Holographic theory for continuous phase transitions: Emergence and symmetry protection of gaplessness

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Two global symmetries are *holoequivalent* if their algebras of local symmetric operators are isomorphic. A holoequivalent class of global symmetries is described by a topological order (TO) in one higher dimension (called symmetry TO), which leads to a symmetry/topological-order (Symm/TO) correspondence. We establish that (1) for systems with a symmetry described by symmetry TO $\mathcal{M}$, their gapped and gapless states are classified by condensable algebras $\mathcal{A}$, formed by elementary excitations in $\mathcal{M}$ with trivial self-/mutual statistics. These so-called $\mathcal{A}$ states can describe symmetry breaking orders, symmetry protected topological orders, symmetry enriched topological orders, gapless critical points, etc. in a unified way. (2) The local low-energy properties of an $\mathcal{A}$ state can be calculated from its reduced symmetry TO $\mathcal{M}_{/\mathcal{A}}$, using holographic modular bootstrap (holoMB) which takes $\mathcal{M}_{/\mathcal{A}}$ as an input. Here $\mathcal{M}_{/\mathcal{A}}$ is obtained from $\mathcal{M}$ by condensing excitations in $\mathcal{A}$. Notably, an $\mathcal{A}$ state must be gapless if $\mathcal{M}_{/\mathcal{A}}$ is nontrivial. This provides a unified understanding of the emergence and symmetry protection of gaplessness that applies to symmetries that are anomalous, higher-form, and/or noninvertible. (3) The relations between condensable algebras constrain the structure of the global phase diagram. We find that, for $1+1$D $\mathbb{Z}_2\times\mathbb{Z}_2'$ symmetry with mixed anomaly, there is a stable continuous transition (deconfined quantum critical point) between the $\mathbb{Z}_2$-breaking-$\mathbb{Z}_2'$-symmetric phase and the $\mathbb{Z}_2$-symmetric-$\mathbb{Z}_2'$-breaking phase. The critical point is the same as a $\mathbb{Z}_4$ symmetry breaking critical point. (4) $1+1$D bosonic systems with S_3 symmetry have four gapped phases with unbroken symmetries S_3, $\mathbb{Z}_3$, $\mathbb{Z}_2$, and $\mathbb{Z}_1$. We find a duality between two transitions $S_3\leftrightarrow\mathbb{Z}_1$ and $\mathbb{Z}_3\leftrightarrow\mathbb{Z}_2$: they are either both first order or both (stably) continuous, and in the latter case, they are described by the same conformal field theory (CFT). (5) The gapped and gapless states for $1+1$D bosonic systems with anomalous S_3 symmetries are obtained as well. For example, anomalous $S_3^{(1)}$ and $S_3^{(2)}$ symmetries can have symmetry protected chiral gapless states with only symmetric irrelevant and marginal operators.

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I. INTRODUCTION

For a long time, Landau's symmetry breaking theory [1,2] was regarded as the standard theory for continuous phase transitions. In particular, it was believed that all continuous phase transitions are spontaneous symmetry breaking transitions, where the symmetry groups for the two phases across a transition have a special relation

$$G_{\text{small}} \subset G_{\text{large}},$$

i.e., the symmetry group for the phase with less symmetry is a strict subgroup of the symmetry group for the phase with more symmetry.

A. Symmetry/topological-order (Symm/TO) correspondence

However, in the last 30 years, increasingly many examples of continuous phase transitions have been discovered in quantum systems whose description is beyond Landau's theory. Continuous quantum phase transitions were found between two states with the same symmetry [3–8] (but different topological orders [9,10]). Continuous quantum phase transitions are also possible between two states with *incompatible* symmetries [11], i.e., the symmetry groups of the two phases across the transition do not have the group-subgroup relation. These "deconfined quantum critical points" (DQCPs) have been found to be related to mutual anomalies between internal and lattice symmetries [12,13]. Even symmetry-breaking transitions with well-defined order parameters are sometimes not described by Landau's symmetry breaking theory [8]. In light of these examples, it appears that many continuous quantum phase transitions are not described by Landau theory, regardless of whether they have symmetry breaking and order parameters or not. There are many situations and mechanisms that can lead to continuous quantum phase transitions that go beyond Landau symmetry breaking theory. It is interesting to ask whether there is a unified theory to understand these various beyond-Landau continuous quantum phase transitions.

To systematically understand gapless critical points at continuous transitions it is fruitful to identify all the emergent symmetries in the gapless states. Emergent symmetry can be very rich and may include 0-symmetry, higher[1] symmetry [14–17], anomalous symmetry [18–21], anomalous higher symmetry [16,17,22–32], beyond-anomalous symmetry [33], noninvertible symmetry [34–40], algebraic higher symmetry [41,42], and/or noninvertible gravitational anomaly

[1] We use "higher symmetry" to cover both higher-form symmetry and higher-group symmetry.

[20,43–48]. Recently, a symmetry/topological-order (Symm/TO) correspondence was proposed [42,49] that can provide a unified description of all those symmetries.

One way to have a unified description of all these symmetries is to restrict to the symmetric sub Hilbert space $\mathcal{V}_{\text{symmetric}}$, which does not have a tensor product decomposition

$$\mathcal{V}_{\text{symmetric}} \neq \bigotimes_i \mathcal{V}_i. \quad (1)$$

Here, $\mathcal{V}_i'$s are local Hilbert spaces on each lattice site. The failure of tensor product decomposition indicates [48] a noninvertible gravitational anomaly [20,43–46]. This leads to the point of view that

$$\text{symmetry (restricted to } \mathcal{V}_{\text{symmetric}}) = \text{noninvertible gravitational anomaly.} \quad (2)$$

For a finite symmetry, its corresponding noninvertible gravitational anomaly is the same as topological order (TO)[2] in one higher dimension, which is referred to as symmetry TO[3] [20,43,46]. This leads to a holographic view of symmetry: Symm/TO correspondence [42,49]

$$\text{symmetry (restricted to } \mathcal{V}_{\text{symmetric}}) = \text{symmetry TO.} \quad (3)$$

This holographic perspective on symmetry in $1+1$D was also discussed in Refs. [38,50,51].

A second way to have a unified description of all emergent symmetries generalizes the idea that, to describe an ordinary symmetry, we can use the conservation law (i.e., the fusion ring) of symmetry charges. To obtain a unified description, we use instead the fusion rings (conservation laws) of both symmetry charges and symmetry defects at an equal footing [49]. The resulting symmetry is called categorical symmetry$^{Ⓗ}$.[4] It is also necessary to include "braiding" properties of symmetry charges/defects [49] which allow us to describe the symmetry actions to have a full description of symmetry.[5]

[2]Here, the topological order in one higher dimension is anomaly-free (i.e., with UV completion). In this paper, the term *topological order* always refers to anomaly-free topological order. Topological order with anomaly will be explicitly referred to as *anomalous topological order*.

[3]A symmetry TO always describes a symmetry in one lower dimension.

[4]Here, we use the term categorical symmetry$^{Ⓗ}$ in the original holographic sense of Refs. [42,49]. However, the term "categorical symmetry" has since been used by many to describe noninvertible symmetry. To avoid possible confusions, we use categorical symmetry$^{Ⓗ}$ in Sans Serif Font with superscript Ⓗ to stress that we use the term in the holographic sense. See also Appendix A for more detailed explanations and discussions on related concepts.

[5]A symmetry is described by the algebra of local symmetric operators. The "braiding" properties are features of such an algebra. See Ref. [33] for details. Such features become the braiding properties in the symmetry TOin Symm/TO correspondence, which leads to the name "braiding" properties. Symmetry charges always has the trivial "braiding" property. Thus, in the ordinary symmetry described by fusion ring of symmetry charge, we do not need to introduce extra data to describe such a trivial "braiding" property.

Thus categorical symmetry$^{Ⓗ}$ has both fusion ring layer and "braiding" layer. Just like ordinary symmetry is described by group, categorical symmetry$^{Ⓗ}$ is described by nondegenerate braided fusion higher (nBF) category (which is referred to as nBF category in short) [42,49]. Here "nondegenerate" indicates that we have included all the symmetry charges and the symmetry defects [33], and "higher" refers to the fact that the symmetry charges and defects can be pointlike, stringlike, etc.

The above holographic view of symmetry and anomaly is motivated variously from anomaly-inflow [52], from the boundary-bulk topological holographic relation [38,42,43,46,47,53–55], from an observation that symmetry protected topological (SPT) order [19,56,57] is closely related to anomaly in one lower dimension [20,22,58], and from an observation that SPT order and anomaly are closely related to braiding [32,59]. This holographic point of view has parallels with the AdS/CFT correspondence [60,61], where a *continuous* G symmetry of a CFT is associated to a G-gauge theory in an *AdS space* in one higher dimension. There are however some important differences between the two. In Symm/TO correspondence, a *finite* G symmetry of a CFT is associated to a G-gauge theory in one higher dimension with *arbitrary metric*.[6] Moreover, in Symm/TO correspondence, the bulk theory is not equivalent to the boundary theory. The bulk topological order (i.e., the symmetry TO) just constrains the boundary dynamics.

We should note that, so far, the Symm/TO correspondence only applies to finite symmetry. For continuous symmetry, we either need to generalize the Symm/TO correspondence, or need to develop a new nonholographic point of view as in Ref. [33]. To that end, there is a third *nonholographic* way to reach a unified description of all emergent symmetries in a gapless state. Here one starts from the point of view that a symmetry is fully described by an algebra of local symmetric operators (LSOs). An ordinary (global) symmetry is characterized by symmetry transformations, which are the commutants of LSOs.[7] These symmetry transformations act on the whole space (or on all the closed subspaces of codimension p for p symmetry), and correspond to the global symmetry transformations. In this approach, we restrict to the symmetric sub Hilbert space $\mathcal{V}_{\text{symmetric}}$, as in the first approach. In this case, we find that the global symmetry transformations act trivially as identity operator. Seemingly, we do not see any global symmetry after the Hilbert space restriction. On the other hand, even after restricting to $\mathcal{V}_{\text{symmetric}}$, symmetry clearly still constrains the low energy dynamics and is physically meaningful. To see the symmetry in this case, Refs. [33,49] considered the so called "commutant patch operators," referred to as "transparent patch operators" in Ref. [33]. Patch commutant operators are operators formed by local symmetric operators (LSOs), acting on one-dimensional, two-dimensional, etc. open subspaces (i.e., patches), and commute with all the LSOs as long as the LSOs are far away from

[6]The metric is arbitrary since G-gauge theory is topological for finite G.

[7]The commutants of local symmetric operators are operators that commute with all the local symmetric operators.

the boundaries of the patches and have no nontrivial linking. Since the commutants of LSOs define global symmetry, we say the commutant patch operators of local symmetric operators define the "patch symmetry" of the system. References [33,49] found that there are two kinds of commutant patch operators: the first kind are global symmetry transformations restricted on the patches, which are called patch symmetry operators. The boundaries of patch symmetry operators corresponds to symmetry defects. The second kind have empty bulk and create neutral charge objects on their boundaries, which are called patch charge operators. The boundaries of patch charge operators corresponds to symmetry charges. We see that, in contrast to global symmetry, patch symmetry treats symmetry charges and symmetry defects at an equal footing. The algebra of commutant patch operators encode the fusion ring and "braiding" properties of symmetry charges/defects, which is conjectured to give rise to a nBF category [33]. Thus the patch symmetry is identical to categorical symmetry$^{Ⓗ}$ and they are both described by nBF category.

We define two symmetries to be *holoequivalent* [42] if the algebras of their local symmetric operators are isomorphic. We define two patch symmetries to be the same if the algebras of their commutant patch operators are isomorphic. This allows us to summarize the above discussions:

$$\begin{aligned}&\text{(generalized) global symmetries (restricted to } \mathcal{V}_{\text{symmetric}})\\ &= \text{categorical symmetries}^{Ⓗ} = \text{patch symmetries}\\ &= \text{holoequivalent classes of global symmetries}\\ &= \text{nBF categories}\\ &= \text{symmetry TOs (for finite symmetry).} \qquad (4)\end{aligned}$$

Here, global symmetry (restricted to $\mathcal{V}_{\text{symmetric}}$) is viewed from the point of view of the algebra of local symmetric operators, and "=" means one-to-one correspondence. We remark that (4) is more precise than (2) and (3).

We see that (generalized) global symmetry is different from categorical symmetry$^{Ⓗ}$ or patch symmetry (which are two names for the same thing). In fact, categorical symmetry$^{Ⓗ}$ (or patch symmetry) only looks at a global symmetry from a local point of view, ignoring the global features [49]. Thus categorical symmetry$^{Ⓗ}$ (or patch symmetry) corresponds to a holoequivalent class of global symmetries. As a result, symmetry TO and nBF category only describe the holoequivalent class of (generalized) global symmetries.

The four terms, categorical symmetry$^{Ⓗ}$, patch symmetry, symmetry TO, and nBF category, describe almost the same thing, but stress on different aspects: categorical symmetry$^{Ⓗ}$ emphasizes on symmetry + dual symmetry (i.e., treating symmetry charges and defects at equal footing); patch symmetry emphasizes on its difference with global symmetry; symmetry TO emphasizes on the holographic picture; nBF category is most accurate. We can use any of them. However, since "categorical symmetry" has been used by many to mean noninvertible global symmetry, in the rest of this paper, we will use *symmetry TO*. We like to remark that symmetry TO can only describe finite symmetries. For continuous symmetries, we need to use nBF categories with infinite objects/morphisms to describe them. Therefore nBF category is a more accurate term. We use the term symmetry TO since it is more easily associated with symmetry and holographic picture.

Let us also point out that the symmetry TO can be used to describe an exact UV symmetry of a lattice model. However, it can also be used to describe an emergent symmetry that appears only at low energies (IR). Consider a system with a separation of energy scale, i.e., some excitations have much higher energies compared to all other excitations which may be gapped or gapless. Well below the energy gap of the high energy excitations, the low energy properties of the system are controlled by an emergent symmetry described by symmetry TO $\mathcal{M}$.

If the low energy excitations are gapped, then they can be described by a fusion n-category $\mathcal{C}$ if the space is n-dimensional. In this case, the emergent symmetry is described by an symmetry TO $\mathcal{M} = \mathcal{Z}(\mathcal{C})$, the "Drinfeld" center of the low energy excitations $\mathcal{C}$ [42,43,46,47]. It was pointed out in Ref. [62] that the emergent higher symmetry contained in $\mathcal{M} = \mathcal{Z}(\mathcal{C})$ is exact, while the emergent 0-symmetry contained in $\mathcal{M} = \mathcal{Z}(\mathcal{C})$ is approximate.

If the low energy excitations are gapless, then the *maximal* emergent symmetry TO [63] may largely characterize the gapless state. We know that the possible gapless states are very rich, and it is hard to believe gapless states can be characterized by their emergent symmetries, if we only consider emergent symmetries described by groups. However, emergent symmetries can be generalized symmetries that are beyond group or beyond higher group. We need to use symmetry TOs to describe those emergent generalized symmetries. In this case, it may be possible that emergent maximal symmetry TO can largely characterize gapless states.

B. Characterizing gapless liquid states using Symm/TO correspondence

In a series of papers [64–66], Kong and Zheng have developed a unified mathematical theory for topological orders and gapless quantum liquids[8] in n-dimensional space-time, based on symmetric monoidal higher category, $\mathcal{QL}^n$, which is called category of quantum liquids. $\mathcal{QL}^n$ for different n are related by delooping $\Sigma_* \mathcal{QL}^n = \mathcal{QL}^{n+1}$ (hypothesis 5.16 in Ref. [64,64]). A quantum liquid L (an object in $\mathcal{QL}^n$) contains two parts: a topological skeleton L_{sk} and a local quantum symmetry L_{lqs}, where the topological skeleton L_{sk} is an anomalous topological order. In 1 + 1D, Kong and Zheng have also developed a theory for gapless boundaries of 2 + 1D topological order based on categories enriched by local quantum symmetry—vertex operator algebra [69–71].

In this paper, we are going to use Symm/TO correspondence and the associated symmetry TO to develop another version of the general unified theory for topological orders and gapless quantum liquids, from a more physical point of view. Our theory is based on the following proposals.

(1) An $n + 1$D symmetric system, when restricted to its symmetric sub-Hilbert space, has a noninvertible gravitational

[8]Here we use the notion of *liquid state* for quantum systems in the sense defined in Refs. [67,68]. We will not discuss nonliquid states, such as fractons.

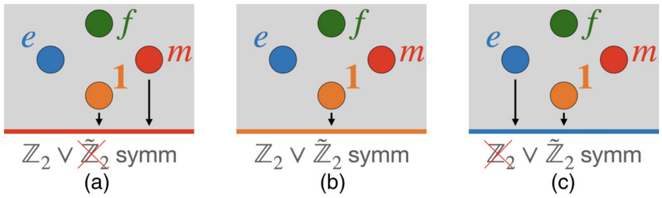


FIG. 1. A 1 + 1D $\mathbb{Z}_2$-symmetric system (which also has a dual $\widetilde{\mathbb{Z}}_2$ symmetry [49]) is a $\mathcal{G}\text{au}_{\mathbb{Z}_2}$ system, i.e., the system has a categorical symmetry[h] $\mathbb{Z}_2 \vee \widetilde{\mathbb{Z}}_2$, which is described by symmetry TO $\mathcal{G}\text{au}_{\mathbb{Z}_2}$—the quantum double of $\mathbb{Z}_2$ group. Physically, the above statement means that the $\mathbb{Z}_2$ symmetric system (when restricted to its symmetric sub-Hilbert space) can be exactly low-energy simulated by a boundary of bulk $\mathbb{Z}_2$ topological order (TO), described by $\mathbb{Z}_2$ gauge theory. The symmetry TO $\mathcal{G}\text{au}_{\mathbb{Z}_2}$ has four anyons $\mathbf{1}, e, m, f = e \otimes m$. The possible condensation-induced states in $\mathcal{G}\text{au}_{\mathbb{Z}_2}$ system are given by the condensable algebras of the symmetry TO, $\mathcal{A} = \mathbf{1}, \mathbf{1} \oplus e, \mathbf{1} \oplus m$. (a) The $\mathbf{1} \oplus m$ state, corresponding to the $\mathbf{1} \oplus m$-condensed boundary, is the $\mathbb{Z}_2$-symmetric state. (c) The $\mathbf{1} \oplus e$ state is the state with spontaneous $\mathbb{Z}_2$ symmetry breaking. (b) The $\mathbf{1}$ state is the gapless critical point at the continuous transition between $\mathbf{1} \oplus m$ state and $\mathbf{1} \oplus e$ state.

anomaly [48], and can be *exactly low-energy simulated* by a boundary of a topological order in one higher dimension [42,48,49]. Such a bulk topological order is called a symmetry TO and is denoted by $\mathcal{M}$. This result allows us to say that the $n + 1$D symmetry is described by a symmetry TO $\mathcal{M}$. We will use $\mathcal{M}$ system to refer to such a system (see Fig. 1). Since the $n + 2$D symmetry TO $\mathcal{M}$ is mathematically described by a nondegenerate braided fusion n-category (called nBF category and also denoted as $\mathcal{M}$) [42,43], we may also say that the $n + 1$D symmetry is described by a nBF category $\mathcal{M}$.

(2) The states of an $\mathcal{M}$ system can be divided into classes labeled by the condensable algebras $\mathcal{A}$ in $\mathcal{M}$, in the sense that a state in a class labeled by $\mathcal{A}$ (called $\mathcal{A}$ state) is exactly low-energy simulated by a boundary of the symmetry TO $\mathcal{M}$ induced by the condensable algebra $\mathcal{A}$ [72], termed an $\mathcal{A}$-condensed boundary (see Fig. 1). This way, condensable algebras can describe, in a unified way, symmetry breaking orders, symmetry protected topological orders, symmetry enriched topological orders, gapless critical points, etc. See Sec. II B for a physical description of condensable algebras.

(3) An $\mathcal{A}$ state can be exactly low-energy simulated by a $\mathbf{1}$-condensed boundary of $\mathcal{M}_{/\mathcal{A}}$, where $\mathcal{M}_{/\mathcal{A}}$[9] is the topological order obtained from $\mathcal{M}$ by condensing the condensable algebra $\mathcal{A}$ [72].[10] $\mathcal{M}_{/\mathcal{A}}$ is referred to as the reduced symmetry TO. As a $\mathbf{1}$-condensed boundary of $\mathcal{M}_{/\mathcal{A}}$, the $\mathcal{A}$ state in $\mathcal{M}$ system has a reduced symmetry TO described by $\mathcal{M}_{/\mathcal{A}}$. So we will also refer to $\mathcal{A}$ state in $\mathcal{M}$ system as an $\mathcal{M}_{/\mathcal{A}}$ state.

We remark that it is possible that $\mathcal{M}_{/\mathcal{A}} = \mathcal{M}_{/\mathcal{A}'} = \mathcal{M}_{\text{reduced}}$ for two different condensible algebras $\mathcal{A}$ and $\mathcal{A}'$. In this case, two different states $\mathcal{A}$ state and $\mathcal{A}'$ state are both referred to as $\mathcal{M}_{\text{reduced}}$ state. As we will see in Sec. II B, $\mathcal{A}$ state and $\mathcal{A}'$ state have the same local low energy properties. Thus $\mathcal{M}_{/\mathcal{A}}$ state is a notion that is useful for gapless states which ignores the global properties. For example, a gapped state always has a trivial reduced symmetry TO $\mathcal{M}_{/\mathcal{A}}$, and a nontivial reduced symmetry TO $\mathcal{M}_{/\mathcal{A}}$ implies gaplessness (see Sec. II C).

(4) We can use symmetry TO to constrain the possible continuous phase transitions. For example, if an $\mathcal{A}_{12}$ state is the critical point for a continuous phase transition between $\mathcal{A}_1$ and $\mathcal{A}_1$ states, then $\mathcal{A}_{12}$ is a subalgebra of both $\mathcal{A}_1$ and $\mathcal{A}_2$ (see Fig. 1).

In the above, we have introduced some important terms (in bold face) that we will use in the rest of this paper. We also used the following notion [42]. Exactly low-energy simulate means that the low energy spectrum in the symmetric sub-Hilbert space is identical to the low energy spectrum of the boundary. It also means that there is a one-to-one correspondence of local symmetric operators in $\mathcal{M}$ system and local operators on the boundary of the symmetry TO $\mathcal{M}$, such that the corresponding operators have identical correlation functions (in the limit that the energy gap of the symmetry TO approaches infinity).

Very often, we can easily compute the reduced symmetry TO $\mathcal{M}_{/\mathcal{A}}$, which allows us to determine if an $\mathcal{M}_{/\mathcal{A}}$ state is gapless or not. If $\mathcal{M}_{/\mathcal{A}}$ is trivial, then the corresponding $\mathcal{M}_{/\mathcal{A}}$ state can be gapped. On the other hand, $\mathcal{M}_{/\mathcal{A}}$ state must be gapless if $\mathcal{M}_{/\mathcal{A}}$ is a nontrivial. (see Sec. II C for a proof.) Moreover, we can constrain its low energy properties using the $\mathcal{M}_{/\mathcal{A}}$. This is a more general version of the familiar notion of *symmetry protected gaplessness*, i.e., condensation patterns in the symmetry TO can determine whether a state is gapless or not.

It is well known that perturbative anomalies for continuous symmetries [18] and perturbative gravitational anomalies [73,74] imply gaplessness [53,75–84]. This can be understood as *perturbative-anomaly protected gaplessness*. Even global anomalies for discrete symmetries may imply gaplessness [57,85–90], which can be understood as *anomalous-symmetry protected gaplessness*. Symmetry fractionalization may also imply gaplessness [91–93], which can be understood as *symmetry-fractionalization protected gaplessness*. Our Symm/TO correspondence provides a unified point of view to understand these different kinds of protected gaplessness.

We want to emphasize that a nontrivial reduced symmetry TO $\mathcal{M}_{/\mathcal{A}}$ is viewed as the reason for gaplessness in this framework. Thus a nontrivial $\mathcal{M}_{/\mathcal{A}}$ represents the emergence of gaplessness. This suggests that the low energy properties of the gapless state are characterized by the reduced symmetry TO $\mathcal{M}_{/\mathcal{A}}$. In other words, we can use reduced symmetry TOs to systematically study, and potentially classify, gapless states and the corresponding quantum field theories in one lower dimensions. In particular, we can use holographic modular bootstrap [48,94,95] to compute the low energy properties of a 1 + 1D gapless state from its reduced symmetry TO $\mathcal{M}_{/\mathcal{A}}$, as we will describe in Sec. II.

C. Organization of the paper

The remainder of this paper is organized as follows. In Sec. II, we flesh out our Symm/TO framework for labeling phases and their phase transitions using condensable algebras.

[9]May be read as "*M slash A*."

[10]Here we assume the topological order $\mathcal{M}_{/\mathcal{A}}$ to have a large energy gap approaching infinity.

We discuss how to identify the patterns of condensation, i.e., the allowed condensable algebras, using a set of number theoretic constraints. Along with the knowledge of boundary partition functions compatible with the bulk topological order, this provides us a pathway to understanding the phase diagram for a system with a given symmetry. In Secs. III–VI, we discuss various examples of anomalous and anomaly-free Abelian and nonAbelian symmetries in 1 + 1D systems. We identify gapped and gapless states allowed by each of these symmetries, and provide a discussion of the gapless theories possible at the phase transitions between these states.

The main results of this paper are summarized in the framed boxes. Gapped and gapless states for 1 + 1D systems with S_3 symmetry (with or without anomaly) and the corresponding condensation patterns in their symmetry TO are summarized in Tables II—IV. The gapless states are potential critical points for continuous transitions between the gapped states.

II. HOLOGRAPHIC THEORY FOR GAPLESS STATES AND FOR CONTINUOUS PHASE TRANSITIONS

In this section, we will formulate a general holographic theory for gapless states and for continuous phase transitions, based on Symm/TO correspondence [42,49]. Later, we will apply Symm/TO correspondence to study some examples. In fact, Symm/TO correspondence also applies to gapped states.

A. Symmetry TO reduction (analogue of symmetry breaking)

For ordinary global symmetry, spontaneous symmetry breaking is a very important notion, which allows us to describe gapped and gapless phases. If we use symmetry TO to describe global symmetry, spontaneous symmetry breaking is replaced by *spontaneous symmetry TO reduction*, or simply *symmetry TO reduction*. Patterns of symmetry TO reduction, physically induced by condensation of excitations, allow us to describe gapped and gapless phases, as well as the critical points at continuous phase transitions. Considered thus, symmetry TO reduction provides more information than spontaneous symmetry breaking. In this section, we will describe symmetry TO reduction in details.

Symm/TO correspondence has the following meaning [42,49], which is the key conjecture used in this paper. A system (i.e., a gapped or gapless lattice Hamiltonian) with a (generalized) global symmetry can be *exactly low-energy simulated* by a boundary (i.e., a boundary Hamiltonian) of a noninvertible topological order $\mathcal{M}$ in one higher dimension. The bulk topological order is referred to as symmetry TO $\mathcal{M}$. This is why we can use symmetry TO $\mathcal{M}$ to describe a symmetry. We will call such a symmetric system as an $\mathcal{M}$ system. We remark that the above Symm/TO correspondence works for anomalous and/or higher and/or noninvertible symmetries. It even works for global symmetries beyond the previous known descriptions. Thus it can be viewed as a most general description of global symmetry.

We have the following mathematical result (see Refs. [69–72] for a summary): *all (gapped or gapless) boundary states of a topological order* $\mathcal{M}$ *are obtained from condensing condensable algebras* $\mathcal{A}$ *of* $\mathcal{M}$. Since different boundary states of $\mathcal{M}$ correspond to different ground states in different $\mathcal{M}$ systems, we can group all gapped or gapless ground states in $\mathcal{M}$ systems into classes labeled by the condensable algebras $\mathcal{A}$, i.e., states in a class labeled by $\mathcal{A}$ correspond to $\mathcal{A}$-condensed boundaries of $\mathcal{M}$. Those states are referred to as $\mathcal{A}$ states. In other words, an $\mathcal{A}$ state in an $\mathcal{M}$ system has a condensation pattern $\mathcal{A}$. After introducing those notions, we can make the following statement. In $\mathcal{M}$ systems, all their gapped and gapless $\mathcal{A}$ states are *exactly low-energy simulated* by the $\mathcal{A}$ condensation-induced boundary states of the symmetry TO $\mathcal{M}$.

B. Reduced symmetry TO (analogue of unbroken symmetry)

Now let us concentrate on a topological order $\mathcal{M}$ and one of its boundary state induced by condensing a condensable algebra $\mathcal{A}$. Such a boundary state corresponds to an $\mathcal{A}$ state in an $\mathcal{M}$ system. The condensable algebra $\mathcal{A}$ is formed by excitations in $\mathcal{M}$ that has trivial self and mutual statistics between them. As a result, excitations in the condensable algebra $\mathcal{A}$ can condense together, which will change the topological order $\mathcal{M}$ to another topological order [72]. We will denote the resulting topological order as $\mathcal{M}_{/\mathcal{A}}$, which will be called a reduced symmetry TO.

Physically a condensable algebra $\mathcal{A}$ corresponds to a set of excitations in $\mathcal{M}$ that can be condensed together, i.e., with trivial self/mutual statistics. Mathematically, a condensable algebra $\mathcal{A}$ is described by a *composite excitation* $\mathbf{1} \oplus a \oplus b \oplus \cdots$,[11] which can be viewed as a "vector space," plus some data describing "multiplication of vectors" in the vector space (see Ref. [72] for a summary, and see Ref. [50] for a detailed discussion of a simple example). For simplicity, in this paper, we will use the "vector space" $\mathbf{1} \oplus a \oplus b \oplus \cdots$ to denote the condensable algebra $\mathcal{A}$, and say $\mathbf{1}$, a, b, etc. belong to the condensable algebra $\mathcal{A}$: $\mathbf{1}, a, b \in \mathcal{A}$. We will always use $\mathbf{1}$ to denote the trivial excitations, i.e., all the excitations that can be created by local symmetric operators. Note that $\mathbf{1}$ can represent a null excitation—the ground state itself—an excitation created by the identity operator.[12]

Roughly speaking, the excitations in $\mathcal{M}_{/\mathcal{A}}$ all comes from the excitations in $\mathcal{M}$: the excitations in $\mathcal{M}$ will become trivial excitations in $\mathcal{M}_{/\mathcal{A}}$, if they are in $\mathcal{A}$ (i.e., if they are condensed). The excitations in $\mathcal{M}$ will be confined in $\mathcal{M}_{/\mathcal{A}}$ (i.e., will disappear), if they have nontrivial mutual statistics with excitations in $\mathcal{A}$ (for more details, see Appendix C).

The condensable algebra $\mathcal{A}$ is called Lagrangian if $\mathcal{M}_{/\mathcal{A}}$ is trivial. In this case, the excitations in $\mathcal{M}$ will be either condensed or confined. A Lagrangian condensable algebra is

[11] A *composite excitation* $\mathbf{1} \oplus a \oplus b \oplus \cdots$ is an excitation where the excitations $\mathbf{1}$, a, b, etc. in the composite happen to have the same energy. For example, the bound state of two spin-1/2 excitations is a composite excitation formed by degenerate spin-0 and spin-1 excitations: spin-$1/2 \otimes$ spin-$1/2 =$ spin-$0 \oplus$ spin-1.

[12] We believe that even in higher dimensions, the various condensation patterns associated to symmetry TO are still classified by condensable algebras $\mathcal{A}$ in the symmetry TO. However, in higher dimensions, the notion of condensable algebras needs to be generalized beyond what is described in this paper.

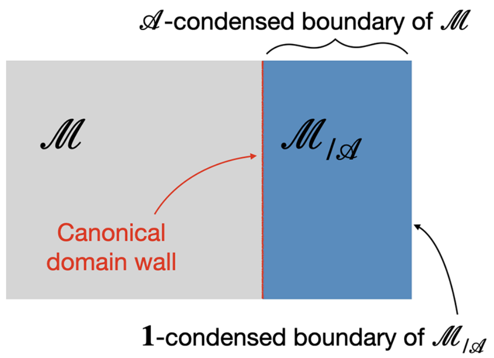


FIG. 2. An $\mathcal{A}$ state in an $\mathcal{M}$ system corresponds to an $\mathcal{A}$-condensed boundary of symmetry TO $\mathcal{M}$. Such a boundary can be obtained by attaching $\mathcal{A}$ condensation induced topological order $\mathcal{M}_{/\mathcal{A}}$ with $\mathbf{1}$-condensed boundary. The $\mathcal{A}$ condensation changes $\mathcal{M}$ to $\mathcal{M}_{/\mathcal{A}}$, causing a *symmetry TO reduction* (an analog of spontaneous symmetry breaking). $\mathcal{M}_{/\mathcal{A}}$ is the *reduced symmetry TO* (an analog of unbroken symmetry) of the $\mathcal{A}$ state.

maximal in some sense, since the induced topological order $\mathcal{M}_{/\mathcal{A}}$ is minimal (i.e., trivial). On the other hand, if the condensable algebra is minimal, $\mathcal{A} = \mathbf{1}$ (i.e., nothing condenses except trivial particle $\mathbf{1}$ which always condenses), the induced topological order $\mathcal{M}_{\mathbf{1}} = \mathcal{M}$ is maximal.

From the $\mathcal{A}$ condensation-induced topological order $\mathcal{M}_{/\mathcal{A}}$, we can have an understanding of an $\mathcal{A}$-condensed boundary of $\mathcal{M}$, which has a special realization as a composite boundary illustrated in Fig. 2. There is a canonical domain wall between $\mathcal{M}$ and $\mathcal{M}_{/\mathcal{A}}$ which is always gapped [72]. Therefore the local low energy properties of the $\mathcal{A}$-condensed boundary of $\mathcal{M}$ is same as the local low energy properties of the $\mathbf{1}$-condensed boundary of $\mathcal{M}_{/\mathcal{A}}$.[13] This result is reasonable and expected, since the $\mathcal{A}$-condensed boundary of $\mathcal{M}$ also implies that excitations outside $\mathcal{A}$ do not condense. For the composite boundary in Fig. 2, the induced topological order $\mathcal{M}_{/\mathcal{A}}$ already has all the condensations for excitations in $\mathcal{A}$, and the $\mathbf{1}$-condensed boundary of $\mathcal{M}_{/\mathcal{A}}$ implies that there is no additional condensation (for excitations outside $\mathcal{A}$).

C. The emergence and the symmetry protection of gaplessness: The 1-condensed boundary of $\mathcal{M}_{/\mathcal{A}}$

It is easy to see that the $\mathbf{1}$-condensed boundary of $\mathcal{M}_{/\mathcal{A}}$ can be gapped if $\mathcal{M}_{/\mathcal{A}}$ is trivial. The $\mathbf{1}$-condensed boundary of $\mathcal{M}_{/\mathcal{A}}$ must be gapless if $\mathcal{M}_{/\mathcal{A}}$ is nontrivial. This result is proposed in Refs. [69–71] and was referred to as topological Wick rotation.

Let us show that a $\mathbf{1}$-condensed boundary of a $2+1$D topological orders $\mathcal{M}$ must be gapless. This result is obtained in Refs. [69–71,96] via some other methods. As pointed out in Refs. [48,97,98], the partition function for a boundary of a $2+1$D topological order is a vector, whose components are labeled by the anyon types of the bulk topological order: $\mathbf{Z}(\tau) = (Z_{\mathbf{1}}(\tau), Z_a(\tau), Z_b(\tau), \cdots)^\top$. Here we have assumed that the space-time at the boundary is a torus, and τ describes the shape of the boundary space-time. Under the modular transformation, vectorlike partition function transforms covariantly [48]:

$$e^{-i2\pi\frac{c_-}{24}} T^{\mathcal{M}} \mathbf{Z}(\tau) = \mathbf{Z}(\tau+1),$$
$$D^{-1} S^{\mathcal{M}} \mathbf{Z}(\tau) = \mathbf{Z}(-1/\tau), \tag{5}$$

where $S^{\mathcal{M}}, T^{\mathcal{M}}$ are the modular data characterizing the bulk topological order $\mathcal{M}$ [9,10] (see Appendix C) and c_- the chiral central center of $\mathcal{M}$. The physics behind the above results were explained in Refs. [48,97,98].

If the $\mathbf{1}$-condensed boundary of the bulk topological order was gapped, the vector-like partition function would be τ independent and would have a form

$$Z_{\mathbf{1}}(\tau) = 1, \quad Z_{a\neq\mathbf{1}}(\tau) = 0, \tag{6}$$

since only $\mathbf{1}$ condenses. Such a partition function cannot be modular covariant, if the bulk topological order is noninvertible. This is because $S^{\mathcal{M}}, T^{\mathcal{M}}$ matrices is more than 1-dimensional for noninvertible topological order, and the $\mathbf{1}$-column of the $S^{\mathcal{M}}$ matrix has a form $(d_{\mathbf{1}}, d_a, d_b, \cdots)^\top$, where d_a is the quantum dimension of type-a bulk anyon and $D = \sqrt{\sum_a d_a^2}$. Since $d_{\mathbf{1}} = 1$, $d_a \geqslant 1$, and $D > 1$, Eq. (5) cannot be satisfied by the vector-like partition function (6). For nontrivial invertible $2+1$D topological order, although the bulk has only the trivial type-$\mathbf{1}$ excitations, the chiral central charge is nonzero, and the $\mathbf{1}$-condensed boundary is always gapless. Thus the $\mathbf{1}$-condensed boundaries of nontrivial $2+1$D topological orders are always gapless.

A similar argument is expected to also work in higher dimensions. The $\mathbf{1}$-condensed boundaries of nontrivial noninvertible topological orders are always gapless. However, in higher dimensions, the $\mathbf{1}$-condensed boundaries of nontrivial invertible topological orders can be gapped, such as the $w_2 w_3$ invertible topological order in $4+1$D [99–108].

D. Canonical boundary

To describe the gapless $\mathbf{1}$-condensed boundaries more precisely, we need to introduce a notion of *local-low-energy equivalence*. Consider a low energy theory $\mathcal{L}$. We can obtain another low energy theory $\mathcal{L}'$ by stacking a gapped state to $\mathcal{L}$. If the gapped state has a nontrivial topological order, the two low energy theories $\mathcal{L}$ and $\mathcal{L}'$ can have different global properties, such as different ground state degeneracies, and different averages of noncontractible loop operators, etc. However, the two low energy theories have the same local correlations for all corresponding local operators (beyond the correlation length of the gapped state). In this case, we say that the two low energy theories $\mathcal{L}$ and $\mathcal{L}'$ are local-low-energy equivalent. Now we can say that the $\mathcal{A}$-condensed boundary of $\mathcal{M}$ is local-low-energy equivalent to the $\mathbf{1}$-condensed boundary of $\mathcal{M}_{/\mathcal{A}}$.

The above discussion leads to the following result. For an $\mathcal{A}$ state in an $\mathcal{M}$ system, there exist a $\mathbf{1}$-condensed boundary state of the $\mathcal{A}$ condensation-induced topological order $\mathcal{M}_{/\mathcal{A}}$, such that the two states are local-low-energy equivalent. Let us note here that there can be many $\mathcal{A}$ states with different

[13]Two systems have the same local low energy properties if there is a correspondence of the local symmetric operators in the two systems, such that the corresponding operators have the same correlation function.

local low energy properties. Similarly, topological order $\mathcal{M}_{/\mathcal{A}}$ can have many different **1**-condensed boundary states with different local low energy properties. What we try to say is that there is an one-to-one correspondence between the $\mathcal{A}$ states and the **1**-condensed boundary states of $\mathcal{M}_{/\mathcal{A}}$, such that the corresponding states have identical local low energy properties. This can be rephrased as $\mathcal{A}$ states in the $\mathcal{M}$ system are local low energy equivalent to **1** states in an $\mathcal{M}_{/\mathcal{A}}$ system.

Some of these states are more stable if they have fewer low energy excitations. Here, we assume that the gapless excitations all have linear dispersion relations. When the velocity of the gapless excitations are all the same, the number low energy excitations can be determined by specific heat. The states with minimal number of low energy excitations are most stable. The most stable **1**-condensed boundaries of $\mathcal{M}$ are called the *canonical boundaries* of $\mathcal{M}$. The local low energy properties of the most stable $\mathcal{A}$ state is same as the local low energy properties of the canonical boundary state of $\mathcal{M}_{/\mathcal{A}}$. We remark that one can also use the number of symmetric relevant operators to define a different notion of "most-stable."

As we have mentioned above that the $\mathcal{A}$ states are not unique. We usually look for the most stable states among the $\mathcal{A}$ states. Note that this aspect is not so different from the notion of "spontaneous $\mathbb{Z}_2$ symmetry breaking state" which does not really refer to a unique state, since we can always stack a gapless $\mathbb{Z}_2$ symmetric state to it while still preserving the fact that the $\mathbb{Z}_2$ symmetry is spontaneously broken. However, the term "spontaneous $\mathbb{Z}_2$ symmetry breaking state" usually refers to the most stable state among these various possible spontaneous $\mathbb{Z}_2$ symmetry breaking states. We use the term $\mathcal{A}$ state in an analogous fashion.

E. Holographic modular bootstrap approach

If a $2+1$D topological order $\mathcal{M}_{/\mathcal{A}}$ is nontrivial, there is an algebraic number theoretical way, also called holographic modular bootstrap (holoMB) approach [48,94,95,109], to determine its gapless boundaries. HoloMB is a generalization of the conventional modular bootstrap [110],

$$Z(\tau) = Z(\tau+1) = Z(-1/\tau), \tag{7}$$

in the sense that holoMB requires additional input data, symmetry TO, that describes (generalized) symmetry. The generalization is given by Eq. (5), and we want to determine the vector-valued partition function from these conditions.

Equation (5) describes a set of algebraic equations. In general, one cannot determine unknown functions $\boldsymbol{Z}(\tau)$ from algebraic equations. However, here a partition function for a given anyon type is the partition function in a certain symmetry charge sector for Hamiltonian with a certain symmetry-twist boundary condition (i.e., in a certain symmetry defect sector):

$$\begin{aligned}&Z_{\text{symm. charge/defect}}(\tau)|_{\text{size } L}\\ &\quad\stackrel{\text{def}}{=} \text{Tr}_{\text{symm. charge}}\, \mathrm{e}^{-\text{Im}(\tau)Lv^{-1}\hat{H}_{\text{symm. defect}}+\mathrm{i}\,\text{Re}(\tau)L\hat{P}},\end{aligned} \tag{8}$$

where L is the size of a one-dimensional ring and v is the velocity of our $1+1$D system. Such a partition function has the form

$$\begin{aligned}Z_a(\tau) &= q^{h_a-c/24}\bar{q}^{\bar{h}_a-\bar{c}/24}\text{Poly}^{\text{nonneg-int}}_{h_a,\bar{h}_a}(q,\bar{q}),\\ q &= \mathrm{e}^{2\pi\,\mathrm{i}\,\tau} \sim \mathrm{e}^{-\beta E},\end{aligned} \tag{9}$$

where h_a, $\bar{h}_a$, c, and $\bar{c}$ are rational numbers, and $\text{Poly}^{\text{nonneg-int}}_{h,\bar{h}}$ is a polynomial of q and $\bar{q}$ with *nonnegative integral coefficients*. In fact, the nonnegative integral coefficients are degeneracies of energy-momentum levels. It appears that the modular covariance conditions (5) can largely determine partition functions that satisfy the "nonnegative-integer" constraint. We note that holomorphic modular bootstrap was developed to solve similar problems [111,112]. Here, we will use a different approach. A $1+1$D gapless boundary conformal field theory (CFT) contains right and left movers, described by conformal characters $\chi^R_i(\tau)$ and $\bar{\chi}^L_j(\bar{\tau})$. Under the modular transformation, the conformal characters transform as

$$\begin{aligned}\widetilde{T}^{ij}_R\chi^R_j(\tau) &= \chi^R_i(\tau+1), & \widetilde{S}^{ij}_R\chi^R_j(\tau) &= \chi^R_i(-1/\tau),\\ \widetilde{T}^{ij}_L\bar{\chi}^L_j(\bar{\tau}) &= \bar{\chi}^L_i(\bar{\tau}+1), & \widetilde{S}^{ij}_L\bar{\chi}^L_j(\bar{\tau}) &= \bar{\chi}^L_i(-1/\bar{\tau}).\end{aligned} \tag{10}$$

The multicomponent partition function for the gapless boundary of $\mathcal{M}_{/\mathcal{A}}$ is given by

$$Z^{\mathcal{M}_{/\mathcal{A}}}_a(\tau) = A^{a,i,j}\chi^R_i(\tau)\bar{\chi}^L_j(\bar{\tau}), \quad A^{a,i,j}\in\mathbb{N}. \tag{11}$$

The modular covariance of $Z^{\mathcal{M}_{/\mathcal{A}}}_a(\tau)$ takes a form

$$\begin{aligned}\mathrm{e}^{-\mathrm{i}\,2\pi\frac{c-}{24}}T^{ab}_{\mathcal{M}_{/\mathcal{A}}}Z^{\mathcal{M}_{/\mathcal{A}}}_b(\tau) &= Z^{\mathcal{M}_{/\mathcal{A}}}_a(\tau+1),\\ D^{-1}S^{ab}_{\mathcal{M}_{/\mathcal{A}}}Z^{\mathcal{M}_{/\mathcal{A}}}_b(\tau) &= Z^{\mathcal{M}_{/\mathcal{A}}}_a(-1/\tau),\end{aligned} \tag{12}$$

where $S_{\mathcal{M}_{/\mathcal{A}}}$ and $T_{\mathcal{M}_{/\mathcal{A}}}$ are the S and T matrices characterizing the bulk topological order $\mathcal{M}_{/\mathcal{A}}$. They constitute the additional input, describing the symmetry TO required in the holoMB approach. Equation (12) can be satisfied if nonnegative integers $A^{a,i,j}$ satisfy

$$\begin{aligned}\mathrm{e}^{-\mathrm{i}\,2\pi\frac{c-}{24}}T^{ab}_{\mathcal{M}_{/\mathcal{A}}}\widetilde{T}^{*ij}_R\widetilde{T}^{*kl}_L A^{b,j,l} &= A^{a,i,k},\\ D^{-1}S^{ab}_{\mathcal{M}_{/\mathcal{A}}}\widetilde{S}^{*ij}_R\widetilde{S}^{*kl}_L A^{b,j,l} &= A^{a,i,k},\end{aligned} \tag{13}$$

or more compactly

$$\begin{aligned}\mathrm{e}^{-\mathrm{i}\,2\pi\frac{c-}{24}}T_{\mathcal{M}_{/\mathcal{A}}}\otimes\widetilde{T}^*_R\otimes\widetilde{T}^*_L\boldsymbol{A} &= \boldsymbol{A},\\ D^{-1}S_{\mathcal{M}_{/\mathcal{A}}}\otimes\widetilde{S}^*_R\otimes\widetilde{S}^*_L\boldsymbol{A} &= \boldsymbol{A},\end{aligned} \tag{14}$$

where we have used the fact that the S and T matrices are symmetric unitary matrices. Comparing Eq. (5) (for gapped boundary where $\mathbb{Z}_a$ are τ independent nonnegative integers) and Eq. (14), we see that the mathematical method to solve for gapped and gapless boundaries are the same. We just need to start with different S and T matrices. In Appendix C, we will describe in more details an algebraic number theoretical method to find nonnegative integer solutions of Eqs. (5) and (14). Appendix C also obtains many additional conditions on Z_a and $A^{a,i,j}$ [see Eq. (C17)].

From the multicomponent partition $Z^{\mathcal{M}_{/\mathcal{A}}}_a(\tau)$, we can obtain the scaling dimensions of operators that carry various representations of the symmetry. Thus the symmetry TO in Symm/TO correspondence allows us to compute properties of gapless state via an algebraic number theoretical method.

To summarize, using Symm/TO correspondence, the properties of gapped and gapless states in systems with (generalized) symmetry can be studied by (1) identifying the corresponding symmetry TO $\mathcal{M}$ that describes the symmetry, (2) computing the condensable algebras $\mathcal{A}$ of $\mathcal{M}$, which classify different reductions of the symmetry TO $\mathcal{M}$ (the reduced symmetry TO is denoted as $\mathcal{M}_{/\mathcal{A}}$, which is analogous to the notion of unbroken symmetry, see Sec. II B), and (3) describing the boundaries induced by condensing $\mathcal{A}$ using holoMB, which correspond to different gapped or gapless states (called $\mathcal{A}$ states) for a given reduced symmetry TO $\mathcal{M}_{/\mathcal{A}}$.

F. From structure of condensable algebra to structure of phase diagram

We have grouped the gapped and gapless states of $\mathcal{M}$ systems into classes labeled by condensable algebras of $\mathcal{M}$. The states in each class labeled by $\mathcal{A}$ are called $\mathcal{A}$ states. If there is a continuous transition between an $\mathcal{A}_1$ state and an $\mathcal{A}_2$ state, the critical point at the transition will be described by an $\mathcal{A}_{12}$ state. The condensable algebra $\mathcal{A}_{12}$ must be a sub algebra of both condensable algebra $\mathcal{A}_1$ and $\mathcal{A}_2$:

$$\mathcal{A}_{12} \subset \mathcal{A}_1, \quad \mathcal{A}_{12} \subset \mathcal{A}_2. \tag{15}$$

This is because as we approach the phase transition boundary, some anyons have increasingly weak affinity to condense. The condensation is absent at the transition and the condensable algebra becomes smaller. For more details, see Appendix B.

To obtain more constraints on the phase diagram from condensable algebras, we introduce a concept of *competing pair*: a pair of anyons (a, b) form a competing pair if they never appear in the same condensable algebra together, but they can appear in condensable algebras separately. In other words, the anyons in a competing pair can condense, but cannot condense together (usually due to the nontrivial mutual statistics between them). Condensing one anyon in a competing pair will uncondense the other. We propose that continuous phase transition is driven by condensing one anyon and uncondensing the other anyon in a competing pair. This implies that if $\mathcal{A}_1$ state and $\mathcal{A}_2$ state are connected by a continuous transition, then the union of $\mathcal{A}_1$ and $\mathcal{A}_2$ should contain a competing pair. If the union of $\mathcal{A}_1$ and $\mathcal{A}_2$ should contain only one competing pair, then the transition is more likely to be stably continuous. For more details, see Appendix B and Sec. V E.

III. 1 + 1D $\mathbb{Z}_2 \times \mathbb{Z}_2'$ SYMMETRY

Let's illustrate the general discussion of the previous section with the example of a $1+1$D system with $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry. Landau's symmetry-breaking framework tells us that the system can spontaneously break the symmetry down to various subgroups of the symmetry group, producing various gapped states.

This symmetry breaking picture can be also be viewed through the lens of symmetry TO. In the same way that $\mathbb{Z}_2$ gauge theory, denoted by $\mathcal{G}au_{\mathbb{Z}_2}$, is the symmetry TO of $\mathbb{Z}_2$ symmetry, for systems with $\mathbb{Z}_2 \times \mathbb{Z}_2'$ symmetry,[14] the symmetry TO is $\mathcal{G}au_{\mathbb{Z}_2\times\mathbb{Z}_2'}$, which refers to the $2+1$D topological order described by $\mathbb{Z}_2 \times \mathbb{Z}_2'$ gauge theory with charge and flux excitations. There are two e anyons (charges), e_1 and e_2, and two m anyons (fluxes), m_1 and m_2, that generate all of the 16 anyons of $\mathcal{G}au_{\mathbb{Z}_2\times\mathbb{Z}_2'}$. The symmetry TO $\mathcal{G}au_{\mathbb{Z}_2\times\mathbb{Z}_2'}$ makes the mod 2 conservation of the flux excitations m_1 and m_2 explicit—this may also be described by the dual symmetry $\widetilde{\mathbb{Z}}_2 \times \widetilde{\mathbb{Z}}_2'$ [49]. To emphasize the dual symmetry, one may denote this symmetry TO as $(\mathbb{Z}_2 \times \mathbb{Z}_2') \vee (\widetilde{\mathbb{Z}}_2 \times \widetilde{\mathbb{Z}}_2')$. We will drop the discussion of dual symmetry in the following for brevity.

Let's consider the possible gapped phases of a $1+1$D system with $\mathbb{Z}_2 \times \mathbb{Z}_2'$ symmetry from the conventional point of view first. We will then translate that into the symmetry TO language.

