Transformation twinning to create isospectral cavities

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Bounded domains have discrete eigenfrequencies/spectra, and cavities with different boundaries and areas have different spectra. A general methodology for isospectral twinning, whereby the spectra of different cavities are made to coincide, is created by combining ideas from across physics including transformation optics, inverse problems, and metamaterial cloaking. We twin a hexagonal drum with a deformed hexagonal drum using a nonsingular coordinate transform that adjusts the deformed shape by mapping a near boundary domain to a zone of heterogeneous anisotropic medium. Splines define the mapping zone for twinning these two drums and we verify isospectrality by a finite-element analysis.

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I. INTRODUCTION

An open challenge across wave physics is to design cavities that are twins, in the sense we shall define, of a different shaped cavity. The ability to design such twinned cavities opens up multiple possibilities in, say, acoustics, of having two different shaped drums or even rooms/auditoria, sounding identical, or of two different elastic components sharing the same vibrational eigenfrequencies. In electromagnetism there are numerous examples of closed cavities [1], and for water waves having vastly different sized experimental wave tanks sharing the same eigenfrequencies would be desirable. In this paper we create such twinned cavities. The twinning we construct is to ensure that both cavities have identical eigenfrequencies, and as such they are isospectral as they share the same spectrum, despite their different boundaries and areas, and furthermore share the same eigenfields within a well-defined portion of the cavities; we consider closed cavities for which the spectrum is discrete [1]. To tackle the challenge of creating twinned cavities we combine ideas from across physics drawing upon transformation optics [2,3], inverse problems [4,5], and metamaterial carpet cloaking [6].

A. Isospectral problems

Isospectral problems in general have a long history, e.g., in the design of isospectral drums [7–9] motivated by the famous question of Kac [10] as to whether one can hear the shape of a drum. Kac was referring to the problem of whether the Laplacian operator in a closed domain, with Dirichlet boundary conditions, could have identical spectra on two distinct planar regions sharing the same area. The question is still open, although there are many results for specific classes and subsets of the problem [11]. These isospectral problems in bounded domains are also related to inverse problems in open space [12]. We use an approach inspired by transformation optics [6] that allows us, in contrast to much of the isospectral literature [13], to not limit ourselves to requiring cavities of the same area. Instead, we modify portions of the original shape to create the isospectral match to the target cavity and in doing so require anisotropic heterogeneous media close to the boundaries.

B. Cloaking theory

We shall also draw upon the theory of cloaking [2]: Cloaking theories are almost invariably focused around scattered fields in unbounded domains with line or point source excitation or incoming plane waves and this is quite separate from the isospectral cavity problem. However, there are very useful concepts that we will utilize: Particularly pertinent is the use of transformation optics in scattering to create mirage [14] or illusion effects [15–17] whereby one object is cloaked such that after scattering it appears to be another object, or where it, or the excitation source, appears to have physically moved.

An important nuance in cloaking revolves around the precise transformation employed and to achieve perfect cloaking a transformation with a singularity is required [2]. This singularity would destroy the discrete nature of the spectrum of the closed cavity should it be employed in the isospectral setting. However, this requirement of extreme material parameters can be relaxed to achieve cloaking over a finite frequency bandwidth by regularizing the transform, i.e., blowing up a small ball instead of a point [18]. Another approach that removes the singularity is carpet cloaking [6] whereby a curved surface maps onto a flat surface with the objective of hiding an object

