).
[21] S. H. Simon, E. H. Rezayi, and N. R. Cooper, Phys. Rev. B **75**, 195306 (2007).
[22] C. Nayak, S. H. Simon, A. Stern, M. Freedman, and S. Das Sarma, Rev. Mod. Phys. **80**, 1083 (2008).
[23] A. Kitaev, Ann. Phys. **303**, 2 (2003).
[24] P. Francesco, P. Mathieu, and D. Sénéchal, *Conformal Field Theory* (Springer-Verlag, New York, 1997).
[25] L. Chen and A. Seidel, Phys. Rev. B **91**, 085103 (2015).
[26] L. Chen, S. Bandyopadhyay, K. Yang, and A. Seidel, Phys. Rev. B **100**, 045136 (2019).
[27] M. Schossler, S. Bandyopadhyay, and A. Seidel, Phys. Rev. B **105**, 155124 (2022).
[28] A. Bochniak, Z. Nussinov, A. Seidel, and G. Ortiz, Commun. Phys. **5**, 171 (2022).
[29] A. C. Balram and J. K. Jain, Phys. Rev. B **93**, 235152 (2016).
[30] Y.-H. Wu, T. Shi, and J. K. Jain, Nano Lett. **17**, 4643 (2017).
[31] A. C. Balram, M. Barkeshli, and M. S. Rudner, Phys. Rev. B **98**, 035127 (2018).
[32] A. C. Balram, S. Mukherjee, K. Park, M. Barkeshli, M. S. Rudner, and J. K. Jain, Phys. Rev. Lett. **121**, 186601 (2018).
[33] W. N. Faugno, A. C. Balram, M. Barkeshli, and J. K. Jain, Phys. Rev. Lett. **123**, 016802 (2019).
[34] W. N. Faugno, T. Zhao, A. C. Balram, T. Jolicoeur, and J. K. Jain, Phys. Rev. B **103**, 085303 (2021).
[35] A. C. Balram and A. Wójs, Phys. Rev. Res. **3**, 033087 (2021).
[36] A. C. Balram, Phys. Rev. B **103**, 155103 (2021).
[37] A. C. Balram, Z. Liu, A. Gromov, and Z. Papić, Phys. Rev. X **12**, 021008 (2022).

[38] R. K. Dora and A. C. Balram, Phys. Rev. B **105**, L241403 (2022).
[39] N. Read, Phys. Rev. Lett. **62**, 86 (1989).
[40] G. Murthy and R. Shankar, Phys. Rev. B **65**, 245309 (2002).
[41] W. Bishara and C. Nayak, Phys. Rev. B **80**, 121302(R) (2009).
[42] S. H. Simon and E. H. Rezayi, Phys. Rev. B **87**, 155426 (2013).
[43] M. R. Peterson and C. Nayak, Phys. Rev. B **87**, 245129 (2013).
[44] I. Sodemann and A. H. MacDonald, Phys. Rev. B **87**, 245425 (2013).
[45] B. Yang, Phys. Rev. B **98**, 201101(R) (2018).
[46] N. Read and E. Rezayi, Phys. Rev. B **54**, 16864 (1996).
[47] G. Ortiz, Z. Nussinov, J. Dukelsky, and A. Seidel, Phys. Rev. B **88**, 165303 (2013).
[48] T. Mazaheri, G. Ortiz, Z. Nussinov, and A. Seidel, Phys. Rev. B **91**, 085115 (2015).
[49] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevB.108.075142 for the derivation of Eqs. (2.21) and (2.54) in the second quantization.
[50] B. A. Bernevig and F. D. M. Haldane, Phys. Rev. B **77**, 184502 (2008).
[51] B. A. Bernevig and F. D. M. Haldane, Phys. Rev. Lett. **100**, 246802 (2008).
[52] B. A. Bernevig and F. D. M. Haldane, Phys. Rev. Lett. **101**, 246806 (2008).
[53] B. A. Bernevig and F. D. M. Haldane, Phys. Rev. Lett. **102**, 066802 (2009).
[54] B. A. Bernevig and N. Regnault, Phys. Rev. Lett. **103**, 206801 (2009).
[55] L. Chen, S. Bandyopadhyay, and A. Seidel, Phys. Rev. B **95**, 195169 (2017).
[56] S. Bandyopadhyay, L. Chen, M. T. Ahari, G. Ortiz, Z. Nussinov, and A. Seidel, Phys. Rev. B **98**, 161118(R) (2018).
[57] M. T. Ahari, S. Bandyopadhyay, Z. Nussinov, A. Seidel, and G. Ortiz, SciPost Phys. **15**, 043 (2023).
[58] J. R. Cruise and A. Seidel, Symmetry **15**, 303 (2023).
[59] R. Thomale, B. Estienne, N. Regnault, and B. A. Bernevig, Phys. Rev. B **84**, 045127 (2011).
[60] A. Di Gioacchino, L. G. Molinari, V. Erba, and P. Rotondo, Phys. Rev. B **95**, 245123 (2017).
[61] J. Flavin and A. Seidel, Phys. Rev. X **1**, 021015 (2011).
[62] M. Levin, B. I. Halperin, and B. Rosenow, Phys. Rev. Lett. **99**, 236806 (2007).
[63] S.-S. Lee, S. Ryu, C. Nayak, and M. P. A. Fisher, Phys. Rev. Lett. **99**, 236807 (2007).
[64] D. T. Son, Phys. Rev. X **5**, 031027 (2015).
[65] P. T. Zucker and D. E. Feldman, Phys. Rev. Lett. **117**, 096802 (2016).
[66] T. Jolicoeur, Phys. Rev. Lett. **99**, 036805 (2007).
[67] R. V. Mishmash, D. F. Mross, J. Alicea, and O. I. Motrunich, Phys. Rev. B **98**, 081107(R) (2018).
[68] M. Yutushui and D. F. Mross, Phys. Rev. B **102**, 195153 (2020).
[69] K. Pakrouski, Phys. Rev. B **104**, 245306 (2021).
[70] M. Schossler, L. Chen, and A. Seidel (unpublished).
[71] M. Huber, M. Grayson, M. Rother, W. Biberacher, W. Wegscheider, and G. Abstreiter, Phys. Rev. Lett. **94**, 016805 (2005).
[72] A. Seidel and K. Yang, Phys. Rev. B **80**, 241309(R) (2009).
[73] C. Wang and D. E. Feldman, Phys. Rev. B **81**, 035318 (2010).
[74] E. H. Rezayi, Phys. Rev. Lett. **119**, 026801 (2017).
[75] S. H. Simon, M. Ippoliti, M. P. Zaletel, and E. H. Rezayi, Phys. Rev. B **101**, 041302(R) (2020).
[76] W. Zhu, D. N. Sheng, and K. Yang, Phys. Rev. Lett. **125**, 146802 (2020).
[77] E. H. Rezayi, K. Pakrouski, and F. D. M. Haldane, Phys. Rev. B **104**, L081407 (2021).
[78] A. Bochniak and G. Ortiz, arXiv:2308.03548 [cond-mat.str-el].
[79] V. S. Shchesnovich, arXiv:1308.3275 [cond-mat.quant-gas].
[80] S. M. Ruiz, Math. Gazette **80**, 579 (1996).