The gapped phases in $1+1$D associated to symmetry group G are classified by the unbroken subgroup H, and possible SPT phases of H [83,113,114]. For $G = \mathbb{Z}_2 \times \mathbb{Z}_2'$, the four nontrivial symmetry-breaking gapped phases are associated to its four proper subgroups $\mathbb{Z}_1$, $\mathbb{Z}_2$, $\mathbb{Z}_2'$, $\mathbb{Z}_2^d$, where $\mathbb{Z}_2^d$ is the "diagonal" $\mathbb{Z}_2$ subgroup. If we present the group $\mathbb{Z}_2 \times \mathbb{Z}_2'$ as $\{(0,0), (0,1), (1,0), (1,1)\}$, then these subgroups are

$$\begin{aligned}
\mathbb{Z}_1 &\simeq \{(0,0)\},\\
\mathbb{Z}_2 &\simeq \{(0,0), (1,0)\},\\
\mathbb{Z}_2' &\simeq \{(0,0), (0,1)\},\\
\mathbb{Z}_2^d &\simeq \{(0,0), (1,1)\}.
\end{aligned}$$

There are no nontrivial $\mathbb{Z}_2$ SPT phases in $1+1$D. However, there is a nontrivial $\mathbb{Z}_2 \times \mathbb{Z}_2'$ SPT phase, the so-called cluster state. So there are a total of six gapped phases. Continuous phase transitions between the symmetry breaking states is straightforward within Landau theory. The transitions between the trivial $\mathbb{Z}_2 \times \mathbb{Z}_2'$ paramagnet phase and the three $\mathbb{Z}_2$ symmetric phase are Ising transitions. The remaining symmetry in the three $\mathbb{Z}_2$ symmetric phases can further spontaneously break via a second Ising transition to reach the $\mathbb{Z}_1$ symmetric phase. In Landau theory, a direct continuous transition between different $\mathbb{Z}_2$-SSB phases is not a possibility since there is no group-subgroup relation between such pairs. The nontrivial SPT, cluster state, also has Ising transitions to the $\mathbb{Z}_2$, $\mathbb{Z}_2'$, $\mathbb{Z}_2^d$ symmetric phases while a direct continuous transition to the $\mathbb{Z}_1$ symmetric phase is not generically possible without fine tuning. Transition from the cluster state to the trivial paramagnet proceeds via an XY-type critical point as was shown by Kramers Wannier transformation in Ref. [115].

Let us now phrase the above discussion in terms of symmetry TO. The symmetry TO of the symmetry group $\mathbb{Z}_2 \times \mathbb{Z}_2'$ is $\mathcal{G}au_{\mathbb{Z}_2\times\mathbb{Z}_2'}$, with the following Lagrangian condensable algebras:

$$\begin{aligned}
&\mathbf{1} \oplus e_1 \oplus e_2 \oplus e_1e_2, \quad \mathbf{1} \oplus e_1 \oplus m_2 \oplus e_1m_2,\\
&\mathbf{1} \oplus m_1 \oplus e_2 \oplus e_2m_1, \quad \mathbf{1} \oplus m_1 \oplus m_2 \oplus m_1m_2,\\
&\mathbf{1} \oplus m_1m_2 \oplus e_1e_2 \oplus f_1f_2, \quad \mathbf{1} \oplus e_2m_1 \oplus e_1m_2 \oplus f_1f_2,
\end{aligned} \tag{16}$$

where $f_1 = e_1 \otimes m_1 = e_1m_1$ and $f_2 = e_2 \otimes m_2 = e_2m_2$. These correspond to gapped boundaries of $\mathcal{G}au_{\mathbb{Z}_2\times\mathbb{Z}_2'}$ and, by our Symm/TO correspondence, to the six gapped phases discussed above. The gapped boundary $\mathbf{1} \oplus m_1 \oplus m_2 \oplus m_1m_2$

[14]The prime on the second $\mathbb{Z}_2$ is used just to explicitly differentiate between the two $\mathbb{Z}_2$ groups for purpose of identification.

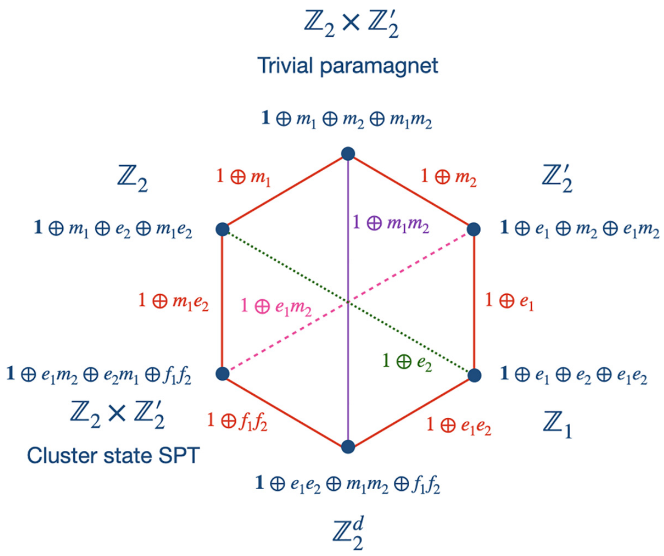


FIG. 3. A $\mathbb{Z}_2 \times \mathbb{Z}_2'$ symmetric system has a symmetry described by $2+1$D $\mathbb{Z}_2 \times \mathbb{Z}_2'$ topological order $\mathcal{G}\mathrm{au}_{\mathbb{Z}_2\times\mathbb{Z}_2'}$. Here, the six Lagrangian condensable algebras of $\mathcal{G}\mathrm{au}_{\mathbb{Z}_2\times\mathbb{Z}_2'}$ are represented by the vertices of the hexagon. The gapped phases they correspond to are described in terms of their symmetry-breaking/SPT order. A connecting edge between any pair represents a non-Lagrangian condensable algebra, which is strictly included in both of them, and therefore corresponds to a phase transition between them. There is also a trivial condensable algebra $\mathbf{1}$, which is not shown in this picture; it corresponds to a multicritical point between any two gapped phases. The inclusion relations between condensable algebras have implication on the structure of phase diagram. For example, the edge labeled by $\mathbf{1} \oplus m_2$ connecting vertices labeled by $\mathbf{1} \oplus m_1 \oplus m_2 \oplus m_1 m_2$ and $\mathbf{1} \oplus e_1 \oplus m_2 \oplus e_1 m_2$ suggests that the former (gapless) $\mathbf{1} \oplus m_2$ state describes a critical point for the stable continuous transition between the latter two gapped states. For this transition to be nonfine-tuned, the gapless $\mathbf{1} \oplus m_2$ state must have only one symmetric relevant operator, which is indeed the case here (see main text for details).

condenses the two m anyons, m_1 and m_2. This phase preserves the $\mathbb{Z}_2 \times \mathbb{Z}_2'$ symmetry since the $\mathbb{Z}_2$ and $\mathbb{Z}_2'$ charges e_1, e_2 remain uncondensed (see the top vertex of Fig. 3). This is the trivial paramagnet phase. The $\mathbf{1} \oplus e_1 \oplus m_2 \oplus e_1 m_2$-condensed boundary corresponds to a $\mathbb{Z}_2'$ symmetric phase since the $\mathbb{Z}_2'$ charge e_2 is uncondensed while the $\mathbb{Z}_2$ charge e_1 is condensed. The $\mathbf{1} \oplus m_1 m_2 \oplus e_1 e_2 \oplus f_1 f_2$ preserves $\mathbb{Z}_2^d$, the diagonal $\mathbb{Z}_2$ symmetry. To see this, note that $e_1 e_2$ is charged under both $\mathbb{Z}_2$ and $\mathbb{Z}_2'$ while it is symmetric under the action of $\mathbb{Z}_2^d$. As a result, condensing $e_1 e_2$ must break both $\mathbb{Z}_2$ and $\mathbb{Z}_2'$ but not $\mathbb{Z}_2^d$. The fact that $m_1 m_2$ is condensed amounts to the same conclusion: we recall that, for a single $\mathbb{Z}_2$ symmetry, condensation of m corresponds to proliferating the disorder operator and hence preserving the $\mathbb{Z}_2$ symmetry. Therefore condensation of $m_1 m_2$ corresponds to preserving the $\mathbb{Z}_2^d$ symmetry. The $\mathbf{1} \oplus e_1 \oplus e_2 \oplus e_1 e_2$-condensed boundary corresponds to a $\mathbb{Z}_2 \times \mathbb{Z}_2'$-SSB phase since both e_1 and e_2 are condensed. The $\mathbf{1} \oplus e_2 m_1 \oplus e_1 m_2 \oplus f_1 f_2$-condensed boundary corresponds to a $1+1$D SPT phase, since none of the $\mathbb{Z}_2 \times \mathbb{Z}_2'$ charges condense and thus $\mathbb{Z}_2 \times \mathbb{Z}_2'$ symmetry is not broken. This corresponds to the SPT state. In fact, we note that this condensable algebra actually involves a proliferation of decorated domain walls [116] since the disorder operator of $\mathbb{Z}_2$, corresponding to m_1, is bound to the charge of $\mathbb{Z}_2'$, corresponding to e_2, and vice versa. In Appendix D, we show that the $\mathbf{1} \oplus e_2 m_1 \oplus e_1 m_2 \oplus f_1 f_2$-condensed boundary is associated with an automorphism in the symmetry TO, which also indicates that $\mathbf{1} \oplus e_2 m_1 \oplus e_1 m_2 \oplus f_1 f_2$-condensed boundary gives rise to an SPT state.

Let us now discuss the possible phase transitions between these gapped phases. Going from the $\mathbf{1} \oplus m_1 \oplus m_2 \oplus m_1 m_2$ phase to the $\mathbf{1} \oplus e_1 \oplus m_2 \oplus e_1 m_2$ phase, the system encounters a continuous phase transition which, in the holographic picture, corresponds to uncondensing m_1 and condensing e_1. In conventional language, this is a phase transition between a $\mathbb{Z}_2 \times \mathbb{Z}_2'$ symmetric phase to a $\mathbb{Z}_2'$-symmetric phase. At this phase transition, only $\mathbf{1} \oplus m_2$ remain condensed. Both e_1, m_1 are uncondensed as well as inequivalent with respect to the condensed particles, which makes the corresponding boundary theory impossible to be gapped, due to their nontrivial mutual statistics. This serves as an argument that indeed the system becomes gapless at this phase transition, i.e., this phase transition is continuous (cf. the top right edge of Fig. 3).

In an analogous manner, one can describe the phase transition from the $\mathbb{Z}_2 \times \mathbb{Z}_2'$-symmetric phase to a $\mathbb{Z}_2$-symmetric phase as uncondensing m_2 and condensing e_2. At the phase transition, only $\mathbf{1} \oplus m_1$ are condensed (cf. the top left edge of Fig. 3). On the other hand, the phase transition from the $\mathbb{Z}_2 \times \mathbb{Z}_2'$ symmetric phase to the $\mathbb{Z}_2^d$ symmetric phase corresponds to uncondensing m_1 and m_2 and condensing $e_1 e_2$. At the phase transition, $\mathbf{1} \oplus m_1 m_2$ are condensed.

At all the phase transitions discussed so far, the condensed anyons form a non-Lagrangian condensable algebra. This is intimately connected with the gaplessness of the critical points, as described in the previous section. Given a symmetry TO, one can in principle obtain all the gapped boundaries that it can support by searching for Lagrangian condensable algebras. The phase transitions between such gapped phases are described by various non-Lagrangian condensable algebras. The example of the $\mathbb{Z}_2 \times \mathbb{Z}_2'$ symmetry discussed here gives us a simple example of such an analysis. The minimal condensable algebra is just $\mathbf{1}$, i.e., condensation of the trivial anyon. Besides this, there are nine other non-Lagrangian condensable algebras which correspond to the various phase transitions between gapped phases associated to the Lagrangian condensable algebras in Eq. (16),

$$\begin{aligned} &\mathbf{1} \oplus e_1, \;\; \mathbf{1} \oplus e_2, \;\; \mathbf{1} \oplus m_1, \;\; \mathbf{1} \oplus m_2, \\ &\mathbf{1} \oplus e_1 e_2, \;\; \mathbf{1} \oplus m_1 m_2, \;\; \mathbf{1} \oplus e_1 m_2, \;\; \mathbf{1} \oplus m_1 e_2, \;\; \mathbf{1} \oplus f_1 f_2. \end{aligned} \tag{17}$$

We already discussed the phase transitions corresponding to the non-Lagrangian condensable algebras $\mathbf{1} \oplus m_1$, $\mathbf{1} \oplus m_2$ and $\mathbf{1} \oplus m_1 m_2$ above. The condensable algebra $\mathbf{1} \oplus e_1$ corresponds to the transition from the $\mathbf{1} \oplus e_1 \oplus m_2 \oplus e_1 m_2$ phase to the $\mathbf{1} \oplus e_1 \oplus e_2 \oplus e_1 e_2$ phase (see the right edge of Fig. 3). Similarly, $\mathbf{1} \oplus e_2$ corresponds to the transition from the $\mathbf{1} \oplus m_1 \oplus e_2 \oplus e_2 m_1$ phase to the $\mathbf{1} \oplus e_1 \oplus e_2 \oplus e_1 e_2$ phase (see the left edge of Fig. 3). On the other hand, $\mathbf{1} \oplus e_1 e_2$ corresponds to the transition from the $\mathbf{1} \oplus e_1 e_2 \oplus m_1 m_2 \oplus f_1 f_2$ phase to the $\mathbf{1} \oplus e_1 \oplus e_2 \oplus e_1 e_2$ phase (see the bottom right edge of Fig. 3). This is a $\mathbb{Z}_2^d$ breaking phase transition in the conventional language.

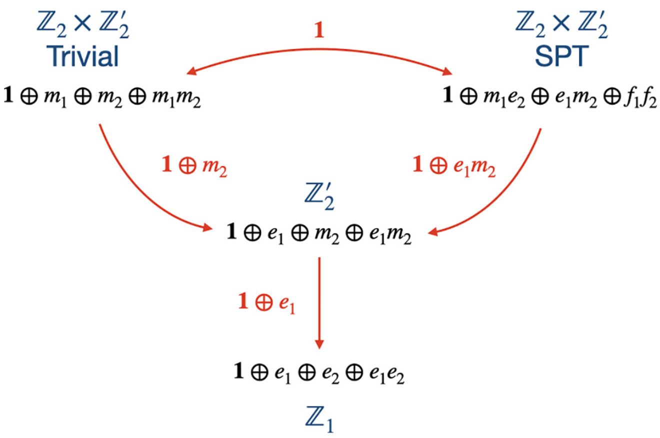


FIG. 4. The arrows denote possible continuous phase transitions. They are labeled by the associated non-Lagrangian condensable algebras that describe the phase transition. The three lower arrows depict possible symmetry breaking cascades from the two $\mathbb{Z}_2\times\mathbb{Z}_2'$-symmetric states, the trivial paramagnet and the cluster state SPT. By replacing $\mathbb{Z}_2'$-symmetric state in the middle of the cascade by $\mathbb{Z}_2$- and $\mathbb{Z}_2^d$-symmetric state, we can obtain two other such symmetry breaking cascades. The top arrow labeled by the minimal condensable algebra $\mathbf{1}$ represents a family of possible continuous quantum phase transition between the trivial symmetric phase and the nontrivial SPT phase [117]. Note that this condensable algebra can also describe fine-tuned continuous phase transitions between any two of the six gapped phases of this system.

Since the minimal condensable algebra $\mathbf{1}$ is the intersection of every pair of Lagrangian condensable algebras, it corresponds to a direct, fine-tuned, phase transition between any two gapped phases. In other words, it describes a multicritical point, e.g., see the top portion of Fig. 4. The gapless $\mathbf{1}$ state is given by the canonical boundary of $(\mathcal{G}\mathrm{au}_{\mathbb{Z}_2\times\mathbb{Z}_2'})_{/\mathbf{1}} = \mathcal{G}\mathrm{au}_{\mathbb{Z}_2\times\mathbb{Z}_2'}$, which are $(c,\bar{c})=(1,1)u(1)$ CFT's. The other nine non-Lagrangian condensable algebras give rise to gapless states that correspond to canonical boundary of $\mathcal{G}\mathrm{au}_{\mathbb{Z}_2}$, since for all of these condensable algebras $\mathcal{A}$, we have

$$\left(\mathcal{G}\mathrm{au}_{\mathbb{Z}_2\times\mathbb{Z}_2'}\right)_{/\mathcal{A}} = \mathcal{G}\mathrm{au}_{\mathbb{Z}_2}. \tag{18}$$

These nine gapless states are therefore all described by $(c,\bar{c})=(\frac{1}{2},\frac{1}{2})$ Ising CFT's (cf. the discussion in Ref. [48]). However, these gapless states are distinct, despite being described by the same CFT, since the assignment of symmetry quantum numbers to the excitations in the CFT is different for each of them.

We see that $1+1$D $\mathbb{Z}_2\times\mathbb{Z}_2'$ symmetric systems can only have two types of "stable" gapless states: $(c,\bar{c})=(1,1)\ u(1)$ CFTs and $(c,\bar{c})=(\frac{1}{2},\frac{1}{2})$ Ising CFTs. From the structure of the condensable algebras, we see that (cf. Fig. 3) the trivial $\mathbb{Z}_2\times\mathbb{Z}_2'$-SPT state (the $\mathbf{1}\oplus m_1\oplus m_2\oplus m_1m_2$ state) and the nontrivial $\mathbb{Z}_2\times\mathbb{Z}_2'$-SPT state (the $\mathbf{1}\oplus e_2m_1\oplus e_1m_2\oplus f_1f_2$ state) can only be connected by the gapless $\mathbf{1}$ state, i.e., by $(c,\bar{c})=(1,1)\ u(1)$ CFTs, since the overlap of the two condensable algebras $\mathbf{1}\oplus m_1\oplus m_2\oplus m_1m_2$ and $\mathbf{1}\oplus e_2m_1\oplus e_1m_2\oplus f_1f_2$ is given by $\mathbf{1}$. This is consistent with the conclusions of Ref. [115]. On the other hand, the nontrivial $\mathbb{Z}_2\times\mathbb{Z}_2'$-SPT state, the $\mathbf{1}\oplus e_2m_1\oplus e_1m_2\oplus f_1f_2$ state, can be connected by gapless states of $(c,\bar{c})=(\frac{1}{2},\frac{1}{2})$ Ising CFT to each of the symmetry breaking states: $\mathbf{1}\oplus m_1m_2\oplus e_1e_2\oplus f_1f_2$, $\mathbf{1}\oplus m_1\oplus e_2\oplus m_1e_2$, and $\mathbf{1}\oplus e_1\oplus m_2\oplus e_1m_2$. The condensable algebras for the corresponding gapless states are $\mathbf{1}\oplus f_1f_2$, $\mathbf{1}\oplus m_1e_2$, and $\mathbf{1}\oplus e_1m_2$, respectively. See also Refs. [118,119] for a different holographic theory for the phase transitions between SPT phases.

To summarize, in Fig. 3, the six Lagrangian condensable algebras (and corresponding gapped phases) are shown along with the nine nontrivial non-Lagrangian condensable algebras. The vertices correspond to the various gapped phases, while the edges describe gapless states of the $1+1$D theory. An edge that connects to a pair of vertices is understood to be describing the gapless critical theory that mediates a phase transition between the two gapped phases. The trivial condensable algebra $\mathbf{1}$ can always mediate a multicritical phase transition between any pair of gapped phases, as noted above. Hence it is not shown in the figure.

In the next section, we contrast this discussion with a $\mathbb{Z}_2\times\mathbb{Z}_2'$ symmetry that has a mixed anomaly. The symmetry TO of such a symmetry is distinct from that of the anomaly-free $\mathbb{Z}_2\times\mathbb{Z}_2'$ symmetry. As was discussed in Ref. [33], the symmetry TO of $\mathbb{Z}_2\times\mathbb{Z}_2'$ symmetry with mixed anomaly is $\mathcal{G}\mathrm{au}_{\mathbb{Z}_4}$. As a result the entire discussion of gapped boundaries and condensable algebras will be completely different from the anomaly-free case.

IV. $1+1$D $\mathbb{Z}_2\times\mathbb{Z}_2'$ SYMMETRY WITH MIXED ANOMALY

In our third example, we consider anomalous $\mathbb{Z}_2\times\mathbb{Z}_2'$ symmetry in $1+1$D. Such an anomaly is characterized by a cocycle ω in $H^3(\mathbb{Z}_2\times\mathbb{Z}_2';\mathbb{R}/\mathbb{Z})=\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$. The middle $\mathbb{Z}_2$ describes the mixed anomaly between the $\mathbb{Z}_2$ and $\mathbb{Z}_2'$ groups. The first and the last $\mathbb{Z}_2$ describe the self anomaly of the $\mathbb{Z}_2$ and the $\mathbb{Z}_2'$ groups, respectively. Thus we can use (m_1, m_{12}, m_2) to label different cocycles ω. We denote an anomalous $\mathbb{Z}_2\times\mathbb{Z}_2'$ symmetry as $(\mathbb{Z}_2\times\mathbb{Z}_2')^\omega=(\mathbb{Z}_2\times\mathbb{Z}_2')^{(m_1m_{12}m_2)}$.

$1+1$D systems with $(\mathbb{Z}_2\times\mathbb{Z}_2')^{(010)}$ symmetry have a gapped state with only the $\mathbb{Z}_2$-symmetry, a gapped state with only the $\mathbb{Z}_2'$ symmetry, and a third gapped state that breaks both the $\mathbb{Z}_2$ and the $\mathbb{Z}_2'$ symmetry. However, there is no gapped state with both the $\mathbb{Z}_2$ and the $\mathbb{Z}_2'$ symmetry due to the anomaly [57]. A state that has the full $\mathbb{Z}_2\times\mathbb{Z}_2'$ symmetry unbroken must be gapless. Such a gapless state happens to be the critical point for the continuous transition between the two gapped states with unbroken $\mathbb{Z}_2$ and unbroken $\mathbb{Z}_2'$ symmetry respectively. Noting that $\mathbb{Z}_2$ and $\mathbb{Z}_2'$ are not related by a group-subgroup relation, we see that this is an example of a continuous phase transition that is beyond the conventional Landau theory of phase transitions.

The critical point with the $\mathbb{Z}_2\times\mathbb{Z}_2'$ symmetry also has other symmetries. The full symmetry of the critical point is described by the symmetry TO $\mathcal{G}\mathrm{au}^{(010)}_{\mathbb{Z}_2\times\mathbb{Z}_2}$, which refers to a $\mathbb{Z}_2\times\mathbb{Z}_2'$ twisted quantum double or a $\mathbb{Z}_2\times\mathbb{Z}_2'$ Dijkgraaf-Witten (DW) gauge theory [120]. The $\mathbb{Z}_2$ symmetry corresponds to the $\mathbb{Z}_2$-gauge charge conservation in the DW theory, while the $\mathbb{Z}_2'$ symmetry corresponds to the $\mathbb{Z}_2'$-gauge charge conservation in the DW theory.

The $\mathcal{G}au^{(010)}_{\mathbb{Z}_2\times\mathbb{Z}_2}$ DW theory also have $\mathbb{Z}_2$ and $\mathbb{Z}_2'$ gauge flux, whose conservation give rise to additional symmetries at the critical point.

In order to discuss the phase diagram and phase transitions of a system with such an anomalous $\mathbb{Z}_2 \times \mathbb{Z}_2'$ symmetry, we will use the fact derived in Refs. [33,121] that the $2+1$D topological order $\mathcal{G}au^{(010)}_{\mathbb{Z}_2\times\mathbb{Z}_2}$ is the same as the $2+1$D topological order $\mathcal{G}au_{\mathbb{Z}_4}$ (i.e., the $\mathbb{Z}_4$ gauge theory with charge excitations). In order words, in $2+1$D, the anomalous $\mathbb{Z}_2 \times \mathbb{Z}_2'$ symmetry and the $\mathbb{Z}_4$ symmetry are equivalent, since they are described by the same symmetry TO. The anyons of $\mathcal{G}au^{(010)}_{\mathbb{Z}_2\times\mathbb{Z}_2'}$ topological order can be mapped to those of $\mathcal{G}au_{\mathbb{Z}_4}$ topological order. This mapping is given as follows:

$$e_1 \to e^2,\;\; e_2 \to m^2,\;\; m_1 \to m,\;\; m_2 \to e, \tag{19}$$

where e and m are the generators of the gauge charge and the gauge flux excitations of $\mathcal{G}au_{\mathbb{Z}_4}$. We argued for this mapping of anyons by studying the patch operators and their associated braided fusion category in Ref. [33]. We found that presence of the mixed anomaly changes the anyon statistics from that described by $\mathcal{G}au_{\mathbb{Z}_2\times\mathbb{Z}_2'}$ to that described by $\mathcal{G}au^{(010)}_{\mathbb{Z}_2\times\mathbb{Z}_2'} = \mathcal{G}au_{\mathbb{Z}_4}$. Supported by this result, we will use the language of $\mathcal{G}au_{\mathbb{Z}_4}$ to describe the phase transitions of a system with $(\mathbb{Z}_2 \times \mathbb{Z}_2')^{(010)}$ symmetry.

Let us first recall what are the different gapped phases that a system with $\mathbb{Z}_4$ symmetry may have from the Ginzburg-Landau mean field theory. Let us introduce two order parameter fields: a complex bosonic field Φ and a real field ϕ. Under the generating transformation of $\mathbb{Z}_4$, they transform as

$$U_{\mathbb{Z}_4}\Phi = e^{i\pi/2}\Phi, \quad U_{\mathbb{Z}_4}\phi = -\phi. \tag{20}$$

Consider the following Ginzburg-Landau functional:

$$F = \int dx\, |\partial\Phi|^2 + u|\Phi|^2 + \frac{1}{2}(\Phi^4 + \text{c.c.}) + 2|\Phi|^4 + |\partial\phi|^2 + w\phi^2 + 2\phi^4 + \frac{1}{2}\phi(\Phi^2 + \text{c.c.}). \tag{21}$$

Since the only subgroups of $\mathbb{Z}_4$ are the trivial group and $\mathbb{Z}_2$, we can have three different gapped phases in total: one with the full $\mathbb{Z}_4$ symmetry ($\Phi = \phi = 0$ when $u, w > 0$), one with unbroken $\mathbb{Z}_2$ symmetry ($\Phi = 0$, $\phi \neq 0$ when $u > 0$, $w < 0$), and one with $\mathbb{Z}_1$ symmetry ($\Phi \neq 0$, $\phi \neq 0$ when $u < 0$) (see Fig. 5).

This shape of the phase boundaries can be understood as follows. When we turn w from positive to negative in the presence of a positive u, we find a minima with nonzero ϕ but still $\Phi = 0$. This is a phase which has an unbroken $\mathbb{Z}_2$ symmetry. The minimum is now at $\phi_* \sim \pm\sqrt{-w}$. This nonzero mean-field value of ϕ turns on the $\phi(\Phi^2 + \text{c.c.})$ term then effectively introduces a modification to the quadratic terms for Φ which is of the form $\sqrt{-w}(\Phi^2 + \text{c.c.})$. Thus we see that for $u < O(\sqrt{-w})$, we transition into a phase with nonzero Φ as well as ϕ. This is the $\mathbb{Z}_1$ symmetric phase. The corresponding phase transition is indicated in green in the bottom right quadrant of Fig. 5. If we are close enough to the phase transition regions, w is small so it is a very good approximation to drop the corresponding higher order terms and only concentrate on the quadratic terms and the $\phi\Phi^2$ term for the mean field phase boundary analysis.

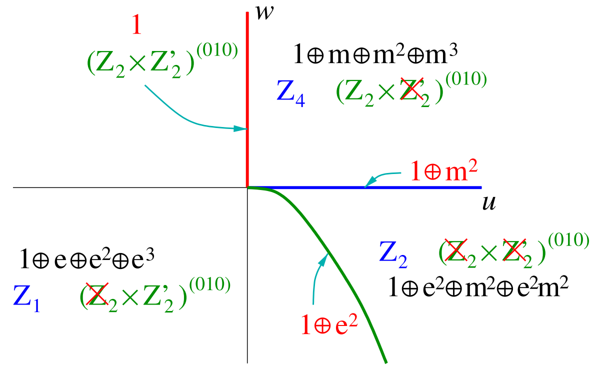


FIG. 5. Mean-field phase diagram for systems with $\mathbb{Z}_4$ symmetry. It has three gapped phases with unbroken $\mathbb{Z}_4$, $\mathbb{Z}_2$, and $\mathbb{Z}_1$ symmetries. The two phase transitions $\mathbb{Z}_4 \to \mathbb{Z}_2$ (marked by $\mathbf{1} \oplus m^2$) and $\mathbb{Z}_2 \to \mathbb{Z}_1$ (marked by $\mathbf{1} \oplus e^2$) are critical points described by the same Ising CFT. The direct phase transition $\mathbb{Z}_4 \to \mathbb{Z}_1$ (marked by $\mathbf{1}$) corresponds to a critical point that does not break the anomalous symmetry $(\mathbb{Z}_2 \times \mathbb{Z}_2')^{(010)}$ and has the full symmetry TO $\mathcal{G}au_{\mathbb{Z}_4}$. It is a critical line that includes the $\mathbb{Z}_4$ parafermion CFT.

Although the two phase transitions $\mathbb{Z}_4 \to \mathbb{Z}_2$ and $\mathbb{Z}_2 \to \mathbb{Z}_1$ correspond to different symmetry breaking pattern, their critical points happen to be described by the same Ising CFT with central charge $(c, \bar{c}) = (\frac{1}{2}, \frac{1}{2})$. The third symmetry breaking pattern $\mathbb{Z}_4 \to \mathbb{Z}_1$ will have a different critical theory. In fact the transition is described by a critical line with central charge $(c, \bar{c}) = (1, 1)$, that includes the $\mathbb{Z}_4$ parafermion CFT [122,123].

Now, from the symmetry TO point of view, the gapped phases of this system are the allowed gapped boundaries of $\mathcal{G}au_{\mathbb{Z}_4}$. Such gapped boundaries are described by the Lagrangian condensable algebras of the symmetry TO $\mathcal{G}au_{\mathbb{Z}_4}$:

$$\mathbf{1} \oplus e \oplus e^2 \oplus e^3, \quad \mathbf{1} \oplus m \oplus m^2 \oplus m^3, \quad \mathbf{1} \oplus e^2 \oplus m^2 \oplus e^2m^2. \tag{22}$$

The Lagrangian condensable algebras match the gapped symmetric and symmetry-breaking phases very well. The first of these condensable algebras, $\mathbf{1} \oplus e \oplus e^2 \oplus e^3$ represents the $\mathbb{Z}_1$ phase that spontaneously breaks the $\mathbb{Z}_4$ symmetry completely. The second, $\mathbf{1} \oplus m \oplus m^2 \oplus m^3$ represents the $\mathbb{Z}_4$-gapped phase that is fully $\mathbb{Z}_4$ symmetric. The last one, $\mathbf{1} \oplus e^2 \oplus m^2 \oplus e^2m^2$ represents the $\mathbb{Z}_2$ gapped phase that breaks $\mathbb{Z}_4$ down to $\mathbb{Z}_2$.

Let us consider these gapped phases in the dual picture with the $(\mathbb{Z}_2 \times \mathbb{Z}_2')^{(010)}$ symmetry. The three gapped phases obtained by breaking $\mathbb{Z}_2$, $\mathbb{Z}_2'$, or $\mathbb{Z}_2 \times \mathbb{Z}_2'$ are denoted as $(\not{\mathbb{Z}}_2 \times \mathbb{Z}_2')^{(010)}$ phase, $(\mathbb{Z}_2 \times \not{\mathbb{Z}}_2')^{(010)}$ phase, $(\not{\mathbb{Z}}_2 \times \not{\mathbb{Z}}_2')^{(010)}$ phase. They have a one-to-one correspondence with the three gapped phases discussed above. How do we identify them? First of all, the condensable algebra $\mathbf{1} \oplus e \oplus e^2 \oplus e^3$ may be written in terms of $\mathbb{Z}_2 \times \mathbb{Z}_2'$ charges and fluxes—see Eq. (19)—as $\mathbf{1} \oplus m_2 \oplus e_1 \oplus e_1m_2$ which indicates a phase that has a broken $\mathbb{Z}_2$ but unbroken $\mathbb{Z}_2'$. Thus the $\mathbb{Z}_1$ phase (in the $\mathbb{Z}_4$ symmetry language) corresponds to the $(\not{\mathbb{Z}}_2 \times \mathbb{Z}_2')^{(010)}$ phase. (see Figs. 5 and 6). Similarly, the condensable algebra $\mathbf{1} \oplus m \oplus m^2 \oplus m^3$ maps on to $\mathbf{1} \oplus m_1 \oplus e_2 \oplus m_1e_2$, which corresponds to the $(\mathbb{Z}_2 \times \not{\mathbb{Z}}_2')^{(010)}$ phase. This phase is mapped to the $\mathbb{Z}_4$

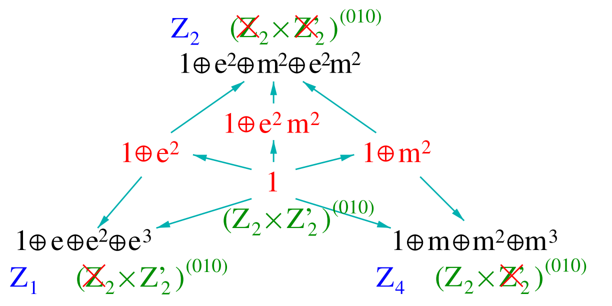


FIG. 6. $\mathbb{Z}_4$-symmetric systems has a symmetry described by $2+1$D $\mathbb{Z}_4$ topological order $\mathcal{G}au_{\mathbb{Z}_4}$ (i.e., a $\mathbb{Z}_4$ gauge theory). The $\mathcal{G}au_{\mathbb{Z}_4}$ topological order has seven condensable algebras and three of them are Lagrangian (the back ones above). They give rise to gapped boundaries. Four of seven are not Lagrangian, and give rise to gapless boundaries. The gapless boundaries may be critical points for the transitions between gapped or gapless boundaries, as indicated above. The arrows indicate the directions of more condensation.

phase. Lastly, the condensable algebra $\mathbf{1}\oplus e^2\oplus m^2\oplus e^2m^2$ maps on to $\mathbf{1}\oplus e_1\oplus e_2\oplus e_1e_2$, which corresponds to the $(\not{\mathbb{Z}}_2\times\not{\mathbb{Z}}_2')^{(010)}$ phase and maps to the $\mathbb{Z}_2$ phase.

Next let us discuss the gapless states that describe the phase transitions of this system. The gapless states are given by condensation patterns described by non-Lagrangian condensable algebras. There are four non-Lagrangian condensable algebras in the $\mathcal{G}au_{\mathbb{Z}_4}$ topological order,

$$\mathbf{1},\quad \mathbf{1}\oplus e^2,\quad \mathbf{1}\oplus m^2,\quad \mathbf{1}\oplus e^2m^2. \tag{23}$$

They map to four non-Lagrangian condensable algebras in the equivalent $\mathcal{G}au^{(010)}_{\mathbb{Z}_2\times\mathbb{Z}_2'}$ topological order

$$\mathbf{1},\quad \mathbf{1}\oplus e_1,\quad \mathbf{1}\oplus e_2,\quad \mathbf{1}\oplus e_1e_2. \tag{24}$$

Thus there are four condensation patterns in the symmetry TO of this system that can give rise to gapless states. We refer to these gapless states as $\mathbf{1}$, $\mathbf{1}\oplus e^2$, $\mathbf{1}\oplus m^2$, and $\mathbf{1}\oplus e^2m^2$ states.

The $1+1$D gapless $\mathbf{1}$ states are given by the canonical boundaries of the $\mathbf{1}$ condensation-induced topological order, which is nothing but the original symmetry TO $\mathcal{G}au_{\mathbb{Z}_4}$. Similarly, $1+1$D gapless $\mathbf{1}\oplus e^2$ and $\mathbf{1}\oplus m^2$ states are given by the canonical boundaries of $(\mathcal{G}au_{\mathbb{Z}_4})_{/\mathbf{1}\oplus e^2}=\mathcal{G}au_{\mathbb{Z}_2}$ and $(\mathcal{G}au_{\mathbb{Z}_4})_{/\mathbf{1}\oplus m^2}=\mathcal{G}au_{\mathbb{Z}_2}$. Last, the $1+1$D gapless $\mathbf{1}\oplus e^2m^2$ state is given by the canonical boundary of $(\mathcal{G}au_{\mathbb{Z}_4})_{/\mathbf{1}\oplus e^2m^2}=\mathcal{M}_{\text{DS}}$, where $\mathcal{M}_{\text{DS}}$ is the double-semion topological order. To see why $(\mathcal{G}au_{\mathbb{Z}_4})_{/\mathbf{1}\oplus e^2m^2}=\mathcal{M}_{\text{DS}}$, we can ask the question: which anyons of $\mathcal{G}au_{\mathbb{Z}_4}$ have trivial mutual statistics with e^2m^2. Out of the 16 anyons of $\mathcal{G}au_{\mathbb{Z}_4}$, there are eight that satisfy this condition:

$$\mathbf{1}, e^2, m^2, e^2m^2, em, e^3m^3, em^3, e^3m.$$

Now since e^2m^2 is condensed in $(\mathcal{G}au_{\mathbb{Z}_4})_{/\mathbf{1}\oplus e^2m^2}$, we should consider the anyons that are related by fusion with e^2m^2 as equivalent,

$$m^2=e^2\cdot e^2m^2,\quad e^2m^2=\mathbf{1}\cdot e^2m^2,$$
$$e^3m^3=em\cdot e^2m^2,\quad e^3m=em^3\cdot e^2m^2.$$

Then we find that the remaining inequivalent anyons are $\mathbf{1}$, e^2, em, and em^3. Computing the self and mutual statistics of these anyons indicates that they correspond to $\mathcal{M}_{\text{DS}}$.

What is the canonical boundary of $\mathcal{G}au_{\mathbb{Z}_4}$? We note that $\mathbf{1}$ is the only condensable algebra in the overlap of two Lagrangian condensable algebras $\mathbf{1}\oplus e\oplus e^2\oplus e^3$ (the $\mathbb{Z}_1$ phase) and $\mathbf{1}\oplus m\oplus m^2\oplus m^3$ (the $\mathbb{Z}_4$ phase). This allows us to conclude that the canonical boundary of $\mathcal{G}au_{\mathbb{Z}_4}$ should describe $\mathbb{Z}_4\to\mathbb{Z}_1$ symmetry breaking transition. Such a transition is described a $(c,\bar{c})=(1,1)$ critical line with only one relevant symmetric operator. Thus the $\mathbb{Z}_4\to\mathbb{Z}_1$ symmetry breaking transition is a stable transition, and the canonical boundaries of $\mathcal{G}au_{\mathbb{Z}_4}$ are described by $(c,\bar{c})=(1,1)u(1)$ CFT. Similarly, we can show that canonical boundaries of $\mathcal{G}au_{\mathbb{Z}_2}$ are described by $(c,\bar{c})=(\frac{1}{2},\frac{1}{2})$ Ising CFT with only one relevant symmetric operator—the critical point of $\mathbb{Z}_2\to\mathbb{Z}_1$ symmetry breaking transition. The canonical boundary of double-semion topological order $\mathcal{M}_{\text{DS}}$ is given by the chiral boson theory Eq. (55) with K matrix given by Eq. (56). So the gapless $\mathbf{1}\oplus e^2m^2$ state, just like the gapless $\mathbf{1}$ state, is also described by $(c,\bar{c})=(1,1)$ $u(1)$ CFT.

After determining the nature of gapless states $\mathbf{1}$, $\mathbf{1}\oplus e^2$, $\mathbf{1}\oplus m^2$, and $\mathbf{1}\oplus e^2m^2$, we consider the harder question: how do these gapless states get connected by RG flow, and what is the structure of the full phase diagram?

The condensable algebra $\mathbf{1}\oplus e^2$ differs from Lagrangian condensable algebras by condensing one excitations. In fact, condensing e changes $\mathbf{1}\oplus e^2$ to $\mathbf{1}\oplus e\oplus e^2\oplus e^3$, and condensing m^2 changes $\mathbf{1}\oplus e^2$ to $\mathbf{1}\oplus e^2\oplus m^2\oplus e^2m^2$. Here (e,m^2), having a nontrivial mutual statistics, form a competing pair. We either have an e condensation that gives rise to the condensable algebra $\mathbf{1}\oplus e\oplus e^2\oplus e^3$, or we have an m^2 condensation that gives rise to the condensable algebra $\mathbf{1}\oplus e^2\oplus m^2\oplus e^2m^2$. However e and m^2 cannot both condense. If we fine tune, we can ensure neither of them condense; that gives rise to the condensable algebra $\mathbf{1}\oplus e^2$. The gapless $\mathbf{1}\oplus e^2$ state is described by $(c,\bar{c})=(\frac{1}{2},\frac{1}{2})$ Ising CFT with only one relevant symmetric operator. The condensable algebra $\mathbf{1}\oplus e^2$ only allows one competing pair (e,m^2). Thus the RG flow in the relevant direction will cause the condensation of the competing pair. This gives rise to the phase diagram Fig. 7(a) near the gapless $\mathbf{1}\oplus e^2$ state. Similarly, the phase diagram near the gapless $\mathbf{1}\oplus m^2$ state is given by Fig. 7(b). The condensable algebra $\mathbf{1}$ differs from the Lagrangian condensable algebras $\mathbf{1}\oplus e\oplus e^2\oplus e^3$ and $\mathbf{1}\oplus e^2\oplus m^2\oplus e^2m^2$ by condensing one excitation. The competing pair involved is (e,m). Since the gapless $\mathbf{1}$ state has only one relevant operator, if that corresponds to this competing pair, the phase diagram near the gapless $\mathbf{1}$ state is given by Fig. 7(c). Putting the three local phase diagram together, we obtain a possible global phase diagram Fig. 8.

However, the gapless $\mathbf{1}$ state also allows competing pairs (e,m^2) and (m,e^2), in addition to (e,m). If instead, it is the competing pair (e,m^2) that corresponds to the relevant direction, the phase diagram will be Fig. 9(a), which implies a stable $\mathbb{Z}_2\to\mathbb{Z}_1$ symmetry breaking transition described by the critical line of $(c,\bar{c})=(1,1)$ $u(1)$ CFT that includes the Ising critical point. On the other hand, if the competing pair (m,e^2) corresponds to the relevant direction, the phase diagram will be Fig. 9(b), which implies a stable $\mathbb{Z}_4\to\mathbb{Z}_2$ symmetry breaking transition described by the critical line of $(c,\bar{c})=(1,1)$ $u(1)$ CFT that involves the Ising critical point.

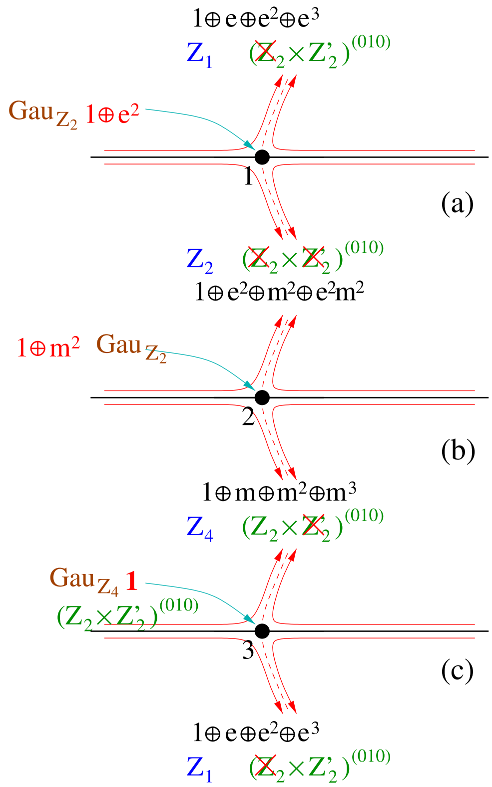


FIG. 7. Possible local phase diagram for systems with $\mathbb{Z}_4$ symmetry, which contains three gapped phases with unbroken symmetries: $\mathbb{Z}_4$, $\mathbb{Z}_2$, and $\mathbb{Z}_1$. The curves with arrows represent the RG flow, and the dots are the RG fixed points that correspond to the critical points of phase transitions. The plane is a space of Hamiltonians with symmetry TO $\mathcal{G}\mathrm{au}_{\mathbb{Z}_4}$ (see Appendix B for detailed discussions). The horizontal line in (a) is a space of Hamiltonians whose ground states have the condensation $\mathcal{A} = \mathbf{1} \oplus e^2$, which is the basin of attraction of the RG fixed point 1. The horizontal line in (b) is a space of Hamiltonians whose ground states have the condensation $\mathcal{A} = \mathbf{1} \oplus m^2$, the basin of attraction of the RG fixed point 2. The horizontal line in (c) is a space of of Hamiltonians whose ground states have the condensation $\mathcal{A} = \mathbf{1}$, the basin of attraction of the RG fixed point 3. The critical point 3 is actually part of a critical line of $(c, \bar{c}) = (1, 1)$ $u(1)$ CFT (the canonical boundary of $\mathcal{G}\mathrm{au}_{\mathbb{Z}_4}$ topological order). The critical point 1 and 2 are the $(c, \bar{c}) = (\frac{1}{2}, \frac{1}{2})$ Ising CFT (the canonical boundary of $\mathcal{G}\mathrm{au}_{\mathbb{Z}_2}$ topological order). We also list the corresponding condensable algebras, for each gapped phase and gapless critical point.

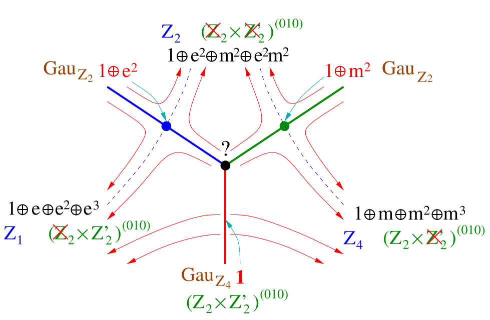


FIG. 8. A possible global phase diagram for systems with $\mathbb{Z}_4$ symmetry or $(\mathbb{Z}_2 \times \mathbb{Z}_2')^{(010)}$ symmetry, which has a similar topology with the mean-field phase diagram Fig. 5. We are not sure about the phase structure near the center of the phase diagram.

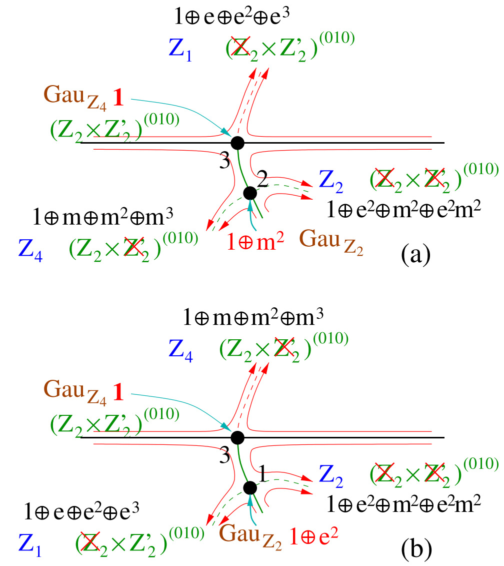


FIG. 9. Other possible local phase diagrams near the gapless **1** state, if the relevant direction corresponds to the condensation of (a) competing pair (e, m^2) or (b) competing pair (m, e^2). This requires the relevant direction of certain $(c, \bar{c}) = (1, 1)$ $u(1)$ CFT's to flow to the $(c, \bar{c}) = (\frac{1}{2}, \frac{1}{2})$ Ising CFT.

The direct phase transition $\mathbb{Z}_4 \to \mathbb{Z}_1$ has been observed which is not a critical line and does not involve Ising critical point. This implies that the competing pair (e, m) corresponds to the relevant direction, and phase diagram Fig. 7(c) is realized. However, since $(c, \bar{c}) = (1, 1)$ $u(1)$ CFT is a critical line with a marginal direction, it is not clear if either of the phase diagrams in Fig. 9 can be realized in some parts of the critical line.

To obtain a concrete global phase diagram for $\mathbb{Z}_4$ symmetric systems, we consider a $\mathbb{Z}_4$ symmetric statistical model on square lattice, which has degree of freedoms (θ_i, ϕ_i), $\theta_i = 0, 1, 2, 3$, $\phi_i = 0, 1, 2, 3$, on site $\boldsymbol{i}$. The energy is given by

$$
\begin{aligned}
E = &-\sum_{\boldsymbol{i}} J_1\big(\delta_{\theta_i,\theta_{i+x}} + \delta_{\theta_i,\theta_{i+y}}\big) \\
&-\sum_{\boldsymbol{i}} J_2\big(\delta_{\mathrm{mod}(\phi_i-\phi_{i+x},2)} + \delta_{\mathrm{mod}(\phi_i-\phi_{i+y},2)}\big) \\
&-\sum_{\boldsymbol{i}} J\delta_{\theta_i,\phi_i} + J_c \sin\left(\frac{\pi(\theta_i-\phi_i)}{2}\right).
\end{aligned}
\tag{25}
$$

The J_c term breaks the $\theta_i \to \mathrm{mod}(-\theta_i, 4)$, $\phi_i \to \mathrm{mod}(-\phi_i, 4)$ symmetry, so the full internal symmetry of the model is Z_4. The J_2 term helps to realize the $\mathbb{Z}_2$ phase.

We use the space-time tensor network renormalization approach [124] to study the above statistical model. In fact, we use a particular version of the tensor network approach which is described in detail in Ref. [56]. We obtain the phase diagram Fig. 10. The lower left of Fig. 10 is the $\mathbb{Z}_4$ phase. The upper left is the $\mathbb{Z}_2$ phase, and the right is the $\mathbb{Z}_1$ phase. The numerical phase diagram qualitatively agrees with Fig. 8.

We have described the gapless states and continuous transitions from the $\mathbb{Z}_4$ symmetry point of view. We can repeat the

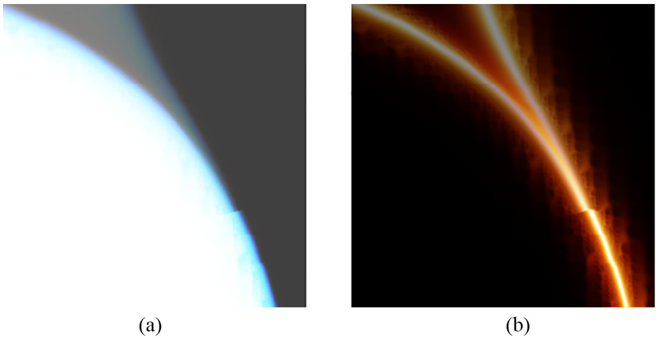


FIG. 10. A phase diagram for the model (25) with $0.993 < J_1\beta < 1.083$ (horizontal axis), $0.533 < J_2\beta < 0.773$ (vertical axis), and $J\beta = J_c\beta = 1.0$. (a) A plot of $1/GSD$, where GSD is obtained from the partition function: $GSD \equiv Z^2(L, L)/Z(L, 2L)$ where $Z(L_1, L_2)$ is the partition function for system of size $L_1 \times L_2$. For gapped quantum systems, GSD happen to be the ground state degeneracy. (b) A plot of central charge c. The central charge c is also obtained from the partition function $Z(L, L_\infty)$, which has a form $e^{-L_\infty[\epsilon L - \frac{2\pi c}{24L} + o(L^{-1})]}$ when $L_\infty \gg L$, where c is the central charge. This way, the central charge c is defined even for noncritical states. The red channel of the colored image is for system of size 64×64, the green channel for 128×128, and the blue channel for 256×256.

above discussion using the $(\mathbb{Z}_2 \times \mathbb{Z}_2')^{(010)}$ symmetry point of view and obtain analogous results.