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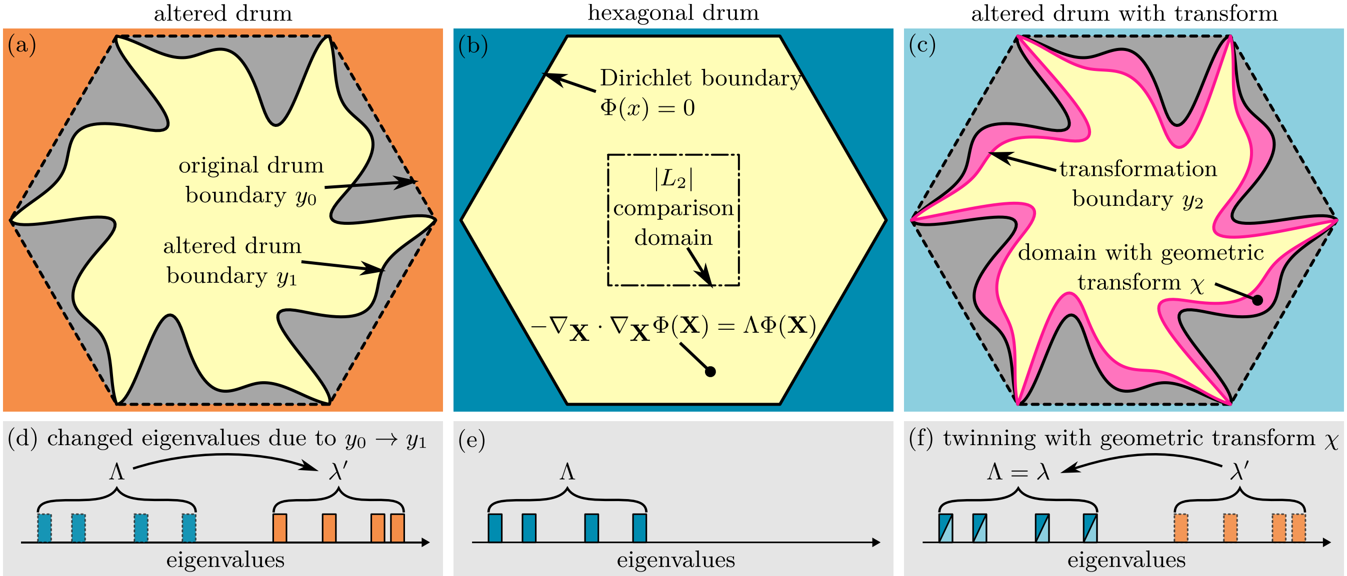

FIG. 1. Principle of transformation twinning for the eigenvalues Λ and eigenfunctions Φ in the Laplacian [Eq. (1)] of a hexagonal drum under Dirichlet boundary conditions. Changing the boundary of a drum (a) with respect to the reference hexagonal drum (b) also results in a change of eigenvalues from Λ to λ' (d). Using a nonsingular coordinate transform [Eq. (5)], a perturbed Laplacian with heterogeneous anisotropic parameters [Eqs. (3) and (4)] is introduced into the transformed domain (pink, between y_1 and y_2) of the altered drum (c). Thus, the eigenvalues λ and eigenfunctions ϕ outside of the transformed domain (unaltered medium) coincide with those of the original hexagonal drum (e) and the two drums are twinned (f).

on a surface. The approach of Ref. [6] is attractive in the isospectral setting as it requires no singularity in the transform and yet is tractable to apply. Hence, we use this transformation idea to deform the boundaries of our target cavity onto those of the reference cavity. Just as in the carpet cloaking approach, this generates a moderately anisotropic and heterogeneous medium. We are able to employ these ideas for isospectral twinning despite it having no scattering field.

C. Inverse problems and boundary measurements

Further, we draw upon the closely related field of inverse problems, [4,19], that aims to uniquely determine some physical parameters, such as an electric conductivity σ within a bounded region Ω, by applying a known static voltage u to the surface $\partial\Omega$ and recording the resulting current, $\sigma\nabla u \cdot \mathbf{n}$, at the boundary with $\mathbf{n}$ as the normal to the surface. Such boundary measurements indeed determine σ [18], but only under certain limited conditions, namely, that σ must be scalar valued, positive, and finite. For the twinned cavities that we design using transformation optics we generate matrix-valued, yet nonsingular, conductivities (although our language differs depending upon the physical setting). This means that the interior field within the cavity cannot be uniquely determined from the boundary measurements. In an isospectral problem this would translate into different eigenfields for the cavities and yet both cavities would share the same spectrum.

II. METHODS

The challenge is exemplified in Fig. 1: A perfect hexagon (original drum) with the zero (Dirichlet) boundary condition on its boundary (y_0) has a discrete spectrum of eigenfrequencies with associated eigenfunctions. Also shown is a highly deformed "hexagon" where each edge has been deformed to have a boundary y_1 (with a Dirichlet boundary condition). The challenge is to have this altered shape have the same eigenvalues as the perfect hexagon. This is achieved by a coordinate transformation in a domain close to boundary which effectively squeezes the missing material into the deformed shape via changing the material parameters. The eigenfield is slightly harder to unequivocally compare between the two shapes, however, this is possible by choosing a domain away from the inner boundary and then comparing the eigenfields by an appropriate norm; here we choose the Euclidean L_2 norm. We use this hexagon and its deformed counterpart as the exemplar upon which to demonstrate our methodology and, without loss of generality, we restrict our analysis to the two-dimensional case. Figures 1(d)–1(f) also show, qualitatively, what we want to achieve: The eigenfrequencies for the perfect hexagon are shifted by the shape change and we want to restore them by using the coordinate transformation.

