Macroscopic dynamics of superfluid ^{3}He with a spatially modulated pair density wave

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We discuss the macroscopic behavior of the superfluid ^{3}He phase (pair density wave phase) with a spatially modulated pair density wave recently observed experimentally. As an order parameter we assume, based on the experimental results and a Landau-type model, a variation of the phase of the macroscopic wave function coupled to spatially modulated plane waves of the macroscopic wave function in the plane perpendicular to the confinement. As a result we find only one true Goldstone mode in orbit space coupling the superfluid aspects to the spatial modulations of the order parameter in the plane of the sample. This coupling is predicted to lead to a propagating mode sharing aspects of second sound and acoustic waves arising from the spatial variations of the order parameter. Due to the coupling of the phase of the order parameter to in-plane spatial modulations, the velocity of first sound becomes anisotropic. In addition, the velocities of first and second sound reflect the static and dynamic coupling terms to the order parameter. Fourth sound in contrast, the velocity of which also becomes anisotropic, couples only to one static cross coupling associated with density variations and not to any dynamic cross-coupling terms. Therefore measurements of the fourth sound velocity could be used to measure quite directly a static cross-coupling term. The situation studied here is thus qualitatively different from that suggested for supersolid ^{4}He for which one has a solid phase with broken translational symmetry associated with a density wave as in a usual solid. As for spin space we obtain, neglecting the tiny magnetic dipole interaction and in the absence of a magnetic field, three pairs of spin waves closely resembling the results of the planar distorted *B* phase. Excitations in spin space in the presence of the magnetic dipole interaction and/or an external magnetic field are also investigated briefly.

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I. INTRODUCTION

In recent experiments on fairly thin layers of superfluid ^{3}He with a typical layer thickness of about 1 μm a new phase [pair density wave (PDW) phase] has been described with a spatially modulated superfluid order parameter in the plane of the layers [1–4]. The experimental techniques used were measurements of the order parameter distortions with NMR [1] and microfabricated fourth sound resonators [2–4]. This was followed up rather recently by a first Ginzburg-Landau-type description [5].

The spatially modulated superfluid phases in thin layers of ^{3}He fall into the category, anticipated a long time ago, for *s*-wave superconductors with a spatially modulated order parameter as predicted by FFLO (Fulde, Ferrell, Larkin, and Ovchinnikov) [6,7]. We note, however, that for these classical superconductors such a phase has apparently never been detected experimentally.

We would like to point out that there is an important difference in the spatial structure of the order parameter between the FF state [6] and the LO state [7]. For the FF state a plane-wave state is used for the order parameter, thereby modulating the phase of the macroscopic wave function, while for the LO state the order parameter has a spatial dependence of the form $\cos(qr)$, effectively modulating the amplitude. Throughout the present paper we will use the plane-wave form of the order-parameter phase and thus treat the PDW state as an FF state. A similar approach has been used before for superfluid ^{3}He/^{4}He mixtures [8].

It is also worthwhile in the present context to mention recent experimental developments related to spatially modulated dipolar supersolids [9–13], which are still of a fairly small aspect ratio from a continuum perspective, but appear as a natural candidate for future examinations of a system with spatially modulated order parameter. Quite recently there has been substantial theoretical progress regarding the FFLO state in the field of cold atom optics making use of the technique of auxiliary field quantum Monte Carlo simulations [14] profiting from methodological advances [15]. It was shown [16] that for the case of a fairly large two-dimensional spin-polarized Fermi gas of attractive fermions strong FFLO correlations emerge in the limit of high density and small spin polarization.

Here we present a macroscopic description of the spatially modulated phases observed recently in thin layers of

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superfluid ^{3}He, for which the phase of the order parameter is spatially modulated. In addition to recent experimental results the work described here has been stimulated by the microscopic descriptions of Aoyama [17] and in the group of Sauls [18].

The spatial variations of the order-parameter phase represent a qualitative difference to all the other phases observed and analyzed for superfluid ^{3}He in the bulk [19–22] as well as in aerogels [23–30], including the superfluid phases, which have been found to exist stably only for sufficiently high magnetic fields, namely, the A_1 phase in the bulk [20,21] and the two new phases found quite recently in strongly anisotropic aerogels in sufficiently high magnetic fields [31].