It is interesting to compare the $\mathbb{Z}_2 \times \mathbb{Z}_2'$ symmetry and $(\mathbb{Z}_2 \times \mathbb{Z}_2')^{(010)}$ symmetry. The $\not{\mathbb{Z}}_2 \times \mathbb{Z}_2' \to \mathbb{Z}_2 \times \not{\mathbb{Z}}_2'$ DQCP-type transition can be a continuous transition described by a gapless **1** state. Such a gapless state is given by the canonical boundary of the symmetry TO $\mathcal{G}au_{\mathbb{Z}_2\times\mathbb{Z}_2'}$, which is the $(c, \bar{c}) = (1, 1)$ Ising$\times$ Ising CFT. Such a gapless state has two $\mathbb{Z}_2 \times \mathbb{Z}_2'$ symmetric relevant operators. So the $\not{\mathbb{Z}}_2 \times \mathbb{Z}_2' \to \mathbb{Z}_2 \times \not{\mathbb{Z}}_2'$ symmetry breaking transition can be a direct continuous transition, but it must be a multicritical point. In contrast, the $(\not{\mathbb{Z}}_2 \times \mathbb{Z}_2')^{(010)} \to (\mathbb{Z}_2 \times \not{\mathbb{Z}}_2')^{(010)}$ symmetry breaking transition is described by the canonical boundary of the symmetry TO $\mathcal{G}au^{(010)}_{\mathbb{Z}_2\times\mathbb{Z}_2'}$, which is a $(c, \bar{c}) = (1, 1)u(1)$ CFT that has only one $(\mathbb{Z}_2 \times \mathbb{Z}_2')^{(010)}$ symmetric relevant operator.

V. 1 + 1D ANOMALY-FREE S_3 SYMMETRY

A. Ginzburg-Landau approach for phases and phase transitions

The description of S_3 symmetry-breaking phases in Ginzburg-Landau theory is based on order parameters that transform nontrivially under the relevant broken symmetries. The group S_3 has two inequivalent nontrivial subgroups, $\mathbb{Z}_2$ and $\mathbb{Z}_3$. In terms of permutations, S_3 is represented as

$$S_3 = \{\mathbb{1}, (1, 2), (2, 3), (1, 3), (1, 2, 3), (1, 3, 2)\} \tag{26}$$

with subgroups

$$\mathbb{Z}_2 \simeq \{\mathbb{1}, (1, 2)\}, \{\mathbb{1}, (1, 3)\}, \{\mathbb{1}, (2, 3)\}, \tag{27}$$

$$\mathbb{Z}_3 \simeq \{\mathbb{1}, (1, 2, 3), (1, 3, 2)\}. \tag{28}$$

The two elements $(1, 2)$ and $(1, 2, 3)$ generate the group S_3. There are two nontrivial representations of this group, a one-dimensional representation and a two-dimensional one. The first one, which we call a_1, may be realized by a real-valued Ising-like field, ϕ^{a_1} that transforms under the generators as

$$(1, 2) \circ \phi^{a_1} = -\phi^{a_1}, \quad (1, 2, 3) \circ \phi^{a_1} = \phi^{a_1}. \tag{29}$$

The condensation of ϕ^{a_1}, which gives it a nonzero vacuum expectation value, breaks S_3 symmetry down to $S_3/\mathbb{Z}_2 = \mathbb{Z}_3$ symmetry. The second nontrivial representation, which we call a_2, may be realized by a complex two-component bosonic field $\Phi^{a_2}_\alpha$, $\alpha = 1, 2$. It transforms under the S_3 generators as

$$(1, 2) \circ \Phi^{a_2} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Phi^{a_2},$$
$$(1, 2, 3) \circ \Phi^{a_2} = \begin{pmatrix} e^{\,i 2\pi/3} & 0 \\ 0 & e^{-i 2\pi/3} \end{pmatrix} \Phi^{a_2}. \tag{30}$$

This representation is fully faithful in its S_3 action. The condensation of Φ^{a_2} satisfying $\Phi^{a_2}_1 = \Phi^{a_2}_2$ breaks S_3 symmetry down to $\mathbb{Z}_2$. The condensation of Φ^{a_2} that does not satisfy $\Phi^{a_2}_1 = \Phi^{a_2}_2$ breaks S_3 symmetry completely down to $\mathbb{Z}_1$ symmetry (i.e., the trivial group). To study the different phases allowed by the symmetry breaking structure, we can work with the following Ginzburg-Landau functional (we have dropped the superscripts a_1, a_2 for readability):

$$\begin{aligned} F[\phi, \Phi_\alpha] = {} & u\phi^2 + \phi^4 + v(|\Phi_1|^2 + |\Phi_2|^2) \\ & + \alpha\big(\Phi_1^3 + \Phi_2^3\big) + \beta\big(\Phi_1^3 - \Phi_2^3\big)\phi \\ & + \gamma\phi^2(|\Phi_1|^2 + |\Phi_2|^2) + |\Phi_1 - \Phi_2^*|^2 \\ & + (|\Phi_1|^2 + |\Phi_2|^2)^2 + \text{c.c.} \end{aligned} \tag{31}$$

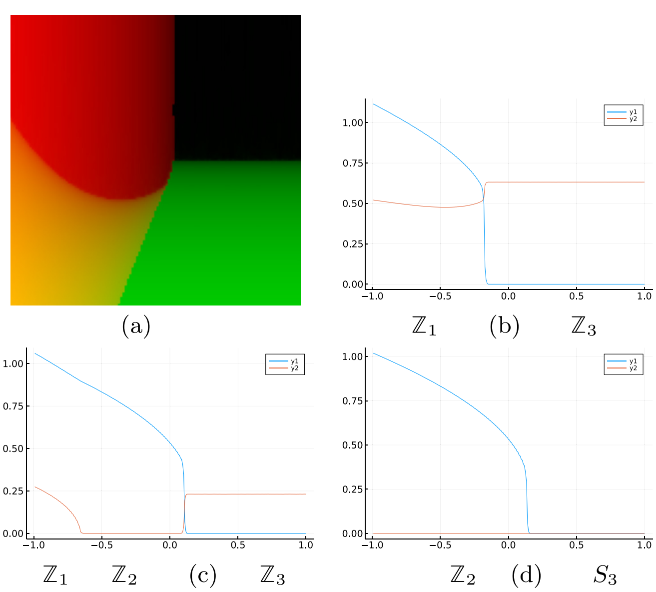


FIG. 11. Mean-field phase diagram of (31) in (u, v) space, for $\alpha = \beta = \gamma = 1$. (a) The horizontal axis is $u \in [-1, 1]$ and the vertical axis is $v \in [-1, 1]$. The red channel corresponds to the value of the S_3 order parameter $\sqrt{|\Phi_1|^2 + |\Phi_2|^2}$. The green channel corresponds to the value of the $\mathbb{Z}_2$ order parameter $\sqrt{|\phi|}$. The black region corresponds to a S_3 symmetric phase, the red region to a $\mathbb{Z}_2$ symmetric phase, the green region to a $\mathbb{Z}_3$ symmetric phase, and the yellow region to a $\mathbb{Z}_1$ phase. [(b)–(d)] Plots of $\sqrt{|\Phi_1|^2 + |\Phi_2|^2}$ (y1) and $|\phi|$ (y2), for $u \in [-1, 1]$, and (b) $v = -0.8$, (c) -0.1, and (d) 0.5.

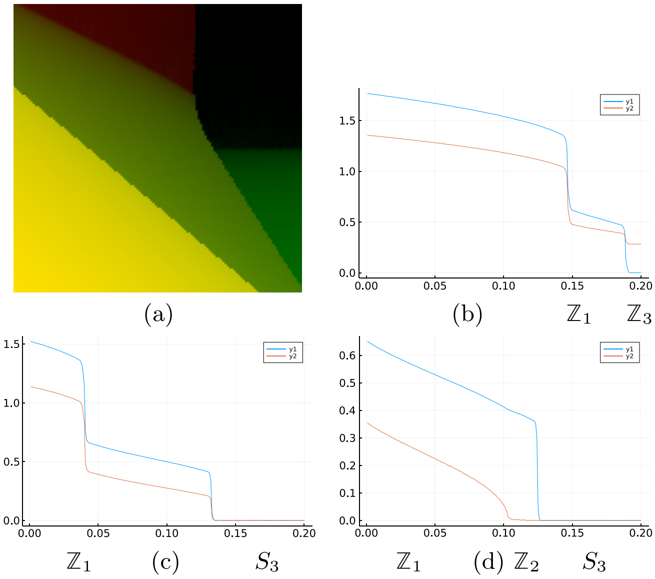


FIG. 12. Mean-field phase diagram of (31) in (u, v) space, for $\alpha = \beta = 1$ and $\gamma = -0.5$. (a) The horizontal axis is $u \in [0, 0.2]$ and the vertical axis is $v \in [-0.2, 0.2]$. The red channel corresponds to the value of the S_3 order parameter $\sqrt{|\Phi_1|^2 + |\Phi_2|^2}$. The green channel corresponds to the value of the $\mathbb{Z}_2$ order parameter $\sqrt{|\phi|}$. The black region corresponds to a S_3 symmetric phase, the red region to a $\mathbb{Z}_2$ symmetric phase, the green region to a $\mathbb{Z}_3$ symmetric phase' and the yellow region to a $\mathbb{Z}_1$ phase. [(b)–(d)] Plots of $\sqrt{|\Phi_1|^2 + |\Phi_2|^2}$ (y1) and $|\phi|$ (y2), for $u \in [0, 0.2]$, and (b) $v = -0.16$, (c) 0.02, and (d) 0.1.

It is straightforward to check that $V[\phi, \Phi_\alpha]$ is symmetric under the action of the two generators of S_3 and hence fully symmetric under S_3 transformations. The mean-field solution is obtained by minimizing this functional with the assumption that the fields are independent of the spatial coordinates. The mean-field phase diagrams are plotted using

$$\mathbb{Z}_2 \text{ order parameter: } |\phi|,$$
$$S_3 \text{ order parameter: } \sqrt{|\Phi_1|^2 + |\Phi_2|^2}. \tag{32}$$

From the mean-field phase diagrams, Figs. 11 and 12 we see that all four symmetry breaking phases, S_3, $\mathbb{Z}_3$, $\mathbb{Z}_1$, and $\mathbb{Z}_1$, are realized. Let us consider the possible phase transitions between the various phases. Landau theory tells us that we should expect continuous phase transitions between pairs of groups that have a group-subgroup relation. The proper subgroups of S_3 are three $\mathbb{Z}_2$ subgroups, a $\mathbb{Z}_3$ subgroup, and the trivial subgroup $\mathbb{Z}_1$. There are five distinct group-subgroup pairs that one can find among these groups:

$$\mathbb{Z}_1 \subset S_3, \quad \mathbb{Z}_1 \subset \mathbb{Z}_2, \quad \mathbb{Z}_1 \subset \mathbb{Z}_3, \quad \mathbb{Z}_2 \subset S_3, \quad \mathbb{Z}_3 \subset S_3.$$

There are two questions one can immediately ask the following.

(1) Are these transitions all distinct?

(2) Are these transitions all stably continuous?

From Landau's theory of phase transitions, one expects that a symmetry-breaking phase transition should depend only on the pair of symmetry groups across this transition. Moreover, we also expect from this point of view that any pair of groups related by a group-subgroup relation should have a corresponding continuous transition between gapped states that have those symmetries. By the same token, a pair of gapped phases that are symmetric under groups not related by a group-subgroup relation are generically expected to have a first-order discontinuous transition; sometimes a continuous transition that is multicritical (i.e., fine-tuned continuous) may also be allowed.

Thus we expect the two transitions $S_3 \leftrightarrow \mathbb{Z}_3$ and $\mathbb{Z}_2 \leftrightarrow \mathbb{Z}_1$ to be stably continuous and identical, since both transitions break a $\mathbb{Z}_2$ symmetry and is controlled by the change of a $\mathbb{Z}_2$ order parameter. However, although the two Ginzburg-Landau theories describing the two transitions are controlled by the same $\mathbb{Z}_2$ order parameter, the Ginzburg-Landau theory for the transition $S_3 \leftrightarrow \mathbb{Z}_3$ has a $\mathbb{Z}_3$ symmetry, while the Ginzburg-Landau theory for the transition $\mathbb{Z}_2 \leftrightarrow \mathbb{Z}_1$ does not have any additional symmetry. Since the $\mathbb{Z}_3$ symmetry has trivial actions on the $\mathbb{Z}_2$ order parameter, the two Ginzburg-Landau theories are actually identical. Therefore Ginzburg-Landau theory predicts that the two transitions $S_3 \leftrightarrow \mathbb{Z}_3$ and $\mathbb{Z}_2 \leftrightarrow \mathbb{Z}_1$ are indeed described by the same CFT. This result is confirmed by numerical calculations and the symmetry TO approach, presented in the next few sections.

We might also expect the two transitions $S_3 \leftrightarrow \mathbb{Z}_3$ and $\mathbb{Z}_2 \leftrightarrow \mathbb{Z}_1$ to be stably continuous and identical, since both transitions break a $\mathbb{Z}_3$ symmetry and is controlled by the change of a $\mathbb{Z}_3$ order parameter. However, the Ginzburg-Landau theory for the transition $S_3 \leftrightarrow \mathbb{Z}_2$ has a $\mathbb{Z}_2$ symmetry, while the Ginzburg-Landau theory for the transition $\mathbb{Z}_3 \leftrightarrow \mathbb{Z}_1$ does not have this symmetry. Also the $\mathbb{Z}_2$ symmetry acts nontrivially on the $\mathbb{Z}_3$ order parameter. Thus the Ginzburg-Landau theories for the two transitions are not really the same. Also, due to the cubic term, the two transitions must be first order at mean-field level. Later, we will see that the fluctuations turn the two first order transitions into stable continuous transitions. The CFT's for the two transitions are different. However, the two CFT's are both constructed from the (6,5) minimal model.

Due to the group-subgroup relation between S_3 and $\mathbb{Z}_1$, we might expect the transition $S_3 \leftrightarrow \mathbb{Z}_1$ to be stably continuous, which is described by a CFT with one relevant direction. But using the symmetry TO approach, we only find three gapless states with one relevant direction and with small central charge less than $(c, \bar{c}) = (1, 1)$, for S_3 symmetric systems. The first gapless state describes the transitions $S_3 \leftrightarrow \mathbb{Z}_3$ and $\mathbb{Z}_2 \leftrightarrow \mathbb{Z}_1$. The second gapless state describes the transition $\mathbb{Z}_3 \leftrightarrow \mathbb{Z}_1$ and the third one describes the transition $S_3 \leftrightarrow \mathbb{Z}_2$. So which gapless state describes the transition $S_3 \leftrightarrow \mathbb{Z}_1$? May be the stable continuous transition $S_3 \leftrightarrow \mathbb{Z}_1$ is described by a CFT with central charge larger than $(c, \bar{c}) = (1, 1)$, or may be the stable continuous transition $S_3 \leftrightarrow \mathbb{Z}_1$ does not exist. In the next section, we perform some numerical calculations to study this issue.

Lastly, we expect the stable transition $\mathbb{Z}_3 \leftrightarrow \mathbb{Z}_2$ to be first order. We know that the transitions $\mathbb{Z}_3 \leftrightarrow \mathbb{Z}_1$ and $\mathbb{Z}_1 \leftrightarrow \mathbb{Z}_2$ can be stably continuous. Can we fine tune a parameter to make the two transitions to coincide and obtain a direct continuous transition $\mathbb{Z}_3 \leftrightarrow \mathbb{Z}_2$? If $\mathbb{Z}_2$ and $\mathbb{Z}_3$ were independent (i.e., if the total symmetry were $\mathbb{Z}_2 \times \mathbb{Z}_3$), then the answer is yes. However, for total symmetry S_3, $\mathbb{Z}_2$, and $\mathbb{Z}_3$ are not independent since $S_3 = \mathbb{Z}_3 \rtimes \mathbb{Z}_2$, so we are not sure. In next section, we find that the transition $\mathbb{Z}_3 \leftrightarrow \mathbb{Z}_2$ can indeed be continuous and multicritical. In fact, the same multicritical point describes both the transitions $S_3 \leftrightarrow \mathbb{Z}_1$ and $\mathbb{Z}_3 \leftrightarrow \mathbb{Z}_2$.

B. Numerical result from tensor network calculations

The three-state Potts model is a well studied statistical model with S_3 symmetry. However, this model only realizes two phases, the S_3-symmetric and the $\mathbb{Z}_2$-symmetric phases. Here we construct an S_3-symmetric statistical model on a square lattice that can realize all four phases: S_3, $\mathbb{Z}_3$, $\mathbb{Z}_2$, and $\mathbb{Z}_1$.

The first model has degrees of freedom (θ_i, s_i), $\theta_i = 0, 1, 2$, $s_i = 0, 1$, on site $\boldsymbol{i}$. The energy is given by

$$\begin{aligned} E = &-\sum_{\boldsymbol{i}} J_1\big(\delta_{\theta_{\boldsymbol{i}},\theta_{\boldsymbol{i}+\boldsymbol{x}}} + \delta_{\theta_{\boldsymbol{i}},\theta_{\boldsymbol{i}+\boldsymbol{y}}}\big) + J_2\big(\delta_{s_{\boldsymbol{i}},s_{\boldsymbol{i}+\boldsymbol{x}}} + \delta_{s_{\boldsymbol{i}},s_{\boldsymbol{i}+\boldsymbol{y}}}\big) \\ &+ \frac{1}{3}J_c \sum_{\boldsymbol{i}=\text{even}} [\text{sgn}(\theta_{\boldsymbol{i}} - \theta_{\boldsymbol{i}+\boldsymbol{x}}) + \text{sgn}(\theta_{\boldsymbol{i}+\boldsymbol{x}} - \theta_{\boldsymbol{i}+\boldsymbol{x}+\boldsymbol{y}}) \\ &+ \text{sgn}(\theta_{\boldsymbol{i}+\boldsymbol{x}+\boldsymbol{y}} - \theta_{\boldsymbol{i}+\boldsymbol{y}}) + \text{sgn}(\theta_{\boldsymbol{i}+\boldsymbol{y}} - \theta_{\boldsymbol{i}})] \\ &\times (s_{\boldsymbol{i}} + s_{\boldsymbol{i}+\boldsymbol{x}} + s_{\boldsymbol{i}+\boldsymbol{x}+\boldsymbol{y}} + s_{\boldsymbol{i}+\boldsymbol{y}} - 2), \end{aligned} \tag{33}$$

where $\text{sgn}(\theta) \equiv \text{mod}(\theta + 1, 3) - 1$. The J_1 and J_2 terms give rise to $q = 3$ Potts model and Ising model. If we view $\theta_{\boldsymbol{i}}$ as a planer vector that can points to three directions separated by 120° degree, then the term $\text{sgn}(\theta_{\boldsymbol{i}} - \theta_{\boldsymbol{i}+\boldsymbol{x}}) + \text{sgn}(\theta_{\boldsymbol{i}+\boldsymbol{x}} - \theta_{\boldsymbol{i}+\boldsymbol{x}+\boldsymbol{y}}) + \text{sgn}(\theta_{\boldsymbol{i}+\boldsymbol{x}+\boldsymbol{y}} - \theta_{\boldsymbol{i}+\boldsymbol{y}}) + \text{sgn}(\theta_{\boldsymbol{i}+\boldsymbol{y}} - \theta_{\boldsymbol{i}})$ has a meaning chirality: it measures whether the vectors turn clockwise or anticlockwise as we go round a square. The coupling of the chirality with the Ising order parameter $s_{\boldsymbol{i}}$ breaks the $S_3 \times \mathbb{Z}_2$ symmetry to S_3 symmetry.

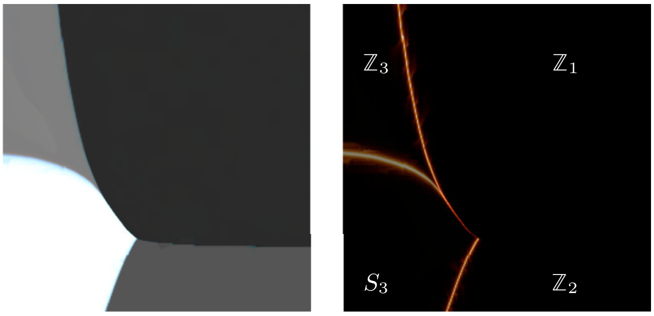
The second S_3 symmetric statistical model has degree of freedoms (θ_i, ϕ_i, s_i), $\theta_i = 0, 1, 2$, $\phi_i = 0, 1, 2$, $s_i = 0, 1$, on site $\boldsymbol{i}$. The energy is given by

$$\begin{aligned} E = &-\sum_{\boldsymbol{i}} J_1\big(\delta_{\theta_{\boldsymbol{i}},\theta_{\boldsymbol{i}+\boldsymbol{x}}} + \delta_{\theta_{\boldsymbol{i}},\theta_{\boldsymbol{i}+\boldsymbol{y}}} + \delta_{\phi_{\boldsymbol{i}},\phi_{\boldsymbol{i}+\boldsymbol{x}}} + \delta_{\phi_{\boldsymbol{i}},\phi_{\boldsymbol{i}+\boldsymbol{y}}}\big) \\ &-\sum_{\boldsymbol{i}} J_2(\delta_{s_{\boldsymbol{i}},s_{\boldsymbol{i}+\boldsymbol{x}}} + \delta_{s_{\boldsymbol{i}},s_{\boldsymbol{i}+\boldsymbol{y}}}) + J\delta_{\theta_{\boldsymbol{i}},\phi_{\boldsymbol{i}}} \\ &+ J_c \sum_{\boldsymbol{i}} \text{sgn}(\theta_{\boldsymbol{i}} - \phi_{\boldsymbol{i}})\Big(s_{\boldsymbol{i}} - \frac{1}{2}\Big). \end{aligned} \tag{34}$$

The J_c term is also a coupling of the chirality with the Ising order parameter $s_{\boldsymbol{i}}$.

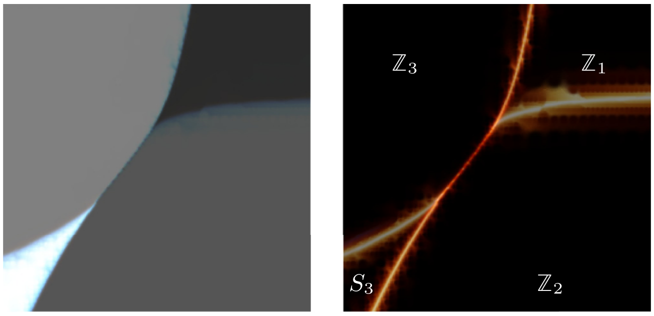
Again, we use a particular version of the tensor network approach in Ref. [56] to study the above two statistical models. We obtain the phase diagrams Figs. 13–15. All three phase diagrams contain all the four phases: S_3 phase in lower left, $\mathbb{Z}_3$ phase in upper left, $\mathbb{Z}_2$ phase in lower right, and $\mathbb{Z}_1$ phase in upper right,

In phase diagram Fig. 13, we see five stable direct transitions $S_3 \leftrightarrow \mathbb{Z}_2$, $S_3 \leftrightarrow \mathbb{Z}_3$, $\mathbb{Z}_3 \leftrightarrow \mathbb{Z}_2$, $\mathbb{Z}_3 \leftrightarrow \mathbb{Z}_1$, and $\mathbb{Z}_2 \leftrightarrow \mathbb{Z}_1$.



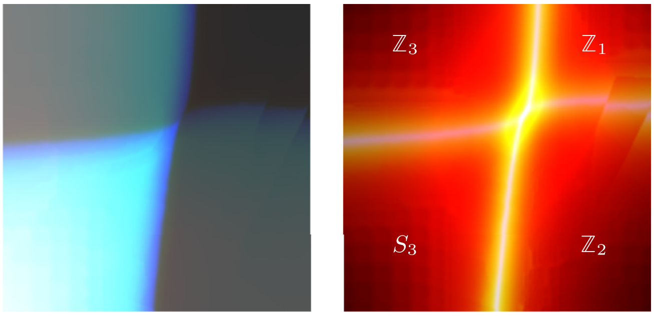
(a) (b)

FIG. 14. A phase diagram for the second model (34) with $0.57 < J_1\beta < 1.29$ (horizontal axis), $0.4775 < J_2\beta < 1.2275$ (vertical axis), and $J\beta = 1 - J_2\beta$, $J_c\beta = 1$. (a) A plot of 1/GSD. (b) A plot of central charge c. The red channel of the colored image is for system of size 64 × 64, the green channel for 128 × 128, and the blue channel for 256 × 256.



(a) (b)

FIG. 13. A phase diagram for the first model (33) with $1.2 < J_1\beta < 1.4$ (horizontal axis), $0.6 < J_2\beta < 1.0$ (vertical axis), and $J_c\beta = 1.5$. (a) A plot of 1/GSD, where $\text{GSD} \equiv Z^2(L, L)/Z(L, 2L)$ and $Z(L_1, L_2)$ is the partition function for system of size $L_1 \times L_2$. (b) A plot of central charge c. $Z(L, L_\infty)$ has a form $e^{-L_\infty[\epsilon L - \frac{2\pi c}{24L} + o(L^{-1})]}$ when $L_\infty \gg L$, where c is the central charge. The red channel of the colored image is for system of size 64 × 64, the green channel for 128 × 128, and the blue channel for 256 × 256.



(a) (b)

FIG. 15. A phase diagram for the first model (33) with $1.02125 < J_1\beta < 1.04125$ (horizontal axis), $0.851875 < J_2\beta < 0.891875$ (vertical axis), and $J_c\beta = 0.5$. (a) A plot of 1/GSD. (b) A plot of central charge c. The red channel of the colored image is for system of size 64 × 64, the green channel for 256 × 256, and the blue channel for 1024 × 1024.

TABLE I. The pointlike excitations and their fusion rules in 2+1D $\mathcal{G}au_{S_3}$ topological order (i.e., S_3 gauge theory with charge excitations). The S_3 group are generated by (1,2) and (1,2,3). Here **1** is the trivial excitation. a_1 and a_2 are pure S_3 charge excitations, where a_1 corresponds to the one-dimensional representation and a_2 the two-dimensional representation of S_3. b and c are pure S_3 flux excitations, where b corresponds to the conjugacy class $\{(1,2,3),(1,3,2)\}$, and c conjugacy class $\{(1,2),(2,3),(1,3)\}$. b_1, b_2, and c_1 are charge-flux bound states. d, s are the quantum dimension and the topological spin of an excitation.

d,s	1,0	1,0	2,0	2,0	$2,\frac{1}{3}$	$2,-\frac{1}{3}$	3,0	$3,\frac{1}{2}$
$\otimes$	$\mathbf{1}$	a_1	a_2	b	b_1	b_2	c	c_1
$\mathbf{1}$	$\mathbf{1}$	a_1	a_2	b	b_1	b_2	c	c_1
a_1	a_1	$\mathbf{1}$	a_2	b	b_1	b_2	c_1	c
a_2	a_2	a_2	$\mathbf{1}\oplus a_1\oplus a_2$	$b_1\oplus b_2$	$b\oplus b_2$	$b\oplus b_1$	$c\oplus c_1$	$c\oplus c_1$
b	b	b	$b_1\oplus b_2$	$\mathbf{1}\oplus a_1\oplus b$	$b_2\oplus a_2$	$b_1\oplus a_2$	$c\oplus c_1$	$c\oplus c_1$
b_1	b_1	b_1	$b\oplus b_2$	$b_2\oplus a_2$	$\mathbf{1}\oplus a_1\oplus b_1$	$b\oplus a_2$	$c\oplus c_1$	$c\oplus c_1$
b_2	b_2	b_2	$b\oplus b_1$	$b_1\oplus a_2$	$b\oplus a_2$	$\mathbf{1}\oplus a_1\oplus b_2$	$c\oplus c_1$	$c\oplus c_1$
c	c	c_1	$c\oplus c_1$	$c\oplus c_1$	$c\oplus c_1$	$c\oplus c_1$	$\mathbf{1}\oplus a_2\oplus b\oplus b_1\oplus b_2$	$a_1\oplus a_2\oplus b\oplus b_1\oplus b_2$
c_1	c_1	c	$c\oplus c_1$	$c\oplus c_1$	$c\oplus c_1$	$c\oplus c_1$	$a_1\oplus a_2\oplus b\oplus b_1\oplus b_2$	$\mathbf{1}\oplus a_2\oplus b\oplus b_1\oplus b_2$

We also computed the central charge $\frac{c+\bar{c}}{2}$, along the transition lines. The nonzero central charges suggest that all the five transitions are stable continuous transitions.

In phase diagram Fig. 14, we see five stable direct transitions $S_3 \leftrightarrow \mathbb{Z}_2$, $S_3 \leftrightarrow \mathbb{Z}_3$, $S_3 \leftrightarrow \mathbb{Z}_1$, $\mathbb{Z}_3 \leftrightarrow \mathbb{Z}_1$, $\mathbb{Z}_2 \leftrightarrow \mathbb{Z}_1$. From the computed central charge, we find that the transitions $S_3 \leftrightarrow \mathbb{Z}_2$, $\mathbb{Z}_3 \leftrightarrow \mathbb{Z}_1$, and $S_3 \leftrightarrow \mathbb{Z}_1$ are stably continuous. The transitions $\mathbb{Z}_2 \leftrightarrow \mathbb{Z}_1$ is first order since the central charge $c = \bar{c} = 0$ along the transition line.

In phase diagram Fig. 15, we reduce J_c from $J_c = 1.5$ in Fig. 13 to $J_c = 0.5$. We see an evidence of a multicritical point connecting the four phases S_3, $\mathbb{Z}_3$, $\mathbb{Z}_2$, and $\mathbb{Z}_1$. More systematic and detailed studies are needed.

Among these stable continuous transitions, the $\mathbb{Z}_2 \leftrightarrow \mathbb{Z}_1$ and $S_3 \leftrightarrow \mathbb{Z}_3$ critical points are the well known Ising critical point, which is described by a conformal field theory (CFT) constructed from (4,3) minimal model. $S_3 \leftrightarrow \mathbb{Z}_2$ critical point is the well known critical point of $q = 3$ Potts model, which is described by a CFT constructed from (6,5) minimal model. However, what are the $\mathbb{Z}_3 \leftrightarrow \mathbb{Z}_2$, $S_3 \leftrightarrow \mathbb{Z}_2$, and $\mathbb{Z}_3 \leftrightarrow \mathbb{Z}_1$ critical points? What is the $(S_3, \mathbb{Z}_2, \mathbb{Z}_1, \mathbb{Z}_3)$ multicritical point? In next section, we will use symmetry TO to understand the above global phase diagrams of S_3 symmetric systems and the critical points. In particular, we will show a duality relation between $S_3 \leftrightarrow \mathbb{Z}_1$ transition and $\mathbb{Z}_3 \leftrightarrow \mathbb{Z}_2$ transition. For example, the existence of a stable continuous transition $S_3 \leftrightarrow \mathbb{Z}_1$ implies the existence of stable continuous transition $\mathbb{Z}_3 \leftrightarrow \mathbb{Z}_2$. The two stable continuous transitions, if they exist, are described by the same CFT.

C. A symmetry TO approach for gapped and gapless phases

$1+1$D S_3 symmetry is described a symmetry TO (i.e., a $2+1$D topological order) whose topological excitations are described by S_3 quantum double $\mathcal{G}au_{S_3}$ (i.e., S_3 gauge theory with both charge and flux excitations, as described in Table I). From Appendix C, we find that a condensable algebra $\mathcal{A} = \bigoplus_{a\in\mathcal{M}} A^a a$ in $\mathcal{M}$ must satisfies

$$\begin{aligned} &A^{\mathbf{1}} = 1, \quad A^a \in \mathbb{N}, \quad A^a = A^{\bar{a}}, \\ &s_a = 0 \ \text{ for } a \in \mathcal{A}, \\ &\frac{\sum_{b\in\mathcal{M}} S^{ab}_{\mathcal{M}} A^b}{\sum_{b\in\mathcal{M}} S^{\mathbf{1}b}_{\mathcal{M}} A^b} = \text{cyclotomic integer for all } a \in \mathcal{M} \\ &A^a \leqslant d_a - \delta(d_a), \\ &A^a A^b \leqslant \sum_c N^{ab}_{\mathcal{M},c} A^c - \delta_{a,\bar{b}}\delta(d_a), \\ &A^a = \sum_b S^{ab}_{\mathcal{M}} A^b \ \text{ if } \mathcal{A} \text{ is Lagrangian.} \end{aligned} \tag{35}$$

Solving the above conditions for $\mathcal{M} = \mathcal{G}au_{S_3}$, we find the following potential condensable algebras:

$$\begin{aligned} &\mathbf{1}\oplus b\oplus c, \quad \mathbf{1}\oplus a_2\oplus c, \quad \mathbf{1}\oplus a_1\oplus 2b, \quad \mathbf{1}\oplus a_1\oplus 2a_2. \\ &\mathbf{1}\oplus b, \quad \mathbf{1}\oplus a_2, \quad \mathbf{1}\oplus a_1, \quad \mathbf{1}. \end{aligned} \tag{36}$$

Since Eq. (35) are only necessary conditions, some of the above $\mathcal{A}$'s may not be valid. For $\mathcal{G}au_{S_3}$, using the following physical considerations, we argue that the above $\mathcal{A}$'s are all valid and describe the actual condensation patterns in physical systems (see Fig. 16).

We know that $1+1$D S_3 symmetric systems can have four gapped phases with unbroken symmetry group S_3, $\mathbb{Z}_3$, $\mathbb{Z}_2$, and $\mathbb{Z}_1$, and they correspond to the four Lagrangian condensable algebras that we find above:

$$\begin{aligned} &\mathbf{1}\oplus b\oplus c \to S_3 \text{ phase}, \quad \mathbf{1}\oplus a_2\oplus c \to \mathbb{Z}_2 \text{ phase}, \\ &\mathbf{1}\oplus a_1\oplus 2b \to \mathbb{Z}_3 \text{ phase}, \quad \mathbf{1}\oplus a_1\oplus 2a_2 \to \mathbb{Z}_1 \text{ phase}. \end{aligned} \tag{37}$$

To understand the above result, we note that a condensation of real field ϕ^{a_1} carrying the one-dimensional representation of S_3 breaks S_3 symmetry down to $\mathbb{Z}_3$ symmetry. Thus the condensation of the corresponding Lagrangian condensable algebra $\mathbf{1}\oplus a_1\oplus 2b$ induces a gapped symmetry breaking phase where the unbroken symmetry is $\mathbb{Z}_3 \subset S_3$. A condensation of complex two-component bosonic field $\Phi^{a_2}_\alpha$, $\alpha = 1$ and 2, carrying the two-dimensional representation of S_3 and satisfying $\Phi^{a_2}_1 = \Phi^{a_2}_2$ breaks S_3 symmetry down to $\mathbb{Z}_2$ symmetry. Thus the condensation of Lagrangian condensable algebra $\mathbf{1}\oplus a_2\oplus c$ induces a gapped symmetry breaking phase where the unbroken symmetry is $\mathbb{Z}_2 \subset S_3$. Similarly,

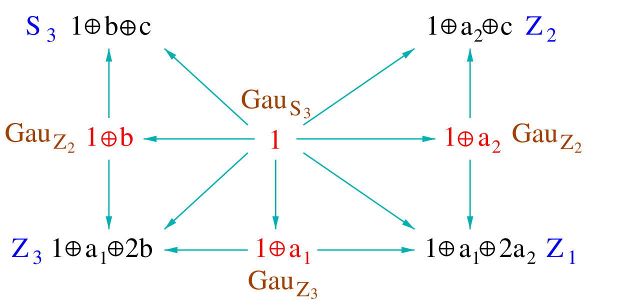


FIG. 16. $1+1$D S_3 symmetry is described by symmetry TO $\mathcal{G}au_{S_3}$ (S_3 gauge theory with charge excitations). The $\mathcal{G}au_{S_3}$ topological order has eight condensable algebras, and four of them are Lagrangian (the black ones above). They give rise to gapped phases as the corresponding gapped boundary states. We also indicate the unbroken symmetry groups of these phases. Four of eight are not Lagrangian, and give rise to gapless states. We also indicate the condensation-induced topological orders whose canonical boundaries give rise to these gapless states. The gapless states are critical points for the transitions between gapped or gapless states, as indicated above. The arrows indicate the embedding maps of condensable algebra and the directions of more condensation. All critical points has only one symmetric local relevant operator and are stable critical points.

the condensation of ϕ^{a_1} and Φ^{a_2} breaks S_3 symmetry down to $\mathbb{Z}_1$ symmetry. So the condensation of Lagrangian condensable algebra $\mathbf{1}\oplus a_1\oplus 2a_2$ induces a gapped symmetry breaking phase where the unbroken symmetry is $\mathbb{Z}_1\subset S_3$.

Since $\mathbf{1}\oplus b\oplus c$, $\mathbf{1}\oplus a_2\oplus c$, $\mathbf{1}\oplus a_1\oplus 2b$, $\mathbf{1}\oplus a_1\oplus 2a_2$ are Lagrangian, i.e., $D_{\mathcal{M}}/d_{\mathcal{A}}=D_{\mathcal{M}_{/\mathcal{A}}}=1$, the corresponding condensation-induced topological orders, $(\mathcal{G}au_{S_3})_{/\mathbf{1}\oplus b\oplus c}$, $(\mathcal{G}au_{S_3})_{/\mathbf{1}\oplus a_2\oplus c}$, $(\mathcal{G}au_{S_3})_{/\mathbf{1}\oplus a_1\oplus 2b}$, $(\mathcal{G}au_{S_3})_{/\mathbf{1}\oplus a_1\oplus 2a_2}$, are all trivial. On the other hand, $\mathbf{1}\oplus a_2$, $\mathbf{1}\oplus b$, $\mathbf{1}\oplus a_1$, $\mathbf{1}$ are not Lagrangian, and their condensation-induced topological orders are not trivial. We have

$$(\mathcal{G}au_{S_3})_{/\mathbf{1}\oplus a_2}=\mathcal{G}au_{\mathbb{Z}_2},\quad (\mathcal{G}au_{S_3})_{/\mathbf{1}\oplus b}=\mathcal{G}au_{\mathbb{Z}_2},$$
$$(\mathcal{G}au_{S_3})_{/\mathbf{1}\oplus a_1}=\mathcal{G}au_{\mathbb{Z}_3},\quad (\mathcal{G}au_{S_3})_{/\mathbf{1}}=\mathcal{G}au_{S_3}. \tag{38}$$

The above results can be understood using the usual Anderson-Higgs condensation in S_3 gauge theory. The $\mathbf{1}\oplus a_2$ condensation correspond to the condensation of the Φ^{a_2} field, which change the topological order $\mathcal{G}au_{S_3}$ described by S_3 gauge theory to a topological order $\mathcal{G}au_{\mathbb{Z}_2}$ described by $\mathbb{Z}_2$ gauge theory. Similarly, the $\mathbf{1}\oplus a_1$ condensation correspond to the condensation of the ϕ^{a_1} field, which change the topological order $\mathcal{G}au_{S_3}$ described by S_3 gauge theory to a topological order $\mathcal{G}au_{\mathbb{Z}_3}$ described by $\mathbb{Z}_3$ gauge theory. $(\mathcal{G}au_{S_3})_{/\mathbf{1}}=\mathcal{G}au_{S_3}$ is obvious since $\mathbf{1}$ is trivial. To understand $(\mathcal{G}au_{S_3})_{/\mathbf{1}\oplus b}=\mathcal{G}au_{\mathbb{Z}_2}$, we note that $\mathcal{G}au_{S_3}$ is invariant under exchanging a_2 and b. Such an automorphism exchanges $\mathbf{1}\oplus a_2$ and $\mathbf{1}\oplus b$. Thus $(\mathcal{G}au_{S_3})_{/\mathbf{1}\oplus b}=(\mathcal{G}au_{S_3})_{/\mathbf{1}\oplus a_2}=\mathcal{G}au_{\mathbb{Z}_2}$.

We can also derive Eq. (38) using the boundary theory of topological order summarized in Appendix C. For example, $(\mathcal{G}au_{S_3})_{/\mathbf{1}\oplus a_2}=\mathcal{G}au_{\mathbb{Z}_2}$ implies that there is a canonical domain wall between $\mathcal{G}au_{S_3}$ and $\mathcal{G}au_{\mathbb{Z}_2}$, that is described by a nonnegative integer matrix (A^{ai}). A^{ai} satisfy Eqs. (C20), (C21), (C23), and (C25), as well as other properties listed in Appendix C.

We find that such (A^{ai}) does exist

$$\left(A^{ai}_{\mathcal{G}au_{S_3}|\mathcal{G}au_{\mathbb{Z}_2}}\right)=\begin{pmatrix} 1&0&0&0&\mathbf{1}\\ 0&0&1&0&a_1\\ 1&0&1&0&a_2\\ 0&0&0&0&b\\ 0&0&0&0&b_1\\ 0&0&0&0&b_2\\ 0&1&0&0&c\\ 0&0&0&1&c_1\\ \mathbf{1}&e&m&f& \end{pmatrix}. \tag{39}$$

In the above $A^{ai}_{\mathcal{G}au_{S_3}|\mathcal{G}au_{\mathbb{Z}_2}}$, the nonzero entries in the first row of (A^{ai}) indicate the condensation of corresponding anyons in $\mathcal{G}au_{\mathbb{Z}_2}$ (i.e., these anyons become $\mathbf{1}$ at the domain wall). Thus the first row of (A^{ai}) gives rise to a condensable algebra in $\mathcal{G}au_{\mathbb{Z}_2}$: $\bigoplus_i A^{\mathbf{1}i}_{\mathcal{G}au_{S_3}|\mathcal{G}au_{\mathbb{Z}_2}}i=\mathbf{1}$, which indicates that the domain wall is a $\mathbf{1}$-condensed boundary of $\mathcal{G}au_{\mathbb{Z}_2}$. This in turn indicates that $\mathcal{G}au_{\mathbb{Z}_2}$ comes from $\mathcal{G}au_{S_3}$ via a condensation.

Similarly, the first column of (A^{ai}) gives rise to a condensable algebra in $\mathcal{G}au_{S_3}$: $\bigoplus_a A^{a\mathbf{1}}_{\mathcal{G}au_{S_3}|\mathcal{G}au_{\mathbb{Z}_2}}a=\mathbf{1}\oplus a_2$, which tells us that the domain wall is a $\mathbf{1}\oplus a_2$-condensed boundary of $\mathcal{G}au_{S_3}$, and the $\mathbf{1}\oplus a_2$ condensation changes $\mathcal{G}au_{S_3}$ to $\mathcal{G}au_{\mathbb{Z}_2}$. This confirms our above result from physical considerations.

$\mathcal{G}au_{S_3}$ and $\mathcal{G}au_{\mathbb{Z}_2}$ has another canonical boundary described by

$$\left(A^{ai}_{\mathcal{G}au_{S_3}|\mathcal{G}au_{\mathbb{Z}_2}}\right)=\begin{pmatrix} 1&0&0&0&\mathbf{1}\\ 0&0&1&0&a_1\\ 0&0&0&0&a_2\\ 1&0&1&0&b\\ 0&0&0&0&b_1\\ 0&0&0&0&b_2\\ 0&1&0&0&c\\ 0&0&0&1&c_1\\ \mathbf{1}&e&m&f& \end{pmatrix}, \tag{40}$$

which describes the condensation of $\mathbf{1}\oplus b$. Such a condensation also changes $\mathcal{G}au_{S_3}$ to $\mathcal{G}au_{\mathbb{Z}_2}$.

Similarly, the canonical domain wall between $\mathcal{G}au_{S_3}$ and $\mathcal{G}au_{\mathbb{Z}_3}$ is given by

$$\left(A^{ai}_{\mathcal{G}au_{S_3}|\mathcal{G}au_{\mathbb{Z}_3}}\right)=\begin{pmatrix} 1&0&0&0&0&0&0&0&0&\mathbf{1}\\ 1&0&0&0&0&0&0&0&0&a_1\\ 0&0&0&1&1&0&0&0&0&a_2\\ 0&1&1&0&0&0&0&0&0&b\\ 0&0&0&0&0&0&0&1&1&b_1\\ 0&0&0&0&0&1&1&0&0&b_2\\ 0&0&0&0&0&0&0&0&0&c\\ 0&0&0&0&0&0&0&0&0&c_1 \end{pmatrix}. \tag{41}$$

From the first row of $(A^{ai}_{\mathcal{G}au_{S_3}|\mathcal{G}au_{\mathbb{Z}_3}})$, we obtain the corresponding condensable algebra in $\mathcal{G}au_{\mathbb{Z}_3}$: $\bigoplus_i A^{\mathbf{1}i}_{\mathcal{G}au_{S_3}|\mathcal{G}au_{\mathbb{Z}_3}}i=\mathbf{1}$, which suggests that $\mathcal{G}au_{\mathbb{Z}_3}$ comes from $\mathcal{G}au_{S_3}$ by a condensation. From the first column of $(A^{ai}_{\mathcal{G}au_{S_3}|\mathcal{G}au_{\mathbb{Z}_3}})$, we obtain the corresponding condensable algebra in $\mathcal{G}au_{S_3}$: $\bigoplus_a A^{a\mathbf{1}}_{\mathcal{G}au_{S_3}|\mathcal{G}au_{\mathbb{Z}_3}}a=\mathbf{1}\oplus a_1$, which tells us that the condensation is given by $\mathbf{1}\oplus a_1$. This again confirms our above result.

Therefore, although we have four non-Lagrangian condensable algebras, we only have three different condensation-induced topological orders, $\mathcal{G}\mathrm{au}_{\mathbb{Z}_2}$, $\mathcal{G}\mathrm{au}_{\mathbb{Z}_3}$, $\mathcal{G}\mathrm{au}_{S_3}$, which correspond to three reduced symmetry TOs and give rise to three types of gapless states. Let us now elaborate on the description of these gapless states from the point of view of symmetry TO.

First, a reduced symmetry TO allows many different gapless states. Here we want to know which is the most stable gapless state with minimal number of relevant operators. If the reduced symmetry TO is trivial, the most stable state with the trivial reduced symmetry TO is gapped. If the reduced symmetry TO is nontrivial, the state with minimal low energy excitations is still gapless. What is this minimal gapless state? How to calculate the canonical boundary of the topological order $\mathcal{M}_{/\mathcal{A}}$, which corresponds to the minimal gapless state with reduced symmetry TO $\mathcal{M}_{/\mathcal{A}}$?

To calculate the canonical boundary, we will use holoMB [48,125–127] which is a generalization of modular bootstrap [37,109,128–133]. Modular bootstrap looks for single-component modular invariant partition functions. HoloMB looks for multicomponent boundary partition functions for a boundary space-time that has a form of torus. The shape of the boundary space-time torus is described by a complex number τ. A multicomponent boundary partition function $Z_\alpha(\tau)$ transform covariantly under the modular transformation, according to the S and T matrices that characterize the bulk topological order [see Eq. (5)]. The physical reason for such a bulk-boundary connection is discussed in Refs. [48,98]. Thus, in contrast to the modular bootstrap, holoMB requires additional input data, the S and T matrices, to describe the symmetry TO.

If $Z_\alpha(\tau)$ is independent of τ, the corresponding boundary is gapped. If $Z_\alpha(\tau)$ depend on τ, the corresponding boundary is gapless. For gapless boundary, the multicomponent partition function $Z_\alpha(\tau)$ is formed by conformal characters of certain CFT. So we look for a CFT, whose conformal characters, after a suitable combination, can form multicomponent boundary partition function that transform under modular transformation according to the bulk S and T matrices. The method of computing a suitable combination is the same as computing gapped boundary (see Appendix C) of some properly constructed topological order. For details see Ref. [48].

Using such a method, we can obtain the properties of the gapless $\mathbf{1}\oplus a_1$ state with reduced symmetry TO $(\mathcal{G}\mathrm{au}_{S_3})_{/\mathbf{1}\oplus a_1} = \mathcal{G}\mathrm{au}_{\mathbb{Z}_3}$. First, we find that one of gapless boundaries of $\mathcal{G}\mathrm{au}_{\mathbb{Z}_3}$ is given by the following multicomponent partition function. Note that the $\mathcal{G}\mathrm{au}_{\mathbb{Z}_3}$ topological order has nine anyons $\mathbf{1}$, e, e^2, m, em, e^2m, m^2, em^2, e^2m^2. The multicomponent boundary partition function for $\mathcal{G}\mathrm{au}_{\mathbb{Z}_3}$ are labeled by these nine anyons:

$$
\begin{aligned}
Z_{\mathbf{1}}^{\mathcal{G}\mathrm{au}_{\mathbb{Z}_3}} &= |\chi_0^{m6}+\chi_3^{m6}|^2 + |\chi_{\frac{2}{5}}^{m6}+\chi_{\frac{7}{5}}^{m6}|^2,\\
Z_{e}^{\mathcal{G}\mathrm{au}_{\mathbb{Z}_3}} &= |\chi_{\frac{2}{3}}^{m6}|^2 + |\chi_{\frac{1}{15}}^{m6}|^2,\\
Z_{e^2}^{\mathcal{G}\mathrm{au}_{\mathbb{Z}_3}} &= |\chi_{\frac{2}{3}}^{m6}|^2 + |\chi_{\frac{1}{15}}^{m6}|^2,\\
Z_{m}^{\mathcal{G}\mathrm{au}_{\mathbb{Z}_3}} &= |\chi_{\frac{2}{3}}^{m6}|^2 + |\chi_{\frac{1}{15}}^{m6}|^2,\\
Z_{me}^{\mathcal{G}\mathrm{au}_{\mathbb{Z}_3}} &= \chi_0^{m6}\bar\chi_{\frac{2}{3}}^{m6} + \chi_3^{m6}\bar\chi_{\frac{2}{3}}^{m6} + \chi_{\frac{2}{5}}^{m6}\bar\chi_{\frac{1}{15}}^{m6} + \chi_{\frac{7}{5}}^{m6}\bar\chi_{\frac{1}{15}}^{m6},\\
Z_{me^2}^{\mathcal{G}\mathrm{au}_{\mathbb{Z}_3}} &= \chi_{\frac{2}{3}}^{m6}\bar\chi_0^{m6} + \chi_{\frac{2}{3}}^{m6}\bar\chi_3^{m6} + \chi_{\frac{1}{15}}^{m6}\bar\chi_{\frac{2}{5}}^{m6} + \chi_{\frac{1}{15}}^{m6}\bar\chi_{\frac{7}{5}}^{m6},\\
Z_{m^2}^{\mathcal{G}\mathrm{au}_{\mathbb{Z}_3}} &= |\chi_{\frac{2}{3}}^{m6}|^2 + |\chi_{\frac{1}{15}}^{m6}|^2,\\
Z_{m^2e}^{\mathcal{G}\mathrm{au}_{\mathbb{Z}_3}} &= \chi_{\frac{2}{3}}^{m6}\bar\chi_0^{m6} + \chi_{\frac{2}{3}}^{m6}\bar\chi_3^{m6} + \chi_{\frac{1}{15}}^{m6}\bar\chi_{\frac{2}{5}}^{m6} + \chi_{\frac{1}{15}}^{m6}\bar\chi_{\frac{7}{5}}^{m6},\\
Z_{m^2e^2}^{\mathcal{G}\mathrm{au}_{\mathbb{Z}_3}} &= \chi_0^{m6}\bar\chi_{\frac{2}{3}}^{m6} + \chi_3^{m6}\bar\chi_{\frac{2}{3}}^{m6} + \chi_{\frac{2}{5}}^{m6}\bar\chi_{\frac{1}{15}}^{m6} + \chi_{\frac{7}{5}}^{m6}\bar\chi_{\frac{1}{15}}^{m6},
\end{aligned} \tag{42}
$$

where $\chi_h^{m6} = \chi_h^{m6}(\tau)$ are conformal characters with conformal dimension h, for (6,5) minimal model. The above result used the expression of S-matrix of (p,q) minimal model in Ref. [134]. Such CFT has a chiral central charge $c = 4/5$.

The above boundary is a $\mathbf{1}$-condensed boundary of the topological order $\mathcal{G}\mathrm{au}_{\mathbb{Z}_3}$. This is because a condensation of an anyon a, will cause the correspond partition function $Z_a(\tau)$ to contain the $|\chi_0^{m6}|^2$ term [see Eq. (43)]. In the above partition, the term $|\chi_0^{m6}|^2$ appears only in $Z_{\mathbf{1}}(\tau)$, and thus the boundary is a $\mathbf{1}$-condensed boundary. Also if a condense, there must be a nontrivial anyon b that has a nontrivial mutual statistics with a. (This is due to the remote-detectability principle of anomaly-free topological order.) The condensation of a will confine the anyon b and cause Z_b to vanish [see Eq. (43)]. This does not happen for the above partition function. Thus there is no condensation of nontrivial anyons.