Numerically we adopt a general approach to transformation so we can tackle arbitrary deformations of the boundaries (e.g., splines) that can describe any object/boundary of the shape and this lends our approach versatility. All of our eigenfunction computations are performed in COMSOL MULTIPHYSICS using standard finite elements to discretize the domains of interest. The respective transformations are introduced into the finite-element simulation via a C library built from the boundary functions (y_0, y_1, and y_2). For our numerical solutions, the eigenfields are normalized to a value range of $[-1, 1]$ with respect to the original drum. These normalized and reoriented regions are then compared based on their L_2 norm [within the centered 8×5 m comparison domain; see Fig. 1(b)], using the numerical integration with

Simpson's rule (Newton-Coates order 2) $\sqrt{\iint u^2 dxdy}$ for the nodal values $u(x, y)$ from the simulation.

A. Derivation of transformed eigenvalue problem

In Ref. [10] Kac wondered whether it is possible to deduce the precise shape of a drum just from hearing the fundamental tone and all the overtones. Mathematically, this reduces to the study of the eigenvalues Λ and associated eigenfields Φ of the Laplacian, e.g.,

$$-\Delta_{\mathbf{X}}\Phi(\mathbf{X}) = \Lambda\Phi(\mathbf{X}) \text{ in a bounded domain } \Pi, \quad (1)$$

where $\Delta_{\mathbf{X}} = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the Laplacian in Cartesian coordinates, and where Ref. [10] considered Dirichlet conditions along the boundary $\partial\Pi$ (Neumann conditions also work). In general, the spectral properties of a shape are linked to its geometry and its material constituent.

Now we introduce a coordinate transformation χ that maps the cavity of shape Π, in our example the undeformed hexagon, onto a transformed cavity of shape π, the deformed hexagon, i.e.,

$$\chi : \mathbf{X} = (x, y) \in \Pi \to \mathbf{s} = [u(x, y), v(x, y)] \in \pi. \quad (2)$$

Using the chain rule, we obtain the counterpart of Eq. (1) in the transformed coordinates as

$$-\nabla_{\mathbf{s}} \cdot [\sigma(\mathbf{s})\nabla_{\mathbf{s}}\phi(\mathbf{s})] = \eta(\mathbf{s})\lambda\phi(\mathbf{s}) \quad (3)$$

in the transformed domain π where $\nabla_{\mathbf{s}}$ is the gradient operator in u, v coordinates, $\phi(\mathbf{s}(\mathbf{X})) = \Phi(\mathbf{X})$, and

$$\sigma(\mathbf{s}) = \mathbf{J}_{\mathbf{sX}}\mathbf{J}_{\mathbf{sX}}^T \det \mathbf{J}_{\mathbf{Xs}} \quad (4)$$

is a matrix-valued, spatially varying parameter [physically related to some artificial anisotropic shear modulus, permittivity, permeability, or mass density, in antiplane shear elastic (SH), transverse magnetic (TM), transverse electric (TE), or pressure acoustic settings, respectively], that can be achieved through effective-medium theory, e.g., with layered media (see Supplemental Material [20]). This parameter σ depends on the Jacobian $\mathbf{J}_{\mathbf{sX}} = \partial(u, v)/\partial(x, y)$ of the geometrical transformation χ. In Eq. (3), $\eta = \det \mathbf{J}_{\mathbf{Xs}}$ is a spatially varying scalar parameter also depending on χ (physically related to some artificial isotropic mass density, permeability, permittivity, or bulk modulus, in SH, TM, TE or pressure acoustic settings, respectively). Eigenvalue problem (3) is solved subject to the Dirichlet condition, $\phi = 0$, on the boundary $\partial\pi$ [or the Neumann condition $\sigma\nabla_s\phi \cdot \mathbf{n} = \mathbf{0}$, with $\mathbf{n}$ the normal to $\partial\pi$, if the Neumann datum was assumed for (1)].