There is also an important aspect that sets apart the superfluid phases described here from the classical FFLO-type description of superconductors [6,7] and from the FF state in superfluid ^{3}He/^{4}He mixtures [8]: in the phases analyzed here in detail the additional spin degrees of freedom of the order parameter can give rise to spin waves.

We use the approach of macroscopic dynamics in the same spirit as it has been applied to other superfluid systems starting with Khalatnikov for superfluid ^{4}He [32]. These applications to the superfluid phases of ^{3}He include in the bulk the A, B, and A_1 phases [33–41] as well as the more recently discovered superfluid phases of ^{3}He in strongly anisotropic aerogels, both without (including, in particular, the polar phase) [42] and in sufficiently high magnetic fields [43].

We emphasize that this technique is not confined to superfluid systems, but has been applied to numerous other condensed matter systems with spontaneously broken continuous symmetries [44–46] including magnetic systems [47], liquid crystals [45,46,48], and solids [44,45]. In some cases the classical hydrodynamic variables, conserved quantities, and variables associated with spontaneously broken continuous symmetries, are supplemented by macroscopic variables, that are not strictly hydrodynamics, but relax on a sufficiently long timescale [32,46]. For the systems investigated here this aspect plays a role for the effects of the magnetic dipole interaction and of small external magnetic fields [34,35].

Finally we note an important difference to the supersolid phase anticipated for solid ^{4}He [49,50]. The order parameter studied there is completely different from that of the superfluid phase of ^{3}He studied here. In particular, for the superfluid solid phase in ^{4}He a displacement field is arising as an independent variable. In the context of hydrodynamics and macroscopic dynamics such a supersolid phase has been studied by Saslow [51] and Liu [52].

The paper is organized as follows. In Sec. II we present the relevant variables due to conservation laws and due to broken symmetries. The macroscopic orbital dynamics of the spatially modulated phase is discussed in detail in Sec. III. This includes, in particular, an analysis of the influence of the spatial modulations on the structure of soundlike excitations. The spin dynamics along with a discussion of the coupling to external magnetic fields and the magnetic dipole interactions is described in Sec. IV. In Sec. V we give a summary.

II. THE RELEVANT MACROSCOPIC VARIABLES FOR THE PDW PHASE

A. Preliminaries

In this paper we will use linearized hydrodynamics [45,46,53] to describe the macroscopic behavior of the spatially modulated superfluid ^{3}He under spatial confinement (PDW phase) as it has been experimentally observed and modeled in the framework of a Landau-type energy recently. We will derive the balance equations describing the behavior of the system in the low-frequency, long-wavelength limit. Low frequencies in this context means small compared to all collisional frequencies while wavelengths are considered to be long if they are large compared to all microscopic lengths. Naturally these conditions for the purely hydrodynamic regime impose rather severe constraints on the frequencies and wave vectors for which this approach is strictly valid. Nevertheless, the hydrodynamic description and its generalization to include variables that relax on a long, but finite timescale have turned out to be rather useful also for the superfluid phases of ^{3}He [21].

The conserved quantities in superfluid ^{3}He are ρ (mass density), ε (energy density), and g_i (momentum density) just as in any normal fluid. The Latin indices refer to vector components in a suitable Cartesian frame (orbital space). All variables are related to the entropy density σ by the Gibbs relation,

$$d\varepsilon_c = T d\sigma + \mu d\rho + v_i^n dg_i \tag{1}$$

thereby defining the thermodynamic quantities, temperature T, chemical potential μ, and normal velocity v_i^n.

B. Hydrodynamic variables due to internal structures that break symmetries

Since the ^{3}He atoms have spin $\frac{1}{2}$ each, there is the magnetization density, s_ν. The frame to describe the orientation of spins is *a priori* not the same as that of, e.g., the flow variables. Therefore it is customary to use in "spin space" a different Cartesian frame indicated by greek indices. It turns out that, neglecting the tiny magnetic dipole-dipole interaction (spin-orbit coupling), the orientations of spin and orbital space are independent and the hydrodynamics in orbital and spin space can be developed separately.

In the absence of a magnetic field s_ν is a conserved quantity, but acquires a source term in its dynamic equation due to the field. The energy density

$$d\varepsilon_s = \chi_{\nu\mu} d\nabla_\mu s_\nu + h'_\nu ds_\nu \equiv h_\nu ds_\nu, \tag{2}$$

where h'_ν is zero in the absence of a magnetic field. In linear order one can simply condense the notation by using the conjugate $h_\nu \equiv h'_\nu - \nabla_\mu \chi_{\nu\mu}$.