We have checked other CFT's with smaller central charges. We find that although these CFT's can be gapless boundaries of $\mathcal{G}\mathrm{au}_{\mathbb{Z}_3}$, but they cannot be $\mathbf{1}$-condensed boundaries of $\mathcal{G}\mathrm{au}_{\mathbb{Z}_3}$. This implies that the above boundary is also a canonical boundary (i.e., a minimal $\mathbf{1}$-condensed boundary) of the topological order $\mathcal{G}\mathrm{au}_{\mathbb{Z}_3}$.

In the above, we obtained the $\mathbf{1}\oplus a_1$-condensed boundary of $\mathcal{G}\mathrm{au}_{S_3}$ via the $\mathbf{1}$-condensed boundary of $\mathcal{G}\mathrm{au}_{\mathbb{Z}_3}$. This works since $\mathcal{G}\mathrm{au}_{\mathbb{Z}_3}$ is the $\mathbf{1}\oplus a_1$ condensation-induced topological order from $\mathcal{G}\mathrm{au}_{S_3}$: $(\mathcal{G}\mathrm{au}_{S_3})_{/\mathbf{1}\oplus a_1} = \mathcal{G}\mathrm{au}_{\mathbb{Z}_3}$. In fact, we can directly obtain $\mathbf{1}\oplus a_1$-condensed boundary of $\mathcal{G}\mathrm{au}_{S_3}$. The topological order $\mathcal{G}\mathrm{au}_{S_3}$ has many gapless boundaries, described by various multicomponent partition functions, labeled by the anyons in $\mathcal{G}\mathrm{au}_{S_3}$. One such multicomponent partition function is given by

$$
\begin{aligned}
Z_{\mathbf{1}}^{\mathcal{G}\mathrm{au}_{S_3}} &= |\chi_0^{m6}+\chi_3^{m6}|^2 + |\chi_{\frac{2}{5}}^{m6}+\chi_{\frac{7}{5}}^{m6}|^2,\\
Z_{a_1}^{\mathcal{G}\mathrm{au}_{S_3}} &= |\chi_0^{m6}+\chi_3^{m6}|^2 + |\chi_{\frac{2}{5}}^{m6}+\chi_{\frac{7}{5}}^{m6}|^2,\\
Z_{a_2}^{\mathcal{G}\mathrm{au}_{S_3}} &= |\chi_{\frac{2}{3}}^{m6}|^2 + |\chi_{\frac{1}{15}}^{m6}|^2,\\
Z_{b}^{\mathcal{G}\mathrm{au}_{S_3}} &= |\chi_{\frac{2}{3}}^{m6}|^2 + |\chi_{\frac{1}{15}}^{m6}|^2,\\
Z_{b_1}^{\mathcal{G}\mathrm{au}_{S_3}} &= \chi_0^{m6}\bar\chi_{\frac{2}{3}}^{m6} + \chi_3^{m6}\bar\chi_{\frac{2}{3}}^{m6} + \chi_{\frac{2}{5}}^{m6}\bar\chi_{\frac{1}{15}}^{m6} + \chi_{\frac{7}{5}}^{m6}\bar\chi_{\frac{1}{15}}^{m6},\\
Z_{b_2}^{\mathcal{G}\mathrm{au}_{S_3}} &= \chi_{\frac{2}{3}}^{m6}\bar\chi_0^{m6} + \chi_{\frac{2}{3}}^{m6}\bar\chi_3^{m6} + \chi_{\frac{1}{15}}^{m6}\bar\chi_{\frac{2}{5}}^{m6} + \chi_{\frac{1}{15}}^{m6}\bar\chi_{\frac{7}{5}}^{m6},\\
Z_{c}^{\mathcal{G}\mathrm{au}_{S_3}} &= 0,\\
Z_{c_1}^{\mathcal{G}\mathrm{au}_{S_3}} &= 0.
\end{aligned} \tag{43}
$$

We see that the term $|\chi_0^{m6}|^2$ appear in and only in $Z_{\mathbf{1}}^{\mathcal{G}au_{S_3}}(\tau)$ and $Z_{a_1}^{\mathcal{G}au_{S_3}}(\tau)$. Thus $\mathbf{1}$ and a_1 condense, or more precisely, the condensable algebra is $\mathbf{1}\oplus a_1$. We also see that $Z_c^{\mathcal{G}au_{S_3}} = Z_{c_1}^{\mathcal{G}au_{S_3}} = 0$. So c and c_1 remain gapped on the boundary. These properties suggest that $\mathbf{1}\oplus a_1$ is a condensable algebra and the above boundary is produced by condensing such a condensable algebra. Such a gapless boundary of $\mathcal{G}au_{S_3}$ is the canonical boundary of $(\mathcal{G}au_{S_3})_{/\mathbf{1}\oplus a_1} = \mathcal{G}au_{\mathbb{Z}_3}$ given in Eq. (42).

Similarly, the gapless $\mathbf{1}\oplus a_2$ state, with reduced symmetry TO $(\mathcal{G}au_{S_3})_{/\mathbf{1}\oplus a_2} = \mathcal{G}au_{\mathbb{Z}_2}$, is given by the canonical boundary of $\mathcal{G}au_{\mathbb{Z}_2}$, which is described by a $(c,\bar c) = (\frac12,\frac12)$ Ising CFT. The corresponding multicomponent boundary partition function is labeled by four anyons $\mathbf{1}, e, m, f$ of $\mathcal{G}au_{\mathbb{Z}_2}$:

$$
\begin{aligned}
Z_{\mathbf{1}}^{\mathcal{G}au_{\mathbb{Z}_2}} &= |\chi_0^{\text{Ising}}|^2 + |\chi_{\frac12}^{\text{Ising}}|^2,\\
Z_e^{\mathcal{G}au_{\mathbb{Z}_2}} &= |\chi_{\frac1{16}}^{\text{Ising}}|^2,\\
Z_m^{\mathcal{G}au_{\mathbb{Z}_2}} &= |\chi_{\frac1{16}}^{\text{Ising}}|^2,\\
Z_f^{\mathcal{G}au_{\mathbb{Z}_2}} &= \chi_0^{\text{Ising}}\bar\chi_{\frac12}^{\text{Ising}} + \chi_{\frac12}^{\text{Ising}}\bar\chi_0^{\text{Ising}},
\end{aligned}
\tag{44}
$$

where $\chi_h^{\text{Ising}} = \chi_h^{\text{Ising}}(\tau)$ are conformal characters with conformal dimension h, for (4,3) minimal model (the Ising CFT). The gapless $\mathbf{1}\oplus b$ state is also described by a $(c,\bar c) = (\frac12,\frac12)$ Ising CFT, since its reduced symmetry TO is also given by $\mathcal{G}au_{\mathbb{Z}_2}$.

Again, the gapless $\mathbf{1}\oplus a_2$ state, with reduced symmetry TO $(\mathcal{G}au_{S_3})_{/\mathbf{1}\oplus a_2} = \mathcal{G}au_{\mathbb{Z}_2}$, can also be given directly by the $\mathbf{1}\oplus a_2$-condensed boundary of $\mathcal{G}au_{S_3}$. Indeed, one of the gapless boundary of $\mathcal{G}au_{S_3}$ is given by the following multicomponent partition function:

$$
\begin{aligned}
Z_{\mathbf{1}}^{\mathcal{G}au_{S_3}} &= |\chi_0^{m4}|^2 + |\chi_{\frac12}^{m4}|^2,\\
Z_{a_1}^{\mathcal{G}au_{S_3}} &= |\chi_{\frac1{16}}^{m4}|^2,\\
Z_{a_2}^{\mathcal{G}au_{S_3}} &= |\chi_0^{m4}|^2 + |\chi_{\frac1{16}}^{m4}|^2 + |\chi_{\frac12}^{m4}|^2,\\
Z_b^{\mathcal{G}au_{S_3}} &= 0, \quad Z_{b_1}^{\mathcal{G}au_{S_3}} = 0,\\
Z_{b_2}^{\mathcal{G}au_{S_3}} &= 0, \quad Z_c^{\mathcal{G}au_{S_3}} = |\chi_{\frac1{16}}^{m4}|^2,\\
Z_{c_1}^{\mathcal{G}au_{S_3}} &= \chi_0^{m4}\bar\chi_{\frac12}^{m4} + \chi_{\frac12}^{m4}\bar\chi_0^{m4}.
\end{aligned}
\tag{45}
$$

The above is a $\mathbf{1}\oplus a_2$-condensed boundary since only $Z_{\mathbf{1}}^{\mathcal{G}au_{S_3}}$ and $Z_{a_2}^{\mathcal{G}au_{S_3}}$ contain $|\chi_0^{m4}|^2$ term.

To obtain the minimal gapless $\mathbf{1}$ state with the full symmetry TO $\mathcal{G}au_{S_3}$, we find that, in addition to the gapless boundary described by Eq. (43), $\mathcal{G}au_{S_3}$ has another gapless boundary described by the following multicomponent partition function, labeled by the anyons in $\mathcal{G}au_{S_3}$:

$$
\begin{aligned}
Z_{\mathbf{1}}^{\mathcal{G}au_{S_3}} &= |\chi_0^{m6}|^2 + |\chi_3^{m6}|^2 + |\chi_{\frac25}^{m6}|^2 + |\chi_{\frac75}^{m6}|^2,\\
Z_{a_1}^{\mathcal{G}au_{S_3}} &= \chi_0^{m6}\bar\chi_3^{m6} + \chi_3^{m6}\bar\chi_0^{m6} + \chi_{\frac25}^{m6}\bar\chi_{\frac75}^{m6} + \chi_{\frac75}^{m6}\bar\chi_{\frac25}^{m6},\\
Z_{a_2}^{\mathcal{G}au_{S_3}} &= |\chi_{\frac23}^{m6}|^2 + |\chi_{\frac1{15}}^{m6}|^2,\\
Z_b^{\mathcal{G}au_{S_3}} &= |\chi_{\frac23}^{m6}|^2 + |\chi_{\frac1{15}}^{m6}|^2,\\
Z_{b_1}^{\mathcal{G}au_{S_3}} &= \chi_0^{m6}\bar\chi_{\frac23}^{m6} + \chi_3^{m6}\bar\chi_{\frac23}^{m6} + \chi_{\frac25}^{m6}\bar\chi_{\frac1{15}}^{m6} + \chi_{\frac75}^{m6}\bar\chi_{\frac1{15}}^{m6},\\
Z_{b_2}^{\mathcal{G}au_{S_3}} &= \chi_{\frac23}^{m6}\bar\chi_0^{m6} + \chi_{\frac23}^{m6}\bar\chi_3^{m6} + \chi_{\frac1{15}}^{m6}\bar\chi_{\frac25}^{m6} + \chi_{\frac1{15}}^{m6}\bar\chi_{\frac75}^{m6},\\
Z_c^{\mathcal{G}au_{S_3}} &= |\chi_{\frac18}^{m6}|^2 + |\chi_{\frac{13}8}^{m6}|^2 + |\chi_{\frac1{40}}^{m6}|^2 + |\chi_{\frac{21}{40}}^{m6}|^2,\\
Z_{c_1}^{\mathcal{G}au_{S_3}} &= \chi_{\frac18}^{m6}\bar\chi_{\frac{13}8}^{m6} + \chi_{\frac{13}8}^{m6}\bar\chi_{\frac18}^{m6} + \chi_{\frac1{40}}^{m6}\bar\chi_{\frac{21}{40}}^{m6} + \chi_{\frac{21}{40}}^{m6}\bar\chi_{\frac1{40}}^{m6},
\end{aligned}
\tag{46}
$$

which is a $(c,\bar c) = (\frac45,\frac45)$ CFT. In contrast to the boundary (43), the above boundary is $\mathbf{1}$-condensed because only $Z_{\mathbf{1}}$ contains the term $|\chi_0^{m6}|^2$. The boundary is $\mathbf{1}$-condensed also because no components of the partition function is zero, i.e., no other topological excitations in $\mathcal{G}au_{S_3}$ remain gapped on the boundary. These gapped topological excitations on the boundary become confined in the condensation-induced topological order $(\mathcal{G}au_{S_3})_{/\mathcal{A}}$. If there are no confined topological excitations, then $\mathcal{A}$ must be trivial $\mathcal{A} = \mathbf{1}$.

We have checked other CFT's with smaller central charges, and find that those CFT's cannot be $\mathbf{1}$-condensed boundaries of the topological order $\mathcal{G}au_{S_3}$ (see Appendix E). Among $(c,\bar c) = (\frac45,\frac45)$ CFT's, the above boundary is the only $\mathbf{1}$-condensed boundary. This implies that the above boundary is also a canonical boundary (i.e., a minimal $\mathbf{1}$-condensed boundary) of the topological order $\mathcal{G}au_{S_3}$, and is the only canonical boundary.

We remark that although the gapless $\mathbf{1}\oplus a_1$ state in Eq. (43) and the gapless $\mathbf{1}$ state in Eq. (46) are both related to (6,5) minimal model with the same central charge $(c,\bar c) = (\frac45,\frac45)$, they are described by different CFT's. For example, from the partition function $Z_{a_1}^{\mathcal{G}au_{S_3}}$, we find that the operator carrying the one-dimensional representation a_1 of S_3 has a minimal scaling dimension of 0 for the $\mathbf{1}\oplus a_1$ state (since a_1 is condensed) and has a minimal scaling dimension of $\frac25+\frac75=\frac95$ for the $\mathbf{1}$ state.

The above three types of gapless states Eqs. (42), (44), and (46), together with four types pf gapped states, are the gapped and gapless phases for systems with S_3 symmetry. They are summarized in Table II.

D. An automorphism in symmetry TO $\mathcal{G}au_{S_3}$

The excitations in the 2 + 1D topological order $\mathcal{G}au_{S_3}$ (i.e., the symmetry TO) are listed in Table I, together with their quantum dimensions, topological spins, and fusion rules. From the table, we see that the symmetry TO $\mathcal{G}au_{S_3}$ has a automorphism that exchange $a_2 \leftrightarrow b$. In fact, the S^{S_3} and T^{S_3} matrices for the $\mathcal{G}au_{S_3}$ topological order are invariant under the exchange $a_2 \leftrightarrow b$. From the correspondence between the condensable algebras and phases of matter (37), we see that the automorphism exchanges S_3 phase with $\mathbb{Z}_2$ phase, and $\mathbb{Z}_3$ phase with $\mathbb{Z}_1$ phase. In other words, the automorphism flips the phase diagrams, Figs. 13–15, as well as Fig. 16, horizontally.

As a result, the transitions $S_3 \leftrightarrow \mathbb{Z}_1$ and $\mathbb{Z}_3 \leftrightarrow \mathbb{Z}_2$ are related, i.e., they are either both first order, both stably con-

TABLE II. Possible gapped and gapless states for systems with S_3 symmetry. The most stable gapped or gapless state with reduced symmetry TO $\mathcal{M}_{/\mathcal{A}}$ is given by the canonical boundary of $\mathcal{M}_{/\mathcal{A}}$.

Condensable algebra $\mathcal{A}$	Reduced symmetry TO $\mathcal{M}_{/\mathcal{A}}$	Most stable low energy state
$\mathbf{1}\oplus a_1\oplus 2a_2$	$(\mathcal{G}\text{au}_{S_3})_{/\mathbf{1}\oplus a_1\oplus 2a_2}=$ trivial	Gapped $\mathbb{Z}_1$ state (S_3 completely broken)
$\mathbf{1}\oplus a_1\oplus 2b$	$(\mathcal{G}\text{au}_{S_3})_{/\mathbf{1}\oplus a_1\oplus 2b}=$ trivial	Gapped $\mathbb{Z}_3$ state (S_3 broken to $\mathbb{Z}_3$)
$\mathbf{1}\oplus a_2\oplus c$	$(\mathcal{G}\text{au}_{S_3})_{/\mathbf{1}\oplus a_2\oplus c}=$ trivial	Gapped $\mathbb{Z}_2$ state (S_3 broken to $\mathbb{Z}_2$)
$\mathbf{1}\oplus b\oplus c$	$(\mathcal{G}\text{au}_{S_3})_{/\mathbf{1}\oplus b\oplus c}=$ trivial	Gapped S_3-symmetric state
$\mathbf{1}\oplus a_1$	$(\mathcal{G}\text{au}_{S_3})_{/\mathbf{1}\oplus a_1}=\mathcal{G}\text{au}_{\mathbb{Z}_3}$	The $(c,\bar{c})=(\frac{4}{5},\frac{4}{5})$ CFT (42)
$\mathbf{1}\oplus a_2$	$(\mathcal{G}\text{au}_{S_3})_{/\mathbf{1}\oplus a_2}=\mathcal{G}\text{au}_{\mathbb{Z}_2}$	The $(c,\bar{c})=(\frac{1}{2},\frac{1}{2})$ Ising CFT (44)
$\mathbf{1}\oplus b$	$(\mathcal{G}\text{au}_{S_3})_{/\mathbf{1}\oplus b}=\mathcal{G}\text{au}_{\mathbb{Z}_2}$	The $(c,\bar{c})=(\frac{1}{2},\frac{1}{2})$ Ising CFT (44)
$\mathbf{1}$	$(\mathcal{G}\text{au}_{S_3})_{/\mathbf{1}}=\mathcal{G}\text{au}_{S_3}$	The $(c,\bar{c})=(\frac{4}{5},\frac{4}{5})$ CFT (46)

tinuous, or both unstably continuous. More precisely, if the transition $S_3\leftrightarrow\mathbb{Z}_1$ is continuous in a S_3 symmetric system, then there is exist another S_3 symmetric system where the transition $\mathbb{Z}_3\leftrightarrow\mathbb{Z}_2$ is also continuous, and the two continuous transitions are described by the same CFT.

E. A symmetry TO approach for phase transitions

We have used symmetry TO approach to study possible gapped and gapless states in S_3 symmetric systems. Now let us discuss a more difficult problem: how are these gapped and gapless states connected by continuous phase transitions and what are the critical points at the transitions? The gapless states Eqs. (42), (44), (46), and others constructed from (5,4) and (7,6) minimal models and $(c,\bar{c})\geqslant(1,1)$ CFTs should describe the (multi)critical points for the transitions between the four gapped phases, S_3 phase, $\mathbb{Z}_3$ phase, $\mathbb{Z}_2$ phase, and $\mathbb{Z}_1$ phase (see Fig. 19). However, which pair of gapped states are connected by which gapless state, as the critical point of the continuous transition?

To address this issue, we first consider the gapless $\mathbf{1}\oplus a_1$ state, which is described by the canonical boundary of $(\mathcal{G}\text{au}_{S_3})_{/\mathbf{1}\oplus a_1}=\mathcal{G}\text{au}_{\mathbb{Z}_3}$ topological order (i.e., 2+1D $\mathbb{Z}_3$ gauge theory). Its multicomponent boundary partition function is given by Eq. (42). From the $|\chi^{m6}_{\frac{2}{5}}+\chi^{m6}_{\frac{7}{5}}|^2$ term in $Z_\mathbf{1}$ in Eq. (42), we see that there is only one $\mathbb{Z}_3$ symmetric relevant operator, which has a scaling dimension $(h,\bar{h})=(\frac{2}{5},\frac{2}{5})$. So the gapless $\mathbf{1}\oplus a_1$ state has only one relevant direction. To see what kind of phase transition the relevant operator induces, we note that the condensable algebra $\mathbf{1}\oplus a_1$ only allows one competing pair (a_2,b). So the single relevant direction must correspond to the switching between the two condensations of the competing pair (a_2,b). This induces a stable continuous phase transition between the $\mathbf{1}\oplus a_1\oplus a_2\oplus\cdots=\mathbf{1}\oplus a_1\oplus 2a_2$ state (the $\mathbb{Z}_1$ state) and $\mathbf{1}\oplus a_1\oplus b\oplus\cdots=\mathbf{1}\oplus a_1\oplus 2b$ (the $\mathbb{Z}_3$ state). The local phase diagram for such transition is given by Fig. 27(a). Thus the $\mathbb{Z}_3\to\mathbb{Z}_1$ symmetry breaking critical point is described by a $(c,\bar{c})=(\frac{4}{5},\frac{4}{5})$ CFT constructed from (6,5) minimal model. This example demonstrates how to use symmetry TO to study continuous phase transitions and their critical points.

Next, we consider the gapless $\mathbf{1}\oplus a_2$ state which is the canonical boundary of $(\mathcal{G}\text{au}_{S_3})_{/\mathbf{1}\oplus a_2}=\mathcal{G}\text{au}_{\mathbb{Z}_2}$ (i.e., 2+1D $\mathbb{Z}_2$ gauge theory). Its multicomponent boundary partition function is given by Eq. (44). From the $|\chi^{\text{Ising}}_{\frac{1}{2}}|^2$ term in $Z_\mathbf{1}$, we see that there is only one $\mathbb{Z}_2$ symmetric relevant operator, with a scaling dimension $(h,\bar{h})=(\frac{1}{2},\frac{1}{2})$. So the gapless $\mathbf{1}\oplus a_2$ state has only one relevant direction. To see which phase transition is induced by the relevant operator, we note that the condensable algebra $\mathbf{1}\oplus a_2$ allows only one competing pair (a_1,c). Thus the gapless $\mathbf{1}\oplus a_2$ state is the critical point for a stable continuous phase transition between $\mathbf{1}\oplus a_2\oplus a_1\oplus\cdots=\mathbf{1}\oplus a_1\oplus 2a_2$ state (the $\mathbb{Z}_1$ state) and $\mathbf{1}\oplus a_2\oplus c\oplus\cdots=\mathbf{1}\oplus a_2\oplus c$ state (the $\mathbb{Z}_2$ state), whose local phase diagram is given in Fig. 27(a).

The gapless $\mathbf{1}\oplus b$ state is similar to the gapless $\mathbf{1}\oplus a_2$ state, due to the automorphism of $\mathcal{G}\text{au}_{S_3}$ that exchange a_2 and b. Both are described by $(c,\bar{c})=(\frac{1}{2},\frac{1}{2})$ Ising CFT (44). The gapless $\mathbf{1}\oplus b$ state also allows only one competing pair (a_1,c), and describes a stable continuous phase transition between $\mathbf{1}\oplus b\oplus a_1\oplus\cdots=\mathbf{1}\oplus a_1\oplus 2b$ state (the $\mathbb{Z}_3$ state) and $\mathbf{1}\oplus b\oplus c\oplus\cdots=\mathbf{1}\oplus b\oplus c$ state (the S_3 state). The $S_3\to\mathbb{Z}_3$ symmetry breaking transition looks different from the $\mathbb{Z}_2\to\mathbb{Z}_1$ symmetry breaking transition. However, the above discussion suggests that the two transitions are described by the same critical theory. This result is supported by the standard Ginzburg-Landau theory.

Last, let us consider the gapless $\mathbf{1}$ state and its neighborhood. The state is given by a canonical boundary of $\mathcal{G}\text{au}_{S_3}$, whose multicomponent boundary partition function is given by Eq. (46). The gapless $\mathbf{1}$ state has only one S_3 symmetric relevant operator with dimension $(h,\bar{h})=(\frac{2}{5},\frac{2}{5})$, as one can see from the $|\chi^{m6}_{\frac{2}{5}}|^2$ term in $Z_\mathbf{1}$. However, the condensable algebra $\mathbf{1}$ allows two competing pairs: (a_2,b) and (a_1,c). Which competing pair corresponds to the relevant direction?

If we assume the competing pair (a_1,c) corresponds to the relevant direction, then on one side of transition, the condensable algebra $\mathbf{1}$ is enlarged to include a_1 condensation: $\mathbf{1}\to\mathbf{1}\oplus a_1\oplus\cdots$. Here $\cdots$ represents the additional condensation, after a_1 condense. Such additional condensations must be compatible with a_1 condensation. We have three possible additional condensations: (1) we may get a gapless $\mathbf{1}\oplus a_1\oplus\cdots=\mathbf{1}\oplus a_1$ state (i.e., no additional condensation); (2) we may get a gapped $\mathbf{1}\oplus a_1\oplus\cdots=\mathbf{1}\oplus a_1\oplus 2a_2$ state (i.e., with additional a_2 condensation); and (3) we may get a gapped $\mathbf{1}\oplus a_1\oplus\cdots=\mathbf{1}\oplus a_1\oplus 2b$ state (i.e., with additional b condensation). On the other side of transition where c condenses, we have two possible additional condensations: (1′) we may get a gapped $\mathbf{1}\oplus c\oplus\cdots=\mathbf{1}\oplus a_2\oplus c$ state (i.e., with additional a_2 condensation); (2′) we may get a

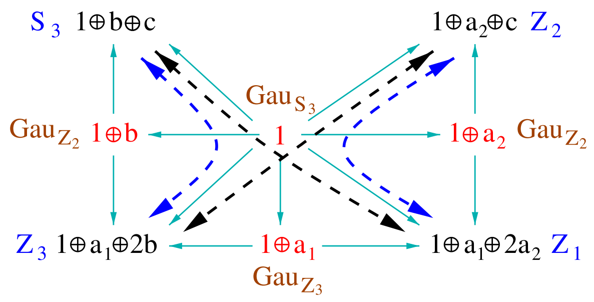


FIG. 17. The gapless **1** state described by Eq. (46) has one relevant direction. If this relevant direction corresponds to the competing pair (a_1, c), the gapless **1** state may be the critical point of the potential continuous transitions represented by the four curved double arrows. These potential continuous transitions are the ones that we cannot rule out at the moment.

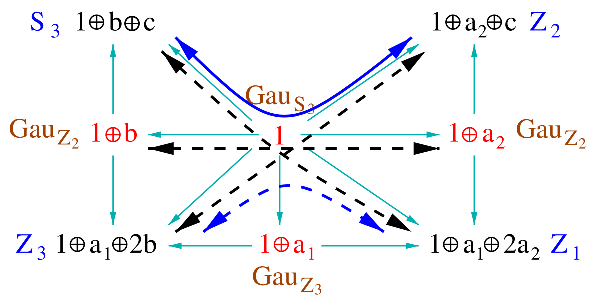


FIG. 18. The gapless **1** state described by Eq. (46) has one relevant direction. If this relevant direction corresponds to the competing pair (a_2, b), the gapless **1** state may be the critical point of the potential continuous transitions represented by the eight curved double-arrows. The dashed curves are the potential continuous transitions that we cannot rule out at the moment. The solid curve is the continuous transition that is known to be realized by three-state Potts model.

gapped $\mathbf{1} \oplus c \oplus \cdots = \mathbf{1} \oplus a_2 \oplus c$ state (i.e., with additional a_2 condensation). Combining the above possibilities, we obtain the following scenarios (see Fig. 17).

(11′) Stable continuous transition between gapless $\mathbf{1} \oplus a_1$ state and $\mathbf{1} \oplus a_2 \oplus c$ state, described by the $(c, \bar{c}) = (\frac{4}{5}, \frac{4}{5})$ CFT (46). Further instability from dangerously irrelevant operators may change the gapless $\mathbf{1} \oplus a_1$ state to $\mathbb{Z}_3$ state or $\mathbb{Z}_1$ state. [Not likely. This scenario assumes that the condensation of c also induce the condensation of a_2. As we switch the condensation of c to the condensation of a_1, the condensation of a_1 is compatible with the condensation of a_2 and does not suppress the condensation of a_2. The condensation of a_2 will destabilize the gapless $\mathbf{1} \oplus a_1$ state and change it to the gapped $\mathbf{1} \oplus a_1 \oplus a_2$ state. This turns the scenario (11′) to scenario (21′)].

(12′) Stable continuous transition between gapless $\mathbf{1} \oplus a_1$ state and $\mathbf{1} \oplus b \oplus c$ state, described by the $(c, \bar{c}) = (\frac{4}{5}, \frac{4}{5})$ CFT (46). Further instability from dangerous irrelevant operators may change the gapless $\mathbf{1} \oplus a_1$ state to $\mathbb{Z}_3$ state or $\mathbb{Z}_1$ state. [Not likely. This scenario assumes that the condensation of c also induce the condensation of b. As we switch the condensation of c to the condensation of a_1, the condensation of a_1 is compatible with the condensation of b and does not suppress the condensation of b. The condensation of b will destabilize the gapless $\mathbf{1} \oplus a_1$ state and change it to the gapped $\mathbf{1} \oplus a_1 \oplus b$ state. This turns the scenario (12′) to scenario (32′)].

(21′) Stable continuous transition between $\mathbf{1} \oplus a_1 \oplus 2a_2$ and $\mathbf{1} \oplus a_2 \oplus c$ states, described the $(c, \bar{c}) = (\frac{4}{5}, \frac{4}{5})$ CFT (46).

(22′) Stable continuous transition between $\mathbf{1} \oplus a_1 \oplus 2a_2$ and $\mathbf{1} \oplus b \oplus c$ states, described by the $(c, \bar{c}) = (\frac{4}{5}, \frac{4}{5})$ CFT (46).

(31′) Stable continuous transition between $\mathbf{1} \oplus a_1 \oplus 2b$ and $\mathbf{1} \oplus a_2 \oplus c$ states, described by the $(c, \bar{c}) = (\frac{4}{5}, \frac{4}{5})$ CFT (46).

(32′) Stable continuous transition between $\mathbf{1} \oplus a_1 \oplus 2b$ and $\mathbf{1} \oplus b \oplus c$ states, described by the $(c, \bar{c}) = (\frac{4}{5}, \frac{4}{5})$ CFT (46).

Some scenarios, (11′) and (12′), are not likely. We remark that the above scenarios may not be mutually exclusive. Different scenarios may be realized at different parts of the phase diagram. We also remark that the scenario (21′), if realized, will represent a nonIsing critical point for the transition between the $\mathbb{Z}_2$ state and $\mathbb{Z}_1$ state. This scenario represents a mechanism that two phases may be connected by different continuous transitions described by different CFTs.

Next, we assume the relevant direction corresponds to the competing pair (a_2, b). On one side of transition, a_2 condenses, which may give rise to the following possible states: (a) the gapless $\mathbf{1} \oplus a_2 \oplus \cdots = \mathbf{1} \oplus a_2$ state; (b) the gapped $\mathbf{1} \oplus a_2 \oplus \cdots = \mathbf{1} \oplus a_1 \oplus 2a_2$ state; and (c) the gapped $\mathbf{1} \oplus a_2 \oplus \cdots = \mathbf{1} \oplus a_2 \oplus c$ state. On one side of transition, b condenses, which may give rise to the following possible states: (a′) the gapless $\mathbf{1} \oplus b \oplus \cdots = \mathbf{1} \oplus b$ state; (b′) the gapped $\mathbf{1} \oplus b \oplus \cdots = \mathbf{1} \oplus a_1 \oplus 2b$ state; (c′) the gapped $\mathbf{1} \oplus b \oplus \cdots = \mathbf{1} \oplus b \oplus c$ state. Combining the above possibilities, we obtain the following scenarios (see Fig. 18).

(aa′) Stable continuous transition between gapless $\mathbf{1} \oplus a_2$ state and gapless $\mathbf{1} \oplus b$ state, described by the $(c, \bar{c}) = (\frac{4}{5}, \frac{4}{5})$ CFT (46). Further instability from dangerous irrelevant operators change the gapless $\mathbf{1} \oplus a_2$ state to $\mathbb{Z}_2$ or $\mathbb{Z}_1$ state, and change the gapless $\mathbf{1} \oplus b$ state to S_3 or $\mathbb{Z}_3$ state.

(ab′) Stable continuous transition between gapless $\mathbf{1} \oplus a_2$ state and gapped $\mathbf{1} \oplus a_1 \oplus 2b$ state, described by the $(c, \bar{c}) = (\frac{4}{5}, \frac{4}{5})$ CFT (46). Further instability from dangerously irrelevant operators change the gapless $\mathbf{1} \oplus a_2$ state to $\mathbb{Z}_2$ or $\mathbb{Z}_1$ state. [Not likely. See discussion in scenario (11′).]

(ac′) Stable continuous transition between gapless $\mathbf{1} \oplus a_2$ state and gapped S_3-$\mathbf{1} \oplus b \oplus c$ state, described by the $(c, \bar{c}) = (\frac{4}{5}, \frac{4}{5})$ CFT (46). Further instability from dangerous irrelevant operators change the gapless $\mathbf{1} \oplus a_2$ state to $\mathbb{Z}_2$ or $\mathbb{Z}_1$ state. [Not likely. See discussion in scenario (11′).]

(ba′) Stable continuous transition between gapped $\mathbf{1} \oplus a_1 \oplus 2a_2$ state and gapless $\mathbf{1} \oplus b$ state, described by the $(c, \bar{c}) = (\frac{4}{5}, \frac{4}{5})$ CFT (46). Further instability from dangerously irrelevant operators change the gapless $\mathbf{1} \oplus b$ state to S_3 or $\mathbb{Z}_3$ state. [Not likely. See discussion in scenario (11′).]

(bb′) Stable continuous transition between gapped $\mathbb{Z}_1$-$\mathbf{1} \oplus a_1 \oplus 2a_2$ state and gapped $\mathbb{Z}_3$-$\mathbf{1} \oplus a_1 \oplus 2b$ state, described by the $(c, \bar{c}) = (\frac{4}{5}, \frac{4}{5})$ CFT (46). [Not valid. Such a transition should be described by a different $(c, \bar{c}) = (\frac{4}{5}, \frac{4}{5})$ CFT (42).]

(bc′) Stable continuous transition between gapped $\mathbb{Z}_1$-$\mathbf{1} \oplus a_1 \oplus 2a_2$ state and gapped S_3-$\mathbf{1} \oplus b \oplus c$ state, described by the $(c, \bar{c}) = (\frac{4}{5}, \frac{4}{5})$ CFT (46).

(ca′) Stable continuous transition between gapped $\mathbb{Z}_2$-$\mathbf{1}\oplus a_2\oplus c$ state and gapless $\mathbf{1}\oplus b$ state, described by the $(c,\bar{c})=(\frac{4}{5},\frac{4}{5})$ CFT (46). Further instability from dangerously irrelevant operators change the gapless $\mathbf{1}\oplus b$ state to S_3 state or $\mathbb{Z}_3$ state. [Not likely. See discussion in scenario (11′).]

(cb′) Stable continuous transition between gapped $\mathbb{Z}_2$-$\mathbf{1}\oplus a_2\oplus c$ state and gapped $\mathbb{Z}_3$-$\mathbf{1}\oplus a_1\oplus 2b$ state, described by the $(c,\bar{c})=(\frac{4}{5},\frac{4}{5})$ CFT (46).

(cc′) Stable continuous transition between gapped $\mathbb{Z}_2$-$\mathbf{1}\oplus a_2\oplus c$ state and gapped S_3-$\mathbf{1}\oplus b\oplus c$ state, described by the $(c,\bar{c})=(\frac{4}{5},\frac{4}{5})$ CFT (46).

We believe that scenario (cc′) is realized in the three-state Potts model, which has a $S_3\leftrightarrow\mathbb{Z}_2$ transition described a $(c,\bar{c})=(\frac{4}{5},\frac{4}{5})$ CFT. We believe such a CFT to be the one given in (46), rather than the $(c,\bar{c})=(\frac{4}{5},\frac{4}{5})$ CFT given in (43).

Note that the stable continuous $S_3\leftrightarrow\mathbb{Z}_1$ transition should be described by a $\mathbf{1}$-condensed boundary of $\mathcal{G}\mathrm{au}_{S_3}$ with one and only one S_3 symmetric operator. The $(c,\bar{c})=(\frac{4}{5},\frac{4}{5})$ CFT (46) is one such $\mathbf{1}$-condensed boundary. But such a CFT is already used to described the stable continuous $S_3\leftrightarrow\mathbb{Z}_2$ transition. We need to find another $\mathbf{1}$-condensed boundary of $\mathcal{G}\mathrm{au}_{S_3}$ to describe the stable continuous $S_3\leftrightarrow\mathbb{Z}_1$ transition.

Summarizing the above result and assuming the scenario (cc′), we obtain several possible global phase diagrams. One of them is Fig. 19(a) and another is Fig. 19(b). Both possibilities are realized by numerical calculations in Figs. 14 and 15. The above two global phase diagrams suggest three tricritical points 5, 5′, and 5″. From the phase diagram, we see that tricritical points 5 and 5′ are connected to the S_3 phase. Thus they are $\mathbf{1}$-condensed boundaries of topological order $\mathcal{G}\mathrm{au}_{S_3}$. From the phase diagram, we also see that tricritical point 5″ connects to both $\mathbb{Z}_3$ and $\mathbb{Z}_2$ phases, and thus has both $\mathbb{Z}_3$ and $\mathbb{Z}_2$ symmetries. Therefore tricritical point 5″ has the full S_3 symmetry and is also a $\mathbf{1}$-condensed boundary of topological order $\mathcal{G}\mathrm{au}_{S_3}$.

The three tricritical points 5, 5′, and 5″ are not the canonical boundaries of $\mathcal{G}\mathrm{au}_{S_3}$, since they have two symmetric relevant operators and are more unstable. By examining other $\mathbf{1}$-condensed boundaries of $\mathcal{G}\mathrm{au}_{S_3}$, we find the following multicomponent partition function:

$$
\begin{aligned}
Z_{\mathbf{1}}^{\mathcal{G}\mathrm{au}_{S_3}} &= \left|\chi_0^{m7}\right|^2+\left|\chi_{\frac{1}{7}}^{m7}\right|^2+\left|\chi_{\frac{5}{7}}^{m7}\right|^2+\left|\chi_{\frac{12}{7}}^{m7}\right|^2 \\
&\quad+\left|\chi_{\frac{22}{7}}^{m7}\right|^2+\left|\chi_{5}^{m7}\right|^2, \\
Z_{a_1}^{\mathcal{G}\mathrm{au}_{S_3}} &= \chi_0^{m7}\bar{\chi}_5^{m7}+\chi_{\frac{1}{7}}^{m7}\bar{\chi}_{\frac{22}{7}}^{m7}+\chi_{\frac{5}{7}}^{m7}\bar{\chi}_{\frac{12}{7}}^{m7}+\chi_{\frac{12}{7}}^{m7}\bar{\chi}_{\frac{5}{7}}^{m7} \\
&\quad+\chi_{\frac{22}{7}}^{m7}\bar{\chi}_{\frac{1}{7}}^{m7}+\chi_5^{m7}\bar{\chi}_0^{m7}, \\
Z_{a_2}^{\mathcal{G}\mathrm{au}_{S_3}} &= \left|\chi_{\frac{4}{3}}^{m7}\right|^2+\left|\chi_{\frac{10}{21}}^{m7}\right|^2+\left|\chi_{\frac{1}{21}}^{m7}\right|^2, \\
Z_{b}^{\mathcal{G}\mathrm{au}_{S_3}} &= \left|\chi_{\frac{4}{3}}^{m7}\right|^2+\left|\chi_{\frac{10}{21}}^{m7}\right|^2+\left|\chi_{\frac{1}{21}}^{m7}\right|^2, \\
Z_{b_1}^{\mathcal{G}\mathrm{au}_{S_3}} &= \chi_{\frac{4}{3}}^{m7}\bar{\chi}_0^{m7}+\chi_{\frac{4}{3}}^{m7}\bar{\chi}_5^{m7}+\chi_{\frac{10}{21}}^{m7}\bar{\chi}_{\frac{1}{7}}^{m7}+\chi_{\frac{10}{21}}^{m7}\bar{\chi}_{\frac{22}{7}}^{m7} \\
&\quad+\chi_{\frac{1}{21}}^{m7}\bar{\chi}_{\frac{5}{7}}^{m7}+\chi_{\frac{1}{21}}^{m7}\bar{\chi}_{\frac{12}{7}}^{m7},
\end{aligned}
$$

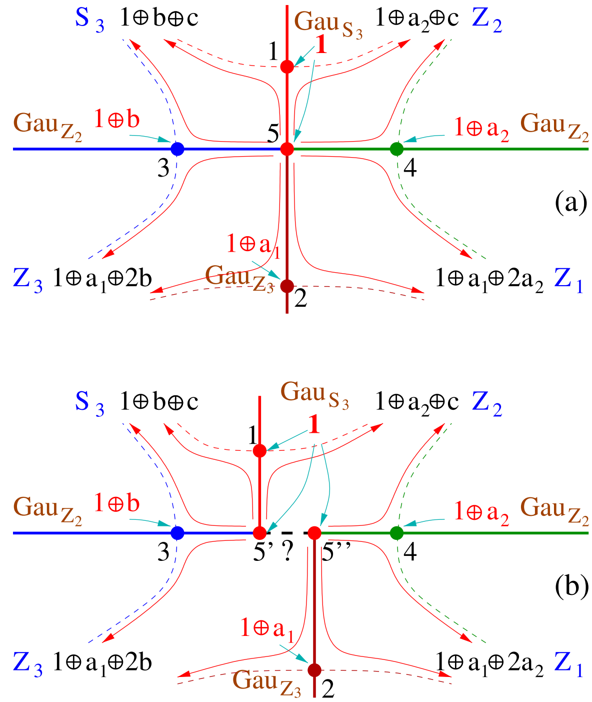


FIG. 19. Two possible global phase diagrams for systems with S_3 symmetry, which contains four gapped phases with unbroken symmetries, S_3, $\mathbb{Z}_3$, $\mathbb{Z}_2$, and $\mathbb{Z}_1$. The curves with arrow represent the RG flow, and the dots are the RG fixed points that correspond to the critical points of phase transitions. The right horizontal line is the space of Hamiltonians whose ground states have a condensation $\mathcal{A}=\mathbf{1}\oplus a_2$ (see Appendix B), which is the basin of attraction of the RG fixed point 4. The left horizontal line is the space of Hamiltonians whose ground states have a condensation $\mathcal{A}=\mathbf{1}\oplus b$, the basin of attraction of the RG fixed point 3. The upper vertical line is the space of Hamiltonians whose ground states have a condensation $\mathcal{A}=\mathbf{1}$, the basin of attraction of the RG fixed point 1. The lower vertical line is the space of Hamiltonians whose ground states have a condensation $\mathcal{A}=\mathbf{1}\oplus a_1$, the basin of attraction of the RG fixed point 2. The critical pointS 3 and 4 are the $(c,\bar{c})=(\frac{1}{2},\frac{1}{2})$ Ising CFT (the canonical boundary of $\mathcal{G}\mathrm{au}_{\mathbb{Z}_2}$ topological order). The critical point 1 is a $(c,\bar{c})=(\frac{4}{5},\frac{4}{5})$ CFT (46) from (6,5) minimal model (the canonical boundary of $\mathcal{G}\mathrm{au}_{S_3}$ topological order). The critical point 2 is another $(c,\bar{c})=(\frac{4}{5},\frac{4}{5})$ CFT (42) also from (6,5) minimal model (the canonical boundary of $\mathcal{G}\mathrm{au}_{\mathbb{Z}_3}$ topological order). The tricritical points 5, 5′, and 5″ are described by gapless $\mathbf{1}$-condensed states with two S_3 symmetric relevant operators. The $(c,\bar{c})=(\frac{5}{6},\frac{5}{6})$ CFT (47) from (7,6) minimal model is one such gapless $\mathbf{1}$-condensed state. We also list the corresponding condensable algebras for these gapped phases and gapless critical points.

$$
\begin{aligned}
Z_{b_2}^{\mathcal{G}\mathrm{au}_{S_3}} &= \chi_0^{m7}\bar{\chi}_{\frac{4}{3}}^{m7}+\chi_{\frac{1}{7}}^{m7}\bar{\chi}_{\frac{10}{21}}^{m7}+\chi_{\frac{5}{7}}^{m7}\bar{\chi}_{\frac{1}{21}}^{m7}+\chi_{\frac{12}{7}}^{m7}\bar{\chi}_{\frac{1}{21}}^{m7} \\
&\quad+\chi_{\frac{22}{7}}^{m7}\bar{\chi}_{\frac{10}{21}}^{m7}+\chi_5^{m7}\bar{\chi}_{\frac{4}{3}}^{m7}, \\
Z_{c}^{\mathcal{G}\mathrm{au}_{S_3}} &= \left|\chi_{\frac{3}{8}}^{m7}\right|^2+\left|\chi_{\frac{1}{56}}^{m7}\right|^2+\left|\chi_{\frac{5}{56}}^{m7}\right|^2+\left|\chi_{\frac{33}{56}}^{m7}\right|^2 \\
&\quad+\left|\chi_{\frac{85}{56}}^{m7}\right|^2+\left|\chi_{\frac{23}{8}}^{m7}\right|^2, \\
Z_{c_1}^{\mathcal{G}\mathrm{au}_{S_3}} &= \chi_{\frac{3}{8}}^{m7}\bar{\chi}_{\frac{23}{8}}^{m7}+\chi_{\frac{1}{56}}^{m7}\bar{\chi}_{\frac{85}{56}}^{m7}+\chi_{\frac{5}{56}}^{m7}\bar{\chi}_{\frac{33}{56}}^{m7}+\chi_{\frac{33}{56}}^{m7}\bar{\chi}_{\frac{5}{56}}^{m7} \\
&\quad+\chi_{\frac{85}{56}}^{m7}\bar{\chi}_{\frac{1}{56}}^{m7}+\chi_{\frac{23}{8}}^{m7}\bar{\chi}_{\frac{3}{8}}^{m7},
\end{aligned}
\tag{47}
$$

which is constructed from the (7,6) minimal model. The above CFT has two relevant operators. It is a candidate CFT for one of the three tricritical points 5, 5′, and 5″, likely the tricritical point 5. We need find more **1**-condensed boundaries of $\mathcal{G}\text{au}_{S_3}$ with two and only two S_3 symmetric relavent operators to describe the other two tricritical points.

We note that the critical points 5, 5′, and 5″, having $\mathbb{Z}_3$ symmetry, are also **1**-condensed boundaries of topological order $\mathcal{G}\text{au}_{\mathbb{Z}_3}$. Indeed, we find the following multicomponent partition function constructed from the (7,6) minimal model, realizing a **1**-condensed boundary of $\mathcal{G}\text{au}_{\mathbb{Z}_3}$:

$$\begin{aligned}
Z_{\mathbf{1}}^{\mathcal{G}\text{au}_{\mathbb{Z}_3}} &= |\chi_0^{m7}|^2 + \chi_0^{m7}\bar\chi_5^{m7} + |\chi_{\frac{1}{7}}^{m7}|^2 + \chi_{\frac{1}{7}}^{m7}\bar\chi_{\frac{22}{7}}^{m7} + |\chi_{\frac{5}{7}}^{m7}|^2 \\
&\quad + \chi_{\frac{5}{7}}^{m7}\bar\chi_{\frac{12}{7}}^{m7} + \chi_{\frac{12}{7}}^{m7}\bar\chi_{\frac{5}{7}}^{m7} + |\chi_{\frac{12}{7}}^{m7}|^2 + \chi_{\frac{22}{7}}^{m7}\bar\chi_{\frac{1}{7}}^{m7} \\
&\quad + |\chi_{\frac{22}{7}}^{m7}|^2 + \chi_5^{m7}\bar\chi_0^{m7} + |\chi_5^{m7}|^2, \\
Z_{e}^{\mathcal{G}\text{au}_{\mathbb{Z}_3}} &= |\chi_{\frac{4}{3}}^{m7}|^2 + |\chi_{\frac{10}{21}}^{m7}|^2 + |\chi_{\frac{1}{21}}^{m7}|^2, \\
Z_{e^2}^{\mathcal{G}\text{au}_{\mathbb{Z}_3}} &= |\chi_{\frac{4}{3}}^{m7}|^2 + |\chi_{\frac{10}{21}}^{m7}|^2 + |\chi_{\frac{1}{21}}^{m7}|^2, \\
Z_{m}^{\mathcal{G}\text{au}_{\mathbb{Z}_3}} &= |\chi_{\frac{4}{3}}^{m7}|^2 + |\chi_{\frac{10}{21}}^{m7}|^2 + |\chi_{\frac{1}{21}}^{m7}|^2, \\
Z_{me}^{\mathcal{G}\text{au}_{\mathbb{Z}_3}} &= \chi_{\frac{4}{3}}^{m7}\bar\chi_0^{m7} + \chi_{\frac{4}{3}}^{m7}\bar\chi_5^{m7} + \chi_{\frac{10}{21}}^{m7}\bar\chi_{\frac{1}{7}}^{m7} + \chi_{\frac{10}{21}}^{m7}\bar\chi_{\frac{22}{7}}^{m7} \\
&\quad + \chi_{\frac{1}{21}}^{m7}\bar\chi_{\frac{5}{7}}^{m7} + \chi_{\frac{1}{21}}^{m7}\bar\chi_{\frac{12}{7}}^{m7}, \\
Z_{me^2}^{\mathcal{G}\text{au}_{\mathbb{Z}_3}} &= \chi_0^{m7}\bar\chi_{\frac{4}{3}}^{m7} + \chi_{\frac{1}{7}}^{m7}\bar\chi_{\frac{10}{21}}^{m7} + \chi_{\frac{5}{7}}^{m7}\bar\chi_{\frac{1}{21}}^{m7} + \chi_{\frac{12}{7}}^{m7}\bar\chi_{\frac{1}{21}}^{m7} \\
&\quad + \chi_{\frac{22}{7}}^{m7}\bar\chi_{\frac{10}{21}}^{m7} + \chi_5^{m7}\bar\chi_{\frac{4}{3}}^{m7}, \\
Z_{m^2}^{\mathcal{G}\text{au}_{\mathbb{Z}_3}} &= |\chi_{\frac{4}{3}}^{m7}|^2 + |\chi_{\frac{10}{21}}^{m7}|^2 + |\chi_{\frac{1}{21}}^{m7}|^2, \\
Z_{m^2e}^{\mathcal{G}\text{au}_{\mathbb{Z}_3}} &= \chi_0^{m7}\bar\chi_{\frac{4}{3}}^{m7} + \chi_{\frac{1}{7}}^{m7}\bar\chi_{\frac{10}{21}}^{m7} + \chi_{\frac{5}{7}}^{m7}\bar\chi_{\frac{1}{21}}^{m7} + \chi_{\frac{12}{7}}^{m7}\bar\chi_{\frac{1}{21}}^{m7} \\
&\quad + \chi_{\frac{22}{7}}^{m7}\bar\chi_{\frac{10}{21}}^{m7} + \chi_5^{m7}\bar\chi_{\frac{4}{3}}^{m7}, \\
Z_{m^2e^2}^{\mathcal{G}\text{au}_{\mathbb{Z}_3}} &= \chi_{\frac{4}{3}}^{m7}\bar\chi_0^{m7} + \chi_{\frac{4}{3}}^{m7}\bar\chi_5^{m7} + \chi_{\frac{10}{21}}^{m7}\bar\chi_{\frac{1}{7}}^{m7} + \chi_{\frac{10}{21}}^{m7}\bar\chi_{\frac{22}{7}}^{m7} \\
&\quad + \chi_{\frac{1}{21}}^{m7}\bar\chi_{\frac{5}{7}}^{m7} + \chi_{\frac{1}{21}}^{m7}\bar\chi_{\frac{12}{7}}^{m7}.
\end{aligned} \tag{48}$$

We also note that three tricritical points 5, 5′, and 5″ can be viewed as a **1**-condensed boundary of topological order $\mathcal{G}\text{au}_{\mathbb{Z}_2}$. We do find the following multicomponent partition function constructed from the (7,6) minimal model, realizing a **1**-condensed boundary of $\mathcal{G}\text{au}_{\mathbb{Z}_2}$:

$$\begin{aligned}
Z_{\mathbf{1}}^{\mathcal{G}\text{au}_{\mathbb{Z}_2}} &= |\chi_0^{m7}|^2 + |\chi_{\frac{1}{7}}^{m7}|^2 + |\chi_{\frac{5}{7}}^{m7}|^2 + |\chi_{\frac{12}{7}}^{m7}|^2 + |\chi_{\frac{22}{7}}^{m7}|^2 \\
&\quad + |\chi_5^{m7}|^2 + |\chi_{\frac{4}{3}}^{m7}|^2 + |\chi_{\frac{10}{21}}^{m7}|^2 + |\chi_{\frac{1}{21}}^{m7}|^2, \\
Z_{e}^{\mathcal{G}\text{au}_{\mathbb{Z}_2}} &= \chi_0^{m7}\bar\chi_5^{m7} + \chi_{\frac{1}{7}}^{m7}\bar\chi_{\frac{22}{7}}^{m7} + \chi_{\frac{5}{7}}^{m7}\bar\chi_{\frac{12}{7}}^{m7} + \chi_{\frac{12}{7}}^{m7}\bar\chi_{\frac{5}{7}}^{m7} \\
&\quad + \chi_{\frac{22}{7}}^{m7}\bar\chi_{\frac{1}{7}}^{m7} + \chi_5^{m7}\bar\chi_0^{m7} + |\chi_{\frac{4}{3}}^{m7}|^2 + |\chi_{\frac{10}{21}}^{m7}|^2 + |\chi_{\frac{1}{21}}^{m7}|^2, \\
Z_{m}^{\mathcal{G}\text{au}_{\mathbb{Z}_2}} &= |\chi_{\frac{3}{8}}^{m7}|^2 + |\chi_{\frac{1}{56}}^{m7}|^2 + |\chi_{\frac{5}{56}}^{m7}|^2 + |\chi_{\frac{33}{56}}^{m7}|^2 + |\chi_{\frac{85}{56}}^{m7}|^2 + |\chi_{\frac{23}{8}}^{m7}|^2, \\
Z_{f}^{\mathcal{G}\text{au}_{\mathbb{Z}_2}} &= \chi_{\frac{3}{8}}^{m7}\bar\chi_{\frac{23}{8}}^{m7} + \chi_{\frac{1}{56}}^{m7}\bar\chi_{\frac{85}{56}}^{m7} + \chi_{\frac{5}{56}}^{m7}\bar\chi_{\frac{33}{56}}^{m7} + \chi_{\frac{33}{56}}^{m7}\bar\chi_{\frac{5}{56}}^{m7} \\
&\quad + \chi_{\frac{85}{56}}^{m7}\bar\chi_{\frac{1}{56}}^{m7} + \chi_{\frac{23}{8}}^{m7}\bar\chi_{\frac{3}{8}}^{m7}.
\end{aligned} \tag{49}$$

The above three multicomponent partition functions are closely related. In fact,

$$\begin{aligned}
Z_{\mathbf{1}}^{\mathcal{G}\text{au}_{S_3}} &+ Z_{a_1}^{\mathcal{G}\text{au}_{S_3}} + 2Z_{a_2}^{\mathcal{G}\text{au}_{S_3}} \\
&= Z_{\mathbf{1}}^{\mathcal{G}\text{au}_{\mathbb{Z}_3}} + Z_{e}^{\mathcal{G}\text{au}_{\mathbb{Z}_3}} + Z_{e^2}^{\mathcal{G}\text{au}_{\mathbb{Z}_3}} \\
&= Z_{\mathbf{1}}^{\mathcal{G}\text{au}_{\mathbb{Z}_2}} + Z_{e}^{\mathcal{G}\text{au}_{\mathbb{Z}_2}}.
\end{aligned} \tag{50}$$

This suggests that the three CFT's, Eqs. (47)–(49), are actually the same CFT. This allows us to conclude that the CFT (47) can be a candidate for one of three tricritical points 5, 5′, and 5″ in Fig. 19. Certainly, it is also possible that the three tricritical points 5, 5′, and 5″ are described by CFT's with $(c, \bar c) \geqslant (1, 1)$.