As the parameter σ in (4) and its counterpart in (6) are both symmetric, the differential operator in (3) is symmetric and its spectrum is real positive. However, to ensure that its resolvent is compact, and thus that $0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_k \cdots$ that tends to $+\infty$, we also require that σ be a positive-definite and bounded matrix. In other words, there should exist two positive real constants m and M such that $0 < m \leqslant \sigma\boldsymbol{\xi} \cdot \boldsymbol{\xi} \leqslant M$, for every vector $\boldsymbol{\xi}$ in $\mathbb{R}^2$. This criterion is satisfied if the eigenvalues of matrix σ are all strictly positive and finite. This raises an important practical point: We can only use a class of nonsingular transforms to preserve the discrete spectrum, and the mathematical criterion is that the eigenvalues of the Jacobian need to be bounded from below and above by strictly positive constants. We now consider nonsingular geometrical transforms that ensure the resolvent of the differential operator in partial differential equation (3) remains compact.

B. Li-Pendry transform

To proceed we borrow the nonsingular geometric transforms of Li and Pendry who introduced them for the design of ground carpet cloaks in electromagnetic scattering problems [6]. As shown in Fig. 1, the region between the outer (y_2) and ground boundary ($y_0 = 0$) is compressed into the region between the outer (y_2) and inner (y_1) boundary. We introduce a transformation which maps the region enclosed between two curves $(x, 0)$ and $(x, y_2(x))$ to the one comprised between $(x, y_1(x))$ and $(x, y_2(x))$. This corresponds to a compression of space from the y_0-y_2 region into the y_1-y_2 region. In Fig. 1, $(x, 0)$ (ground boundary) is mapped on $(x, y_1(x))$ (inner boundary) and $(x, y_2(x))$ (outer boundary) is fixed pointwise, of the form

$$\begin{aligned} u &= x, \\ v &= \alpha(x)y + \beta(x), \end{aligned} \quad (5)$$

where $\alpha(x) = (y_2 - y_1)/y_2$ and $\beta(x) = y_1$. The transformed parameter σ is then easily deduced from (4) by computing the Jacobian matrix $\mathbf{J}_{\mathbf{sX}}$ associated with (5), and similarly for η is deduced from the determinant of the Jacobian.

For our exemplar, the hexagon and its deformed counterpart, we need to do a coordinate transformation where the deformations are identical on each face, bar a shift and rotation. Typically, the transformation concept, as applied to carpet cloaks [6], is for flat reference boundaries (see Supplemental Material [20] for curved reference boundaries $y_0 \neq 0$), whereas here we have flat walls at certain angles with respect to the horizontal axis. We thus proceed as follows: First, we design the transformation for the wall parallel to the x axis, and deduce parameter σ for the other five carpets through the formula

$$\mathbf{R}(\theta)\sigma(\mathbf{s})\mathbf{R}(-\theta), \quad (6)$$

where $\mathbf{R}(\theta)$ is the rotation matrix through a counterclockwise angle θ. In the present case, we consider an hexagonal drum so $\theta = n\pi/3$, with $n = 1, \ldots, 5$. We note that parameter η in Eq. (3) is not affected by the rotation. One need not assume in general that the n carpet cloaks constituting the polygonal drum are identical, but the $n - 1$ carpets associated with walls not parallel to the x axis should be deduced from the transformation design via a rotation of the given angle. We further note that the matrix in Eq. (6) is symmetric, as required, since a is symmetric and $\mathbf{R}(-\theta) = \mathbf{R}^T(\theta)$.

III. NUMERICAL RESULTS

In Fig. 2 we show the results for three modes, with respective eigenfrequencies and L_2 norm results. In the perfect hexagonal drum [Fig. 2(b)] the eigenmodes retain the symmetry of the shape and we chose three modes: one mode with simpler features, one with a rapidly oscillating geometrical mode, and another multipolar mode. The deformed hexagon [Fig. 2(a)], as expected, has dramatically different eigenmodes