VI. 1 + 1D ANOMALOUS S_3 SYMMETRY

In 1 + 1D, the anomalies for S_3 symmetry are classified by $H^3(S_3; \mathbb{R}/\mathbb{Z}) = \mathbb{Z}_3 \times \mathbb{Z}_2 \simeq \mathbb{Z}_6$.[19] We label those anomalies by $m \in \{0, 1, 2, 3, 4, 5\}$. The symmetry TO for an anomalous S_3 symmetry, $S_3^{(m)}$, is given by a topological order $\mathcal{G}\text{au}_{S_3}^{(m)}$ that is described in the IR limit by the 2 + 1D Dijkgraaf-Witten gauge theory [120] with gauge charges. In this section, we will use these symmetry TOs to study the 1 + 1D gapped and gapless states with anomalous S_3 symmetry.

A. Anomalous $S_3^{(1)}$ symmetry

The $\mathcal{G}\text{au}_{S_3}^{(1)}$ topological order has anyons given by

$$\begin{array}{rcccccccc}
\text{anyons}: & \mathbf{1} & a_1 & a_2 & b & b_1 & b_2 & c & c_1 \\
d_a: & 1 & 1 & 2 & 2 & 2 & 2 & 3 & 3 \\
s_a: & 0 & 0 & 0 & \frac{1}{9} & \frac{4}{9} & \frac{7}{9} & \frac{1}{4} & \frac{3}{4}
\end{array} \tag{51}$$

The potential condensable algebras of $\mathcal{G}\text{au}_{S_3}^{(1)}$ topological order are given by

$$\mathbf{1} \oplus a_1 \oplus 2a_2, \quad \mathbf{1} \oplus a_2, \quad \mathbf{1} \oplus a_1, \quad \mathbf{1}. \tag{52}$$

The condensable algebra $\mathbf{1} \oplus a_1 \oplus 2a_2$ is Lagrangian, and gives rise to a gapped state that break the $S_3^{(1)}$ symmetry completely. This is the only gapped state allowed by the anomalous $S_3^{(1)}$ symmetry.

The potential condensable algebra $\mathbf{1} \oplus a_2$ is not Lagrangian. If it is a valid condensable algebra, its condensation will induce a 2 + 1D topological order, that has a canonical domain wall with $\mathcal{G}\text{au}_{S_3}^{(1)}$. Indeed, we find a canonical domain wall between $\mathcal{G}\text{au}_{S_3}^{(1)}$ and $\mathcal{M}_{\text{DS}}$. Here $\mathcal{M}_{\text{DS}}$ is the double-semion topological order, which has excitations $\mathbf{1}, b, s_+, s_-$ with spins $s_a = 0, 0, \frac{1}{4}, \frac{3}{4}$. The canonical domain wall is given by

$$\left(A^{ai}_{\mathcal{G}\text{au}_{S_3}^{(1)}|\mathcal{M}_{\text{DS}}}\right) = \begin{pmatrix}
1 & 0 & 0 & 0 & \mathbf{1} \\
0 & 1 & 0 & 0 & a_1 \\
1 & 1 & 0 & 0 & a_2 \\
0 & 0 & 0 & 0 & b \\
0 & 0 & 0 & 0 & b_1 \\
0 & 0 & 0 & 0 & b_2 \\
0 & 0 & 0 & 1 & c \\
0 & 0 & 1 & 0 & c_1 \\
\mathbf{1} & b & s_+ & s_- &
\end{pmatrix}. \tag{53}$$

From the first row and the first column of $A^{ai}_{\mathcal{G}\text{au}^{(1)}_{S_3}|\mathcal{M}_{\text{DS}}}$, we can see that $\mathcal{M}_{\text{DS}}$ is induced from $\mathcal{G}\text{au}^{(1)}_{S_3}$ via a condensation of $\mathbf{1}\oplus a_2$. This indicates that $\mathbf{1}\oplus a_2$ is a valid condensable algebra, and its condensation induced topological order is

$$\big(\mathcal{G}\text{au}^{(1)}_{S_3}\big)_{/\mathbf{1}\oplus a_2} = \mathcal{M}_{\text{DS}}. \tag{54}$$

The $\mathbf{1}\oplus a_2$ state is the canonical boundary of $\mathcal{M}_{\text{DS}}$, which breaks the S_3 symmetry down to $\mathbb{Z}_2$ symmetry. Despite the symmetry breaking, such a state still must be gapless. To see this, we note that $\mathbf{1}\oplus a_2$ state actually breaks the anomalous $S_3^{(1)}$ symmetry down to anomalous $\mathbb{Z}_2^{(1)}$ symmetry (as implied by the double-semion topological order $\mathcal{M}_{\text{DS}}$). Since the unbroken $\mathbb{Z}_2$ symmetry is anomalous, the $\mathbf{1}\oplus a_2$ state must be gapless since it does not break the anomalous $\mathbb{Z}_2$ symmetry. Such a gapless state is described by the following Lagrangian:

$$\mathcal{L} = \frac{(K^{-1})_{IJ}}{4\pi}\partial_x\phi_I\partial_t\phi_J - V_{IJ}\partial_x\phi_I\partial_x\phi_J,$$
$$u_I = \mathrm{e}^{\,\mathrm{i}\phi_I} \text{ generate all local operators,} \tag{55}$$

with

$$K = (K_{IJ}) = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}. \tag{56}$$

The condensable algebra $\mathbf{1}\oplus a_1$ can induce a 2 + 1D topological order $\mathcal{M}_{K_{04;3}}$, where $\mathcal{M}_{K_{04;3}}$ is an Abelian topological order described by the K matrix [135–137]

$$K_{04;3} = \begin{pmatrix} 0 & 3 \\ 3 & 4 \end{pmatrix}. \tag{57}$$

The nine anyons in $\mathcal{M}_{K_{04;3}}$ have the following spins: 0, 0, 0, $\frac{1}{9}$, $\frac{1}{9}$, $\frac{4}{9}$, $\frac{4}{9}$, $\frac{7}{9}$, and $\frac{7}{9}$. Indeed, we find a canonical domain wall between $\mathcal{G}\text{au}^{(1)}_{S_3}$ and $\mathcal{M}_{K_{04;3}}$ given by

$$\big(A^{ai}_{\mathcal{G}\text{au}^{(1)}_{S_3}|\mathcal{M}_{K_{04;3}}}\big) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & a_2 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & b \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & b_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & b_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_1 \end{pmatrix}. \tag{58}$$

From the first row and the first column of $A^{ai}_{\mathcal{G}\text{au}^{(1)}_{S_3}|\mathcal{M}_{K_{04;3}}}$, we can see that $\mathcal{M}_{K_{04;3}}$ is induced from $\mathcal{G}\text{au}^{(1)}_{S_3}$ via a condensation of $\mathbf{1}\oplus a_1$. This indicates that $\mathbf{1}\oplus a_1$ is a valid condensable algebra, and its condensation-induced topological order is

$$\big(\mathcal{G}\text{au}^{(1)}_{S_3}\big)_{/\mathbf{1}\oplus a_1} = \mathcal{M}_{K_{04;3}}. \tag{59}$$

The $\mathbf{1}\oplus a_1$ state is the canonical boundary of $\mathcal{M}_{K_{04;3}}$, which breaks the anomalous $S_3^{(1)}$ symmetry down to anomalous $\mathbb{Z}_3^{(1)}$ symmetry (as indicated by its symmetry TO $\mathcal{M}_{K_{04;3}}$). The $\mathbf{1}\oplus a_1$ state must be gapless since it does not break the anomalous $\mathbb{Z}_3$ symmetry. Such a gapless state is described by the Lagrangian (55) with K given by Eq. (57).

The $\mathbf{1}$ state is the canonical boundary of $\mathcal{G}\text{au}^{(1)}_{S_3}$, which has the full symmetry TO $\mathcal{G}\text{au}^{(1)}_{S_3}$. Such a state must be gapless since it has a nontrivial reduced symmetry TO. The gapless state is described by the following multicomponent partition function:

$$\begin{aligned}
Z_{\mathbf{1}}^{\mathcal{G}\text{au}^{(1)}_{S_3}} &= \chi^{so(9)_2\times u(1)_2\times\bar{u}(1)_2\times\bar{E}(8)_1}_{1,0;1,0;1,0} + \chi^{so(9)_2\times u(1)_2\times\bar{u}(1)_2\times\bar{E}(8)_1}_{2,1;2,\frac{1}{4};2,-\frac{1}{4}},\\
Z_{a_1}^{\mathcal{G}\text{au}^{(1)}_{S_3}} &= \chi^{so(9)_2\times u(1)_2\times\bar{u}(1)_2\times\bar{E}(8)_1}_{1,0;2,\frac{1}{4};2,-\frac{1}{4}} + \chi^{so(9)_2\times u(1)_2\times\bar{u}(1)_2\times\bar{E}(8)_1}_{2,1;1,0;1,0},\\
Z_{a_2}^{\mathcal{G}\text{au}^{(1)}_{S_3}} &= \chi^{so(9)_2\times u(1)_2\times\bar{u}(1)_2\times\bar{E}(8)_1}_{6,1;1,0;1,0} + \chi^{so(9)_2\times u(1)_2\times\bar{u}(1)_2\times\bar{E}(8)_1}_{6,1;2,\frac{1}{4};2,-\frac{1}{4}},\\
Z_{b}^{\mathcal{G}\text{au}^{(1)}_{S_3}} &= \chi^{so(9)_2\times u(1)_2\times\bar{u}(1)_2\times\bar{E}(8)_1}_{5,\frac{10}{9};1,0;1,0} + \chi^{so(9)_2\times u(1)_2\times\bar{u}(1)_2\times\bar{E}(8)_1}_{5,\frac{10}{9};2,\frac{1}{4};2,-\frac{1}{4}},\\
Z_{b_1}^{\mathcal{G}\text{au}^{(1)}_{S_3}} &= \chi^{so(9)_2\times u(1)_2\times\bar{u}(1)_2\times\bar{E}(8)_1}_{8,\frac{4}{9};1,0;1,0} + \chi^{so(9)_2\times u(1)_2\times\bar{u}(1)_2\times\bar{E}(8)_1}_{8,\frac{4}{9};2,\frac{1}{4};2,-\frac{1}{4}},\\
Z_{b_2}^{\mathcal{G}\text{au}^{(1)}_{S_3}} &= \chi^{so(9)_2\times u(1)_2\times\bar{u}(1)_2\times\bar{E}(8)_1}_{7,\frac{7}{9};1,0;1,0} + \chi^{so(9)_2\times u(1)_2\times\bar{u}(1)_2\times\bar{E}(8)_1}_{7,\frac{7}{9};2,\frac{1}{4};2,-\frac{1}{4}},\\
Z_{c}^{\mathcal{G}\text{au}^{(1)}_{S_3}} &= \chi^{so(9)_2\times u(1)_2\times\bar{u}(1)_2\times\bar{E}(8)_1}_{3,\frac{1}{2};1,0;2,-\frac{1}{4}} + \chi^{so(9)_2\times u(1)_2\times\bar{u}(1)_2\times\bar{E}(8)_1}_{4,1;2,\frac{1}{4};1,0},\\
Z_{c_1}^{\mathcal{G}\text{au}^{(1)}_{S_3}} &= \chi^{so(9)_2\times u(1)_2\times\bar{u}(1)_2\times\bar{E}(8)_1}_{3,\frac{1}{2};2,\frac{1}{4};1,0} + \chi^{so(9)_2\times u(1)_2\times\bar{u}(1)_2\times\bar{E}(8)_1}_{4,1;1,0;2,-\frac{1}{4}}.
\end{aligned} \tag{60}$$

Here $\chi^{\text{CFT}_1\times\text{CFT}_2\times\cdots}_{a_1,h_1;a_2,h_2;\cdots}$ is product of conformal characters of CFT_i for the primary fields labeled by a_i with scaling dimension h_i. For example,

$$\chi^{so(9)_2\times u(1)_2\times\bar{u}(1)_2\times\bar{E}(8)_1}_{2,1;2,\frac{1}{4};2,-\frac{1}{4}} = \chi^{so(9)_2}_{2,1}(\tau)\chi^{u(1)_2}_{2,\frac{1}{4}}(\tau)\chi^{\bar{u}(1)_2}_{2,-\frac{1}{4}}(\bar{\tau})\chi^{\bar{E}(8)_1}(\bar{\tau}), \tag{61}$$

where $\chi^{so(9)_2}_{2,1}(\tau)$ is the conformal character of $so(9)_2$ CFT, for the second primary field with scaling dimension $h = 1$; $\chi^{u(1)_2}_{2,\frac{1}{4}}(\tau)$ is the conformal character of $u(1)_2$ CFT [the chiral boson theory described by K matrix $K = (2)$], for the second primary field with scaling dimension $h = \frac{1}{4}$; $\chi^{\bar{u}(1)_2}_{2,-\frac{1}{4}}(\bar{\tau})$ is the conformal character of $\bar{u}(1)_2$ CFT [the antichiral boson theory described by K matrix $K = (-2)$], for the second primary field with scaling dimension $h = \frac{1}{4}$; $\chi^{\bar{E}(8)_1}$ is the conformal character of $\bar{E}(8)_1$ CFT [the complex conjugate of $E(8)$ level-1 Kac-Moody algebra]. The $\bar{E}(8)_1$ CFT has only one primary field (the identity), whose index is suppressed.

Equation (60) describes a gapless state that does not break the anomalous $S_3^{(1)}$ symmetry (or more precisely, does not maximally condense and trivialize the symmetry TO $\mathcal{G}\text{au}^{(1)}_{S_3}$). The gapless state is described by a $so(9)_2\times u(1)_2\times\bar{u}(1)_2\times\bar{E}(8)_1$ CFT with central charge $(c,\bar{c}) = (9,9)$. Such a CFT is chiral, where right movers and left movers have different dynamics. In particular, the right movers are described by a $so(9)$ level-2 CFT and a U(1) level-2 CFT [i.e., K matrix $K = (2)$]. The left movers are described by a U(1) level-2 CFT [i.e., K matrix $K = (-2)$] and a $E(8)$ level-1 CFT. Such a combined CFT corresponds to a gapless state with no $S_3^{(1)}$ symmetric relevant perturbations; it only has $S_3^{(1)}$ symmetric irrelevant and marginal perturbations. We remark that the primary field for the conformal character (61) in $Z_{\mathbf{1}}^{\mathcal{G}\text{au}^{(1)}_{S_3}}$ appears to be a symmetric relavent operator since its scaling dimension $h+\bar{h} = 1+\frac{1}{4}+\frac{1}{4} = \frac{3}{2} < 2$. However, this operator has $h-\bar{h} = 1+\frac{1}{4}-\frac{1}{4} = 1$, and hence describes

TABLE III. Possible gapped and gapless states for systems with anomalous $S_3^{(1)}$ symmetry.

Condensable algebra $\mathcal{A}$	Reduced symmetry TO $\mathcal{M}_{/\mathcal{A}}$	Stable low energy state
$\mathbf{1}\oplus a_1\oplus 2a_2$	$(\mathcal{G}\mathrm{au}_{S_3}^{(1)})_{/\mathbf{1}\oplus a_1\oplus a_2}=$ trivial	S_3 symmetry breaking gapped $\mathbb{Z}_1$ state
$\mathbf{1}\oplus a_2$	$(\mathcal{G}\mathrm{au}_{S_3}^{(1)})_{/\mathbf{1}\oplus a_2}=\mathcal{M}_{\mathrm{DS}}$	$K=\begin{pmatrix}2&0\\0&-2\end{pmatrix}$ chiral boson theory (55)
$\mathbf{1}\oplus a_1$	$(\mathcal{G}\mathrm{au}_{S_3}^{(1)})_{/\mathbf{1}\oplus a_1}=\mathcal{M}_{K_{04;3}}$	$K=\begin{pmatrix}0&3\\3&4\end{pmatrix}$ chiral boson theory (55)
$\mathbf{1}$	$(\mathcal{G}\mathrm{au}_{S_3}^{(1)})_{/\mathbf{1}}=\mathcal{G}\mathrm{au}_{S_3}^{(1)}$	$(c,\bar{c})=(9,9)$ CFT $so(9)_2\times u(1)_2\times\bar{u}(1)_2\times\bar{E}(8)_1$ (60)

a chiral operator. We recall that a chiral operator, such as $\psi_R\psi_R'$ that couples two free right-moving fermions, cannot open an energy gap even when they are formally relevant. In this paper, we regard them as irrelevant. Thus the anomalous $S_3^{(1)}$ symmetry can give rise to a symmetry protected chiral gapless phase. The gapped and gapless phases for systems with anomalous $S_3^{(1)}$ symmetry is summarized in Table III.

B. Anomalous $S_3^{(2)}$ symmetry

The $\mathcal{G}\mathrm{au}_{S_3}^{(2)}$ topological order has anyons given by

$$\begin{array}{ccccccccc} \text{anyons}: & \mathbf{1} & a_1 & a_2 & b & b_1 & b_2 & c & c_1\\ d_a: & 1 & 1 & 2 & 2 & 2 & 2 & 3 & 3\\ s_a: & 0 & 0 & 0 & \frac{2}{9} & \frac{5}{9} & \frac{8}{9} & 0 & \frac{1}{2}\end{array} \tag{62}$$

The potential condensable algebras of $\mathcal{G}\mathrm{au}_{S_3}^{(2)}$ topological order are given by

$$\mathbf{1}\oplus a_1\oplus 2a_2,\quad \mathbf{1}\oplus a_2\oplus c,\quad \mathbf{1}\oplus a_2,\quad \mathbf{1}\oplus a_1,\quad \mathbf{1}. \tag{63}$$

The condensable algebra $\mathbf{1}\oplus a_1\oplus 2a_2$ is Lagrangian, and gives rise to a gapped state that break the $S_3^{(2)}$ symmetry completely. The condensable algebra $\mathbf{1}\oplus a_2\oplus c$ is also Lagrangian, and gives rise to a gapped state that break the $S_3^{(2)}$ symmetry down to anomaly-free $\mathbb{Z}_2$ symmetry. These are the only two gapped states allowed by the anomalous $S_3^{(2)}$ symmetry.

The condensable algebra $\mathbf{1}\oplus a_2$ is not Lagrangian. We find a canonical domain wall between $\mathcal{G}\mathrm{au}_{S_3}^{(2)}$ and $\mathcal{G}\mathrm{au}_{\mathbb{Z}_2}$:

$$\left(A^{ai}_{\mathcal{G}\mathrm{au}_{S_3}^{(2)}|\mathcal{G}\mathrm{au}_{\mathbb{Z}_2}}\right)=\begin{pmatrix}1&0&0&0&\mathbf{1}\\0&1&0&0&a_1\\1&1&0&0&a_2\\0&0&0&0&b\\0&0&0&0&b_1\\0&0&0&0&b_2\\0&0&1&0&c\\0&0&0&1&c_1\\\mathbf{1}&e&m&f&\end{pmatrix}, \tag{64}$$

which tells us that $\mathcal{G}\mathrm{au}_{\mathbb{Z}_2}$ is induced from $\mathcal{G}\mathrm{au}_{S_3}^{(2)}$ via a condensation of $\mathbf{1}\oplus a_2$. The $\mathbf{1}\oplus a_2$ state is the canonical boundary of $\mathcal{M}_{\mathbb{Z}_2}$, which breaks the anomalous $S_3^{(2)}$ symmetry down to $\mathbb{Z}_2\vee\widetilde{\mathbb{Z}}_2$ symmetry. Such a state must be gapless and is described by $(c,\bar{c})=(\frac{1}{2},\frac{1}{2})$ Ising CFT [48].

The condensable algebra $\mathbf{1}\oplus a_1$ is not Lagrangian. We find a canonical domain wall between $\mathcal{G}\mathrm{au}_{S_3}^{(2)}$ and $\mathcal{M}_{-K_{04;3}}$:

$$\left(A^{ai}_{\mathcal{G}\mathrm{au}_{S_3}^{(2)}|\mathcal{M}_{-K_{04;3}}}\right)=\begin{pmatrix}1&0&0&0&0&0&0&0&0&\mathbf{1}\\1&0&0&0&0&0&0&0&0&a_1\\0&1&1&0&0&0&0&0&0&a_2\\0&0&0&0&0&0&0&1&1&b\\0&0&0&0&0&1&1&0&0&b_1\\0&0&0&1&1&0&0&0&0&b_2\\0&0&0&0&0&0&0&0&0&c\\0&0&0&0&0&0&0&0&0&c_1\end{pmatrix}, \tag{65}$$

which suggests that $\mathcal{M}_{-K_{04;3}}$ is induced from $\mathcal{G}\mathrm{au}_{S_3}^{(2)}$ via a condensation of $\mathbf{1}\oplus a_1$. The $\mathbf{1}\oplus a_1$ state is the canonical boundary of $\mathcal{M}_{-K_{04;3}}$, which breaks the anomalous $S_3^{(2)}$ symmetry down to anomalous $\mathbb{Z}_3^{(1)}$ symmetry. The $\mathbf{1}\oplus a_1$ state must be gapless since it does not break the anomalous $\mathbb{Z}_3$ symmetry. Such a gapless state is described by the Lagrangian (55) with K given by the negative of the K matrix in Eq. (57).

The gapless $\mathbf{1}$ state is a canonical boundary of $\mathcal{G}\mathrm{au}_{S_3}^{(2)}$. What are the properties of such a gapless state? It turns out that a canonical boundary of $\mathcal{G}\mathrm{au}_{S_3}^{(2)}$ is given by the $(c,\bar{c})=(8,8)$ $E(8)_1\times\overline{so}(9)_2$ CFT. In other words, the right movers are described by $E(8)_1$ current algebra and the left movers are described by $so(9)_2$ current algebra. The $E(8)_1$ current algebra has only one conformal character which is modular invariant. The $so(9)_2$ current algebra has $(c,\bar{c})=(0,8)$ and eight conformal characters with the following quantum dimensions (d_a) and scaling dimensions ($\bar{h}_a$)

$$\begin{array}{ccccccccc} \text{characters}: & \mathbf{1} & \bar{a}_1 & \bar{a}_2 & \bar{b} & \bar{b}_1 & \bar{b}_2 & \bar{c} & \bar{c}_1\\ d_a: & 1 & 1 & 2 & 2 & 2 & 2 & 3 & 3\\ \bar{h}_a: & 0 & 1 & 1 & \frac{7}{9} & \frac{4}{9} & \frac{10}{9} & 0 & \frac{1}{2}\end{array} \tag{66}$$

The above quantum dimensions d_a and scaling dimensions($-\bar{h}_a$ mod 1) exactly match those of anyons in $\mathcal{G}\mathrm{au}_{S_3}^{(2)}$ [see Eq. (62)]. Thus the eight conformal characters of $so(9)_2$ transform according to the S and T matrices of $\mathcal{G}\mathrm{au}_{S_3}^{(2)}$. We add the $E(8)_1$ to make $(c,\bar{c})=(8,8)$. This matches the central charge of $\mathcal{G}\mathrm{au}_{S_3}^{(2)}$ that satisfies $c=\bar{c}$. This is why the $E(8)_1\times\overline{so}(9)_2$ CFT is a canonical boundary of $\mathcal{G}\mathrm{au}_{S_3}^{(2)}$. In particular, the multicomponent partition function for the canonical boundary is given by

$$Z_{\mathbf{1}}^{\mathcal{G}\mathrm{au}_{S_3}^{(2)}}=\chi_{1,0}^{E(8)_1\times\overline{so}(9)_2},$$
$$Z_{a_1}^{\mathcal{G}\mathrm{au}_{S_3}^{(2)}}=\chi_{2,-1}^{E(8)_1\times\overline{so}(9)_2},$$
$$Z_{a_2}^{\mathcal{G}\mathrm{au}_{S_3}^{(2)}}=\chi_{6,-1}^{E(8)_1\times\overline{so}(9)_2},$$

TABLE IV. Possible gapped and gapless states for systems with anomalous $S_3^{(2)}$ symmetry.

Condensable algebra $\mathcal{A}$	Reduced symmetry TO $\mathcal{M}_{/\mathcal{A}}$	Stable low energy state
$\mathbf{1}\oplus a_1\oplus 2a_2$	$(\mathcal{G}\mathrm{au}_{S_3}^{(2)})_{/\mathbf{1}\oplus a_1\oplus 2a_2}=$ trivial	S_3 symmetry breaking gapped $\mathbb{Z}_1$ state
$\mathbf{1}\oplus a_2\oplus c$	$(\mathcal{G}\mathrm{au}_{S_3}^{(2)})_{/\mathbf{1}\oplus a_2\oplus c}=$ trivial	gapped $\mathbb{Z}_2$ state
$\mathbf{1}\oplus a_2$	$(\mathcal{G}\mathrm{au}_{S_3}^{(2)})_{/\mathbf{1}\oplus a_2}=\mathcal{G}\mathrm{au}_{\mathbb{Z}_2}$	$(c,\bar c)=(\frac{1}{2},\frac{1}{2})$ Ising CFT
$\mathbf{1}\oplus a_1$	$(\mathcal{G}\mathrm{au}_{S_3}^{(2)})_{/\mathbf{1}\oplus a_1}=\mathcal{M}_{-K_{04;3}}$	$K=-\begin{pmatrix}0&3\\3&4\end{pmatrix}$ chiral boson theory (55)
$\mathbf{1}$	$(\mathcal{G}\mathrm{au}_{S_3}^{(2)})_{/\mathbf{1}}=\mathcal{G}\mathrm{au}_{S_3}^{(2)}$	$(c,\bar c)=(8,8)$ CFT $E(8)_1\times\overline{so}(9)_2$

$$
\begin{aligned}
Z_b^{\mathcal{G}\mathrm{au}_{S_3}^{(2)}} &= \chi_{7,-\frac{7}{9}}^{E(8)_1\times\overline{so}(9)_2}\\
Z_{b_1}^{\mathcal{G}\mathrm{au}_{S_3}^{(2)}} &= \chi_{8,-\frac{4}{9}}^{E(8)_1\times\overline{so}(9)_2},\\
Z_{b_2}^{\mathcal{G}\mathrm{au}_{S_3}^{(2)}} &= \chi_{5,-\frac{10}{9}}^{E(8)_1\times\overline{so}(9)_2},\\
Z_{c}^{\mathcal{G}\mathrm{au}_{S_3}^{(2)}} &= \chi_{4,-1}^{E(8)_1\times\overline{so}(9)_2},\\
Z_{c_1}^{\mathcal{G}\mathrm{au}_{S_3}^{(2)}} &= \chi_{3,-\frac{1}{2}}^{E(8)_1\times\overline{so}(9)_2}.
\end{aligned}
\tag{67}
$$

We would like to point out that the $E(8)_1\times\overline{so}(9)_2$ CFT has no $S_3^{(2)}$ symmetric relevant operators, since the $\mathbf{1}$-component of the multicomponent partition function is given by $Z_{\mathbf{1}}^{\mathcal{G}\mathrm{au}_{S_3}^{(2)}}=\chi_{1,0}^{E(8)_1\times\overline{so}(9)_2}=\chi^{E(8)_1}(\tau)\bar\chi_{1,0}^{so(9)_2}(\bar\tau)$. Apart from the identity operator [the primary field with $(h,\bar h)=(0,0)$], other nonchiral operators (the descendant fields of the current algebra) in this sector have scaling dimensions at least $(h,\bar h)=(1,1)$. The operators are at most marginal. Thus the gapless $\mathbf{1}$-state with the full symmetry TO $\mathcal{G}\mathrm{au}_{S_3}^{(2)}$ is a gapless state that has no unstable deformations, but has marginal deformations. The gapped and gapless phases for systems with anomalous $S_3^{(2)}$ symmetry is summarized in Table IV.

C. Anomalous $S_3^{(3)}$ symmetry

The $\mathcal{G}\mathrm{au}_{S_3}^{(3)}$ topological order has anyons given by

$$
\begin{array}{ccccccccc}
\text{anyons}: & \mathbf{1} & a_1 & a_2 & b & b_1 & b_2 & c & c_1\\
d_a: & 1 & 1 & 2 & 2 & 2 & 2 & 3 & 3\\
s_a: & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{2}{3} & \frac{1}{4} & \frac{3}{4}
\end{array}
\tag{68}
$$

The potential condensable algebras of $\mathcal{G}\mathrm{au}_{S_3}^{(3)}$ topological order are given by

$$
\begin{aligned}
&\mathbf{1}\oplus a_1\oplus 2a_2,\quad \mathbf{1}\oplus a_1\oplus 2b,\\
&\mathbf{1}\oplus a_2,\quad \mathbf{1}\oplus b,\quad \mathbf{1}\oplus a_1,\quad \mathbf{1}.
\end{aligned}
\tag{69}
$$

The condensable algebra $\mathbf{1}\oplus a_1\oplus 2a_2$ is Lagrangian, and gives rise to a gapped state that break the $S_3^{(3)}$ symmetry completely. The condensable algebra $\mathbf{1}\oplus a_1\oplus 2b$ is also Lagrangian, and gives rise to a gapped state that break the $S_3^{(3)}$ symmetry down to anomaly-free $\mathbb{Z}_3$ symmetry. These are the only two gapped states allowed by the anomalous $S_3^{(3)}$ symmetry.

The condensable algebra $\mathbf{1}\oplus a_2$ is not Lagrangian. We find a canonical domain wall between $\mathcal{G}\mathrm{au}_{S_3}^{(3)}$ and $\mathcal{M}_{\mathrm{DS}}$:

$$
\left(A^{ai}_{\mathcal{G}\mathrm{au}_{S_3}^{(3)}|\mathcal{M}_{\mathrm{DS}}}\right)=\begin{pmatrix}
1&0&0&0&\mathbf{1}\\
0&1&0&0&a_1\\
1&1&0&0&a_2\\
0&0&0&0&b\\
0&0&0&0&b_1\\
0&0&0&0&b_2\\
0&0&1&0&c\\
0&0&0&1&c_1\\
\mathbf{1}&b&s_+&s_-&
\end{pmatrix},
\tag{70}
$$

which tells us that $\mathcal{M}_{\mathrm{DS}}$ is induced from $\mathcal{G}\mathrm{au}_{S_3}^{(3)}$ via a condensation of $\mathbf{1}\oplus a_2$. The $\mathbf{1}\oplus a_2$ state is the canonical boundary of $\mathcal{M}_{\mathrm{DS}}$, which breaks the anomalous $S_3^{(3)}$ symmetry down to anomalous $\mathbb{Z}_2^{(1)}$ symmetry. Such a state must be gapless and is described by the Lagrangian (55) with K given by Eq. (56).

The $\mathbf{1}\oplus b$ condensation is similar to the $\mathbf{1}\oplus a_2$ condensation discussed above, due to a $a_2\leftrightarrow b$ automorphism of $\mathcal{G}\mathrm{au}_{S_3}^{(3)}$ topological order. The $\mathbf{1}\oplus b$ state has the full anomalous $S_3^{(3)}$ symmetry where the a_2 excitations are gapped (i.e., the S_3 charges, carrying the two-dimensional representation, are gapped). Such a gapless state is described by the Lagrangian (55) with K given by Eq. (56). We note that $\mathbf{1}$ state also has the full anomalous $S_3^{(3)}$ symmetry. However, in $\mathbf{1}$ state, the a_2 excitations are gapless. So the $\mathbf{1}$ state and $\mathbf{1}\oplus b$ state actually have different symmetry breaking patterns, despite both states have the full anomalous $S_3^{(3)}$ symmetry.

The condensable algebra $\mathbf{1}\oplus a_1$ is not Lagrangian. We find a canonical domain wall between $\mathcal{G}\mathrm{au}_{S_3}^{(3)}$ and $\mathcal{G}\mathrm{au}_{\mathbb{Z}_3}$:

$$
\left(A^{ai}_{\mathcal{G}\mathrm{au}_{S_3}^{(3)}|\mathcal{G}\mathrm{au}_{\mathbb{Z}_3}}\right)=\begin{pmatrix}
1&0&0&0&0&0&0&0&0&\mathbf{1}\\
1&0&0&0&0&0&0&0&0&a_1\\
0&1&1&0&0&0&0&0&0&a_2\\
0&0&0&1&1&0&0&0&0&b\\
0&0&0&0&0&1&1&0&0&b_1\\
0&0&0&0&0&0&0&1&1&b_2\\
0&0&0&0&0&0&0&0&0&c\\
0&0&0&0&0&0&0&0&0&c_1
\end{pmatrix},
\tag{71}
$$

which suggests that $\mathcal{G}\mathrm{au}_{\mathbb{Z}_3}$ is induced from $\mathcal{G}\mathrm{au}_{S_3}^{(3)}$ via a condensation of $\mathbf{1}\oplus a_1$. The $\mathbf{1}\oplus a_1$ state is the canonical boundary of $\mathcal{G}\mathrm{au}_{\mathbb{Z}_3}$, which breaks the anomalous $S_3^{(3)}$

symmetry down to $\mathbb{Z}_3 \vee \widetilde{\mathbb{Z}}_3$ symmetry. The $\mathbf{1} \oplus a_1$ state must be gapless. Such a gapless state is described by the $(c, \bar{c}) = (\frac{4}{5}, \frac{4}{5})$ CFT constructed from (6,5) minimal model [see Eq. (42)].

The $\mathbf{1}$ state is the canonical boundary of $\mathcal{G}\mathrm{au}_{S_3}^{(3)}$, which has the full symmetry TO $\mathcal{G}\mathrm{au}_{S_3}^{(3)}$, which is gapless. The gapless state is described by the following multicomponent partition function:

$$
\begin{aligned}
Z_{\mathbf{1}}^{\mathcal{G}\mathrm{au}_{S_3}^{(3)}} &= \chi^{m6\times u(1)_2\times\bar{m}6\times\bar{u}(1)_2}_{1,0;1,0;1,0;1,0} + \chi^{m6\times u(1)_2\times\bar{m}6\times\bar{u}(1)_2}_{1,0;2,\frac{1}{4};5,-3;2,-\frac{1}{4}} + \chi^{m6\times u(1)_2\times\bar{m}6\times\bar{u}(1)_2}_{5,3;1,0;5,-3;1,0} + \chi^{m6\times u(1)_2\times\bar{m}6\times\bar{u}(1)_2}_{5,3;2,\frac{1}{4};1,0;2,-\frac{1}{4}} \\
&\quad + \chi^{m6\times u(1)_2\times\bar{m}6\times\bar{u}(1)_2}_{6,\frac{2}{5};1,0;6,-\frac{2}{5};1,0} + \chi^{m6\times u(1)_2\times\bar{m}6\times\bar{u}(1)_2}_{6,\frac{2}{5};2,\frac{1}{4};10,-\frac{7}{5};2,-\frac{1}{4}} + \chi^{m6\times u(1)_2\times\bar{m}6\times\bar{u}(1)_2}_{10,\frac{7}{5};1,0;10,-\frac{7}{5};1,0} + \chi^{m6\times u(1)_2\times\bar{m}6\times\bar{u}(1)_2}_{10,\frac{7}{5};2,\frac{1}{4};6,-\frac{2}{5};2,-\frac{1}{4}}, \\
Z_{a_1}^{\mathcal{G}\mathrm{au}_{S_3}^{(3)}} &= \chi^{m6\times u(1)_2\times\bar{m}6\times\bar{u}(1)_2}_{1,0;1,0;5,-3;1,0} + \chi^{m6\times u(1)_2\times\bar{m}6\times\bar{u}(1)_2}_{1,0;2,\frac{1}{4};1,0;2,-\frac{1}{4}} + \chi^{m6\times u(1)_2\times\bar{m}6\times\bar{u}(1)_2}_{5,3;1,0;1,0;1,0} + \chi^{m6\times u(1)_2\times\bar{m}6\times\bar{u}(1)_2}_{5,3;2,\frac{1}{4};5,-3;2,-\frac{1}{4}} \\
&\quad + \chi^{m6\times u(1)_2\times\bar{m}6\times\bar{u}(1)_2}_{6,\frac{2}{5};1,0;10,-\frac{7}{5};1,0} + \chi^{m6\times u(1)_2\times\bar{m}6\times\bar{u}(1)_2}_{6,\frac{2}{5};2,\frac{1}{4};6,-\frac{2}{5};2,-\frac{1}{4}} + \chi^{m6\times u(1)_2\times\bar{m}6\times\bar{u}(1)_2}_{10,\frac{7}{5};1,0;6,-\frac{2}{5};1,0} + \chi^{m6\times u(1)_2\times\bar{m}6\times\bar{u}(1)_2}_{10,\frac{7}{5};2,\frac{1}{4};10,-\frac{7}{5};2,-\frac{1}{4}}, \\
Z_{a_2}^{\mathcal{G}\mathrm{au}_{S_3}^{(3)}} &= \chi^{m6\times u(1)_2\times\bar{m}6\times\bar{u}(1)_2}_{3,\frac{2}{3};1,0;3,-\frac{2}{3};1,0} + \chi^{m6\times u(1)_2\times\bar{m}6\times\bar{u}(1)_2}_{3,\frac{2}{3};2,\frac{1}{4};3,-\frac{2}{3};2,-\frac{1}{4}} + \chi^{m6\times u(1)_2\times\bar{m}6\times\bar{u}(1)_2}_{8,\frac{1}{15};1,0;8,-\frac{1}{15};1,0} + \chi^{m6\times u(1)_2\times\bar{m}6\times\bar{u}(1)_2}_{8,\frac{1}{15};2,\frac{1}{4};8,-\frac{1}{15};2,-\frac{1}{4}}, \\
Z_{b}^{\mathcal{G}\mathrm{au}_{S_3}^{(3)}} &= \chi^{m6\times u(1)_2\times\bar{m}6\times\bar{u}(1)_2}_{3,\frac{2}{3};1,0;3,-\frac{2}{3};1,0} + \chi^{m6\times u(1)_2\times\bar{m}6\times\bar{u}(1)_2}_{3,\frac{2}{3};2,\frac{1}{4};3,-\frac{2}{3};2,-\frac{1}{4}} + \chi^{m6\times u(1)_2\times\bar{m}6\times\bar{u}(1)_2}_{8,\frac{1}{15};1,0;8,-\frac{1}{15};1,0} + \chi^{m6\times u(1)_2\times\bar{m}6\times\bar{u}(1)_2}_{8,\frac{1}{15};2,\frac{1}{4};8,-\frac{1}{15};2,-\frac{1}{4}}, \\
Z_{b_1}^{\mathcal{G}\mathrm{au}_{S_3}^{(3)}} &= \chi^{m6\times u(1)_2\times\bar{m}6\times\bar{u}(1)_2}_{1,0;1,0;3,-\frac{2}{3};1,0} + \chi^{m6\times u(1)_2\times\bar{m}6\times\bar{u}(1)_2}_{1,0;2,\frac{1}{4};3,-\frac{2}{3};2,-\frac{1}{4}} + \chi^{m6\times u(1)_2\times\bar{m}6\times\bar{u}(1)_2}_{5,3;1,0;3,-\frac{2}{3};1,0} + \chi^{m6\times u(1)_2\times\bar{m}6\times\bar{u}(1)_2}_{5,3;2,\frac{1}{4};3,-\frac{2}{3};2,-\frac{1}{4}} \\
&\quad + \chi^{m6\times u(1)_2\times\bar{m}6\times\bar{u}(1)_2}_{6,\frac{2}{5};1,0;8,-\frac{1}{15};1,0} + \chi^{m6\times u(1)_2\times\bar{m}6\times\bar{u}(1)_2}_{6,\frac{2}{5};2,\frac{1}{4};8,-\frac{1}{15};2,-\frac{1}{4}} + \chi^{m6\times u(1)_2\times\bar{m}6\times\bar{u}(1)_2}_{10,\frac{7}{5};1,0;8,-\frac{1}{15};1,0} + \chi^{m6\times u(1)_2\times\bar{m}6\times\bar{u}(1)_2}_{10,\frac{7}{5};2,\frac{1}{4};8,-\frac{1}{15};2,-\frac{1}{4}}, \\
Z_{b_2}^{\mathcal{G}\mathrm{au}_{S_3}^{(3)}} &= \chi^{m6\times u(1)_2\times\bar{m}6\times\bar{u}(1)_2}_{3,\frac{2}{3};1,0;1,0;1,0} + \chi^{m6\times u(1)_2\times\bar{m}6\times\bar{u}(1)_2}_{3,\frac{2}{3};1,0;5,-3;1,0} + \chi^{m6\times u(1)_2\times\bar{m}6\times\bar{u}(1)_2}_{3,\frac{2}{3};2,\frac{1}{4};1,0;2,-\frac{1}{4}} + \chi^{m6\times u(1)_2\times\bar{m}6\times\bar{u}(1)_2}_{3,\frac{2}{3};2,\frac{1}{4};5,-3;2,-\frac{1}{4}} \\
&\quad + \chi^{m6\times u(1)_2\times\bar{m}6\times\bar{u}(1)_2}_{8,\frac{1}{15};1,0;6,-\frac{2}{5};1,0} + \chi^{m6\times u(1)_2\times\bar{m}6\times\bar{u}(1)_2}_{8,\frac{1}{15};1,0;10,-\frac{7}{5};1,0} + \chi^{m6\times u(1)_2\times\bar{m}6\times\bar{u}(1)_2}_{8,\frac{1}{15};2,\frac{1}{4};6,-\frac{2}{5};2,-\frac{1}{4}} + \chi^{m6\times u(1)_2\times\bar{m}6\times\bar{u}(1)_2}_{8,\frac{1}{15};2,\frac{1}{4};10,-\frac{7}{5};2,-\frac{1}{4}}, \\
Z_{c}^{\mathcal{G}\mathrm{au}_{S_3}^{(3)}} &= \chi^{m6\times u(1)_2\times\bar{m}6\times\bar{u}(1)_2}_{2,\frac{1}{8};1,0;4,-\frac{13}{8};2,-\frac{1}{4}} + \chi^{m6\times u(1)_2\times\bar{m}6\times\bar{u}(1)_2}_{2,\frac{1}{8};2,\frac{1}{4};2,-\frac{1}{8};1,0} + \chi^{m6\times u(1)_2\times\bar{m}6\times\bar{u}(1)_2}_{4,\frac{13}{8};1,0;2,-\frac{1}{8};2,-\frac{1}{4}} + \chi^{m6\times u(1)_2\times\bar{m}6\times\bar{u}(1)_2}_{4,\frac{13}{8};2,\frac{1}{4};4,-\frac{13}{8};1,0} \\
&\quad + \chi^{m6\times u(1)_2\times\bar{m}6\times\bar{u}(1)_2}_{7,\frac{1}{40};1,0;9,-\frac{21}{40};2,-\frac{1}{4}} + \chi^{m6\times u(1)_2\times\bar{m}6\times\bar{u}(1)_2}_{7,\frac{1}{40};2,\frac{1}{4};7,-\frac{1}{40};1,0} + \chi^{m6\times u(1)_2\times\bar{m}6\times\bar{u}(1)_2}_{9,\frac{21}{40};1,0;7,-\frac{1}{40};2,-\frac{1}{4}} + \chi^{m6\times u(1)_2\times\bar{m}6\times\bar{u}(1)_2}_{9,\frac{21}{40};2,\frac{1}{4};9,-\frac{21}{40};1,0}, \\
Z_{c_1}^{\mathcal{G}\mathrm{au}_{S_3}^{(3)}} &= \chi^{m6\times u(1)_2\times\bar{m}6\times\bar{u}(1)_2}_{2,\frac{1}{8};1,0;2,-\frac{1}{8};2,-\frac{1}{4}} + \chi^{m6\times u(1)_2\times\bar{m}6\times\bar{u}(1)_2}_{2,\frac{1}{8};2,\frac{1}{4};4,-\frac{13}{8};1,0} + \chi^{m6\times u(1)_2\times\bar{m}6\times\bar{u}(1)_2}_{4,\frac{13}{8};1,0;4,-\frac{13}{8};2,-\frac{1}{4}} + \chi^{m6\times u(1)_2\times\bar{m}6\times\bar{u}(1)_2}_{4,\frac{13}{8};2,\frac{1}{4};2,-\frac{1}{8};1,0} \\
&\quad + \chi^{m6\times u(1)_2\times\bar{m}6\times\bar{u}(1)_2}_{7,\frac{1}{40};1,0;7,-\frac{1}{40};2,-\frac{1}{4}} + \chi^{m6\times u(1)_2\times\bar{m}6\times\bar{u}(1)_2}_{7,\frac{1}{40};2,\frac{1}{4};9,-\frac{21}{40};1,0} + \chi^{m6\times u(1)_2\times\bar{m}6\times\bar{u}(1)_2}_{9,\frac{21}{40};1,0;9,-\frac{21}{40};2,-\frac{1}{4}} + \chi^{m6\times u(1)_2\times\bar{m}6\times\bar{u}(1)_2}_{9,\frac{21}{40};2,\frac{1}{4};7,-\frac{1}{40};1,0}.
\end{aligned}
\tag{72}
$$

Equation (72) describes a gapless state that does not break the anomalous $S_3^{(3)}$ symmetry. The gapless state is described by a $m6 \times u(1)_2 \times \bar{m}6 \times \bar{u}(1)_2$ CFT with central charge $(c, \bar{c}) = (\frac{9}{5}, \frac{9}{5})$. Such a CFT is nonchiral, where right movers and left movers have the dynamics. In particular, the right movers (and left movers) are described by a (6,5) minimal model CFT (denoted as $m6$) and a U(1) level-2 CFT [i.e., K matrix $K = (2)$]. Such a CFT corresponds a gapless state with one $S_3^{(3)}$ symmetric relevant operator. Thus the CFT may describe a stable continuous phase transition. The gapped and gapless phases for systems with anomalous $S_3^{(3)}$ symmetry is summarized in Table V.

VII. SUMMARY

It is well known that symmetry and anomaly can constrain the low energy properties of quantum systems. However, even given a symmetry and/or an anomaly, there still can be a lot of allowed possible low energy properties, which are hard to organize and hard to understand. In this paper, we used Symm/TO correspondence proposed in Refs. [33,42,48,49], to view symmetry and anomaly from a new point of view, and also to place them in a more generalized framework. This allows us to organize the low energy properties according to the condensation patterns and their reduced symmetry TO. These patterns of condensations can describe, in a unified way, symmetry breaking phases, symmetry enriched topological phases, symmetry protected topological phases, and gapless critical points connecting these phases. These patterns of symmetry TO reductions, and the associated gapped/gapless phases, are classified by the condensable algebras $\mathcal{A}$ in the symmetry TO $\mathcal{M}$.

In order to similarly study phases and symmetry in n-dimensional space for $n > 1$, the theory of condensable algebra needs to be further developed. In some sense, a condensable algebra should correspond to an n-dimensional domain wall in a topological order in $(n+1)$-dimensional space, which describes a symmetry for a quantum system

TABLE V. Possible gapped and gapless states for systems with anomalous $S_3^{(3)}$ symmetry.

condensable algebra $\mathcal{A}$	reduced symmetry TO $\mathcal{M}_{/\mathcal{A}}$	Stable low energy state
$\mathbf{1}\oplus a_1\oplus 2a_2$	$(\mathcal{G}\text{au}_{S_3}^{(3)})_{/\mathbf{1}\oplus a_1\oplus 2a_2}=$ trivial	S_3 symmetry breaking gapped $\mathbb{Z}_1$ state
$\mathbf{1}\oplus a_1\oplus 2b$	$(\mathcal{G}\text{au}_{S_3}^{(3)})_{/\mathbf{1}\oplus a_1\oplus 2b}=$ trivial	Gapped $\mathbb{Z}_3$ state
$\mathbf{1}\oplus a_2$	$(\mathcal{G}\text{au}_{S_3}^{(3)})_{/\mathbf{1}\oplus a_2}=\mathcal{M}_{\text{DS}}$	$K=\begin{pmatrix}2&0\\0&-2\end{pmatrix}$ chiral boson theory (55)
$\mathbf{1}\oplus b$	$(\mathcal{G}\text{au}_{S_3}^{(3)})_{/\mathbf{1}\oplus b}=\mathcal{M}_{\text{DS}}$	$K=\begin{pmatrix}2&0\\0&-2\end{pmatrix}$ chiral boson theory (55)
$\mathbf{1}\oplus a_1$	$(\mathcal{G}\text{au}_{S_3}^{(3)})_{/\mathbf{1}\oplus a_1}=\mathcal{G}\text{au}_{\mathbb{Z}_3}$	$(c,\bar{c})=(\frac{4}{5},\frac{4}{5})$ CFT (42)
$\mathbf{1}$	$(\mathcal{G}\text{au}_{S_3}^{(3)})_{/\mathbf{1}}=\mathcal{G}\text{au}_{S_3}^{(3)}$	$(c,\bar{c})=(\frac{9}{5},\frac{9}{5})$ CFT $m6\times u(1)_2\times\bar{m}6\times\bar{u}(1)_2$ (72)

in n spatial dimensions. These domain walls are necessarily descendant excitations [i.e., formed by the condensation of $(n-1)$-dimensional, $(n-2)$-dimensional, etc. excitations]. Under such a generalization of condensable algebra, one must also include topological orders in n-dimensional space without any symmetry. This is because condensation of trivial excitations in the symmetry TO [i.e., topological order in $(n+1)$-dimensional space] can give rise to topological order in n-dimensional space. Condensation of nontrivial excitations, on the other hand, can give rise to symmetry enriched topological order in n-dimensional space. In this way, an appropriately generalized analog of condensable algebra should be able to describe symmetry-enriched topologically ordered gapped phases of quantum systems in $n>1$ spatial dimensions.

For gapless states, the possible low energy properties with a reduced symmetry TO $\mathcal{M}_{/\mathcal{A}}$ are the same as the possible low energy properties of the $\mathbf{1}$-condensed boundary of $\mathcal{M}_{/\mathcal{A}}$. In the language of bulk topological order, this refers to the $\mathbf{1}$-condensed boundary of the topological order induced from $\mathcal{M}$ via the condensation of $\mathcal{A}$, which we denote by $\mathcal{M}_{/\mathcal{A}}$. We find that possible low energy properties, such as scaling dimensions, are determined by the reduced symmetry TO $\mathcal{M}_{/\mathcal{A}}$, and can be computed using an algebraic number theoretical method.

Different condensable algebras $\mathcal{A}$'s of $\mathcal{M}$ can give rise to the same reduced symmetry TO $\mathcal{M}_{/\mathcal{A}}$, which implies that different patterns of condensation associated to a symmetry TO can give rise to the same set of low energy properties. This allows us to show that some seemingly different continuous quantum phase transitions are described by the same critical theory. It appears that Symm/TO correspondence is a powerful way to use symmetry TO (also referred to as categorical symmetry$^{ⓗ}$ before) to study, or even to classify, gapless quantum states and the associated quantum field theories (up to local low-energy equivalence). In higher than $1+1$D, similar techniques are lacking, partly due to a lack of systematic understanding of gapped boundaries of $3+1$D and higher topological orders. This constitutes one major direction for future research.

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APPENDIX A: SOME REMARKS ON THE TERM CATEGORICAL SYMMETRY$^{ⓗ}$

To address some comments from referee about the term categorical symmetry$^{ⓗ}$, we make some remarks here. The term categorical symmetry$^{ⓗ}$ was introduced in 2019 [49], which is a way to describe a symmetry by viewing it as a (noninvertible) gravitational anomaly [48], or by including both symmetry charges and symmetry defects *at an equal footing*. We stress that *at equal footing* is the key here. If we only include symmetry charges, and use the fusion ring (i.e., conservation law) of symmetry charges to describe the symmetry, it will lead to a group theory (or fusion category) description of symmetry.[15] If we only use symmetry transformations (or symmetry defects) to describe the symmetry, it will also lead to a group theory (or fusion category [34–40,138,139]) description of symmetry. On the other hand, if we include both symmetry charges and symmetry defects at an equal footing, and use the fusion rings (i.e., conservation laws) of symmetry charges and symmetry defects to describe the symmetry, we find that we also need to include the "braiding" properties of symmetry charges and symmetry defects. Thus we need to use noninvertible topological order in one higher dimension (called symmetry TO), or more precisely, "nondegenerate braided fusion n-category" to describe such a structure, if the system is in n-dimensional space. This way, the categorical symmetry$^{ⓗ}$ point of view leads to Symm/TO correspondence [48,49] (see Fig. 20).

In an earlier work Ref. [69], categorical symmetry$^{ⓗ}$ appeared in $1+1$D CFT as the ambient category of enriched fusion category of all the topological defect lines and is, at the same time, the category of modules over a chiral or nonchiral symmetry (i.e., a VOA or a full field algebra). Topological field theory (TFT) in one higher dimension was also used in Ref. [140] to discuss a duality relation in $1+1$D Ising model.

[15] We do not need to include the "braiding" properties of symmetry charges, since their are always trivial for anomaly-free and anomalous symmetries.

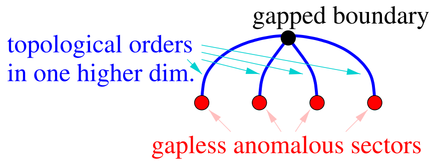

FIG. 20. Symm/TO correspondence where (emergent) symmetry is viewed as anomaly (Fig. 1 in Ref. [126]).

Later, "categorical symmetry" was also used to refer to "noninvertible symmetry" (which was called "fusion category symmetry," a term first introduced also in 2019 [38]). "Simons Collaboration on Global Categorical Symmetries" founded in 2021 used the term "categorical symmetry" with "noninvertible" meaning.

In 2021, motivated by Ref. [141], Ref. [142] introduced "symmetry TFT", which is closely related to "symmetry TO." A possible difference is that, for example, in 2 + 1D, the $\mathbb{Z}_2 \times \mathbb{Z}_2$-DW theory and the $\mathbb{Z}_4$-gauge theory are usually regarded as different field theories. Thus they may be viewed as different TFT's, but they correspond to the same symmetry TO. In other words, symmetry TFT may carry extra information about the field theory representations, which is not needed here.

In 2022, "topological symmetry" was introduced [143]. "Topological symmetry" corresponds to a pair (ρ, σ), where σ is the symmetry TO discussed above, and ρ a gapped boundary of the symmetry TO: $\text{bulk}(\rho) = \sigma$ (see Fig. 21). The pair (ρ, σ) describes a (generalized) symmetry in a quantum field theory F (using the notations in Ref. [42]): $F \cong \rho \boxtimes_\sigma \tilde{F}$, where $\tilde{F}$ is a boundary of σ, i.e., $\text{bulk}(\tilde{F}) = \sigma$.

In 2020, Ref. [42] also used a similar pair $(\tilde{\mathcal{R}}, \text{bulk}(\mathcal{C}))$ (see Fig. 22) to describe an anomaly-free algebraic higher symmetry (i.e., noninvertible higher symmetry), where $\text{bulk}(\mathcal{C})$ is a nondegenerate braided fusion higher category (i.e., a symmetry TO corresponding to σ in the above) and $\tilde{\mathcal{R}}$ is a local fusion higher category that satisfy $\mathcal{Z}(\tilde{\mathcal{R}}) = \text{bulk}(\mathcal{C})$ (i.e., a gapped boundary of the symmetry TO corresponding to ρ in the above). Reference [42] used the pair $(\tilde{\mathcal{R}}, \text{bulk}(\mathcal{C}))$ to classify symmetry protected topological orders and symmetry enriched topological orders with the anomaly-free algebraic higher symmetry. Here $\mathcal{C}$ corresponds to $\tilde{F}$ in the above. In Ref. [42], $\tilde{\mathcal{R}}$ in the pair is assumed to be a local fusion higher category. Since, $\text{bulk}(\mathcal{C}) = \text{bulk}(\tilde{\mathcal{R}})$, Ref. [42] usually used $\tilde{\mathcal{R}}$, or its dual $\mathcal{R}$, to describe the symmetry, which is referred to as algebraic higher symmetry (that is anomaly-free since $\mathcal{R}$ and $\tilde{\mathcal{R}}$ are assumed to be local fusion higher categories).

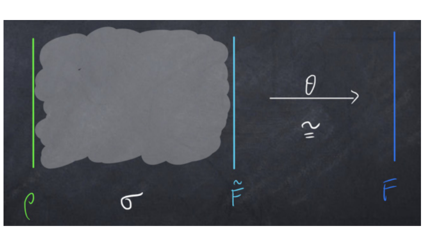

FIG. 21. A pair (ρ, σ) describes a "topological symmetry" in an anomaly-free field theory F (Fig. 1 in Ref. [143]).

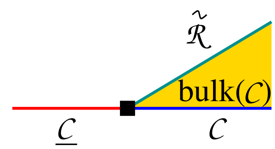

FIG. 22. In Ref. [42], Fig. 21 was drawn as the above, where $F = \underline{\mathcal{C}}$, $\tilde{F} = \mathcal{C}$, $\rho = \tilde{\mathcal{R}}$, and $\sigma = \text{bulk}(\mathcal{C})$. (See Figs. 24 and 29 in Ref. [42]).

It turns out that the pair (ρ, σ) also appeared in a 2015 work [46,47], but was interpreted differently as a morphism from $\tilde{F}$ to F, which leads an equivalence $F \overset{\theta}{\cong} \rho \boxtimes_\sigma \tilde{F}$ (see Fig. 21). Such an equivalence corresponds to a symmetry described by (ρ, σ), as pointed out in Refs. [42,143].

More specifically, in Refs. [46,47], morphism between $n+1$D (gapped or gapless) quantum field theories $\mathcal{C}_n$ and $\mathcal{D}_n$ with (nonperturbative) gravitational anomalies [43] are studied. The partition function of a $n+1$D anomalous field theory $\mathcal{D}_n$ on space-time M^{n+1} is define only after we view M^{n+1} as a boundary of N^{n+2} (like Wess-Zumino-Witten theory [77,144]) and is denoted as

$$Z(\mathcal{D}_n; M^{n+1}, N^{n+2}), \quad M^{n+1} = \partial N^{n+2}. \tag{A1}$$

The morphism is a *topological domain wall* $(f_{n-1}^{(1)}, f_n^{(0)})$ between anomalous field theories (see Fig. 23, (4.3) of Ref. [46]), *where* $f_{n-1}^{(1)}$ *is invertible*. Since $f_{n-1}^{(1)}$ is invertible, the morphism $(f_{n-1}^{(1)}, f_n^{(0)})$ (i.e., the presence of topological domain wall) gives rise to an equivalence relation (see (4.3) in Ref. [46], which is called a decomposition in Ref. [63]):

$$\mathcal{D}_n \overset{f_{n-1}^{(1)}}{\cong} f_n^{(0)} \boxtimes_{\mathcal{Z}_n(\mathcal{C}_n)} \mathcal{C}_n, \tag{A2}$$

where $\mathcal{D}_n$ and $f_n^{(0)} \boxtimes_{\mathcal{Z}_n(\mathcal{C}_n)} \mathcal{C}_n$ have the same partition function [63]

$$Z(\mathcal{D}_n; M^{n+1}, N^{n+2}) = Z(f_n^{(0)} \boxtimes_{\mathcal{Z}_n(\mathcal{C}_n)} \mathcal{C}_n; M^{n+1}, N^{n+2}), \tag{A3}$$

which is the defining property of the equivalence relation or the decomposition (A2) [63]. Since the domain wall $(f_{n-1}^{(1)}, f_n^{(0)})$ is topological and $f_{n-1}^{(1)}$ is invertible, the above implies that the two anomalous field theories $\mathcal{C}_n$ and $\mathcal{D}_n$ have the same *local low energy properties* defined in footnote 13. Some explicit examples of such local low energy equivalence were discussed in Ref. [62,63].

The equivalence relation or the decomposition, (A2) and (A3), reveals the symmetry described by the pair $(f_n^{(0)}, \mathcal{Z}_n(\mathcal{C}_n))$ [62,63], in the anomalous field theories $\mathcal{D}_n$

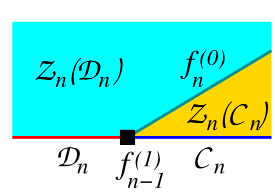

FIG. 23. A morphism between $n+1$D (gapped or gapless) quantum field theories $\mathcal{C}_n$ and $\mathcal{D}_n$ with (noninvertible) gravitational anomalies. See (4.3) of Ref. [46].

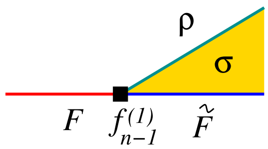

FIG. 24. When $\mathcal{D}_n \equiv F$ is anomaly-free, Fig. 23 becomes the above, which is another way to represent Fig. 21.

and $\mathcal{C}_n$. Thus morphism between (anomalous) quantum field theories defined in Ref. [46] corresponds to symmetry.

In the special case when $\mathcal{D}_n$ is an anomaly-free field theory (denoted as F), its center (i.e., the bulk) $\mathcal{Z}_n(\mathcal{D}_n) = \text{bulk}(\mathcal{D}_n)$ is trivial and Fig. 23 becomes Fig. 24, if we rename $\mathcal{C}_n$ as $\tilde{F}$, $\mathcal{Z}_n(\mathcal{C}_n)$ as σ, and $f_n^{(0)}$ as ρ. Due to the equivalence relation (A2) and (A3), the pair (ρ, σ) can be viewed as a symmetry of an anomaly-free field theory F as pointed out in Refs. [42,143]. In other words, the equivalence relation or the decomposition, (A2) and (A3), reveals the symmetry (ρ, σ) in the anomaly-free field theory F and in the anomalous field theory $\tilde{F}$. Reference [63] used the equivalence relation or decomposition, (A2) and (A3), to identify maximal categorical symmetry$^{ⓗ}$ in an anomaly-free field theory F. Note that the anomalous field theory $\tilde{F}$ corresponds to anomaly-free field theory F restricted in the symmetric sub-Hilbert space $\mathcal{V}_{\text{symmetric}}$ [49].

The categorical symmetry$^{ⓗ}$ of the anomaly-free field theory F only corresponds to the σ in the pair. We see a clear distinction between categorical symmetry$^{ⓗ}$ and symmetry: a categorical symmetry$^{ⓗ}$ is a holoequivalent class of symmetries, as indicated by (4). (Two symmetries, (ρ, σ) and (ρ', σ'), are holoequivalent if $\sigma \cong \sigma'$ [42].)

We have been using categorical symmetry$^{ⓗ}$ to just mean "including both symmetry charges and symmetry defects *at an equal footing*, and including their "braiding" properties." This leads to Symm/TO correspondence. However, "categorical symmetry" has since been used to mean different things. This causes some confusions.

Another source of confusion comes from the fact that categorical symmetry$^{ⓗ}$ has several equivalent descriptions, that emphasis different aspects of Symm/TO correspondence: a categorical symmetry$^{ⓗ}$ can be, equivalently, described by the following.

(1) A noninvertible gravitational anomaly [48] (see Fig. 20). Here we view symmetry by restricting to symmetric sub-Hilbert space. The symmetric sub-Hilbert space does not have a tensor product decomposition $\mathcal{V}_{\text{symm}} \neq \bigotimes_i \mathcal{V}_i$, where $\mathcal{V}_i'$s are vector spaces on lattice sites i. This implies a noninvertible gravitational anomaly, and thus a symmetry can be described by a noninvertible gravitational anomaly.

(2) A symmetry + dual symmetry + braiding [49]. Conservation (i.e., the fusion ring) of symmetry charges corresponds to symmetry. Conservation (i.e., the fusion ring) of symmetry defects corresponds to dual-symmetry. Here we treat symmetry charges and symmetry defects at an equal footing. The fusion ring of symmetry charges/defects corresponds to "symmetry" in categorical symmetry$^{ⓗ}$. The braiding properties of symmetry charges/defects corresponds to "categorical" in categorical symmetry$^{ⓗ}$. In fact, the term categorical symmetry$^{ⓗ}$ is a parallel generalization of the term "anomalous symmetry." The fusion ring of the symmetry charges correspond to "symmetry" in "anomalous symmetry" and the braiding properties of symmetry defects corresponds to "anomalous" in "anomalous symmetry" [32,59].

(3) A topological order in one higher dimension (symmetry TO) [42,49]. This is because gravitational anomaly = topological order in one higher dimension [43].

(4) A part of topological skeleton introduced in Ref. [64].

(5) A nondegenerate braided fusion higher category [38,42]. This is because topological order is described by nondegenerate braided fusion higher category. We use a short name "nBF category" to refer to "nondegenerate braided fusion higher category." Thus nBF category, replacing group and higher group, is used to describe (generalized) symmetry. This leads to a unified frame work to classify spontaneous symmetry breaking order, topological order, symmetry protect topological order, symmetry enriched topological order, etc. in any dimension [42].

(6) An equivalence class of algebras of commutant patch operators (also called transparent patch operators) [33,66]. This is a nonholographic point of view that does not go to one higher dimension, and leads to the notion of patch symmetry [see Eq. (4)]. Here a symmetry is defined via the algebra of local symmetric operators. Using an algebra formed by commutant patch operators (that are constructed from local symmetric operators and define the patch symmetry), we can compute a nondegenerate braided fusion category that describes a categorical symmetry$^{ⓗ}$.

Let us use some simple examples to illustrate the notion of categorical symmetry$^{ⓗ}$. In 1 + 1D Ising model with $\mathbb{Z}_2$ symmetry. The $\mathbb{Z}_2$ symmetry charge is denoted by e and $\mathbb{Z}_2$ symmetry defect is denoted by m. The $\mathbb{Z}_2$ symmetry is described by the fusion ring $e \otimes e = \mathbf{1}$ (the conservation law). The $\mathbb{Z}_2$ symmetry is also described by transformation law $U^2 = \text{id}$. On the other hand, the categorical symmetry$^{ⓗ}$ of the Ising model is described by the fusion ring of $\mathbb{Z}_2$ symmetry charges $e \otimes e = \mathbf{1}$ and the fusion ring of $\mathbb{Z}_2$ symmetry defects $m \otimes m = \mathbf{1}$, as well as a nontrivial "braiding" property between e and m. In other words, categorical symmetry$^{ⓗ}$ treats the symmetry charges and symmetry transformations (or symmetry defects) at equal footing. Such a treatment leads to nBF category description of symmetry (instead of group description of symmetry).

In an 1 + 1D model with $\mathbb{Z}_4$ symmetry. The $\mathbb{Z}_4$ symmetry charges are denoted by e^k, $k = 1, 2, 3$, and $\mathbb{Z}_4$ symmetry defects are denoted by m^k, $k = 1, 2, 3$. The $\mathbb{Z}_4$ symmetry is described by the fusion ring $e \otimes e \otimes e \otimes e = \mathbf{1}$ (the conservation law). The $\mathbb{Z}_4$ symmetry is also described by transformation law $U^4 = \text{id}$, or $m \otimes m \otimes m \otimes m = \mathbf{1}$. In contrast, the categorical symmetry$^{ⓗ}$ of the $\mathbb{Z}_4$ model is described by the fusion ring of $\mathbb{Z}_4$ symmetry charges $e \otimes e \otimes e \otimes e = \mathbf{1}$ and the fusion ring of $\mathbb{Z}_4$ symmetry defects $m \otimes m \otimes m \otimes m = \mathbf{1}$, as well as a nontrivial "braiding" property between e and m.

However, if we call $e^2 = e_1$ the charge and $m = m_1$ the defect of first $\mathbb{Z}_2$-symmetry, and $m^2 = e_2$ the charge and $e = m_2$ the defect of second $\mathbb{Z}_2'$-symmetry, then the same categorical symmetry$^{ⓗ}$ will describe $\mathbb{Z}_2 \times \mathbb{Z}_2'$ symmetry with a

TABLE VI. The first row is the classification of 2+1D topological orders [up to $E(8)$ invertible topological order] for bosonic systems with no symmetry, up to 10 types of anyons. This leads to a classification of 2+1D symmetry TOs, which classify all the 1+1D global symmetries up to holo-equivalence (the second row). Such a classification include all finite-group symmetries with potential anomalies (the third row). It also includes beyond-group symmetries, such as the Fibonacci symmetry in Fig. 25.

No. of anyon types (rank)	1	2	3	4	5	6	7	8	9	10	11
No. of 2 + 1D topological orders (MTC)	1	4	12	18	10	50	28	64	81	76	44
No. of symmetry TOs (MTC in trivial Witt class)	1	0	0	3	0	0	0	6	6	3	0
No. of finite-group symmetries (with anomaly ω)	1	0	0	$2_{\mathbb{Z}_2^\omega}$	0	0	0	$6_{S_3^\omega}$	$3_{\mathbb{Z}_3^\omega}$	0	0

mixed anomaly [33]. This example demonstrates a difference between the usual symmetry point of view and categorical symmetry$^{\textcircled{h}}$ point of view. The categorical symmetry$^{\textcircled{h}}$ point of view allows us to see certain relations more easily.

Since many people use "categorical symmetry" to mean noninvertible symmetry, in this paper, we will use an equivalent notion *symmetry TO* to refer to categorical symmetry$^{\textcircled{h}}$, hoping to avoid confusions. In fact, symmetry TO (i.e., categorical symmetry$^{\textcircled{h}}$) is the "Drinfeld" center of global symmetry or fusion category symmetry.

Recently, modular tensor categories with up to 11 types of anyons were classified [145]. This leads to a classification of 1+1D generalized symmetries with 11 or fewer symmetry charges/defects, via the classification of 2+1D symmetry TOs up to rank 11 (see Table VI). For example, for global symmetries with four types of symmetry charges/defects, the three holo-equivalence classes (which contain only one symmetry each in this case) are (1) $\mathbb{Z}_2$ symmetry where the symmetry TO is the 2+1D $\mathbb{Z}_2$ gauge theory; (2) anomalous $\mathbb{Z}_2$ symmetry where the symmetry TO is the double-semion topological order; (3) Fibonacci symmetry where the symmetry TO is the double-Fibonacci topological order (see Fig. 25). From Table VI, we also see a clear distinction between generic TO which may not allow gapped boundary and symmetry TO which allows gapped boundary.

APPENDIX B: STRUCTURE OF PHASE DIAGRAM

Condensable algebras $\mathcal{A}$ have many relations, such as algebra-subalgebra relation, overlap relations, etc. These relations can constrain the phase diagram of condensation patterns

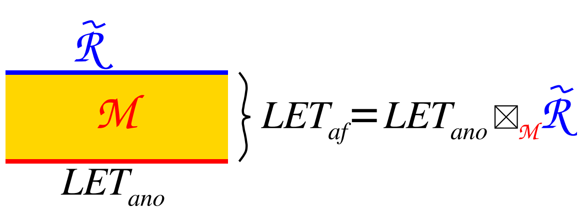

FIG. 25. A 1+1D lattice model with emergent Fibonacci symmetry at low energies. The 1+1D lattice model is constructed from a slab of 2+1D lattice. In the bulk, we have a commuting-projector Hamiltonian that realizes a double-Fibonacci topological order [146] with large energy gap. The top boundary $\tilde{R}$ is a gapped boundary of the double-Fibonacci topological order with large energy gap. The lower boundary is described by an anomalous low energy theory LET$_{ano}$. The low energy theory LET$_{af}$ of the slab has an emergent Fibonacci symmetry below the energy gaps of the bulk and top boundary.

$\mathcal{A}$ in $\mathcal{M}$ symmetric systems. To describe such a phase diagram, let $\mathcal{X}_{\mathcal{M}}$ be the space of all $\mathcal{M}$ systems (which is called moduli space), that have liquid ground states [67,68]. $\mathcal{X}_{\mathcal{M}}$ is parametrized by the coupling constants in the Hamiltonians with the symmetry.

The moduli space $\mathcal{X}_{\mathcal{M}}$ can be divided in to many regions, each described by a different condensation $\mathcal{A}$, which will be called $\mathcal{A}$ phase. The state in the $\mathcal{A}$ phase will be called $\mathcal{A}$ state. Here the $\mathcal{A}$ phase can be gapped. The $\mathcal{A}$ phase can also be gapless, in which case, the gapless $\mathcal{A}$ state has no symmetric relevant operators.

The boundary between two regions of condensations $\mathcal{A}_1$ and $\mathcal{A}_2$ describes the phase transition between $\mathcal{A}_1$ and $\mathcal{A}_2$. If the phase transition is first order, at the boundary, the system has degenerate ground states: one described by $\mathcal{A}_1$ condensation and the other by $\mathcal{A}_2$ condensation. If the phase transition is continuous, at the boundary, the system has a condensation described by $\mathcal{A}_c$. Starting in the $\mathcal{A}_1$ phase, as we approach the $\mathcal{A}_c$ boundary, certain condensation becomes weaker and weaker. At the boundary, we reach a smaller condensation $\mathcal{A}_c \subset \mathcal{A}_1$. Similarly, we have $\mathcal{A}_c \subset \mathcal{A}_2$. Thus A stable continuous transition between $\mathcal{A}_1$ and $\mathcal{A}_2$ phases is described by a critical point with $\mathcal{A}_c$ condensation that satisfies

$$\mathcal{A}_c \subset \mathcal{A}_1, \quad \mathcal{A}_c \subset \mathcal{A}_2. \tag{B1}$$

The gapless $\mathcal{A}_c$ state has only one symmetric relevant operator. Here $\mathcal{A}_1$ and $\mathcal{A}_2$ can be the same, in which case, the gapless $\mathcal{A}_c$ state describes a continuous transition between the same phase.

To summarize, an $\mathcal{A}$ state is with no symmetric relevant operator form the stable phases. A gapless $\mathcal{A}_c$ state is with one and only one symmetric relevant operator describe stable continuous transitions between stable phases. Similarly, the gapless $\mathcal{A}_t$ state is with two and only two symmetric relevant operators are tricritical points, a kind of multicritical points.

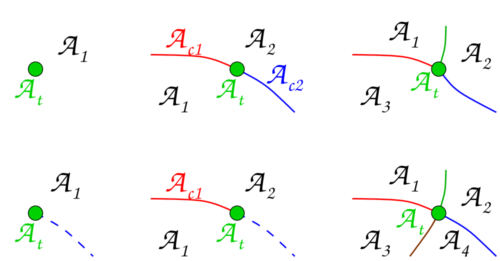

FIG. 26. Some possible structures of the phases near a tricritical point $\mathcal{A}_t$ (the dot). The solid curves are continuous transitions (described by $\mathcal{A}_{c1}$, $\mathcal{A}_{c2}$, etc.). The dashed curves are first-order transitions.

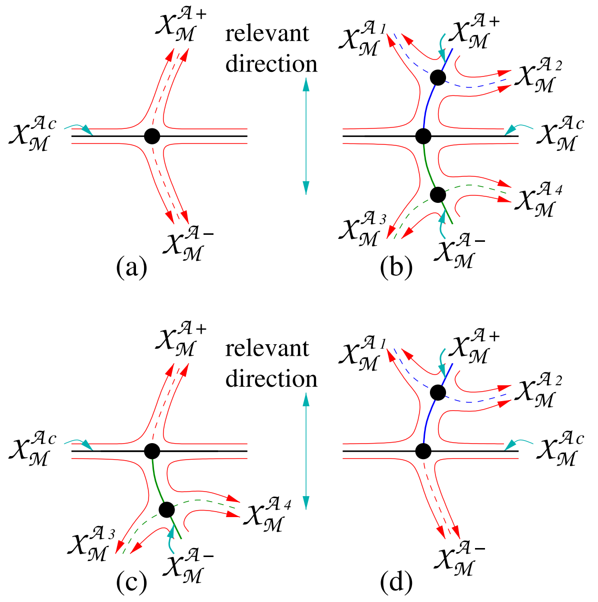

FIG. 27. The curves with arrow represent the RG flow. The dots represent the fixed points of the RG flow. The plane is a subspace of the total moduli space $\mathcal{X}_{\mathcal{M}}$ (the space formed by $\mathcal{M}$ systems). Let $\mathcal{X}_{\mathcal{M}}^{\mathcal{A}}$ be the subspace formed by $\mathcal{A}$ states. The horizontal line is a subspace of $\mathcal{X}_{\mathcal{M}}^{\mathcal{A}_c}$. In (a), the subspaces $\mathcal{X}_{\mathcal{M}}^{\mathcal{A}+}$ and $\mathcal{X}_{\mathcal{M}}^{\mathcal{A}-}$ are upper and lower half planes. In (b), the subspaces $\mathcal{X}_{\mathcal{M}}^{\mathcal{A}+}$ and $\mathcal{X}_{\mathcal{M}}^{\mathcal{A}-}$ are two marked curves. (c) and (d) are combinations of (a) and (b).

Some structures of the phases near a tricritical point are described in Fig. 26.

Let us consider a stable continuous transition, whose critical point is described by a gapless $\mathcal{A}_c$ state that has one symmetric relevant operator. The possible renormalization-group (RG) flows are presented schematically in Fig. 27. However, what are the resulting states after a long RG flow? To address this question, let us introduce a notion of allowed competing pair for the condensable algebra $\mathcal{A}_c$, which is a pair of excitations a_+ and a_- with nontrivial mutual statistics, but they both have trivial mutual statistics with respect to $\mathcal{A}_c$.[16] If a_+ condenses, a_- will be confined and uncondense. If a_- condenses, a_+ will be confined and uncondense. Thus we can imagine that there is some parameter ε in the Hamiltonian that controls whether a_+ condenses or a_- condenses. For example, we could have a situation that $\varepsilon > 0$ causes a_+ to condense and $\varepsilon < 0$ causes a_- to condense. a_+ and a_- cannot both condense due to their nontrivial mutual statistics, but a_+ and a_- can be both uncondensed. Let us assume this can happen only if we fine tune ε, so as to set $\varepsilon = 0$ (otherwise, we would have had a stable gapless phase).

If the gapless $\mathcal{A}_c$ state only allows one competing pair, then the two different condensations of the one competing pair should correspond to the relevant direction. However if the gapless $\mathcal{A}_c$ state allows several competing pairs, then the only relevant direction should correspond to one of these competing pairs. With these considerations, we propose that *the switching between two different condensations of a competing pair is the basic mechanism for continuous phase transition. The resulting two condensable algebras $\mathcal{A}_+$ and $\mathcal{A}_-$ from the two condensations must contain the condensing particle and must contain $\mathcal{A}_c$ as a sub algebra.*

To be more concrete, let us assume the competing pair (a_+, a_-) corresponds to the relevant direction. After the condensation of a_+, or a_-, the condensable algebra $\mathcal{A}_c$ will change to $\mathcal{A}_+ = \mathcal{A}_c \oplus a_+ \oplus + \cdots$ or $\mathcal{A}_- = \mathcal{A}_c \oplus a_- \oplus + \cdots$, where $\cdots$ represent any additional excitations that condense together with the a_+ or a_- condensations.

Now, we need to consider several cases separately. If $\mathcal{A}_+$ and $\mathcal{A}_-$ are Lagrangian, then the switching between two different condensations of the competing pair (a_+, a_-) will cause a stable continuous phase transition between the $\mathcal{A}_+$ and $\mathcal{A}_-$ states. We will have a local phase diagram as in Fig. 27(a), where we have assumed that the parameter ε mentioned above has an overlap with the relevant direction of the RG flow.

If $\mathcal{A}_+$ and $\mathcal{A}_-$ are both non-Lagrangian, then the switching between two different condensations of the competing pair (a_+, a_-) will cause a continuous phase transition between the gapless $\mathcal{A}_+$ and the gapless $\mathcal{A}_-$ states. Let us further assume that both $\mathcal{A}_+$ and $\mathcal{A}_-$ states have one relevant operator [if neither has a relevant operator, the local phase diagram will be given by Fig. 27(a) as in the previous case]. In this case, the continuous transition will be multicritical. The local phase diagram will be controlled by the relevant operator and the dangerously irrelevant operator, a mechanism discussed in Ref. [147]. The unstable $\mathcal{A}_+$ state can become $\mathcal{A}_1$ state or $\mathcal{A}_2$ state. The unstable $\mathcal{A}_-$ state can become $\mathcal{A}_3$ state or $\mathcal{A}_4$ state. Thus we find the phase diagram shown schematically in Fig. 27(b).

From the phase diagram Fig. 27(b), we see that there are stable continuous transitions $\mathcal{A}_1 \leftrightarrow \mathcal{A}_3$ and $\mathcal{A}_2 \leftrightarrow \mathcal{A}_4$. Whether we get an $\mathcal{A}_1 \leftrightarrow \mathcal{A}_3$ transition or an $\mathcal{A}_2 \leftrightarrow \mathcal{A}_4$ transition is controlled by dangerously irrelevant operators. From the phase diagram, we also see a direct continuous transition $\mathcal{A}_1 \leftrightarrow \mathcal{A}_4$ and a direct continuous transition $\mathcal{A}_2 \leftrightarrow \mathcal{A}_3$. These two transitions are not stable and are controlled by a multicritical point. The critical points for all the four transitions $\mathcal{A}_1 \leftrightarrow \mathcal{A}_3$, $\mathcal{A}_2 \leftrightarrow \mathcal{A}_4$ $\mathcal{A}_1 \leftrightarrow \mathcal{A}_4$, and $\mathcal{A}_2 \leftrightarrow \mathcal{A}_3$, are described by the same critical theory with condensation pattern $\mathcal{A}_c$ and with only one relevant operator. How can a critical theory with only one relevant operator sometimes describe stable continuous transition, and other times describe multicritical continuous transition? This is because sometimes tuning dangerously irrelevant operators can also cause phase transitions. Thus a critical theory with only one relevant operator can sometimes describe a multicritical point.

If $\mathcal{A}_+$ is Lagrangian but $\mathcal{A}_-$ is non-Lagrangian, the local phase diagram will be a combination of the above two cases, and is given by Fig. 27(c). Similarly, if $\mathcal{A}_+$ is non-Lagrangian but $\mathcal{A}_-$ is Lagrangian, we get a phase diagram Fig. 27(d).

From the above discussion, we see that the properties of a continuous phase transition are not only determined by the number of relevant operators of the critical point, as we usually expect, they are also determined by the condensation

[16]More precisely, an allowed competing pair (a_+, a_-) for a condensable algebra $\mathcal{A}_c$ has the following defining properties: (1) a_+ can be added to $\mathcal{A}_c$ to generate a larger condensable algebra and so does a_-. (2) a_+ and a_- cannot be added together to generate a larger condensable algebra.

pattern $\mathcal{A}_c$ of the critical point. In particular, the number of condensations needed to change $\mathcal{A}_c$ into Lagrangian condensable algebra will strongly influence the critical properties. Compared to Landau symmetry breaking theory, the holographic theory replaces the group-subgroup relation by the relations of condensable algebras. The new theory applies to beyond-Landau continuous phase transitions, as well as noninvertible symmetries (i.e., algebraic higher symmetries), as we discuss in the main text.

APPENDIX C: ALGEBRAIC NUMBER THEORETICAL METHOD TO CALCULATE CONDENSABLE ALGEBRAS AND GAPPED/GAPLESS BOUNDARIES/DOMAIN WALLS

We have see that the symmetry TO $\mathcal{M}$, its condensable algebras $\mathcal{A}$, and the induced topological orders $\mathcal{M}_{/\mathcal{A}}$ are very important in understanding the patterns of possible condensations and the allowed gaplessness by the symmetry TO $\mathcal{M}$. For $1+1$D symmetry, there are some simple relations among $\mathcal{A}$, $\mathcal{M}$, and $\mathcal{M}_{/\mathcal{A}}$, where $\mathcal{M}$, and $\mathcal{M}_{/\mathcal{A}}$ are viewed $2+1$D as topological orders.

Let us use a, b, and c to label the anyons in $\mathcal{M}$. As a $2+1$D topological order, $\mathcal{M}$ is characterized by modular matrices $\widetilde{S}_{\mathcal{M}} = (\widetilde{S}^{ab}_{\mathcal{M}})$ and $\widetilde{T}_{\mathcal{M}} = (\widetilde{T}^{ab}_{\mathcal{M}})$, whose indices are labeled by the anyons. $\widetilde{S}_{\mathcal{M}}$ and $\widetilde{T}_{\mathcal{M}}$ are unitary matrices that generate a representation of $SL(2,\mathbb{Z}_n)$, where n is the smallest integer that satisfy $\widetilde{T}^n_{\mathcal{M}} = \text{id}$. We call n as the order of $\widetilde{T}_{\mathcal{M}}$ and denote it as $n = \text{ord}(\widetilde{T}_{\mathcal{M}})$. $\widetilde{T}_{\mathcal{M}}$ is a diagonal matrix and $\widetilde{S}_{\mathcal{M}}$ is a symmetric matrix.

From $\widetilde{S}_{\mathcal{M}}$ and $\widetilde{T}_{\mathcal{M}}$, we define normalized S^{cat} and T^{cat} matrices and unitary S and T matrices

$$S^{\text{cat}}_{\mathcal{M}} = \widetilde{S}_{\mathcal{M}}/\widetilde{S}^{\mathbf{11}}_{\mathcal{M}}, \quad T^{\text{cat}}_{\mathcal{M}} = \widetilde{T}_{\mathcal{M}}/\widetilde{T}^{\mathbf{11}}_{\mathcal{M}}.$$
$$S_{\mathcal{M}} = S^{\text{cat}}_{\mathcal{M}}/D_{\mathcal{M}}, \quad T_{\mathcal{M}} = T^{\text{cat}}_{\mathcal{M}}. \tag{C1}$$

Let d_a be the quantum dimension of anyon a, which is given by $d_a = (S^{\text{cat}}_{\mathcal{M}})^{a\mathbf{1}}$. Let s_a be the topological spin of anyon a, which is given by $\mathrm{e}^{\mathrm{i}2\pi s_a} = (T^{\text{cat}}_{\mathcal{M}})^{aa}$. The total dimension of $\mathcal{M}$ is defined as $D^2_{\mathcal{M}} \equiv \sum_{a\in\mathcal{M}} d_a^2$. Also let $d_{\mathcal{A}}$ be the quantum dimension of the condensable algebra $\mathcal{A}$, i.e., if

$$\mathcal{A} = \bigoplus_{a\in\mathcal{M}} A^a a \tag{C2}$$

then $d_{\mathcal{A}} = \sum_a A^a d_a$. We also have a particle to antiparticle conjugation $a \to \bar{a}$. Similarly, we use i, j, and k to label the anyons in $\mathcal{M}_{/\mathcal{A}}$. Following the above, we can define $\widetilde{S}_{\mathcal{M}_{/\mathcal{A}}} = (\widetilde{S}^{ij}_{\mathcal{M}_{/\mathcal{A}}})$, $\widetilde{T}_{\mathcal{M}_{/\mathcal{A}}} = (\widetilde{T}^{ij}_{\mathcal{M}_{/\mathcal{A}}})$, $S^{\text{cat}}_{\mathcal{M}_{/\mathcal{A}}}$, $T^{\text{cat}}_{\mathcal{M}_{/\mathcal{A}}}$, $S_{\mathcal{M}_{/\mathcal{A}}}$, $T_{\mathcal{M}_{/\mathcal{A}}}$, as well as d_i, s_i, and $D^2_{\mathcal{M}_{/\mathcal{A}}}$. Then we have the following properties.

(1) The distinct s_i's form a sub set of $\{s_a \mid a \in \mathcal{M}\}$.

(2) $(S^{\text{cat}}_{\mathcal{M}})^{ab}$, $(T^{\text{cat}}_{\mathcal{M}})^{aa}$, $D^2_{\mathcal{M}}$, d_a, and $d_{\mathcal{A}}$ are cyclotomic integers, whose conductors divide $\text{ord}(T^{\text{cat}}_{\mathcal{M}})$. $D_{\mathcal{M}}$ is a real cyclotomic integer whose conductors divide $\text{ord}(\widetilde{T}_{\mathcal{M}})$ (assuming $\widetilde{S}^{\mathbf{11}}_{\mathcal{M}}$ is real).

(3) $(S^{\text{cat}}_{\mathcal{M}_{/\mathcal{A}}})^{ij}$, $(T^{\text{cat}}_{\mathcal{M}_{/\mathcal{A}}})^{ii}$, $D^2_{\mathcal{M}_{/\mathcal{A}}}$, and d_i are cyclotomic integers, whose conductors divide $\text{ord}(T^{\text{cat}}_{\mathcal{M}_{/\mathcal{A}}})$. $D_{\mathcal{M}_{/\mathcal{A}}}$ is a real cyclotomic integer whose conductors divide $\text{ord}(\widetilde{T}_{\mathcal{M}_{/\mathcal{A}}})$ (assuming $\widetilde{S}^{\mathbf{11}}_{\mathcal{M}_{/\mathcal{A}}}$ is real).

(4) $D_{\mathcal{M}} = D_{\mathcal{M}_{/\mathcal{A}}} d_{\mathcal{A}}$.

(5) A^a in $\mathcal{A}$ are nonnegative integers, $A^a = A^{\bar{a}}$, and $A^{\mathbf{1}} = 1$.

(6) For $a \in \mathcal{A}$ (i.e., for $A^a \neq 0$), the corresponding $s_a = 0$ mod 1, i.e., the anyons in $\mathcal{A}$ are all bosonic.

(7) if $a, b \in \mathcal{A}$, then at least one of the fusion products in $a \otimes b$ must be contained $\mathcal{A}$, i.e., $\exists c \in \mathcal{A}$ such that $a \otimes b = c \oplus \cdots$.

Now, let us assume $\mathcal{A}$ to be Lagrangian, then the $\mathcal{A}$-condensed boundary of $\mathcal{M}$ is gapped. Let us use x to label the (simple) excitations on the gapped boundary. If we bring a bulk excitation a to such a boundary, it will become a (composite) boundary excitation X

$$X = \oplus M^a_x x, \quad M^a_x \in \mathbb{N}. \tag{C3}$$

Then A^a is given by $A^a = M^a_{\mathbf{1}}$. In other words, $A^a \neq 0$ means that a condenses on the boundary (i.e., the bulk a can become the null excitation $\mathbf{1}$ on the boundary). M^a_x satisfies

$$\sum_c N^{ab}_{\mathcal{M},c} M^c_x = \sum_{y,z} M^a_y M^b_z K^{yz}_x, \tag{C4}$$

where $N^{ab}_{\mathcal{M},c}$ describes the fusion ring of the bulk excitations in $\mathcal{M}$ and K^{xy}_z describes the fusion ring of the boundary excitations. By rewriting $\sum_y = \sum_{y=\mathbf{1}} + \sum_{y\neq\mathbf{1}}$, we find

$$\sum_c N^{ab}_{\mathcal{M},c} M^c_x \geqslant A^a M^b_x. \tag{C5}$$

Let $\bar{A}^a \equiv \sum_{x\neq\mathbf{1}} M^a_x$ and do $\sum_{x\neq\mathbf{1}}$ to the above, we obtain (noticing $\bar{A}^{\mathbf{1}} = 0$)

$$\sum_{c\neq\mathbf{1}} N^{ab}_{\mathcal{M},c} \bar{A}^c \geqslant A^a \bar{A}^b. \tag{C6}$$

Taking $x = \mathbf{1}$, Eq. (C4) reduces to

$$\sum_c N^{ab}_{\mathcal{M},c} A^c = A^a A^b + \sum_{x\neq\mathbf{1}} M^a_x M^b_{\bar{x}}. \tag{C7}$$

Since $M^a_x \geqslant 0$, we obtain an additional condition on A^a

$$\sum_c N^{ab}_{\mathcal{M},c} A^c \geqslant A^a A^b. \tag{C8}$$

We can try to obtain a stronger condition, by showing $\sum_{x\neq\mathbf{1}} M^a_x M^b_{\bar{x}}$ is equal or larger than a positive integer. Summing over x, Eq. (C4) implies

$$\sum_c N^{ab}_{\mathcal{M},c}(A^c + \bar{A}^c) \geqslant A^a A^b + A^a \bar{A}^b + A^b \bar{A}^a. \tag{C9}$$

Combining the above two equations, we find

$$\sum_{x\neq\mathbf{1}} M^a_x M^b_{\bar{x}} \geqslant A^a \bar{A}^b + A^b \bar{A}^a - \sum_{c\neq\mathbf{1}} N^{ab}_{\mathcal{M},c} \bar{A}^c. \tag{C10}$$

Taking $b = \bar{a}$ in Eq. (C7), we find

$$\sum_c N^{a\bar{a}}_{\mathcal{M},c} A^c \geqslant (A^a)^2 + \bar{A}^a. \tag{C11}$$

From the conservation of quantum dimensions, we have

$$d_a = \sum_x M^a_x d_x = A^a + \sum_{x\neq\mathbf{1}} M^a_x d_x, \tag{C12}$$

which implies

$$\delta(d_a) \leqslant \bar{A}^a \leqslant d_a - A^a, \tag{C13}$$

where $\delta(d_a)$ is defined as

$$\delta(d) = \begin{cases} 0 & \text{if } d \in \mathbb{N}, \\ 1 & \text{if } d \notin \mathbb{N}. \end{cases} \tag{C14}$$

Let us define $\bar{A}^a_{\text{max}}$ to be the largest integer that is less than both $d_a - A^a$ and $\sum_c N^{a\bar{a}}_{\mathcal{M},c} A^c - (A^a)^2$. Note that both terms must be larger than $\delta(d_a)$ and $\bar{A}^a_{\text{max}}$ is equal or larger than $\delta(d_a)$:

$$\bar{A}^a_{\text{max}} \geqslant \delta(d_a). \tag{C15}$$

Substituting $\bar{A}^a \leqslant \bar{A}^a_{\text{max}}$ into Eq. (C10), we obtain

$$\begin{aligned}\sum_{x\neq \mathbf{1}} M^a_x M^b_{\bar{x}} &\geqslant \max\left(0, A^a\bar{A}^b + A^b\bar{A}^a - \sum_{c\neq\mathbf{1}} N^{ab}_{\mathcal{M},c}\bar{A}^c_{\text{max}}\right) \\ &\geqslant \max\left(0, A^a\delta(d_b) + A^b\delta(d_a) - \sum_{c\neq\mathbf{1}} N^{ab}_{\mathcal{M},c}\bar{A}^c_{\text{max}}\right).\end{aligned} \tag{C16}$$

For Lagrangian $\mathcal{A}$, the condensation of $\mathcal{A}$ give rises to a gapped boundary. Reference [48] gave a physical picture of the multicomponent τ-independent partition function Z_a of the corresponding gapped boundary. From such a physical picture, we find that $A^a = Z_a$ and satisfies Eq. (5). Summarizing the above discussions, we see that

$$\begin{aligned} &\boldsymbol{A} = S_{\mathcal{M}}\boldsymbol{A}, \quad \boldsymbol{A} = T_{\mathcal{M}}\boldsymbol{A}, \\ &A^a \leqslant d_a - \delta(d_a), \quad A^a A^{\bar{a}} \leqslant \sum_c N^{a\bar{a}}_{\mathcal{M},c} A^c - \delta(d_a), \\ &A^a A^b \leqslant \sum_c N^{ab}_{\mathcal{M},c} A^c, \end{aligned} \tag{C17}$$

where $\boldsymbol{A} = (A^{\mathbf{1}}, A^a, \dots)^\top$. The last condition can be improved to

$$\begin{aligned} A^a A^b \leqslant \sum_c N^{ab}_{\mathcal{M},c} A^c - \max\bigg(0, A^a\delta(d_b) + A^b\delta(d_a) \\ - \sum_{c\neq\mathbf{1}} N^{ab}_{\mathcal{M},c}\bar{A}^c_{\text{max}}\bigg). \end{aligned} \tag{C18}$$

Now, let us assume $\mathcal{A}$ not to be Lagrangian. In this case, $\mathcal{M}_{/\mathcal{A}}$ is nontrivial. Let us consider the domain wall between $\mathcal{M}$ and $\mathcal{M}_{/\mathcal{A}}$. Such a domain wall can be viewed as a boundary of $\mathcal{M} \boxtimes \overline{\mathcal{M}}_{/\mathcal{A}}$ topological order form by stacking $\mathcal{M}$ and the spatial reflection of $\mathcal{M}_{/\mathcal{A}}$. Since the domain wall, and hence the boundary, is gapped, there must be a Lagrangian condensable algebra $\mathcal{A}_{\mathcal{M}\boxtimes\overline{\mathcal{M}}_{/\mathcal{A}}}$ in $\mathcal{M} \boxtimes \overline{\mathcal{M}}_{/\mathcal{A}}$, whose condensation gives rise to the boundary. Let

$$\mathcal{A}_{\mathcal{M}\boxtimes\overline{\mathcal{M}}_{/\mathcal{A}}} = \bigoplus_{a\in\mathcal{M},\ i\in\mathcal{M}_{/\mathcal{A}}} A^{ai}\, a \otimes i, \tag{C19}$$

then the matrix $A = (A^{ai})$ satisfies

$$\begin{aligned} &S_{\mathcal{M}}A = AS_{\mathcal{M}_{/\mathcal{A}}},\ T_{\mathcal{M}}A = AT_{\mathcal{M}_{/\mathcal{A}}},\ A^{ai} \leqslant d_a d_i - \delta(d_a d_i), \\ &A^{ai}A^{bj} \leqslant \sum_{c,k} N^{ab}_{\mathcal{M},c} N^{ij}_{\mathcal{M}_{/\mathcal{A}},k} A^{ck} - \delta_{a,\bar{b}}\delta_{i,\bar{j}}\delta(d_a d_i), \end{aligned} \tag{C20}$$

where Eqs. (C13) and (C11) are used. The above conditions only require the domain wall between $\mathcal{M}$ and $\mathcal{M}_{/\mathcal{A}}$ to be gapped. However, since $\mathcal{M}$ and $\mathcal{M}_{/\mathcal{A}}$ are related by a condensation of $\mathcal{A}$, there is a special domain wall (called the canonical domain wall) such that all the excitation in $\mathcal{M}_{/\mathcal{A}}$ can pass through the domain wall to go into $\mathcal{M}$ without leaving any nontrivial excitations on the wall. For the canonical domain wall, the corresponding A^{ai} must satisfy the following condition:

$$\text{for any } i, \text{ there exists an } a \text{ such that } A^{ai} \neq 0. \tag{C21}$$

The canonical domain wall can be viewed as $\mathcal{A}_{\mathcal{M}\to\mathcal{M}_{/\mathcal{A}}}$-condensed boundary of $\mathcal{M}$ with

$$\mathcal{A}_{\mathcal{M}\to\mathcal{M}_{/\mathcal{A}}} = \bigoplus A^{a\mathbf{1}} a. \tag{C22}$$

We note that anyon a in $\mathcal{M}$ condenses on the canonical domain wall between $\mathcal{M}$ and $\mathcal{M}_{/\mathcal{A}}$, if and only if $A^{a\mathbf{1}} \neq 0$. This implies that

$$\mathcal{A}_{\mathcal{M}\to\mathcal{M}_{/\mathcal{A}}} = \mathcal{A}, \quad A^a = A^{a\mathbf{1}}. \tag{C23}$$

The domain wall can also be viewed as $\mathcal{A}_{\mathcal{M}_{/\mathcal{A}}\to\mathcal{M}}$-condensed boundary of $\mathcal{M}_{/\mathcal{A}}$ with

$$\mathcal{A}_{\mathcal{M}_{/\mathcal{A}}\to\mathcal{M}} = \bigoplus A^{\mathbf{1}i} i. \tag{C24}$$

Since $\mathcal{M}_{/\mathcal{A}}$ comes from a condensation of $\mathcal{M}$, the canonical domain wall must be an **1**-condensed boundary of $\mathcal{M}_{/\mathcal{A}}$, i.e.,

$$\mathcal{A}_{\mathcal{M}_{/\mathcal{A}}\to\mathcal{M}} = \mathbf{1}, \quad A^{\mathbf{1}i} = \delta_{\mathbf{1},i}. \tag{C25}$$

We can obtain more conditions on A^a. From Eq. (C20), we find

$$\frac{D_{\mathcal{M}_{/\mathcal{A}}}}{D_{\mathcal{M}}} \sum_{b\in\mathcal{M}} \left(S^{\text{cat}}_{\mathcal{M}}\right)^{ab} A^{bi} = \sum_{j\in\mathcal{M}_{/\mathcal{A}}} A^{aj} \left(S^{\text{cat}}_{\mathcal{M}_{/\mathcal{A}}}\right)^{ji}, \tag{C26}$$

which implies

$$\frac{\sum_{b\in\mathcal{M}} \left(S^{\text{cat}}_{\mathcal{M}}\right)^{ab} A^{bi}}{\sum_{b\in\mathcal{M}} d_b A^b} = \text{cyclotomic integer} \quad \text{for all } a \in \mathcal{M},\ \ i \in \mathcal{M}_{/\mathcal{A}}. \tag{C27}$$

In particular, $\mathcal{A} = \bigoplus_a A^a a$ must satisfies

$$\frac{\sum_{b\in\mathcal{M}} \left(S^{\text{cat}}_{\mathcal{M}}\right)^{ab} A^{b}}{\sum_{b\in\mathcal{M}} d_b A^b} = \text{cyclotomic integer for all } a \in \mathcal{M}. \tag{C28}$$

From Eq. (C20), we also obtain

$$\begin{aligned} &A^a \leqslant d_a - \delta(d_a), \\ &A^a A^b \leqslant \sum_c N^{ab}_{\mathcal{M},c} A^c - \delta_{a,\bar{b}}\delta(d_a). \end{aligned} \tag{C29}$$

These conditions can help us to find possible condensable algebras $\mathcal{A} = \bigoplus_a A^a a$, which we call different condensation patterns of the system. These conditions can also help us to find possible condensation-induced topological orders $\mathcal{M}_{/\mathcal{A}}$, which determine the low energy properties of the gapless $\mathcal{A}$ state. When combined with conformal character of CFT's [see Eq. (14)], these conditions allow us to obtain gapless (and gapped) boundaries of topological order $\mathcal{M}$ and $\mathcal{M}_{/\mathcal{A}}$. This represents an algebraic number theoretical way to calculate properties of critical points.