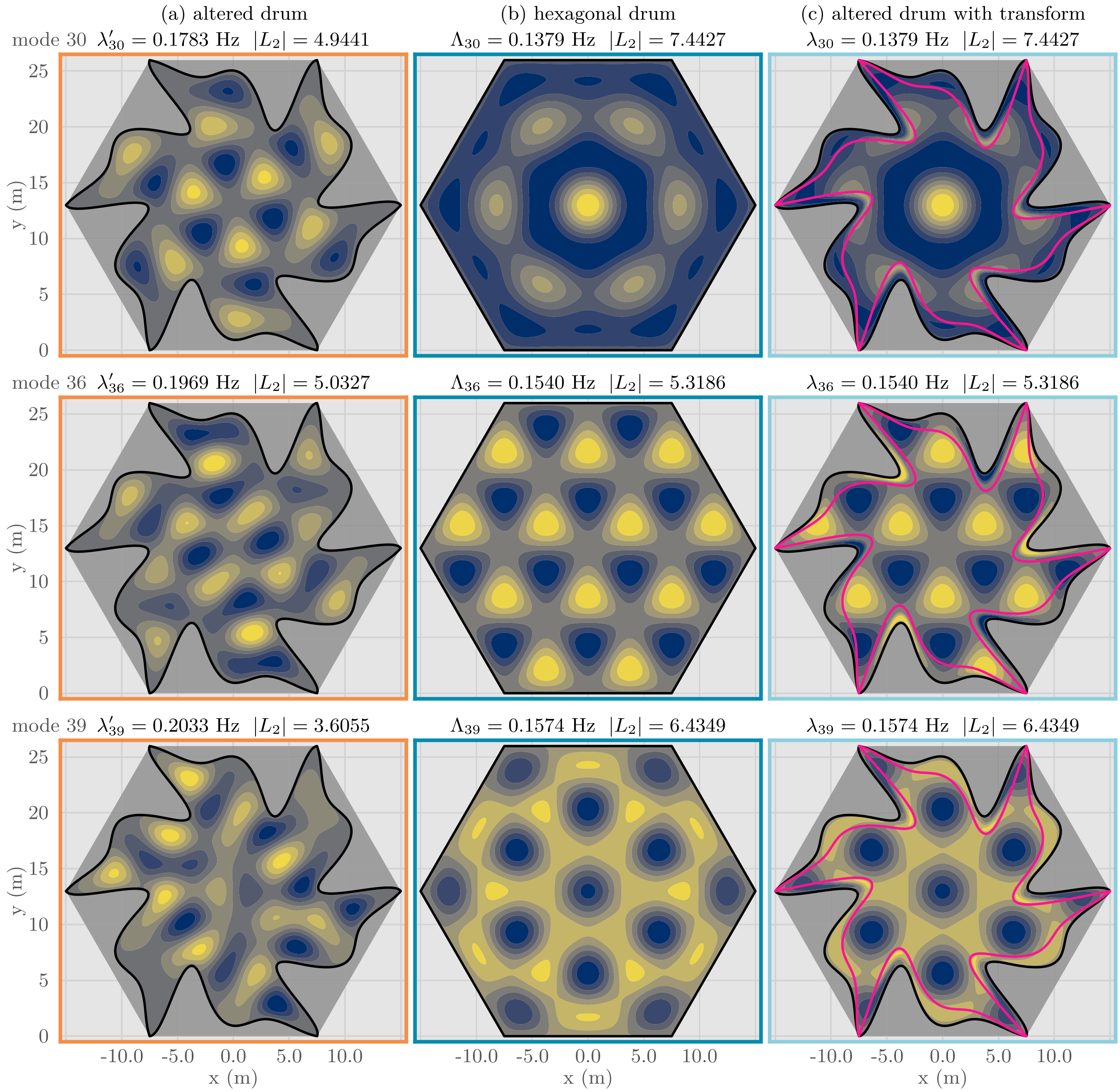

FIG. 2. Selection of modes for a comparison of three different drums (assuming wave speed $c = 1$ ms^{-1}, as $\sigma = \eta = 1$ in the unaltered medium). The altered drum (a) shows different eigenmodes and eigenfrequencies compared to the hexagonal drum (b), caused by the deformed Dirichlet boundary (black line). By introducing a coordinate transform within a boundary layer (between the black and pink line) of the altered drum (c), the respective eigenfrequencies and eigenmodes coincide with those of the hexagonal drum. Eigenfrequencies and $|L_2|$ are provided above each panel.

that bear little relation to that of the perfect hexagon. The eigenfrequencies are similarly poorly matched to those of the perfect hexagon. Turning to the deformed hexagon, with the transformation applied, we see the eigenfrequencies coincide with those of the perfect hexagon and, within the nontransformed region [see Fig. 1(c)], the L_2 norm gives very good agreement: The cavities are twinned despite the severity of the deformation and the change of area enclosed. The modes shown are very typical and the accuracy follows across many tens of modes with Fig. 3 showing the first 50 eigenfrequencies. The presented data correspond to a frequency range of 0–0.23 Hz, but we can twin the entire spectrum in theory, the only restriction being computational resources and numerical accuracy. One way towards achieving twinning in an experiment would be the use of a layered medium, with parameters derived from effective-medium theory. We provide a proof of concept for this approach in the Supplemental Material [20].