APPENDIX D: SPT ORDER AS AUTOMORPHISM OF HOLOCAT SYMMETRY

In a $1+1$D $\mathbb{Z}_2\times\mathbb{Z}_2'$-symmetric system, its holocat symmetry is described symmetry TO $\mathcal{G}au_{\mathbb{Z}_2\times\mathbb{Z}_2'}$ (i.e., a $2+1$D $\mathbb{Z}_2\times\mathbb{Z}_2'$-gauge theory). Let us elaborate on one of the boundary of $\mathcal{G}au_{\mathbb{Z}_2\times\mathbb{Z}_2'}$, the $\mathbf{1}\oplus e_2m_1\oplus e_1m_2\oplus f_1f_2$-condensed boundary, to make contact with Refs. [41,42], where a classification of SPT order for finite symmetries, higher symmetries, and algebraic higher symmetries[17] was given in terms of certain automorphisms of the corresponding symmetry TO. The argument is based on the fact that the boundary of a bulk topological order can be changed by stacking with a domain wall of the bulk TO. It is assumed that all the changes of a gapped boundary phase to other gapped boundary phases can be obtained this way. The gapped boundaries that correspond to trivial and nontrivial SPT orders form a group, i.e., they all have inverses. This motivates the association of SPT orders with certain invertible domain walls in the bulk topological order which correspond to automorphisms of the TO [41,42]. The $\mathbb{Z}_2\times\mathbb{Z}_2'$-SPT state, the $\mathbf{1}\oplus e_2m_1\oplus e_1m_2\oplus f_1f_2$ state, furnishes a simple example of this result.

Let us spell this out in more detail. According to Refs. [41,42], all anomaly-free generalized symmetries are described and classified by local fusion higher category $\mathcal{R}$ formed by the symmetry charges. For $1+1$D $\mathbb{Z}_2\times\mathbb{Z}_2'$ symmetry, $\mathcal{R}$ is a fusion 1-category consisting of the anyons $(\mathbf{1},e_1,e_2,e_1e_2)$ as objects. The dual symmetry $\widetilde{\mathbb{Z}}_2\times\widetilde{\mathbb{Z}}_2'$ is described by the dual fusion 1-category, $\widetilde{\mathcal{R}}$, similarly formed by $(\mathbf{1},m_1,m_2,m_1m_2)$. The symmetry TO of $\mathcal{R}$ and $\widetilde{\mathcal{R}}$ are the same and is given by their Drinfeld center

$$\mathcal{G}au_{\mathbb{Z}_2\times\mathbb{Z}_2'}=\mathcal{Z}(\mathcal{R})=\mathcal{Z}(\widetilde{\mathcal{R}}). \tag{D1}$$

Since *center is bulk* [43,46,47], the above expression means that the $2+1$D TO $\mathcal{G}au_{\mathbb{Z}_2\times\mathbb{Z}_2'}$ has two gapped boundaries with excitations described by $\mathcal{R}$ and $\widetilde{\mathcal{R}}$. The $\mathcal{R}$-boundary is induced by condensing $\mathcal{A}_{\mathcal{R}}=\mathbf{1}\oplus m_1\oplus m_2\oplus m_1m_2$. The $\widetilde{\mathcal{R}}$ boundary is induced by condensing $\mathcal{A}_{\widetilde{\mathcal{R}}}=\mathbf{1}\oplus e_1\oplus e_2\oplus e_1e_2$. The $\mathcal{A}_{\mathcal{R}}$-condensed boundary is the $\mathbb{Z}_2\times\mathbb{Z}_2'$-symmetric state with trivial SPT order. The gapped state with nontrivial SPT order is given by condensation of $\alpha(\mathcal{A}_{\mathcal{R}})$ on the boundary, where α is an automorphism of the bulk topological order $\mathcal{G}au_{\mathbb{Z}_2\times\mathbb{Z}_2'}$ that satisfies $\alpha(\mathcal{A}_{\widetilde{\mathcal{R}}})=\mathcal{A}_{\widetilde{\mathcal{R}}}$. In other words, the automorphism acts trivially on the so-called *electric* Lagrangian condensable algebra. This requirement for the automorphism α is required so that it does not alter the description of the symmetry on the boundary system. For our example $\mathcal{G}au_{\mathbb{Z}_2\times\mathbb{Z}_2'}$, one of the automorphisms is $(e_1\leftrightarrow e_2, m_1\leftrightarrow m_2)$, which exchanges $\mathbb{Z}_2$ and $\mathbb{Z}_2'$ and changes $\mathbf{1}\oplus e_1\oplus e_2\oplus e_1e_2$ to $\mathbf{1}\oplus e_2\oplus e_1\oplus e_1e_2$. However, such an automorphism changes the symmetry since, for instance, $\mathbb{Z}_2$ symmetry may correspond to spin rotation while $\mathbb{Z}_2'$ to charge conjugation. So we need to exclude such automorphisms.[18] By observation, we find another nontrivial automorphism of $\mathcal{G}au_{\mathbb{Z}_2\times\mathbb{Z}_2'}$:

$$\alpha(e_1)=e_1,\quad \alpha(e_2)=e_2,\quad \alpha(m_1)=e_2m_1,\quad \alpha(m_2)=e_1m_2,$$

which maps $\mathcal{A}_{\widetilde{\mathcal{R}}}$ to $\mathcal{A}_{\widetilde{\mathcal{R}}}$ and maps $\mathcal{A}_{\mathcal{R}}$ to

$$\begin{aligned}\alpha(\mathbf{1}\oplus m_1\oplus m_2\oplus m_1m_2)&\\ =\alpha(\mathbf{1})\oplus\alpha(m_1)\oplus\alpha(m_2)\oplus\alpha(m_1)\alpha(m_2)&\\ =\mathbf{1}\oplus e_2m_1\oplus e_1m_2\oplus f_1f_2.&\end{aligned} \tag{D2}$$

Thus the gapped $\mathbf{1}\oplus e_2m_1\oplus e_1m_2\oplus f_1f_2$ state is a nontrivial $\mathbb{Z}_2\times\mathbb{Z}_2'$-SPT state.

APPENDIX E: GAPLESS BOUNDARIES OF $\mathcal{G}au_{S_3}$ WITH CENTRAL CHARGE $(c,\bar{c})\leqslant(\frac{5}{6},\frac{5}{6})$

In this section, we list all the multicomponent boundary partition functions with central charge $(c,\bar{c})\leqslant(\frac{5}{6},\frac{5}{6})$ for the $2+1$D topological order $\mathcal{G}au_{S_3}$. These gapless boundaries are described by CFT's constructed from minimal models. Here χ_h^{m4} are conformal characters with conformal dimension h, for (4,3) minimal model. χ_h^{m5} are conformal characters for (5,4) minimal model, etc.

We can determine the condensable algebra $\mathcal{A}$ that produces the boundary by examine the appearances of $|\chi_0^{m\#}|^2$ term. From partition function $Z_{\mathbf{1}}$, we can also determine the number of relevant operators.

$\mathbf{1}\oplus a_2$-condensed boundary with 1 relevant operator:

$$\begin{aligned}Z_{\mathbf{1}}&=|\chi_0^{m4}|^2+|\chi_{\frac12}^{m4}|^2,\\ Z_{a_1}&=|\chi_{\frac1{16}}^{m4}|^2,\\ Z_{a_2}&=|\chi_0^{m4}|^2+|\chi_{\frac1{16}}^{m4}|^2+|\chi_{\frac12}^{m4}|^2,\\ Z_b&=0,\\ Z_{b_1}&=0,\\ Z_{b_2}&=0,\\ Z_c&=|\chi_{\frac1{16}}^{m4}|^2,\\ Z_{c_1}&=\chi_0^{m4}\bar{\chi}_{\frac12}^{m4}+\chi_{\frac12}^{m4}\bar{\chi}_0^{m4},\end{aligned} \tag{E1}$$

$$\begin{aligned}Z_{\mathbf{1}}&=|\chi_0^{m4}|^2+|\chi_{\frac1{16}}^{m4}|^2+|\chi_{\frac12}^{m4}|^2,\\ Z_{a_1}&=|\chi_0^{m4}|^2+|\chi_{\frac1{16}}^{m4}|^2+|\chi_{\frac12}^{m4}|^2,\\ Z_{a_2}&=2|\chi_0^{m4}|^2+2|\chi_{\frac1{16}}^{m4}|^2+2|\chi_{\frac12}^{m4}|^2,\\ Z_b&=0,\\ Z_{b_1}&=0,\\ Z_{b_2}&=0,\\ Z_c&=0,\\ Z_{c_1}&=0.\end{aligned} \tag{E2}$$

[17] Also known as noninvertible symmetries.

[18] Note that, here, we view $\mathbf{1}\oplus e_1\oplus e_2\oplus e_1e_2$ and $\mathbf{1}\oplus e_2\oplus e_1\oplus e_1e_2$ as unequal condensable algebras: $\mathbf{1}\oplus e_1\oplus e_2\oplus e_1e_2\neq\mathbf{1}\oplus e_2\oplus e_1\oplus e_1e_2$.

1 $\oplus$ b-condensed boundary with 1 relevant operator:

$$\begin{aligned}
Z_{\mathbf{1}} &= |\chi_0^{m4}|^2 + |\chi_{\frac{1}{2}}^{m4}|^2,\\
Z_{a_1} &= |\chi_{\frac{1}{16}}^{m4}|^2,\\
Z_{a_2} &= 0,\\
Z_b &= |\chi_0^{m4}|^2 + |\chi_{\frac{1}{16}}^{m4}|^2 + |\chi_{\frac{1}{2}}^{m4}|^2,\\
Z_{b_1} &= 0,\\
Z_{b_2} &= 0,\\
Z_c &= |\chi_{\frac{1}{16}}^{m4}|^2,\\
Z_{c_1} &= \chi_0^{m4}\bar{\chi}_{\frac{1}{2}}^{m4} + \chi_{\frac{1}{2}}^{m4}\bar{\chi}_0^{m4},
\end{aligned} \tag{E3}$$

$$\begin{aligned}
Z_{\mathbf{1}} &= |\chi_0^{m4}|^2 + |\chi_{\frac{1}{16}}^{m4}|^2 + |\chi_{\frac{1}{2}}^{m4}|^2,\\
Z_{a_1} &= |\chi_0^{m4}|^2 + |\chi_{\frac{1}{16}}^{m4}|^2 + |\chi_{\frac{1}{2}}^{m4}|^2,\\
Z_{a_2} &= 0,\\
Z_b &= 2|\chi_0^{m4}|^2 + 2|\chi_{\frac{1}{16}}^{m4}|^2 + 2|\chi_{\frac{1}{2}}^{m4}|^2,\\
Z_{b_1} &= 0,\\
Z_{b_2} &= 0,\\
Z_c &= 0,\\
Z_{c_1} &= 0,
\end{aligned} \tag{E4}$$

$$\begin{aligned}
Z_{\mathbf{1}} &= |\chi_0^{m4}|^2 + |\chi_{\frac{1}{16}}^{m4}|^2 + |\chi_{\frac{1}{2}}^{m4}|^2,\\
Z_{a_1} &= 0,\\
Z_{a_2} &= |\chi_0^{m4}|^2 + |\chi_{\frac{1}{16}}^{m4}|^2 + |\chi_{\frac{1}{2}}^{m4}|^2,\\
Z_b &= 0,\\
Z_{b_1} &= 0,\\
Z_{b_2} &= 0,\\
Z_c &= |\chi_0^{m4}|^2 + |\chi_{\frac{1}{16}}^{m4}|^2 + |\chi_{\frac{1}{2}}^{m4}|^2,\\
Z_{c_1} &= 0,
\end{aligned} \tag{E5}$$

$$\begin{aligned}
Z_{\mathbf{1}} &= |\chi_0^{m4}|^2 + |\chi_{\frac{1}{16}}^{m4}|^2 + |\chi_{\frac{1}{2}}^{m4}|^2,\\
Z_{a_1} &= 0,\\
Z_{a_2} &= 0,\\
Z_b &= |\chi_0^{m4}|^2 + |\chi_{\frac{1}{16}}^{m4}|^2 + |\chi_{\frac{1}{2}}^{m4}|^2,\\
Z_{b_1} &= 0,\\
Z_{b_2} &= 0,\\
Z_c &= |\chi_0^{m4}|^2 + |\chi_{\frac{1}{16}}^{m4}|^2 + |\chi_{\frac{1}{2}}^{m4}|^2,\\
Z_{c_1} &= 0.
\end{aligned} \tag{E6}$$

1 $\oplus$ a_2-condensed boundary with 2 relevant operators:

$$\begin{aligned}
Z_{\mathbf{1}} &= |\chi_0^{m5}|^2 + |\chi_{\frac{1}{10}}^{m5}|^2 + |\chi_{\frac{3}{5}}^{m5}|^2 + |\chi_{\frac{3}{2}}^{m5}|^2,\\
Z_{a_1} &= |\chi_{\frac{7}{16}}^{m5}|^2 + |\chi_{\frac{3}{80}}^{m5}|^2,\\
Z_{a_2} &= |\chi_0^{m5}|^2 + |\chi_{\frac{1}{10}}^{m5}|^2 + |\chi_{\frac{3}{5}}^{m5}|^2 + |\chi_{\frac{3}{2}}^{m5}|^2 + |\chi_{\frac{7}{16}}^{m5}|^2\\
&\quad + |\chi_{\frac{3}{80}}^{m5}|^2,\\
Z_b &= 0,\\
Z_{b_1} &= 0,\\
Z_{b_2} &= 0,\\
Z_c &= |\chi_{\frac{7}{16}}^{m5}|^2 + |\chi_{\frac{3}{80}}^{m5}|^2,\\
Z_{c_1} &= \chi_0^{m5}\bar{\chi}_{\frac{3}{2}}^{m5} + \chi_{\frac{1}{10}}^{m5}\bar{\chi}_{\frac{3}{5}}^{m5} + \chi_{\frac{3}{5}}^{m5}\bar{\chi}_{\frac{1}{10}}^{m5} + \chi_{\frac{3}{2}}^{m5}\bar{\chi}_0^{m5},
\end{aligned} \tag{E7}$$

$$\begin{aligned}
Z_{\mathbf{1}} &= |\chi_0^{m5}|^2 + |\chi_{\frac{1}{10}}^{m5}|^2 + |\chi_{\frac{3}{5}}^{m5}|^2 + |\chi_{\frac{3}{2}}^{m5}|^2 + |\chi_{\frac{7}{16}}^{m5}|^2 + |\chi_{\frac{3}{80}}^{m5}|^2,\\
Z_{a_1} &= |\chi_0^{m5}|^2 + |\chi_{\frac{1}{10}}^{m5}|^2 + |\chi_{\frac{3}{5}}^{m5}|^2 + |\chi_{\frac{3}{2}}^{m5}|^2 + |\chi_{\frac{7}{16}}^{m5}|^2 + |\chi_{\frac{3}{80}}^{m5}|^2,\\
Z_{a_2} &= 2|\chi_0^{m5}|^2 + 2|\chi_{\frac{1}{10}}^{m5}|^2 + 2|\chi_{\frac{3}{5}}^{m5}|^2 + 2|\chi_{\frac{3}{2}}^{m5}|^2 + 2|\chi_{\frac{7}{16}}^{m5}|^2\\
&\quad + 2|\chi_{\frac{3}{80}}^{m5}|^2,\\
Z_b &= 0,\\
Z_{b_1} &= 0,\\
Z_{b_2} &= 0,\\
Z_c &= 0,\\
Z_{c_1} &= 0.
\end{aligned} \tag{E8}$$

1 $\oplus$ b-condensed boundary with two relevant operators:

$$\begin{aligned}
Z_{\mathbf{1}} &= |\chi_0^{m5}|^2 + |\chi_{\frac{1}{10}}^{m5}|^2 + |\chi_{\frac{3}{5}}^{m5}|^2 + |\chi_{\frac{3}{2}}^{m5}|^2,\\
Z_{a_1} &= |\chi_{\frac{7}{16}}^{m5}|^2 + |\chi_{\frac{3}{80}}^{m5}|^2,\\
Z_{a_2} &= 0,\\
Z_b &= |\chi_0^{m5}|^2 + |\chi_{\frac{1}{10}}^{m5}|^2 + |\chi_{\frac{3}{5}}^{m5}|^2 + |\chi_{\frac{3}{2}}^{m5}|^2 + |\chi_{\frac{7}{16}}^{m5}|^2 + |\chi_{\frac{3}{80}}^{m5}|^2,\\
Z_{b_1} &= 0,\\
Z_{b_2} &= 0,\\
Z_c &= |\chi_{\frac{7}{16}}^{m5}|^2 + |\chi_{\frac{3}{80}}^{m5}|^2,\\
Z_{c_1} &= \chi_0^{m5}\bar{\chi}_{\frac{3}{2}}^{m5} + \chi_{\frac{1}{10}}^{m5}\bar{\chi}_{\frac{3}{5}}^{m5} + \chi_{\frac{3}{5}}^{m5}\bar{\chi}_{\frac{1}{10}}^{m5} + \chi_{\frac{3}{2}}^{m5}\bar{\chi}_0^{m5},
\end{aligned} \tag{E9}$$

$$\begin{aligned}
Z_{\mathbf{1}} &= |\chi_0^{m5}|^2 + |\chi_{\frac{1}{10}}^{m5}|^2 + |\chi_{\frac{3}{5}}^{m5}|^2 + |\chi_{\frac{3}{2}}^{m5}|^2 + |\chi_{\frac{7}{16}}^{m5}|^2 + |\chi_{\frac{3}{80}}^{m5}|^2,\\
Z_{a_1} &= |\chi_0^{m5}|^2 + |\chi_{\frac{1}{10}}^{m5}|^2 + |\chi_{\frac{3}{5}}^{m5}|^2 + |\chi_{\frac{3}{2}}^{m5}|^2 + |\chi_{\frac{7}{16}}^{m5}|^2 + |\chi_{\frac{3}{80}}^{m5}|^2,\\
Z_{a_2} &= 0,\\
Z_b &= 2|\chi_0^{m5}|^2 + 2|\chi_{\frac{1}{10}}^{m5}|^2 + 2|\chi_{\frac{3}{5}}^{m5}|^2 + 2|\chi_{\frac{3}{2}}^{m5}|^2 + 2|\chi_{\frac{7}{16}}^{m5}|^2\\
&\quad + 2|\chi_{\frac{3}{80}}^{m5}|^2,\\
Z_{b_1} &= 0,\\
Z_{b_2} &= 0,\\
Z_c &= 0,\\
Z_{c_1} &= 0,
\end{aligned} \tag{E10}$$

$$\begin{aligned}
Z_{\mathbf{1}} &= |\chi_0^{m5}|^2 + |\chi_{\frac{1}{10}}^{m5}|^2 + |\chi_{\frac{3}{5}}^{m5}|^2 + |\chi_{\frac{3}{2}}^{m5}|^2 + |\chi_{\frac{7}{16}}^{m5}|^2 \\
&\quad + |\chi_{\frac{3}{80}}^{m5}|^2, \\
Z_{a_1} &= 0, \\
Z_{a_2} &= |\chi_0^{m5}|^2 + |\chi_{\frac{1}{10}}^{m5}|^2 + |\chi_{\frac{3}{5}}^{m5}|^2 + |\chi_{\frac{3}{2}}^{m5}|^2 + |\chi_{\frac{7}{16}}^{m5}|^2 \\
&\quad + |\chi_{\frac{3}{80}}^{m5}|^2, \\
Z_b &= 0, \\
Z_{b_1} &= 0, \\
Z_{b_2} &= 0, \\
Z_c &= |\chi_0^{m5}|^2 + |\chi_{\frac{1}{10}}^{m5}|^2 + |\chi_{\frac{3}{5}}^{m5}|^2 + |\chi_{\frac{3}{2}}^{m5}|^2 + |\chi_{\frac{7}{16}}^{m5}|^2 \\
&\quad + |\chi_{\frac{3}{80}}^{m5}|^2, \\
Z_{c_1} &= 0,
\end{aligned} \tag{E11}$$

$$\begin{aligned}
Z_{\mathbf{1}} &= |\chi_0^{m5}|^2 + |\chi_{\frac{1}{10}}^{m5}|^2 + |\chi_{\frac{3}{5}}^{m5}|^2 + |\chi_{\frac{3}{2}}^{m5}|^2 + |\chi_{\frac{7}{16}}^{m5}|^2 \\
&\quad + |\chi_{\frac{3}{80}}^{m5}|^2, \\
Z_{a_1} &= 0, \\
Z_{a_2} &= 0, \\
Z_b &= |\chi_0^{m5}|^2 + |\chi_{\frac{1}{10}}^{m5}|^2 + |\chi_{\frac{3}{5}}^{m5}|^2 + |\chi_{\frac{3}{2}}^{m5}|^2 + |\chi_{\frac{7}{16}}^{m5}|^2 \\
&\quad + |\chi_{\frac{3}{80}}^{m5}|^2, \\
Z_{b_1} &= 0, \\
Z_{b_2} &= 0, \\
Z_c &= |\chi_0^{m5}|^2 + |\chi_{\frac{1}{10}}^{m5}|^2 + |\chi_{\frac{3}{5}}^{m5}|^2 + |\chi_{\frac{3}{2}}^{m5}|^2 + |\chi_{\frac{7}{16}}^{m5}|^2 \\
&\quad + |\chi_{\frac{3}{80}}^{m5}|^2, \\
Z_{c_1} &= 0.
\end{aligned} \tag{E12}$$

$\mathbf{1} \oplus a_2$-condensed boundary with three relevant operators:

$$\begin{aligned}
Z_{\mathbf{1}} &= |\chi_0^{m6}|^2 + |\chi_{\frac{2}{3}}^{m6}|^2 + |\chi_3^{m6}|^2 + |\chi_{\frac{2}{5}}^{m6}|^2 + |\chi_{\frac{1}{15}}^{m6}|^2 \\
&\quad + |\chi_{\frac{7}{5}}^{m6}|^2, \\
Z_{a_1} &= |\chi_{\frac{1}{8}}^{m6}|^2 + |\chi_{\frac{13}{8}}^{m6}|^2 + |\chi_{\frac{1}{40}}^{m6}|^2 + |\chi_{\frac{21}{40}}^{m6}|^2, \\
Z_{a_2} &= |\chi_0^{m6}|^2 + |\chi_{\frac{1}{8}}^{m6}|^2 + |\chi_{\frac{2}{3}}^{m6}|^2 + |\chi_{\frac{13}{8}}^{m6}|^2 + |\chi_3^{m6}|^2 \\
&\quad + |\chi_{\frac{2}{5}}^{m6}|^2 + |\chi_{\frac{1}{40}}^{m6}|^2 + |\chi_{\frac{1}{15}}^{m6}|^2 + |\chi_{\frac{21}{40}}^{m6}|^2 + |\chi_{\frac{7}{5}}^{m6}|^2, \\
Z_b &= 0, \\
Z_{b_1} &= 0, \\
Z_{b_2} &= 0, \\
Z_c &= \chi_0^{m6}\bar{\chi}_3^{m6} + |\chi_{\frac{2}{3}}^{m6}|^2 + \chi_3^{m6}\bar{\chi}_0^{m6} + \chi_{\frac{2}{5}}^{m6}\bar{\chi}_{\frac{7}{5}}^{m6} \\
&\quad + |\chi_{\frac{1}{15}}^{m6}|^2 + \chi_{\frac{7}{5}}^{m6}\bar{\chi}_{\frac{2}{5}}^{m6}, \\
Z_{c_1} &= \chi_{\frac{1}{8}}^{m6}\bar{\chi}_{\frac{13}{8}}^{m6} + \chi_{\frac{13}{8}}^{m6}\bar{\chi}_{\frac{1}{8}}^{m6} + \chi_{\frac{1}{40}}^{m6}\bar{\chi}_{\frac{21}{40}}^{m6} + \chi_{\frac{21}{40}}^{m6}\bar{\chi}_{\frac{1}{40}}^{m6}.
\end{aligned} \tag{E13}$$

$\mathbf{1} \oplus b$-condensed boundary with three relevant operators:

$$\begin{aligned}
Z_{\mathbf{1}} &= |\chi_0^{m6}|^2 + |\chi_{\frac{2}{3}}^{m6}|^2 + |\chi_3^{m6}|^2 + |\chi_{\frac{2}{5}}^{m6}|^2 + |\chi_{\frac{1}{15}}^{m6}|^2 \\
&\quad + |\chi_{\frac{7}{5}}^{m6}|^2, \\
Z_{a_1} &= |\chi_{\frac{1}{8}}^{m6}|^2 + |\chi_{\frac{13}{8}}^{m6}|^2 + |\chi_{\frac{1}{40}}^{m6}|^2 + |\chi_{\frac{21}{40}}^{m6}|^2, \\
Z_{a_2} &= 0, \\
Z_b &= |\chi_0^{m6}|^2 + |\chi_{\frac{1}{8}}^{m6}|^2 + |\chi_{\frac{2}{3}}^{m6}|^2 + |\chi_{\frac{13}{8}}^{m6}|^2 + |\chi_3^{m6}|^2 \\
&\quad + |\chi_{\frac{2}{5}}^{m6}|^2 + |\chi_{\frac{1}{40}}^{m6}|^2 + |\chi_{\frac{1}{15}}^{m6}|^2 + |\chi_{\frac{21}{40}}^{m6}|^2 + |\chi_{\frac{7}{5}}^{m6}|^2, \\
Z_{b_1} &= 0, \\
Z_{b_2} &= 0, \\
Z_c &= \chi_0^{m6}\bar{\chi}_3^{m6} + |\chi_{\frac{2}{3}}^{m6}|^2 + \chi_3^{m6}\bar{\chi}_0^{m6} + \chi_{\frac{2}{5}}^{m6}\bar{\chi}_{\frac{7}{5}}^{m6} \\
&\quad + |\chi_{\frac{1}{15}}^{m6}|^2 + \chi_{\frac{7}{5}}^{m6}\bar{\chi}_{\frac{2}{5}}^{m6}, \\
Z_{c_1} &= \chi_{\frac{1}{8}}^{m6}\bar{\chi}_{\frac{13}{8}}^{m6} + \chi_{\frac{13}{8}}^{m6}\bar{\chi}_{\frac{1}{8}}^{m6} + \chi_{\frac{1}{40}}^{m6}\bar{\chi}_{\frac{21}{40}}^{m6} + \chi_{\frac{21}{40}}^{m6}\bar{\chi}_{\frac{1}{40}}^{m6},
\end{aligned} \tag{E14}$$

$$\begin{aligned}
Z_{\mathbf{1}} &= |\chi_0^{m6}|^2 + \chi_0^{m6}\bar{\chi}_3^{m6} + 2|\chi_{\frac{2}{3}}^{m6}|^2 + \chi_3^{m6}\bar{\chi}_0^{m6} + |\chi_3^{m6}|^2 \\
&\quad + |\chi_{\frac{2}{5}}^{m6}|^2 + \chi_{\frac{2}{5}}^{m6}\bar{\chi}_{\frac{7}{5}}^{m6} + 2|\chi_{\frac{1}{15}}^{m6}|^2 + \chi_{\frac{7}{5}}^{m6}\bar{\chi}_{\frac{2}{5}}^{m6} + |\chi_{\frac{7}{5}}^{m6}|^2, \\
Z_{a_1} &= |\chi_0^{m6}|^2 + \chi_0^{m6}\bar{\chi}_3^{m6} + 2|\chi_{\frac{2}{3}}^{m6}|^2 + \chi_3^{m6}\bar{\chi}_0^{m6} + |\chi_3^{m6}|^2 \\
&\quad + |\chi_{\frac{2}{5}}^{m6}|^2 + \chi_{\frac{2}{5}}^{m6}\bar{\chi}_{\frac{7}{5}}^{m6} + 2|\chi_{\frac{1}{15}}^{m6}|^2 + \chi_{\frac{7}{5}}^{m6}\bar{\chi}_{\frac{2}{5}}^{m6} + |\chi_{\frac{7}{5}}^{m6}|^2, \\
Z_{a_2} &= 0, \\
Z_b &= 2|\chi_0^{m6}|^2 + 2\chi_0^{m6}\bar{\chi}_3^{m6} + 4|\chi_{\frac{2}{3}}^{m6}|^2 + 2\chi_3^{m6}\bar{\chi}_0^{m6} + 2|\chi_3^{m6}|^2 \\
&\quad + 2|\chi_{\frac{2}{5}}^{m6}|^2 + 2\chi_{\frac{2}{5}}^{m6}\bar{\chi}_{\frac{7}{5}}^{m6} + 4|\chi_{\frac{1}{15}}^{m6}|^2 + 2\chi_{\frac{7}{5}}^{m6}\bar{\chi}_{\frac{2}{5}}^{m6} + 2|\chi_{\frac{7}{5}}^{m6}|^2, \\
Z_{b_1} &= 0, \\
Z_{b_2} &= 0, \\
Z_c &= 0, \\
Z_{c_1} &= 0.
\end{aligned} \tag{E15}$$

$\mathbf{1} \oplus a_1$-condensed boundary with one relevant operator:

$$\begin{aligned}
Z_{\mathbf{1}} &= |\chi_0^{m6}|^2 + \chi_0^{m6}\bar{\chi}_3^{m6} + \chi_3^{m6}\bar{\chi}_0^{m6} + |\chi_3^{m6}|^2 + |\chi_{\frac{2}{5}}^{m6}|^2 \\
&\quad + \chi_{\frac{2}{5}}^{m6}\bar{\chi}_{\frac{7}{5}}^{m6} + \chi_{\frac{7}{5}}^{m6}\bar{\chi}_{\frac{2}{5}}^{m6} + |\chi_{\frac{7}{5}}^{m6}|^2, \\
Z_{a_1} &= |\chi_0^{m6}|^2 + \chi_0^{m6}\bar{\chi}_3^{m6} + \chi_3^{m6}\bar{\chi}_0^{m6} + |\chi_3^{m6}|^2 + |\chi_{\frac{2}{5}}^{m6}|^2 \\
&\quad + \chi_{\frac{2}{5}}^{m6}\bar{\chi}_{\frac{7}{5}}^{m6} + \chi_{\frac{7}{5}}^{m6}\bar{\chi}_{\frac{2}{5}}^{m6} + |\chi_{\frac{7}{5}}^{m6}|^2, \\
Z_{a_2} &= 2|\chi_{\frac{2}{3}}^{m6}|^2 + 2|\chi_{\frac{1}{15}}^{m6}|^2, \quad Z_b = 2|\chi_{\frac{2}{3}}^{m6}|^2 + 2|\chi_{\frac{1}{15}}^{m6}|^2, \\
Z_{b_1} &= 2\chi_0^{m6}\bar{\chi}_{\frac{2}{3}}^{m6} + 2\chi_3^{m6}\bar{\chi}_{\frac{2}{3}}^{m6} + 2\chi_{\frac{2}{5}}^{m6}\bar{\chi}_{\frac{1}{15}}^{m6} + 2\chi_{\frac{7}{5}}^{m6}\bar{\chi}_{\frac{1}{15}}^{m6}, \\
Z_{b_2} &= 2\chi_{\frac{2}{3}}^{m6}\bar{\chi}_0^{m6} + 2\chi_{\frac{2}{3}}^{m6}\bar{\chi}_3^{m6} + 2\chi_{\frac{1}{15}}^{m6}\bar{\chi}_{\frac{2}{5}}^{m6} + 2\chi_{\frac{1}{15}}^{m6}\bar{\chi}_{\frac{7}{5}}^{m6}, \\
Z_c &= 0, \quad Z_{c_1} = 0,
\end{aligned} \tag{E16}$$

$$\begin{aligned}
Z_{\mathbf{1}} &= |\chi_0^{m6}|^2 + \chi_0^{m6}\bar{\chi}_3^{m6} + 2|\chi_{\frac{2}{3}}^{m6}|^2 + \chi_3^{m6}\bar{\chi}_0^{m6} + |\chi_3^{m6}|^2 \\
&\quad + |\chi_{\frac{2}{5}}^{m6}|^2 + \chi_{\frac{2}{5}}^{m6}\bar{\chi}_{\frac{7}{5}}^{m6} + 2|\chi_{\frac{1}{15}}^{m6}|^2 + \chi_{\frac{7}{5}}^{m6}\bar{\chi}_{\frac{2}{5}}^{m6} + |\chi_{\frac{7}{5}}^{m6}|^2, \\
Z_{a_1} &= 0,
\end{aligned}$$

$$\begin{aligned}
Z_{a_2} &= |\chi_0^{m6}|^2 + \chi_0^{m6}\bar{\chi}_3^{m6} + 2|\chi_{\frac{2}{3}}^{m6}|^2 + \chi_3^{m6}\bar{\chi}_0^{m6} + |\chi_3^{m6}|^2 \\
&\quad + |\chi_{\frac{2}{5}}^{m6}|^2 + \chi_{\frac{2}{5}}^{m6}\bar{\chi}_{\frac{7}{5}}^{m6} + 2|\chi_{\frac{1}{15}}^{m6}|^2 + \chi_{\frac{7}{5}}^{m6}\bar{\chi}_{\frac{2}{5}}^{m6} + |\chi_{\frac{7}{5}}^{m6}|^2, \\
Z_b &= 0, \\
Z_{b_1} &= 0, \\
Z_{b_2} &= 0, \\
Z_c &= |\chi_0^{m6}|^2 + \chi_0^{m6}\bar{\chi}_3^{m6} + 2|\chi_{\frac{2}{3}}^{m6}|^2 + \chi_3^{m6}\bar{\chi}_0^{m6} + |\chi_3^{m6}|^2 \\
&\quad + |\chi_{\frac{2}{5}}^{m6}|^2 + \chi_{\frac{2}{5}}^{m6}\bar{\chi}_{\frac{7}{5}}^{m6} + 2|\chi_{\frac{1}{15}}^{m6}|^2 + \chi_{\frac{7}{5}}^{m6}\bar{\chi}_{\frac{2}{5}}^{m6} + |\chi_{\frac{7}{5}}^{m6}|^2, \\
Z_{c_1} &= 0,
\end{aligned} \tag{E17}$$

$$\begin{aligned}
Z_{\mathbf{1}} &= |\chi_0^{m6}|^2 + \chi_0^{m6}\bar{\chi}_3^{m6} + 2|\chi_{\frac{2}{3}}^{m6}|^2 + \chi_3^{m6}\bar{\chi}_0^{m6} + |\chi_3^{m6}|^2 \\
&\quad + |\chi_{\frac{2}{5}}^{m6}|^2 + \chi_{\frac{2}{5}}^{m6}\bar{\chi}_{\frac{7}{5}}^{m6} + 2|\chi_{\frac{1}{15}}^{m6}|^2 + \chi_{\frac{7}{5}}^{m6}\bar{\chi}_{\frac{2}{5}}^{m6} + |\chi_{\frac{7}{5}}^{m6}|^2, \\
Z_{a_1} &= 0, \\
Z_{a_2} &= 0, \\
Z_b &= |\chi_0^{m6}|^2 + \chi_0^{m6}\bar{\chi}_3^{m6} + 2|\chi_{\frac{2}{3}}^{m6}|^2 + \chi_3^{m6}\bar{\chi}_0^{m6} + |\chi_3^{m6}|^2 \\
&\quad + |\chi_{\frac{2}{5}}^{m6}|^2 + \chi_{\frac{2}{5}}^{m6}\bar{\chi}_{\frac{7}{5}}^{m6} + 2|\chi_{\frac{1}{15}}^{m6}|^2 + \chi_{\frac{7}{5}}^{m6}\bar{\chi}_{\frac{2}{5}}^{m6} + |\chi_{\frac{7}{5}}^{m6}|^2, \\
Z_{b_1} &= 0, \\
Z_{b_2} &= 0, \\
Z_c &= |\chi_0^{m6}|^2 + \chi_0^{m6}\bar{\chi}_3^{m6} + 2|\chi_{\frac{2}{3}}^{m6}|^2 + \chi_3^{m6}\bar{\chi}_0^{m6} + |\chi_3^{m6}|^2 \\
&\quad + |\chi_{\frac{2}{5}}^{m6}|^2 + \chi_{\frac{2}{5}}^{m6}\bar{\chi}_{\frac{7}{5}}^{m6} + 2|\chi_{\frac{1}{15}}^{m6}|^2 + \chi_{\frac{7}{5}}^{m6}\bar{\chi}_{\frac{2}{5}}^{m6} + |\chi_{\frac{7}{5}}^{m6}|^2, \\
Z_{c_1} &= 0.
\end{aligned} \tag{E18}$$

1-condensed boundary with one relevant operator:

$$\begin{aligned}
Z_{\mathbf{1}} &= |\chi_0^{m6}|^2 + |\chi_3^{m6}|^2 + |\chi_{\frac{2}{5}}^{m6}|^2 + |\chi_{\frac{7}{5}}^{m6}|^2, \\
Z_{a_1} &= \chi_0^{m6}\bar{\chi}_3^{m6} + \chi_3^{m6}\bar{\chi}_0^{m6} + \chi_{\frac{2}{5}}^{m6}\bar{\chi}_{\frac{7}{5}}^{m6} + \chi_{\frac{7}{5}}^{m6}\bar{\chi}_{\frac{2}{5}}^{m6}, \\
Z_{a_2} &= |\chi_{\frac{2}{3}}^{m6}|^2 + |\chi_{\frac{1}{15}}^{m6}|^2, \\
Z_b &= |\chi_{\frac{2}{3}}^{m6}|^2 + |\chi_{\frac{1}{15}}^{m6}|^2, \\
Z_{b_1} &= \chi_0^{m6}\bar{\chi}_{\frac{2}{3}}^{m6} + \chi_3^{m6}\bar{\chi}_{\frac{2}{3}}^{m6} + \chi_{\frac{2}{5}}^{m6}\bar{\chi}_{\frac{1}{15}}^{m6} + \chi_{\frac{7}{5}}^{m6}\bar{\chi}_{\frac{1}{15}}^{m6}, \\
Z_{b_2} &= \chi_{\frac{2}{3}}^{m6}\bar{\chi}_0^{m6} + \chi_{\frac{2}{3}}^{m6}\bar{\chi}_3^{m6} + \chi_{\frac{1}{15}}^{m6}\bar{\chi}_{\frac{2}{5}}^{m6} + \chi_{\frac{1}{15}}^{m6}\bar{\chi}_{\frac{7}{5}}^{m6}, \\
Z_c &= |\chi_{\frac{1}{8}}^{m6}|^2 + |\chi_{\frac{13}{8}}^{m6}|^2 + |\chi_{\frac{1}{40}}^{m6}|^2 + |\chi_{\frac{21}{40}}^{m6}|^2, \\
Z_{c_1} &= \chi_{\frac{1}{8}}^{m6}\bar{\chi}_{\frac{13}{8}}^{m6} + \chi_{\frac{13}{8}}^{m6}\bar{\chi}_{\frac{1}{8}}^{m6} + \chi_{\frac{1}{40}}^{m6}\bar{\chi}_{\frac{21}{40}}^{m6} + \chi_{\frac{21}{40}}^{m6}\bar{\chi}_{\frac{1}{40}}^{m6},
\end{aligned} \tag{E19}$$

$$\begin{aligned}
Z_{\mathbf{1}} &= |\chi_0^{m6}|^2 + |\chi_{\frac{1}{8}}^{m6}|^2 + |\chi_{\frac{2}{3}}^{m6}|^2 + |\chi_{\frac{13}{8}}^{m6}|^2 + |\chi_3^{m6}|^2 \\
&\quad + |\chi_{\frac{2}{5}}^{m6}|^2 + |\chi_{\frac{1}{40}}^{m6}|^2 + |\chi_{\frac{1}{15}}^{m6}|^2 + |\chi_{\frac{21}{40}}^{m6}|^2 + |\chi_{\frac{7}{5}}^{m6}|^2, \\
Z_{a_1} &= 0, \\
Z_{a_2} &= |\chi_0^{m6}|^2 + |\chi_{\frac{1}{8}}^{m6}|^2 + |\chi_{\frac{2}{3}}^{m6}|^2 + |\chi_{\frac{13}{8}}^{m6}|^2 + |\chi_3^{m6}|^2 \\
&\quad + |\chi_{\frac{2}{5}}^{m6}|^2 + |\chi_{\frac{1}{40}}^{m6}|^2 + |\chi_{\frac{1}{15}}^{m6}|^2 + |\chi_{\frac{21}{40}}^{m6}|^2 + |\chi_{\frac{7}{5}}^{m6}|^2, \\
Z_b &= 0, \\
Z_{b_1} &= 0, \\
Z_{b_2} &= 0, \\
Z_c &= |\chi_0^{m6}|^2 + |\chi_{\frac{1}{8}}^{m6}|^2 + |\chi_{\frac{2}{3}}^{m6}|^2 + |\chi_{\frac{13}{8}}^{m6}|^2 + |\chi_3^{m6}|^2 \\
&\quad + |\chi_{\frac{2}{5}}^{m6}|^2 + |\chi_{\frac{1}{40}}^{m6}|^2 + |\chi_{\frac{1}{15}}^{m6}|^2 + |\chi_{\frac{21}{40}}^{m6}|^2 + |\chi_{\frac{7}{5}}^{m6}|^2, \\
Z_{c_1} &= 0.
\end{aligned} \tag{E20}$$

1 $\oplus$ a_2-condensed boundary with three relevant operators:

$$\begin{aligned}
Z_{\mathbf{1}} &= |\chi_0^{m6}|^2 + |\chi_{\frac{2}{3}}^{m6}|^2 + |\chi_3^{m6}|^2 + |\chi_{\frac{2}{5}}^{m6}|^2 + |\chi_{\frac{1}{15}}^{m6}|^2 + |\chi_{\frac{7}{5}}^{m6}|^2, \\
Z_{a_1} &= \chi_0^{m6}\bar{\chi}_3^{m6} + |\chi_{\frac{2}{3}}^{m6}|^2 + \chi_3^{m6}\bar{\chi}_0^{m6} + \chi_{\frac{2}{5}}^{m6}\bar{\chi}_{\frac{7}{5}}^{m6} + |\chi_{\frac{1}{15}}^{m6}|^2 \\
&\quad + \chi_{\frac{7}{5}}^{m6}\bar{\chi}_{\frac{2}{5}}^{m6}, \\
Z_{a_2} &= |\chi_0^{m6}|^2 + \chi_0^{m6}\bar{\chi}_3^{m6} + 2|\chi_{\frac{2}{3}}^{m6}|^2 + \chi_3^{m6}\bar{\chi}_0^{m6} + |\chi_3^{m6}|^2 \\
&\quad + |\chi_{\frac{2}{5}}^{m6}|^2 + \chi_{\frac{2}{5}}^{m6}\bar{\chi}_{\frac{7}{5}}^{m6} + 2|\chi_{\frac{1}{15}}^{m6}|^2 + \chi_{\frac{7}{5}}^{m6}\bar{\chi}_{\frac{2}{5}}^{m6} + |\chi_{\frac{7}{5}}^{m6}|^2, \\
Z_b &= 0, \\
Z_{b_1} &= 0, \\
Z_{b_2} &= 0, \\
Z_c &= |\chi_{\frac{1}{8}}^{m6}|^2 + |\chi_{\frac{13}{8}}^{m6}|^2 + |\chi_{\frac{1}{40}}^{m6}|^2 + |\chi_{\frac{21}{40}}^{m6}|^2, \\
Z_{c_1} &= \chi_{\frac{1}{8}}^{m6}\bar{\chi}_{\frac{13}{8}}^{m6} + \chi_{\frac{13}{8}}^{m6}\bar{\chi}_{\frac{1}{8}}^{m6} + \chi_{\frac{1}{40}}^{m6}\bar{\chi}_{\frac{21}{40}}^{m6} + \chi_{\frac{21}{40}}^{m6}\bar{\chi}_{\frac{1}{40}}^{m6},
\end{aligned} \tag{E21}$$

$$\begin{aligned}
Z_{\mathbf{1}} &= |\chi_0^{m6}|^2 + |\chi_{\frac{1}{8}}^{m6}|^2 + |\chi_{\frac{2}{3}}^{m6}|^2 + |\chi_{\frac{13}{8}}^{m6}|^2 + |\chi_3^{m6}|^2 \\
&\quad + |\chi_{\frac{2}{5}}^{m6}|^2 + |\chi_{\frac{1}{40}}^{m6}|^2 + |\chi_{\frac{1}{15}}^{m6}|^2 + |\chi_{\frac{21}{40}}^{m6}|^2 + |\chi_{\frac{7}{5}}^{m6}|^2, \\
Z_{a_1} &= 0, \\
Z_{a_2} &= 0, \\
Z_b &= |\chi_0^{m6}|^2 + |\chi_{\frac{1}{8}}^{m6}|^2 + |\chi_{\frac{2}{3}}^{m6}|^2 + |\chi_{\frac{13}{8}}^{m6}|^2 + |\chi_3^{m6}|^2 \\
&\quad + |\chi_{\frac{2}{5}}^{m6}|^2 + |\chi_{\frac{1}{40}}^{m6}|^2 + |\chi_{\frac{1}{15}}^{m6}|^2 + |\chi_{\frac{21}{40}}^{m6}|^2 + |\chi_{\frac{7}{5}}^{m6}|^2, \\
Z_{b_1} &= 0, \\
Z_{b_2} &= 0, \\
Z_c &= |\chi_0^{m6}|^2 + |\chi_{\frac{1}{8}}^{m6}|^2 + |\chi_{\frac{2}{3}}^{m6}|^2 + |\chi_{\frac{13}{8}}^{m6}|^2 + |\chi_3^{m6}|^2 \\
&\quad + |\chi_{\frac{2}{5}}^{m6}|^2 + |\chi_{\frac{1}{40}}^{m6}|^2 + |\chi_{\frac{1}{15}}^{m6}|^2 + |\chi_{\frac{21}{40}}^{m6}|^2 + |\chi_{\frac{7}{5}}^{m6}|^2, \\
Z_{c_1} &= 0.
\end{aligned} \tag{E22}$$

1 $\oplus$ b-condensed boundary with three relevant operators:

$$\begin{aligned}
Z_{\mathbf{1}} &= |\chi_0^{m6}|^2 + |\chi_{\frac{2}{3}}^{m6}|^2 + |\chi_3^{m6}|^2 + |\chi_{\frac{2}{5}}^{m6}|^2 + |\chi_{\frac{1}{15}}^{m6}|^2 + |\chi_{\frac{7}{5}}^{m6}|^2, \\
Z_{a_1} &= \chi_0^{m6}\bar{\chi}_3^{m6} + |\chi_{\frac{2}{3}}^{m6}|^2 + \chi_3^{m6}\bar{\chi}_0^{m6} + \chi_{\frac{2}{5}}^{m6}\bar{\chi}_{\frac{7}{5}}^{m6} + |\chi_{\frac{1}{15}}^{m6}|^2 \\
&\quad + \chi_{\frac{7}{5}}^{m6}\bar{\chi}_{\frac{2}{5}}^{m6}, \\
Z_{a_2} &= 0, \\
Z_b &= |\chi_0^{m6}|^2 + \chi_0^{m6}\bar{\chi}_3^{m6} + 2|\chi_{\frac{2}{3}}^{m6}|^2 + \chi_3^{m6}\bar{\chi}_0^{m6} + |\chi_3^{m6}|^2 \\
&\quad + |\chi_{\frac{2}{5}}^{m6}|^2 + \chi_{\frac{2}{5}}^{m6}\bar{\chi}_{\frac{7}{5}}^{m6} + 2|\chi_{\frac{1}{15}}^{m6}|^2 + \chi_{\frac{7}{5}}^{m6}\bar{\chi}_{\frac{2}{5}}^{m6} + |\chi_{\frac{7}{5}}^{m6}|^2,
\end{aligned}$$

$$
\begin{aligned}
Z_{b_1} &= 0,\\
Z_{b_2} &= 0,\\
Z_c &= |\chi^{m6}_{\frac{1}{8}}|^2 + |\chi^{m6}_{\frac{13}{8}}|^2 + |\chi^{m6}_{\frac{1}{40}}|^2 + |\chi^{m6}_{\frac{21}{40}}|^2,\\
Z_{c_1} &= \chi^{m6}_{\frac{1}{8}}\bar{\chi}^{m6}_{\frac{13}{8}} + \chi^{m6}_{\frac{13}{8}}\bar{\chi}^{m6}_{\frac{1}{8}} + \chi^{m6}_{\frac{1}{40}}\bar{\chi}^{m6}_{\frac{21}{40}} + \chi^{m6}_{\frac{21}{40}}\bar{\chi}^{m6}_{\frac{1}{40}},
\end{aligned} \tag{E23}
$$

$$
\begin{aligned}
Z_{\mathbf{1}} &= |\chi^{m6}_0|^2 + |\chi^{m6}_{\frac{1}{8}}|^2 + |\chi^{m6}_{\frac{2}{3}}|^2 + |\chi^{m6}_{\frac{13}{8}}|^2 + |\chi^{m6}_3|^2\\
&\quad + |\chi^{m6}_{\frac{2}{5}}|^2 + |\chi^{m6}_{\frac{1}{40}}|^2 + |\chi^{m6}_{\frac{1}{15}}|^2 + |\chi^{m6}_{\frac{21}{40}}|^2 + |\chi^{m6}_{\frac{7}{5}}|^2,\\
Z_{a_1} &= |\chi^{m6}_0|^2 + |\chi^{m6}_{\frac{1}{8}}|^2 + |\chi^{m6}_{\frac{2}{3}}|^2 + |\chi^{m6}_{\frac{13}{8}}|^2 + |\chi^{m6}_3|^2\\
&\quad + |\chi^{m6}_{\frac{2}{5}}|^2 + |\chi^{m6}_{\frac{1}{40}}|^2 + |\chi^{m6}_{\frac{1}{15}}|^2 + |\chi^{m6}_{\frac{21}{40}}|^2 + |\chi^{m6}_{\frac{7}{5}}|^2,\\
Z_{a_2} &= 2|\chi^{m6}_0|^2 + 2|\chi^{m6}_{\frac{1}{8}}|^2 + 2|\chi^{m6}_{\frac{2}{3}}|^2 + 2|\chi^{m6}_{\frac{13}{8}}|^2 + 2|\chi^{m6}_3|^2\\
&\quad + 2|\chi^{m6}_{\frac{2}{5}}|^2 + 2|\chi^{m6}_{\frac{1}{40}}|^2 + 2|\chi^{m6}_{\frac{1}{15}}|^2 + 2|\chi^{m6}_{\frac{21}{40}}|^2 + 2|\chi^{m6}_{\frac{7}{5}}|^2,\\
Z_b &= 0,\\
Z_{b_1} &= 0,\\
Z_{b_2} &= 0,\\
Z_c &= 0,\\
Z_{c_1} &= 0,
\end{aligned} \tag{E24}
$$

$$
\begin{aligned}
Z_{\mathbf{1}} &= |\chi^{m6}_0|^2 + |\chi^{m6}_{\frac{1}{8}}|^2 + |\chi^{m6}_{\frac{2}{3}}|^2 + |\chi^{m6}_{\frac{13}{8}}|^2 + |\chi^{m6}_3|^2\\
&\quad + |\chi^{m6}_{\frac{2}{5}}|^2 + |\chi^{m6}_{\frac{1}{40}}|^2 + |\chi^{m6}_{\frac{1}{15}}|^2 + |\chi^{m6}_{\frac{21}{40}}|^2 + |\chi^{m6}_{\frac{7}{5}}|^2,\\
Z_{a_1} &= |\chi^{m6}_0|^2 + |\chi^{m6}_{\frac{1}{8}}|^2 + |\chi^{m6}_{\frac{2}{3}}|^2 + |\chi^{m6}_{\frac{13}{8}}|^2 + |\chi^{m6}_3|^2\\
&\quad + |\chi^{m6}_{\frac{2}{5}}|^2 + |\chi^{m6}_{\frac{1}{40}}|^2 + |\chi^{m6}_{\frac{1}{15}}|^2 + |\chi^{m6}_{\frac{21}{40}}|^2 + |\chi^{m6}_{\frac{7}{5}}|^2,\\
Z_{a_2} &= 0,\\
Z_b &= 2|\chi^{m6}_0|^2 + 2|\chi^{m6}_{\frac{1}{8}}|^2 + 2|\chi^{m6}_{\frac{2}{3}}|^2 + 2|\chi^{m6}_{\frac{13}{8}}|^2 + 2|\chi^{m6}_3|^2\\
&\quad + 2|\chi^{m6}_{\frac{2}{5}}|^2 + 2|\chi^{m6}_{\frac{1}{40}}|^2 + 2|\chi^{m6}_{\frac{1}{15}}|^2 + 2|\chi^{m6}_{\frac{21}{40}}|^2 + 2|\chi^{m6}_{\frac{7}{5}}|^2,\\
Z_{b_1} &= 0,\\
Z_{b_2} &= 0,\\
Z_c &= 0,\\
Z_{c_1} &= 0.
\end{aligned} \tag{E25}
$$