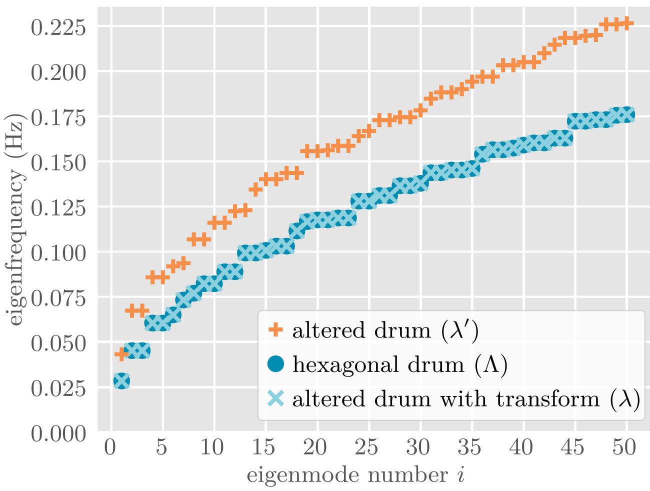

FIG. 3. Numerical comparison of the first 50 computed eigenfrequencies (some of which are degenerate) of the hexagonal drum vs the altered and the twinned drum assuming wave speed $c = 1$ ms^{-1}, as $\sigma = \eta = 1$ in the unaltered medium. The eigenfrequencies of the altered drum are generally higher than the hexagonal drum. With the transformation, however, the eigenfrequencies of the altered drum match those of the hexagonal drum closely and achieve twinning. The mean absolute difference between twinned and original drum is $3.51 \times 10^{-8} \pm 3.50 \times 10^{-8}$ Hz, and $3.85 \times 10^{-2} \pm 9.37 \times 10^{-3}$ Hz between the altered and original drum, for a mesh with 2 500 000 elements (mean ± standard deviation).

IV. CONCLUSION

In this paper, the recent theories of transformation optics have been applied, not to scattering, but to spectral problems. In doing so we have shown that we can match the spectrum for strongly different shapes in closed domains, while limiting the transformation to a boundary region. This opens the path to several applications; the spectrum is essential in terms of energy transport for photonic waveguides and crystal fibers. The ideas presented here could be used to retrofit, or repair, waveguides using metamaterial regions of transformed material to create, or recreate, the eigenfrequencies and eigenmodes desired. Similarly one could envisage cavity devices that, through manufacture or damage, are not operating at the right frequency or which require tunability and for which this transformation methodology would allow eigenfrequencies to be designed for. In vibration control an example would be shifting the eigenfrequencies of slender bridges to avoid unwanted swaying and resonance effects [42]. This also has implications for designed and tunable resonance, in particular to decrease a cavity area while maintaining the same resonance properties.

On a practical level the effectiveness and quality of cloaking for optics and electromagnetism is hard to quantify and assess, and one reason we were drawn to the identification of the spectrum is that the isospectral problem is a robust, and unequivocal, approach to assess the quality of carpet cloaks (independent of boundary conditions, wave simulations, perfectly matched layers, etc.). The agreement of the twinned spectra is a robust measure of cloaking quality. The approach here paves the way towards numerous extensions to other wave systems such as the full Maxwell system [2,3,43,44] or to acoustics [38,45] and water waves [46,47] and also to others such as elasticity where perfect cloaking is not available [41], but ideas around direct lattice transformation [48] and near cloaking [49] could be adapted, to perforated domains, heterogeneous or anisotropic cavities, and will motivate experiments and devices.

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[1] D. A. Hill, *Electromagnetic Fields in Cavities: Deterministic and Statistical Theories* (Wiley, Hoboken, NJ, 2009).
[2] J. B. Pendry, D. Schurig, and D. R. Smith, Controlling electromagnetic fields, Science **312**, 1780 (2006).
[3] U. Leonhardt and T. Philbin, *Geometry and Light: The Science of Invisibility* (Courier Corporation, North Chelmsford, MA, 2010).
[4] R. Kohn and M. Vogelius, Determining conductivity by boundary measurements, Commun. Pure Appl. Math. **37**, 289 (1984).
[5] A. Greenleaf, M. Lassas, and G. Uhlmann, Anisotropic conductivities that cannot be detected by EIT, Physiol. Meas. **24**, 413 (2003).
[6] J. Li and J. B. Pendry, Hiding under the Carpet: A New Strategy for Cloaking, Phys. Rev. Lett. **101**, 203901 (2008).
[7] T. A. Driscoll and H. P. W. Gottlieb, Isospectral shapes with Neumann and alternating boundary conditions, Phys. Rev. E **68**, 016702 (2003).
[8] H. Wu, D. W. L. Sprung, and J. Martorell, Numerical investigation of isospectral cavities built from triangles, Phys. Rev. E **51**, 703 (1995).
[9] T. A. Driscoll, Eigenmodes of isospectral drums, SIAM Rev. **39**, 1 (1997).
[10] M. Kac, Can one hear the shape of a drum? Am. Math. Mon. **73**, 1 (1966).
[11] C. Gordon, D. L. Webb, and S. Wolpert, One cannot hear the shape of a drum, Bull. Am. Math. Soc. **27**, 134 (1992).
[12] B. Sleeman, The inverse problem of acoustic scattering, IMA J. Appl. Math. **29**, 113 (1982).
[13] H. Isozaki, *Inverse Spectral and Scattering Theory: An Introduction* (Springer, Berlin, 2020).
[14] F. Zolla, S. Guenneau, A. Nicolet, and J. Pendry, Electromagnetic analysis of cylindrical invisibility cloaks and the mirage effect, Opt. Lett. **32**, 1069 (2007).
[15] Y. Lai, J. Ng, H. Y. Chen, D. Z. Han, J. J. Xiao, Z.-Q. Zhang, and C. T. Chan, Illusion Optics: The Optical Transformation of an Object into Another Object, Phys. Rev. Lett. **102**, 253902 (2009).
[16] A. Diatta and S. Guenneau, Non-singular cloaks allow mimesis, J. Opt. **13**, 024012 (2011).

[17] W. Kan, B. Liang, X. Zhu, R. Li, X. Zou, H. Wu, J. Yang, and J. Cheng, Acoustic illusion near boundaries of arbitrary curved geometry, Sci. Rep. **3**, 1427 (2013).

[18] R. V. Kohn, H. Shen, M. S. Vogelius, and M. I. Weinstein, Cloaking via change of variables in electric impedance tomography, Inverse Probl. **24**, 015016 (2008).

[19] J. M. Lee and G. Uhlmann, Determining anisotropic real-analytic conductivities by boundary measurements, Commun. Pure Appl. Math. **42**, 1097 (1989).

[20] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevB.108.064209 for details on the data analysis, an expanded eigenfrequency range, some notes on polygonal cavities and boundary definition with spline curves, a proof-of-concept design for a cavity with an anisotropic medium derived via effective-medium theory (including a note on homogenized Maxwell's equations), notes on homogenization theory for eigenvalue problems, and numerical proof-of-concept twinning using a layered medium, which includes Refs. [21–41].

[21] A. Meurer, C. P. Smith, M. Paprocki, O. Čertík, S. B. Kirpichev, M. Rocklin, A. Kumar, S. Ivanov, J. K. Moore, S. Singh, T. Rathnayake, S. Vig, B. E. Granger, R. P. Muller, F. Bonazzi, H. Gupta, S. Vats, F. Johansson, F. Pedregosa, M. J. Curry *et al.*, SymPy: symbolic computing in Python, PeerJ Comput. Sci. **3**, e103 (2017).

[22] C. R. Harris, K. J. Millman, S. J. van der Walt, R. Gommers, P. Virtanen, D. Cournapeau, E. Wieser, J. Taylor, S. Berg, N. J. Smith, R. Kern, M. Picus, S. Hoyer, M. H. van Kerkwijk, M. Brett, A. Haldane, J. F. del Río, M. Wiebe, P. Peterson, P. Gérard-Marchant *et al.*, Array programming with NumPy, Nature (London) **585**, 357 (2020).

[23] P. Virtanen, R. Gommers, T. E. Oliphant, M. Haberland, T. Reddy, D. Cournapeau, E. Burovski, P. Peterson, W. Weckesser, J. Bright, S. J. van der Walt, M. Brett, J. Wilson, K. J. Millman, N. Mayorov, A. R. J. Nelson, E. Jones, R. Kern, E. Larson, C. J. Carey *et al.*, SciPy 1.0: Fundamental algorithms for scientific computing in Python, Nat. Methods **17**, 261 (2020).

[24] J. D. Hunter, Matplotlib: A 2D graphics environment, Comput. Sci. Eng. **9**, 90 (2007).

[25] S. O'Brien and J. B. Pendry, Photonic band-gap effects and magnetic activity in dielectric composites, J. Phys.: Condens. Matter **14**, 4035 (2002).