$\mathbf{1} \oplus a_2$-condensed boundary with four relevant operators:

$$
\begin{aligned}
Z_{\mathbf{1}} &= |\chi^{m7}_0|^2 + |\chi^{m7}_{\frac{1}{7}}|^2 + |\chi^{m7}_{\frac{5}{7}}|^2 + |\chi^{m7}_{\frac{12}{7}}|^2 + |\chi^{m7}_{\frac{22}{7}}|^2\\
&\quad + |\chi^{m7}_5|^2 + |\chi^{m7}_{\frac{4}{3}}|^2 + |\chi^{m7}_{\frac{10}{21}}|^2 + |\chi^{m7}_{\frac{1}{21}}|^2,\\
Z_{a_1} &= |\chi^{m7}_{\frac{3}{8}}|^2 + |\chi^{m7}_{\frac{1}{56}}|^2 + |\chi^{m7}_{\frac{5}{56}}|^2 + |\chi^{m7}_{\frac{33}{56}}|^2 + |\chi^{m7}_{\frac{85}{56}}|^2\\
&\quad + |\chi^{m7}_{\frac{23}{8}}|^2,\\
Z_{a_2} &= |\chi^{m7}_0|^2 + |\chi^{m7}_{\frac{1}{7}}|^2 + |\chi^{m7}_{\frac{5}{7}}|^2 + |\chi^{m7}_{\frac{12}{7}}|^2 + |\chi^{m7}_{\frac{22}{7}}|^2\\
&\quad + |\chi^{m7}_5|^2 + |\chi^{m7}_{\frac{3}{8}}|^2 + |\chi^{m7}_{\frac{1}{56}}|^2 + |\chi^{m7}_{\frac{5}{56}}|^2 + |\chi^{m7}_{\frac{33}{56}}|^2\\
&\quad + |\chi^{m7}_{\frac{85}{56}}|^2 + |\chi^{m7}_{\frac{23}{8}}|^2 + |\chi^{m7}_{\frac{4}{3}}|^2 + |\chi^{m7}_{\frac{10}{21}}|^2 + |\chi^{m7}_{\frac{1}{21}}|^2,\\
Z_b &= 0,\\
Z_{b_1} &= 0,\\
Z_{b_2} &= 0,\\
Z_c &= \chi^{m7}_0\bar{\chi}^{m7}_5 + \chi^{m7}_{\frac{1}{7}}\bar{\chi}^{m7}_{\frac{22}{7}} + \chi^{m7}_{\frac{5}{7}}\bar{\chi}^{m7}_{\frac{12}{7}} + \chi^{m7}_{\frac{12}{7}}\bar{\chi}^{m7}_{\frac{5}{7}}\\
&\quad + \chi^{m7}_{\frac{22}{7}}\bar{\chi}^{m7}_{\frac{1}{7}} + \chi^{m7}_5\bar{\chi}^{m7}_0 + |\chi^{m7}_{\frac{4}{3}}|^2 + |\chi^{m7}_{\frac{10}{21}}|^2 + |\chi^{m7}_{\frac{1}{21}}|^2,\\
Z_{c_1} &= \chi^{m7}_{\frac{3}{8}}\bar{\chi}^{m7}_{\frac{23}{8}} + \chi^{m7}_{\frac{1}{56}}\bar{\chi}^{m7}_{\frac{85}{56}} + \chi^{m7}_{\frac{5}{56}}\bar{\chi}^{m7}_{\frac{33}{56}} + \chi^{m7}_{\frac{33}{56}}\bar{\chi}^{m7}_{\frac{5}{56}}\\
&\quad + \chi^{m7}_{\frac{85}{56}}\bar{\chi}^{m7}_{\frac{1}{56}} + \chi^{m7}_{\frac{23}{8}}\bar{\chi}^{m7}_{\frac{3}{8}}.
\end{aligned} \tag{E26}
$$

$\mathbf{1} \oplus b$-condensed boundary with four relevant operators:

$$
\begin{aligned}
Z_{\mathbf{1}} &= |\chi^{m7}_0|^2 + |\chi^{m7}_{\frac{1}{7}}|^2 + |\chi^{m7}_{\frac{5}{7}}|^2 + |\chi^{m7}_{\frac{12}{7}}|^2 + |\chi^{m7}_{\frac{22}{7}}|^2\\
&\quad + |\chi^{m7}_5|^2 + |\chi^{m7}_{\frac{4}{3}}|^2 + |\chi^{m7}_{\frac{10}{21}}|^2 + |\chi^{m7}_{\frac{1}{21}}|^2,\\
Z_{a_1} &= |\chi^{m7}_{\frac{3}{8}}|^2 + |\chi^{m7}_{\frac{1}{56}}|^2 + |\chi^{m7}_{\frac{5}{56}}|^2 + |\chi^{m7}_{\frac{33}{56}}|^2 + |\chi^{m7}_{\frac{85}{56}}|^2\\
&\quad + |\chi^{m7}_{\frac{23}{8}}|^2,\\
Z_{a_2} &= 0,\\
Z_b &= |\chi^{m7}_0|^2 + |\chi^{m7}_{\frac{1}{7}}|^2 + |\chi^{m7}_{\frac{5}{7}}|^2 + |\chi^{m7}_{\frac{12}{7}}|^2 + |\chi^{m7}_{\frac{22}{7}}|^2\\
&\quad + |\chi^{m7}_5|^2 + |\chi^{m7}_{\frac{3}{8}}|^2 + |\chi^{m7}_{\frac{1}{56}}|^2 + |\chi^{m7}_{\frac{5}{56}}|^2 + |\chi^{m7}_{\frac{33}{56}}|^2\\
&\quad + |\chi^{m7}_{\frac{85}{56}}|^2 + |\chi^{m7}_{\frac{23}{8}}|^2 + |\chi^{m7}_{\frac{4}{3}}|^2 + |\chi^{m7}_{\frac{10}{21}}|^2 + |\chi^{m7}_{\frac{1}{21}}|^2,\\
Z_{b_1} &= 0,\\
Z_{b_2} &= 0,\\
Z_c &= \chi^{m7}_0\bar{\chi}^{m7}_5 + \chi^{m7}_{\frac{1}{7}}\bar{\chi}^{m7}_{\frac{22}{7}} + \chi^{m7}_{\frac{5}{7}}\bar{\chi}^{m7}_{\frac{12}{7}} + \chi^{m7}_{\frac{12}{7}}\bar{\chi}^{m7}_{\frac{5}{7}}\\
&\quad + \chi^{m7}_{\frac{22}{7}}\bar{\chi}^{m7}_{\frac{1}{7}} + \chi^{m7}_5\bar{\chi}^{m7}_0 + |\chi^{m7}_{\frac{4}{3}}|^2 + |\chi^{m7}_{\frac{10}{21}}|^2 + |\chi^{m7}_{\frac{1}{21}}|^2,\\
Z_{c_1} &= \chi^{m7}_{\frac{3}{8}}\bar{\chi}^{m7}_{\frac{23}{8}} + \chi^{m7}_{\frac{1}{56}}\bar{\chi}^{m7}_{\frac{85}{56}} + \chi^{m7}_{\frac{5}{56}}\bar{\chi}^{m7}_{\frac{33}{56}} + \chi^{m7}_{\frac{33}{56}}\bar{\chi}^{m7}_{\frac{5}{56}}\\
&\quad + \chi^{m7}_{\frac{85}{56}}\bar{\chi}^{m7}_{\frac{1}{56}} + \chi^{m7}_{\frac{23}{8}}\bar{\chi}^{m7}_{\frac{3}{8}},
\end{aligned} \tag{E27}
$$

$$
\begin{aligned}
Z_{\mathbf{1}} &= |\chi^{m7}_0|^2 + \chi^{m7}_0\bar{\chi}^{m7}_5 + |\chi^{m7}_{\frac{1}{7}}|^2 + \chi^{m7}_{\frac{1}{7}}\bar{\chi}^{m7}_{\frac{22}{7}} + |\chi^{m7}_{\frac{5}{7}}|^2\\
&\quad + \chi^{m7}_{\frac{5}{7}}\bar{\chi}^{m7}_{\frac{12}{7}} + \chi^{m7}_{\frac{12}{7}}\bar{\chi}^{m7}_{\frac{5}{7}} + |\chi^{m7}_{\frac{12}{7}}|^2 + \chi^{m7}_{\frac{22}{7}}\bar{\chi}^{m7}_{\frac{1}{7}}\\
&\quad + |\chi^{m7}_{\frac{22}{7}}|^2 + \chi^{m7}_5\bar{\chi}^{m7}_0 + |\chi^{m7}_5|^2 + 2|\chi^{m7}_{\frac{4}{3}}|^2\\
&\quad + 2|\chi^{m7}_{\frac{10}{21}}|^2 + 2|\chi^{m7}_{\frac{1}{21}}|^2,\\
Z_{a_1} &= |\chi^{m7}_0|^2 + \chi^{m7}_0\bar{\chi}^{m7}_5 + |\chi^{m7}_{\frac{1}{7}}|^2 + \chi^{m7}_{\frac{1}{7}}\bar{\chi}^{m7}_{\frac{22}{7}} + |\chi^{m7}_{\frac{5}{7}}|^2\\
&\quad + \chi^{m7}_{\frac{5}{7}}\bar{\chi}^{m7}_{\frac{12}{7}} + \chi^{m7}_{\frac{12}{7}}\bar{\chi}^{m7}_{\frac{5}{7}} + |\chi^{m7}_{\frac{12}{7}}|^2 + \chi^{m7}_{\frac{22}{7}}\bar{\chi}^{m7}_{\frac{1}{7}}\\
&\quad + |\chi^{m7}_{\frac{22}{7}}|^2 + \chi^{m7}_5\bar{\chi}^{m7}_0 + |\chi^{m7}_5|^2 + 2|\chi^{m7}_{\frac{4}{3}}|^2\\
&\quad + 2|\chi^{m7}_{\frac{10}{21}}|^2 + 2|\chi^{m7}_{\frac{1}{21}}|^2,\\
Z_{a_2} &= 0,\\
Z_b &= 2|\chi^{m7}_0|^2 + 2\chi^{m7}_0\bar{\chi}^{m7}_5 + 2|\chi^{m7}_{\frac{1}{7}}|^2 + 2\chi^{m7}_{\frac{1}{7}}\bar{\chi}^{m7}_{\frac{22}{7}} + 2|\chi^{m7}_{\frac{5}{7}}|^2
\end{aligned}
$$

$$
\begin{aligned}
&+2\chi^{m7}_{\frac{5}{7}}\bar{\chi}^{m7}_{\frac{12}{7}}+2\chi^{m7}_{\frac{12}{7}}\bar{\chi}^{m7}_{\frac{5}{7}}+2|\chi^{m7}_{\frac{12}{7}}|^2+2\chi^{m7}_{\frac{22}{7}}\bar{\chi}^{m7}_{\frac{1}{7}}\\
&+2|\chi^{m7}_{\frac{22}{7}}|^2+2\chi^{m7}_{5}\bar{\chi}^{m7}_{0}+2|\chi^{m7}_{5}|^2+4|\chi^{m7}_{\frac{4}{3}}|^2\\
&+4|\chi^{m7}_{\frac{10}{21}}|^2+4|\chi^{m7}_{\frac{1}{21}}|^2,\\
Z_{b_1}&=0,\\
Z_{b_2}&=0,\\
Z_c&=0,\\
Z_{c_1}&=0. \qquad \text{(E28)}
\end{aligned}
$$

$\mathbf{1}\oplus a_1$-condensed boundary with two relevant operators:

$$
\begin{aligned}
Z_{\mathbf{1}}&=|\chi^{m7}_{0}|^2+\chi^{m7}_{0}\bar{\chi}^{m7}_{5}+|\chi^{m7}_{\frac{1}{7}}|^2+\chi^{m7}_{\frac{1}{7}}\bar{\chi}^{m7}_{\frac{22}{7}}+|\chi^{m7}_{\frac{5}{7}}|^2\\
&+\chi^{m7}_{\frac{5}{7}}\bar{\chi}^{m7}_{\frac{12}{7}}+\chi^{m7}_{\frac{12}{7}}\bar{\chi}^{m7}_{\frac{5}{7}}+|\chi^{m7}_{\frac{12}{7}}|^2+\chi^{m7}_{\frac{22}{7}}\bar{\chi}^{m7}_{\frac{1}{7}}\\
&+|\chi^{m7}_{\frac{22}{7}}|^2+\chi^{m7}_{5}\bar{\chi}^{m7}_{0}+|\chi^{m7}_{5}|^2,\\
Z_{a_1}&=|\chi^{m7}_{0}|^2+\chi^{m7}_{0}\bar{\chi}^{m7}_{5}+|\chi^{m7}_{\frac{1}{7}}|^2+\chi^{m7}_{\frac{1}{7}}\bar{\chi}^{m7}_{\frac{22}{7}}+|\chi^{m7}_{\frac{5}{7}}|^2\\
&+\chi^{m7}_{\frac{5}{7}}\bar{\chi}^{m7}_{\frac{12}{7}}+\chi^{m7}_{\frac{12}{7}}\bar{\chi}^{m7}_{\frac{5}{7}}+|\chi^{m7}_{\frac{12}{7}}|^2+\chi^{m7}_{\frac{22}{7}}\bar{\chi}^{m7}_{\frac{1}{7}}\\
&+|\chi^{m7}_{\frac{22}{7}}|^2+\chi^{m7}_{5}\bar{\chi}^{m7}_{0}+|\chi^{m7}_{5}|^2,\\
Z_{a_2}&=2|\chi^{m7}_{\frac{4}{3}}|^2+2|\chi^{m7}_{\frac{10}{21}}|^2+2|\chi^{m7}_{\frac{1}{21}}|^2,\\
Z_b&=2|\chi^{m7}_{\frac{4}{3}}|^2+2|\chi^{m7}_{\frac{10}{21}}|^2+2|\chi^{m7}_{\frac{1}{21}}|^2,\\
Z_{b_1}&=2\chi^{m7}_{\frac{4}{3}}\bar{\chi}^{m7}_{0}+2\chi^{m7}_{\frac{4}{3}}\bar{\chi}^{m7}_{5}+2\chi^{m7}_{\frac{10}{21}}\bar{\chi}^{m7}_{\frac{1}{7}}+2\chi^{m7}_{\frac{10}{21}}\bar{\chi}^{m7}_{\frac{22}{7}}\\
&+2\chi^{m7}_{\frac{1}{21}}\bar{\chi}^{m7}_{\frac{5}{7}}+2\chi^{m7}_{\frac{1}{21}}\bar{\chi}^{m7}_{\frac{12}{7}},\\
Z_{b_2}&=2\chi^{m7}_{0}\bar{\chi}^{m7}_{\frac{4}{3}}+2\chi^{m7}_{\frac{1}{7}}\bar{\chi}^{m7}_{\frac{10}{21}}+2\chi^{m7}_{\frac{5}{7}}\bar{\chi}^{m7}_{\frac{1}{21}}+2\chi^{m7}_{\frac{12}{7}}\bar{\chi}^{m7}_{\frac{1}{21}}\\
&+2\chi^{m7}_{\frac{22}{7}}\bar{\chi}^{m7}_{\frac{10}{21}}+2\chi^{m7}_{5}\bar{\chi}^{m7}_{\frac{4}{3}},\\
Z_c&=0,\\
Z_{c_1}&=0. \qquad \text{(E29)}
\end{aligned}
$$

$\mathbf{1}$-condensed boundary with two relevant operators:

$$
\begin{aligned}
Z_{\mathbf{1}}&=|\chi^{m7}_{0}|^2+|\chi^{m7}_{\frac{1}{7}}|^2+|\chi^{m7}_{\frac{5}{7}}|^2+|\chi^{m7}_{\frac{12}{7}}|^2+|\chi^{m7}_{\frac{22}{7}}|^2\\
&+|\chi^{m7}_{5}|^2,\\
Z_{a_1}&=\chi^{m7}_{0}\bar{\chi}^{m7}_{5}+\chi^{m7}_{\frac{1}{7}}\bar{\chi}^{m7}_{\frac{22}{7}}+\chi^{m7}_{\frac{5}{7}}\bar{\chi}^{m7}_{\frac{12}{7}}+\chi^{m7}_{\frac{12}{7}}\bar{\chi}^{m7}_{\frac{5}{7}}\\
&+\chi^{m7}_{\frac{22}{7}}\bar{\chi}^{m7}_{\frac{1}{7}}+\chi^{m7}_{5}\bar{\chi}^{m7}_{0},\\
Z_{a_2}&=|\chi^{m7}_{\frac{4}{3}}|^2+|\chi^{m7}_{\frac{10}{21}}|^2+|\chi^{m7}_{\frac{1}{21}}|^2,\\
Z_b&=|\chi^{m7}_{\frac{4}{3}}|^2+|\chi^{m7}_{\frac{10}{21}}|^2+|\chi^{m7}_{\frac{1}{21}}|^2,\\
Z_{b_1}&=\chi^{m7}_{\frac{4}{3}}\bar{\chi}^{m7}_{0}+\chi^{m7}_{\frac{4}{3}}\bar{\chi}^{m7}_{5}+\chi^{m7}_{\frac{10}{21}}\bar{\chi}^{m7}_{\frac{1}{7}}+\chi^{m7}_{\frac{10}{21}}\bar{\chi}^{m7}_{\frac{22}{7}}\\
&+\chi^{m7}_{\frac{1}{21}}\bar{\chi}^{m7}_{\frac{5}{7}}+\chi^{m7}_{\frac{1}{21}}\bar{\chi}^{m7}_{\frac{12}{7}},\\
Z_{b_2}&=\chi^{m7}_{0}\bar{\chi}^{m7}_{\frac{4}{3}}+\chi^{m7}_{\frac{1}{7}}\bar{\chi}^{m7}_{\frac{10}{21}}+\chi^{m7}_{\frac{5}{7}}\bar{\chi}^{m7}_{\frac{1}{21}}+\chi^{m7}_{\frac{12}{7}}\bar{\chi}^{m7}_{\frac{1}{21}}\\
&+\chi^{m7}_{\frac{22}{7}}\bar{\chi}^{m7}_{\frac{10}{21}}+\chi^{m7}_{5}\bar{\chi}^{m7}_{\frac{4}{3}},\\
Z_c&=|\chi^{m7}_{\frac{3}{8}}|^2+|\chi^{m7}_{\frac{1}{56}}|^2+|\chi^{m7}_{\frac{5}{56}}|^2+|\chi^{m7}_{\frac{33}{56}}|^2+|\chi^{m7}_{\frac{85}{56}}|^2\\
&+|\chi^{m7}_{\frac{23}{8}}|^2,\\
Z_{c_1}&=\chi^{m7}_{\frac{3}{8}}\bar{\chi}^{m7}_{\frac{23}{8}}+\chi^{m7}_{\frac{1}{56}}\bar{\chi}^{m7}_{\frac{85}{56}}+\chi^{m7}_{\frac{5}{56}}\bar{\chi}^{m7}_{\frac{33}{56}}+\chi^{m7}_{\frac{33}{56}}\bar{\chi}^{m7}_{\frac{5}{56}}\\
&+\chi^{m7}_{\frac{85}{56}}\bar{\chi}^{m7}_{\frac{1}{56}}+\chi^{m7}_{\frac{23}{8}}\bar{\chi}^{m7}_{\frac{3}{8}}, \qquad \text{(E30)}\\
Z_{\mathbf{1}}&=|\chi^{m7}_{0}|^2+|\chi^{m7}_{\frac{1}{7}}|^2+|\chi^{m7}_{\frac{5}{7}}|^2+|\chi^{m7}_{\frac{12}{7}}|^2+|\chi^{m7}_{\frac{22}{7}}|^2\\
&+|\chi^{m7}_{5}|^2+|\chi^{m7}_{\frac{3}{8}}|^2+|\chi^{m7}_{\frac{1}{56}}|^2+|\chi^{m7}_{\frac{5}{56}}|^2+|\chi^{m7}_{\frac{33}{56}}|^2\\
&+|\chi^{m7}_{\frac{85}{56}}|^2+|\chi^{m7}_{\frac{23}{8}}|^2+|\chi^{m7}_{\frac{4}{3}}|^2+|\chi^{m7}_{\frac{10}{21}}|^2+|\chi^{m7}_{\frac{1}{21}}|^2,\\
Z_{a_1}&=0,\\
Z_{a_2}&=|\chi^{m7}_{0}|^2+|\chi^{m7}_{\frac{1}{7}}|^2+|\chi^{m7}_{\frac{5}{7}}|^2+|\chi^{m7}_{\frac{12}{7}}|^2+|\chi^{m7}_{\frac{22}{7}}|^2\\
&+|\chi^{m7}_{5}|^2+|\chi^{m7}_{\frac{3}{8}}|^2+|\chi^{m7}_{\frac{1}{56}}|^2+|\chi^{m7}_{\frac{5}{56}}|^2+|\chi^{m7}_{\frac{33}{56}}|^2\\
&+|\chi^{m7}_{\frac{85}{56}}|^2+|\chi^{m7}_{\frac{23}{8}}|^2+|\chi^{m7}_{\frac{4}{3}}|^2+|\chi^{m7}_{\frac{10}{21}}|^2+|\chi^{m7}_{\frac{1}{21}}|^2,\\
Z_b&=0,\\
Z_{b_1}&=0,\\
Z_{b_2}&=0,\\
Z_c&=|\chi^{m7}_{0}|^2+|\chi^{m7}_{\frac{1}{7}}|^2+|\chi^{m7}_{\frac{5}{7}}|^2+|\chi^{m7}_{\frac{12}{7}}|^2+|\chi^{m7}_{\frac{22}{7}}|^2\\
&+|\chi^{m7}_{5}|^2+|\chi^{m7}_{\frac{3}{8}}|^2+|\chi^{m7}_{\frac{1}{56}}|^2+|\chi^{m7}_{\frac{5}{56}}|^2+|\chi^{m7}_{\frac{33}{56}}|^2\\
&+|\chi^{m7}_{\frac{85}{56}}|^2+|\chi^{m7}_{\frac{23}{8}}|^2+|\chi^{m7}_{\frac{4}{3}}|^2+|\chi^{m7}_{\frac{10}{21}}|^2+|\chi^{m7}_{\frac{1}{21}}|^2,\\
Z_{c_1}&=0. \qquad \text{(E31)}
\end{aligned}
$$

$\mathbf{1}\oplus a_2$-condensed boundary with four relevant operators:

$$
\begin{aligned}
Z_{\mathbf{1}}&=|\chi^{m7}_{0}|^2+|\chi^{m7}_{\frac{1}{7}}|^2+|\chi^{m7}_{\frac{5}{7}}|^2+|\chi^{m7}_{\frac{12}{7}}|^2+|\chi^{m7}_{\frac{22}{7}}|^2\\
&+|\chi^{m7}_{5}|^2+|\chi^{m7}_{\frac{4}{3}}|^2+|\chi^{m7}_{\frac{10}{21}}|^2+|\chi^{m7}_{\frac{1}{21}}|^2,\\
Z_{a_1}&=\chi^{m7}_{0}\bar{\chi}^{m7}_{5}+\chi^{m7}_{\frac{1}{7}}\bar{\chi}^{m7}_{\frac{22}{7}}+\chi^{m7}_{\frac{5}{7}}\bar{\chi}^{m7}_{\frac{12}{7}}+\chi^{m7}_{\frac{12}{7}}\bar{\chi}^{m7}_{\frac{5}{7}}+\chi^{m7}_{\frac{22}{7}}\bar{\chi}^{m7}_{\frac{1}{7}}\\
&+\chi^{m7}_{5}\bar{\chi}^{m7}_{0}+|\chi^{m7}_{\frac{4}{3}}|^2+|\chi^{m7}_{\frac{10}{21}}|^2+|\chi^{m7}_{\frac{1}{21}}|^2,\\
Z_{a_2}&=|\chi^{m7}_{0}|^2+\chi^{m7}_{0}\bar{\chi}^{m7}_{5}+|\chi^{m7}_{\frac{1}{7}}|^2+\chi^{m7}_{\frac{1}{7}}\bar{\chi}^{m7}_{\frac{22}{7}}+|\chi^{m7}_{\frac{5}{7}}|^2\\
&+\chi^{m7}_{\frac{5}{7}}\bar{\chi}^{m7}_{\frac{12}{7}}+\chi^{m7}_{\frac{12}{7}}\bar{\chi}^{m7}_{\frac{5}{7}}+|\chi^{m7}_{\frac{12}{7}}|^2+\chi^{m7}_{\frac{22}{7}}\bar{\chi}^{m7}_{\frac{1}{7}},\\
&+|\chi^{m7}_{\frac{22}{7}}|^2+\chi^{m7}_{5}\bar{\chi}^{m7}_{0}+|\chi^{m7}_{5}|^2+2|\chi^{m7}_{\frac{4}{3}}|^2+2|\chi^{m7}_{\frac{10}{21}}|^2\\
&+2|\chi^{m7}_{\frac{1}{21}}|^2,\\
Z_b&=0,\\
Z_{b_1}&=0,\\
Z_{b_2}&=0,\\
Z_c&=|\chi^{m7}_{\frac{3}{8}}|^2+|\chi^{m7}_{\frac{1}{56}}|^2+|\chi^{m7}_{\frac{5}{56}}|^2+|\chi^{m7}_{\frac{33}{56}}|^2+|\chi^{m7}_{\frac{85}{56}}|^2+|\chi^{m7}_{\frac{23}{8}}|^2,\\
Z_{c_1}&=\chi^{m7}_{\frac{3}{8}}\bar{\chi}^{m7}_{\frac{23}{8}}+\chi^{m7}_{\frac{1}{56}}\bar{\chi}^{m7}_{\frac{85}{56}}+\chi^{m7}_{\frac{5}{56}}\bar{\chi}^{m7}_{\frac{33}{56}}+\chi^{m7}_{\frac{33}{56}}\bar{\chi}^{m7}_{\frac{5}{56}}\\
&+\chi^{m7}_{\frac{85}{56}}\bar{\chi}^{m7}_{\frac{1}{56}}+\chi^{m7}_{\frac{23}{8}}\bar{\chi}^{m7}_{\frac{3}{8}}, \qquad \text{(E32)}
\end{aligned}
$$

$$\begin{aligned}
Z_{\mathbf{1}} &= |\chi^{m7}_{0}|^2+|\chi^{m7}_{\frac{1}{7}}|^2+|\chi^{m7}_{\frac{5}{7}}|^2+|\chi^{m7}_{\frac{12}{7}}|^2+|\chi^{m7}_{\frac{22}{7}}|^2 \\
&\quad+|\chi^{m7}_{5}|^2+|\chi^{m7}_{\frac{3}{8}}|^2+|\chi^{m7}_{\frac{1}{56}}|^2+|\chi^{m7}_{\frac{5}{56}}|^2+|\chi^{m7}_{\frac{33}{56}}|^2 \\
&\quad+|\chi^{m7}_{\frac{85}{56}}|^2+|\chi^{m7}_{\frac{23}{8}}|^2+|\chi^{m7}_{\frac{4}{3}}|^2+|\chi^{m7}_{\frac{10}{21}}|^2+|\chi^{m7}_{\frac{1}{21}}|^2, \\
Z_{a_1} &= 0, \\
Z_{a_2} &= 0, \\
Z_{b} &= |\chi^{m7}_{0}|^2+|\chi^{m7}_{\frac{1}{7}}|^2+|\chi^{m7}_{\frac{5}{7}}|^2+|\chi^{m7}_{\frac{12}{7}}|^2+|\chi^{m7}_{\frac{22}{7}}|^2 \\
&\quad+|\chi^{m7}_{5}|^2+|\chi^{m7}_{\frac{3}{8}}|^2+|\chi^{m7}_{\frac{1}{56}}|^2+|\chi^{m7}_{\frac{5}{56}}|^2+|\chi^{m7}_{\frac{33}{56}}|^2 \\
&\quad+|\chi^{m7}_{\frac{85}{56}}|^2+|\chi^{m7}_{\frac{23}{8}}|^2+|\chi^{m7}_{\frac{4}{3}}|^2+|\chi^{m7}_{\frac{10}{21}}|^2+|\chi^{m7}_{\frac{1}{21}}|^2, \\
Z_{b_1} &= 0, \\
Z_{b_2} &= 0, \\
Z_{c} &= |\chi^{m7}_{0}|^2+|\chi^{m7}_{\frac{1}{7}}|^2+|\chi^{m7}_{\frac{5}{7}}|^2+|\chi^{m7}_{\frac{12}{7}}|^2+|\chi^{m7}_{\frac{22}{7}}|^2 \\
&\quad+|\chi^{m7}_{5}|^2+|\chi^{m7}_{\frac{3}{8}}|^2+|\chi^{m7}_{\frac{1}{56}}|^2+|\chi^{m7}_{\frac{5}{56}}|^2+|\chi^{m7}_{\frac{33}{56}}|^2 \\
&\quad+|\chi^{m7}_{\frac{85}{56}}|^2+|\chi^{m7}_{\frac{23}{8}}|^2+|\chi^{m7}_{\frac{4}{3}}|^2+|\chi^{m7}_{\frac{10}{21}}|^2+|\chi^{m7}_{\frac{1}{21}}|^2, \\
Z_{c_1} &= 0.
\end{aligned} \tag{E33}$$

$\mathbf{1} \oplus b$-condensed boundary with four relevant operators:

$$\begin{aligned}
Z_{\mathbf{1}} &= |\chi^{m7}_{0}|^2+|\chi^{m7}_{\frac{1}{7}}|^2+|\chi^{m7}_{\frac{5}{7}}|^2+|\chi^{m7}_{\frac{12}{7}}|^2+|\chi^{m7}_{\frac{22}{7}}|^2 \\
&\quad+|\chi^{m7}_{5}|^2+|\chi^{m7}_{\frac{4}{3}}|^2+|\chi^{m7}_{\frac{10}{21}}|^2+|\chi^{m7}_{\frac{1}{21}}|^2, \\
Z_{a_1} &= \chi^{m7}_{0}\bar{\chi}^{m7}_{5}+\chi^{m7}_{\frac{1}{7}}\bar{\chi}^{m7}_{\frac{22}{7}}+\chi^{m7}_{\frac{5}{7}}\bar{\chi}^{m7}_{\frac{12}{7}}+\chi^{m7}_{\frac{12}{7}}\bar{\chi}^{m7}_{\frac{5}{7}} \\
&\quad+\chi^{m7}_{\frac{22}{7}}\bar{\chi}^{m7}_{\frac{1}{7}}+\chi^{m7}_{5}\bar{\chi}^{m7}_{0}+|\chi^{m7}_{\frac{4}{3}}|^2+|\chi^{m7}_{\frac{10}{21}}|^2+|\chi^{m7}_{\frac{1}{21}}|^2, \\
Z_{a_2} &= 0, \\
Z_{b} &= |\chi^{m7}_{0}|^2+\chi^{m7}_{0}\bar{\chi}^{m7}_{5}+|\chi^{m7}_{\frac{1}{7}}|^2+\chi^{m7}_{\frac{1}{7}}\bar{\chi}^{m7}_{\frac{22}{7}}+|\chi^{m7}_{\frac{5}{7}}|^2 \\
&\quad+\chi^{m7}_{\frac{5}{7}}\bar{\chi}^{m7}_{\frac{12}{7}}+\chi^{m7}_{\frac{12}{7}}\bar{\chi}^{m7}_{\frac{5}{7}}+|\chi^{m7}_{\frac{12}{7}}|^2+\chi^{m7}_{\frac{22}{7}}\bar{\chi}^{m7}_{\frac{1}{7}}+|\chi^{m7}_{\frac{22}{7}}|^2 \\
&\quad+\chi^{m7}_{5}\bar{\chi}^{m7}_{0}+|\chi^{m7}_{5}|^2+2|\chi^{m7}_{\frac{4}{3}}|^2+2|\chi^{m7}_{\frac{10}{21}}|^2+2|\chi^{m7}_{\frac{1}{21}}|^2, \\
Z_{b_1} &= 0, \\
Z_{b_2} &= 0, \\
Z_{c} &= |\chi^{m7}_{\frac{3}{8}}|^2+|\chi^{m7}_{\frac{1}{56}}|^2+|\chi^{m7}_{\frac{5}{56}}|^2+|\chi^{m7}_{\frac{33}{56}}|^2+|\chi^{m7}_{\frac{85}{56}}|^2+|\chi^{m7}_{\frac{23}{8}}|^2, \\
Z_{c_1} &= \chi^{m7}_{\frac{3}{8}}\bar{\chi}^{m7}_{\frac{23}{8}}+\chi^{m7}_{\frac{1}{56}}\bar{\chi}^{m7}_{\frac{85}{56}}+\chi^{m7}_{\frac{5}{56}}\bar{\chi}^{m7}_{\frac{33}{56}}+\chi^{m7}_{\frac{33}{56}}\bar{\chi}^{m7}_{\frac{5}{56}}+\chi^{m7}_{\frac{85}{56}}\bar{\chi}^{m7}_{\frac{1}{56}} \\
&\quad+\chi^{m7}_{\frac{23}{8}}\bar{\chi}^{m7}_{\frac{3}{8}},
\end{aligned} \tag{E34}$$

$$\begin{aligned}
Z_{\mathbf{1}} &= |\chi^{m7}_{0}|^2+\chi^{m7}_{0}\bar{\chi}^{m7}_{5}+|\chi^{m7}_{\frac{1}{7}}|^2+\chi^{m7}_{\frac{1}{7}}\bar{\chi}^{m7}_{\frac{22}{7}}+|\chi^{m7}_{\frac{5}{7}}|^2 \\
&\quad+\chi^{m7}_{\frac{5}{7}}\bar{\chi}^{m7}_{\frac{12}{7}}+\chi^{m7}_{\frac{12}{7}}\bar{\chi}^{m7}_{\frac{5}{7}}+|\chi^{m7}_{\frac{12}{7}}|^2+\chi^{m7}_{\frac{22}{7}}\bar{\chi}^{m7}_{\frac{1}{7}} \\
&\quad+|\chi^{m7}_{\frac{22}{7}}|^2+\chi^{m7}_{5}\bar{\chi}^{m7}_{0}+|\chi^{m7}_{5}|^2+2|\chi^{m7}_{\frac{4}{3}}|^2+2|\chi^{m7}_{\frac{10}{21}}|^2 \\
&\quad+2|\chi^{m7}_{\frac{1}{21}}|^2, \\
Z_{a_1} &= 0, \\
Z_{a_2} &= |\chi^{m7}_{0}|^2+\chi^{m7}_{0}\bar{\chi}^{m7}_{5}+|\chi^{m7}_{\frac{1}{7}}|^2+\chi^{m7}_{\frac{1}{7}}\bar{\chi}^{m7}_{\frac{22}{7}}+|\chi^{m7}_{\frac{5}{7}}|^2 \\
&\quad+\chi^{m7}_{\frac{5}{7}}\bar{\chi}^{m7}_{\frac{12}{7}}+\chi^{m7}_{\frac{12}{7}}\bar{\chi}^{m7}_{\frac{5}{7}}+|\chi^{m7}_{\frac{12}{7}}|^2+\chi^{m7}_{\frac{22}{7}}\bar{\chi}^{m7}_{\frac{1}{7}} \\
&\quad+|\chi^{m7}_{\frac{22}{7}}|^2+\chi^{m7}_{5}\bar{\chi}^{m7}_{0}+|\chi^{m7}_{5}|^2+2|\chi^{m7}_{\frac{4}{3}}|^2+2|\chi^{m7}_{\frac{10}{21}}|^2 \\
&\quad+2|\chi^{m7}_{\frac{1}{21}}|^2, \\
Z_{b} &= 0, \\
Z_{b_1} &= 0, \\
Z_{b_2} &= 0, \\
Z_{c} &= |\chi^{m7}_{0}|^2+\chi^{m7}_{0}\bar{\chi}^{m7}_{5}+|\chi^{m7}_{\frac{1}{7}}|^2+\chi^{m7}_{\frac{1}{7}}\bar{\chi}^{m7}_{\frac{22}{7}}+|\chi^{m7}_{\frac{5}{7}}|^2 \\
&\quad+\chi^{m7}_{\frac{5}{7}}\bar{\chi}^{m7}_{\frac{12}{7}}+\chi^{m7}_{\frac{12}{7}}\bar{\chi}^{m7}_{\frac{5}{7}}+|\chi^{m7}_{\frac{12}{7}}|^2+\chi^{m7}_{\frac{22}{7}}\bar{\chi}^{m7}_{\frac{1}{7}} \\
&\quad+|\chi^{m7}_{\frac{22}{7}}|^2+\chi^{m7}_{5}\bar{\chi}^{m7}_{0}+|\chi^{m7}_{5}|^2+2|\chi^{m7}_{\frac{4}{3}}|^2+2|\chi^{m7}_{\frac{10}{21}}|^2 \\
&\quad+2|\chi^{m7}_{\frac{1}{21}}|^2, \\
Z_{c_1} &= 0,
\end{aligned} \tag{E35}$$

$$\begin{aligned}
Z_{\mathbf{1}} &= |\chi^{m7}_{0}|^2+\chi^{m7}_{0}\bar{\chi}^{m7}_{5}+|\chi^{m7}_{\frac{1}{7}}|^2+\chi^{m7}_{\frac{1}{7}}\bar{\chi}^{m7}_{\frac{22}{7}}+|\chi^{m7}_{\frac{5}{7}}|^2 \\
&\quad+\chi^{m7}_{\frac{5}{7}}\bar{\chi}^{m7}_{\frac{12}{7}}+\chi^{m7}_{\frac{12}{7}}\bar{\chi}^{m7}_{\frac{5}{7}}+|\chi^{m7}_{\frac{12}{7}}|^2+\chi^{m7}_{\frac{22}{7}}\bar{\chi}^{m7}_{\frac{1}{7}} \\
&\quad+|\chi^{m7}_{\frac{22}{7}}|^2+\chi^{m7}_{5}\bar{\chi}^{m7}_{0}+|\chi^{m7}_{5}|^2+2|\chi^{m7}_{\frac{4}{3}}|^2+2|\chi^{m7}_{\frac{10}{21}}|^2 \\
&\quad+2|\chi^{m7}_{\frac{1}{21}}|^2, \\
Z_{a_1} &= 0, \\
Z_{a_2} &= 0, \\
Z_{b} &= |\chi^{m7}_{0}|^2+\chi^{m7}_{0}\bar{\chi}^{m7}_{5}+|\chi^{m7}_{\frac{1}{7}}|^2+\chi^{m7}_{\frac{1}{7}}\bar{\chi}^{m7}_{\frac{22}{7}}+|\chi^{m7}_{\frac{5}{7}}|^2 \\
&\quad+\chi^{m7}_{\frac{5}{7}}\bar{\chi}^{m7}_{\frac{12}{7}}+\chi^{m7}_{\frac{12}{7}}\bar{\chi}^{m7}_{\frac{5}{7}}+|\chi^{m7}_{\frac{12}{7}}|^2+\chi^{m7}_{\frac{22}{7}}\bar{\chi}^{m7}_{\frac{1}{7}} \\
&\quad+|\chi^{m7}_{\frac{22}{7}}|^2+\chi^{m7}_{5}\bar{\chi}^{m7}_{0}+|\chi^{m7}_{5}|^2+2|\chi^{m7}_{\frac{4}{3}}|^2+2|\chi^{m7}_{\frac{10}{21}}|^2 \\
&\quad+2|\chi^{m7}_{\frac{1}{21}}|^2, \\
Z_{b_1} &= 0, \\
Z_{b_2} &= 0, \\
Z_{c} &= |\chi^{m7}_{0}|^2+\chi^{m7}_{0}\bar{\chi}^{m7}_{5}+|\chi^{m7}_{\frac{1}{7}}|^2+\chi^{m7}_{\frac{1}{7}}\bar{\chi}^{m7}_{\frac{22}{7}}+|\chi^{m7}_{\frac{5}{7}}|^2 \\
&\quad+\chi^{m7}_{\frac{5}{7}}\bar{\chi}^{m7}_{\frac{12}{7}}+\chi^{m7}_{\frac{12}{7}}\bar{\chi}^{m7}_{\frac{5}{7}}+|\chi^{m7}_{\frac{12}{7}}|^2+\chi^{m7}_{\frac{22}{7}}\bar{\chi}^{m7}_{\frac{1}{7}} \\
&\quad+|\chi^{m7}_{\frac{22}{7}}|^2+\chi^{m7}_{5}\bar{\chi}^{m7}_{0}+|\chi^{m7}_{5}|^2+2|\chi^{m7}_{\frac{4}{3}}|^2+2|\chi^{m7}_{\frac{10}{21}}|^2 \\
&\quad+2|\chi^{m7}_{\frac{1}{21}}|^2, \\
Z_{c_1} &= 0,
\end{aligned} \tag{E36}$$

$$\begin{aligned}
Z_{\mathbf{1}} &= |\chi^{m7}_{0}|^2+|\chi^{m7}_{\frac{1}{7}}|^2+|\chi^{m7}_{\frac{5}{7}}|^2+|\chi^{m7}_{\frac{12}{7}}|^2+|\chi^{m7}_{\frac{22}{7}}|^2 \\
&\quad+|\chi^{m7}_{5}|^2+|\chi^{m7}_{\frac{3}{8}}|^2+|\chi^{m7}_{\frac{1}{56}}|^2+|\chi^{m7}_{\frac{5}{56}}|^2+|\chi^{m7}_{\frac{33}{56}}|^2 \\
&\quad+|\chi^{m7}_{\frac{85}{56}}|^2+|\chi^{m7}_{\frac{23}{8}}|^2+|\chi^{m7}_{\frac{4}{3}}|^2+|\chi^{m7}_{\frac{10}{21}}|^2+|\chi^{m7}_{\frac{1}{21}}|^2, \\
Z_{a_1} &= |\chi^{m7}_{0}|^2+|\chi^{m7}_{\frac{1}{7}}|^2+|\chi^{m7}_{\frac{5}{7}}|^2+|\chi^{m7}_{\frac{12}{7}}|^2+|\chi^{m7}_{\frac{22}{7}}|^2
\end{aligned}$$

$$
\begin{aligned}
&+|\chi_5^{m7}|^2+|\chi_{\frac{3}{8}}^{m7}|^2+|\chi_{\frac{1}{56}}^{m7}|^2+|\chi_{\frac{5}{56}}^{m7}|^2+|\chi_{\frac{33}{56}}^{m7}|^2\\
&+|\chi_{\frac{85}{56}}^{m7}|^2+|\chi_{\frac{23}{8}}^{m7}|^2+|\chi_{\frac{4}{3}}^{m7}|^2+|\chi_{\frac{10}{21}}^{m7}|^2+|\chi_{\frac{1}{21}}^{m7}|^2,\\
Z_{a_2}&=2|\chi_0^{m7}|^2+2|\chi_{\frac{1}{7}}^{m7}|^2+2|\chi_{\frac{5}{7}}^{m7}|^2+2|\chi_{\frac{12}{7}}^{m7}|^2+2|\chi_{\frac{22}{7}}^{m7}|^2\\
&+2|\chi_5^{m7}|^2+2|\chi_{\frac{3}{8}}^{m7}|^2+2|\chi_{\frac{1}{56}}^{m7}|^2+2|\chi_{\frac{5}{56}}^{m7}|^2+2|\chi_{\frac{33}{56}}^{m7}|^2\\
&+2|\chi_{\frac{85}{56}}^{m7}|^2+2|\chi_{\frac{23}{8}}^{m7}|^2+2|\chi_{\frac{4}{3}}^{m7}|^2+2|\chi_{\frac{10}{21}}^{m7}|^2+2|\chi_{\frac{1}{21}}^{m7}|^2,\\
Z_b&=0,\\
Z_{b_1}&=0,\\
Z_{b_2}&=0,\\
Z_c&=0,\\
Z_{c_1}&=0,
\end{aligned}
\tag{E37}
$$

$$
\begin{aligned}
Z_{\mathbf{1}}&=|\chi_0^{m7}|^2+|\chi_{\frac{1}{7}}^{m7}|^2+|\chi_{\frac{5}{7}}^{m7}|^2+|\chi_{\frac{12}{7}}^{m7}|^2+|\chi_{\frac{22}{7}}^{m7}|^2\\
&+|\chi_5^{m7}|^2+|\chi_{\frac{3}{8}}^{m7}|^2+|\chi_{\frac{1}{56}}^{m7}|^2+|\chi_{\frac{5}{56}}^{m7}|^2+|\chi_{\frac{33}{56}}^{m7}|^2\\
&+|\chi_{\frac{85}{56}}^{m7}|^2+|\chi_{\frac{23}{8}}^{m7}|^2+|\chi_{\frac{4}{3}}^{m7}|^2+|\chi_{\frac{10}{21}}^{m7}|^2+|\chi_{\frac{1}{21}}^{m7}|^2,\\
Z_{a_1}&=|\chi_0^{m7}|^2+|\chi_{\frac{1}{7}}^{m7}|^2+|\chi_{\frac{5}{7}}^{m7}|^2+|\chi_{\frac{12}{7}}^{m7}|^2+|\chi_{\frac{22}{7}}^{m7}|^2\\
&+|\chi_5^{m7}|^2+|\chi_{\frac{3}{8}}^{m7}|^2+|\chi_{\frac{1}{56}}^{m7}|^2+|\chi_{\frac{5}{56}}^{m7}|^2+|\chi_{\frac{33}{56}}^{m7}|^2\\
&+|\chi_{\frac{85}{56}}^{m7}|^2+|\chi_{\frac{23}{8}}^{m7}|^2+|\chi_{\frac{4}{3}}^{m7}|^2+|\chi_{\frac{10}{21}}^{m7}|^2+|\chi_{\frac{1}{21}}^{m7}|^2,\\
Z_{a_2}&=0,\\
Z_b&=2|\chi_0^{m7}|^2+2|\chi_{\frac{1}{7}}^{m7}|^2+2|\chi_{\frac{5}{7}}^{m7}|^2+2|\chi_{\frac{12}{7}}^{m7}|^2+2|\chi_{\frac{22}{7}}^{m7}|^2\\
&+2|\chi_5^{m7}|^2+2|\chi_{\frac{3}{8}}^{m7}|^2+2|\chi_{\frac{1}{56}}^{m7}|^2+2|\chi_{\frac{5}{56}}^{m7}|^2+2|\chi_{\frac{33}{56}}^{m7}|^2\\
&+2|\chi_{\frac{85}{56}}^{m7}|^2+2|\chi_{\frac{23}{8}}^{m7}|^2+2|\chi_{\frac{4}{3}}^{m7}|^2+2|\chi_{\frac{10}{21}}^{m7}|^2+2|\chi_{\frac{1}{21}}^{m7}|^2,\\
Z_{b_1}&=0,\\
Z_{b_2}&=0,\\
Z_c&=0,\\
Z_{c_1}&=0.
\end{aligned}
\tag{E38}
$$

APPENDIX F: 1 + 1D NONINVERTIBLE SYMMETRY $\widetilde{S}_3$: DUAL SYMMETRY OF S_3

In Refs. [42,49], a 1 + 1D model with a noninvertible symmetry, denoted as $\widetilde{S}_3$, is constructed. The model has degrees of freedom on the links ij, which are labeled by the S_3 group elements $g_{ij} \in S_3$. The $\widetilde{S}_3$ symmetry transformation are generated by

$$W_R = \mathrm{Tr}\left(\prod_i R(g_{i,i+1})\right) \tag{F1}$$

for all irreducible representations R of S_3, i.e., $R = \mathbf{1}, a_1, a_2$. Using

$$W_R W_{R'} = \mathrm{Tr}\left(\prod_i R(g_{i,i+1}) \otimes R'(g_{i,i+1})\right), \tag{F2}$$

we find that the symmetry transformations satisfy the following algebra:

$$
\begin{aligned}
&W_{\mathbf{1}}W_{\mathbf{1}}=W_{\mathbf{1}},\quad W_{\mathbf{1}}W_{a_1}=W_{a_1},\quad W_{\mathbf{1}}W_{a_2}=W_{a_2},\\
&W_{a_1}W_{\mathbf{1}}=W_{a_1},\quad W_{a_1}W_{a_1}=W_{\mathbf{1}},\quad W_{a_1}W_{a_2}=W_{a_2},\\
&W_{a_2}W_{\mathbf{1}}=W_{a_2},\quad W_{a_2}W_{a_1}=W_{a_2},\\
&W_{a_2}W_{a_2}=W_{\mathbf{1}}+W_{a_1}+W_{a_2}.
\end{aligned}
\tag{F3}
$$

For example, $R = a_2$ is a two-dimensional irreducible representation of S_3. $a_2 \otimes a_2$ is a four-dimensional reducible representation of S_3, which is a direction sum of a one-dimensional trivial representation $\mathbf{1}$, a one-dimensional nontrivial representation a_1, and a two-dimensional irreducible representation a_2: $a_2 \otimes a_2 = \mathbf{1} \oplus a_1 \oplus a_2$. This leads to the last expression in the above.

The algebra for the symmetry transformations is not a group algebra like $W_R W_{R'} = W_{R''}$. The composition of two a_2 symmetry transformations $W_{a_2}W_{a_2} = W_{\mathbf{1}} + W_{a_1} + W_{a_2}$ makes the $\widetilde{S}_3$ symmetry noninvertible. Such kind of symmetry was referred to as algebraic symmetry, or fusion category symmetry, etc.

References [42,49] showed that $\widetilde{S}_3$ and S_3 symmetries are equivalent symmetries, i.e., they have the same holocat symmetry. Reference [33] shows that the symmetries with the same holocat symmetry have isomorphic algebras of local symmetric operators, which is the meaning of equivalence. A holocat symmetry is nothing but an isomorphic class of algebras of local symmetric operators.

From the holographic point of view, both 1 + 1D S_3 and $\widetilde{S}_3$ symmetry are described by the same 2 + 1D topological order $\mathcal{G}\mathrm{au}_{S_3}$. In other words, systems with S_3 symmetry are *exactly locally reproduced* by boundaries of $\mathcal{G}\mathrm{au}_{S_3}$ topological orders, in the sense that the local symmetric operators for a system have identical correlations with the local symmetric operators for the corresponding boundary. Similarly, systems with $\widetilde{S}_3$ symmetry are also *exactly locally reproduced* by boundaries of $\mathcal{G}\mathrm{au}_{S_3}$ topological orders.

The charges of S_3 symmetry correspond to a_1 and a_2 anyons in $\mathcal{G}\mathrm{au}_{S_3}$, while the S_3 symmetry transformations correspond to string operators that produce b and c anyons in $\mathcal{G}\mathrm{au}_{S_3}$ (at the string ends). Similarly, the charges of $\widetilde{S}_3$ symmetry correspond to b and c anyons in $\mathcal{G}\mathrm{au}_{S_3}$, while the $\widetilde{S}_3$ symmetry transformations correspond to string operators that produce a_1 and a_2 anyons in $\mathcal{G}\mathrm{au}_{S_3}$.

Certainly, we can also divide the anyons in $\mathcal{G}\mathrm{au}_{S_3}$ differently. Call some of them charges of a symmetry and others as the transformation of the symmetry. This way, we get a different symmetry or an anomalous symmetry, or even a symmetry beyond anomaly.

This example demonstrates that the notions of symmetry and anomaly are not essential notions that reflect the physical properties of a quantum system. They are notions that depend on our point of views to look at the system. Different angles to look at the same system will leads to different points of view. In contrast, holocat symmetry reflects the essence of (anomalous) symmetries. It more directly reflects the physical properties of a quantum system.

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