[26] A. Bensoussan, J.-L. Lions, and G. Papanicolaou, *Asymptotic Analysis for Periodic Structures* (American Mathematical Society, Providence, RI, 1978).

[27] G. W. Milton, *The Theory of Composites* (Cambridge University Press, Cambridge, UK, 2002).

[28] D. Petiteau, S. Guenneau, M. Bellieud, M. Zerrad, and C. Amra, Spectral effectiveness of engineered thermal cloaks in the frequency regime, Sci. Rep. **4**, 7386 (2014).

[29] L. Pomot, C. Payan, M. Remillieux, and S. Guenneau, Acoustic cloaking: Geometric transform, homogenization and a genetic algorithm, Wave Motion **92**, 102413 (2020).

[30] S. Guenneau and F. Zolla, Homogenization of three-dimensional finite photonic crystals, Prog. Electromagn. Res. **27**, 91 (2000).

[31] G. Kristensson and N. Wellander, Homogenization of the maxwell equations at fixed frequency, SIAM J. Appl. Math. **64**, 170 (2003).

[32] S. Guenneau, F. Zolla, and A. Nicolet, Homogenization of 3D finite photonic crystals with heterogeneous permittivity and permeability, Waves Random Complex Media **17**, 653 (2007).

[33] A. Bossavit, Solving Maxwell equations in a closed cavity, and the question of "spurious modes", IEEE Trans. Magn. **26**, 702 (1990).

[34] F. Zolla, G. Renversez, A. Nicolet, B. Kuhlmey, S. R. Guenneau, and D. Felbacq, *Foundations of Photonic Crystal Fibres* (World Scientific, Singapore, 2005).

[35] P. Monk, *Finite Element Methods for Maxwell's Equations* (Oxford University Press, Oxford, UK, 2003).

[36] R. V. Craster, J. Kaplunov, and A. V. Pichugin, High-frequency homogenization for periodic media, Proc. R. Soc. A **466**, 2341 (2010).

[37] G. Allaire and C. Conca, Bloch wave homogenization and spectral asymptotic analysis, J. Math. Pures Appl. **77**, 153 (1998).

[38] B.-I. Popa, L. Zigoneanu, and S. A. Cummer, Experimental Acoustic Ground Cloak in Air, Phys. Rev. Lett. **106**, 253901 (2011).

[39] T. Ergin, N. Stenger, P. Brenner, J. B. Pendry, and M. Wegener, Three-dimensional invisibility cloak at optical wavelengths, Science **328**, 337 (2010).

[40] J.-P. Berenger, Perfectly matched layer for the FDTD solution of wave-structure interaction problems, IEEE Trans. Antennas Propag. **44**, 110 (1996).

[41] G. W. Milton, M. Briane, and J. R. Willis, On cloaking for elasticity and physical equations with a transformation invariant form, New J. Phys. **8**, 248 (2006).

[42] M. Brun, A. Movchan, I. Jones, and R. McPhedran, Bypassing shake, rattle and roll, Phys. World **26**, 32 (2013).

[43] A. Nicolet, J.-F. Remacle, B. Meys, A. Genon, and W. Legros, Transformation methods in computational electromagnetism, J. Appl. Phys. **75**, 6036 (1994).

[44] R. Fleury, F. Monticone, and A. Alù, Invisibility and Cloaking: Origins, Present, and Future Perspectives, Phys. Rev. Appl. **4**, 037001 (2015).

[45] C. Faure, O. Richoux, S. Félix, and V. Pagneux, Experiments on metasurface carpet cloaking for audible acoustics, Appl. Phys. Lett. **108**, 064103 (2016).

[46] C. P. Berraquero, A. Maurel, P. Petitjeans, and V. Pagneux, Experimental realization of a water-wave metamaterial shifter, Phys. Rev. E **88**, 051002(R) (2013).

[47] G. Dupont, O. Kimmoun, B. Molin, S. Guenneau, and S. Enoch, Numerical and experimental study of an invisibility carpet in a water channel, Phys. Rev. E **91**, 023010 (2015).

[48] T. Bückmann, M. Kadic, R. Schittny, and M. Wegener, Mechanical cloak design by direct lattice transformation, Proc. Natl. Acad. Sci. USA **112**, 4930 (2015).

[49] D. E. Quadrelli, R. Craster, M. Kadic, and F. Braghin, Elastic wave near-cloaking, Extreme Mech. Lett. **44**, 101262 (2021).