In superfluid ^{3}He the neutral He atoms combine to form Cooper pairs similar to those found in superconductors which can be viewed as composite bosons. While the electrons in superconductors are in a spin-singlet s-wave state, the He atoms are in a spin-triplet p-wave state. This fact clearly distinguishes the two situations: The pair of electrons has no internal structure, but the pair of He atoms is intrinsically anisotropic. Because of the spin-triplet and p-wave pairing the order parameter $T_{\nu j}$ has to be a complex 3×3 matrix whose expectation value can in general be written as [39,54]

$$\langle T_{\nu j}(\mathbf{c}, \mathbf{r})\rangle = \mathcal{F}(|\mathbf{c}|)A_{\nu j}(\mathbf{r}),$$
$$A_{\nu j}A^*_{\nu j} = 1, \tag{3}$$

where ν is an index in spin space, j an index in orbital space, $\mathbf{r}$ is the position vector of the center of gravity, and $\mathbf{c}$ is the relative vector between the two He atoms.

The normalization amplitude $\mathcal{F}$ describes the degree of ordering and is considered a microscopic variable, which does not appear in the macroscopic dynamics of the system.

The order parameter for the homogeneous planar distorted B phase (PDB) reads [1,2,5]

$$A^B_{\nu j}(z) = \big[\Delta_h(z)\big(e^x_\nu e^x_j + e^y_\nu e^y_j\big) + \Delta_v(z)e^z_\nu e^z_j\big]\exp(i\tilde{\varphi}), \tag{4}$$

where $e^{x,y,z}_j$ and $e^{x,y,z}_\nu$ are the components of a Cartesian triad of unit vectors in orbit and spin space, respectively. The relative orientation of the two triads is arbitrary.

The z direction (in orbit space) is normal to the confinement layer plane, given by the x-y plane. The horizontal gap in this plane is $\Delta_h(z)$, while $\Delta_v(z)$ is the vertical gap along the z direction. The superfluid phase is $\tilde{\varphi}$. It is important to note that the phase variation is odd under time reversal.

Equation (4) contains the planar B phase as a special case for $\Delta_v \equiv 0$ and the isotropic B phase for $\Delta_v = \Delta_h$. The order parameter $A^B_{\nu j}(z)$ is invariant under $SO(2)_{L_z+S_z}$ rotations of the orbital and spin coordinates about the z axis. It factorizes in the spin and orbit space variables.

In the PDW phase there is a periodic variation of the superfluid order parameter in the layer plane and the full order parameter can be written as

$$A_{\nu j}(z, \mathbf{r}_\perp) = A^B_{\nu j}(z)\exp(-i\,\mathbf{q}\cdot\mathbf{r}_h) \tag{5}$$

[5] with the two-dimensional position vector $\mathbf{r}_h = (x, y)$ and where $\mathbf{q}$ is related to the momentum density of the superfluid condensate [17]. It has the same symmetry signatures as a velocity, changing sign under time reversal as well as under spatial inversion. We take $\mathbf{q}$ as a constant on the macroscopic level, i.e., fluctuations have already relaxed to zero on a microscopic timescale.

In the following we neglect biaxiality effects and assume uniaxial symmetry coming exclusively from $\mathbf{q}$ lying in the plane of the sample, and with $\hat{z}$ the layer normal.

The vector $\mathbf{q} = (q_1, q_2)$ defines the spatial structure of the modulations in the PDW phase, e.g., $q_1 = q_x$, $q_2 = q_y$ for a quadratic structure and $q_1 = \frac{\sqrt{3}}{2}q_x + \frac{1}{2}q_y$, $q_2 = -\frac{\sqrt{3}}{2}q_x + \frac{1}{2}q_y$ for a hexagonal one.

The order parameter in Eq. (5) contains the effective phase $\varphi = \tilde{\varphi} - \mathbf{q}\cdot\mathbf{r}_h$ that consists of a part related to gauge transformations and a part due to periodic in-plane variations. Thus, both operations, gauge transformations, $\tilde{\varphi} \to \tilde{\varphi}_0 + \delta\tilde{\varphi}$, and translations in the plane, $\mathbf{r}_h \to \mathbf{r}^0_h + \mathbf{u}_h$, will change the order parameter $A_{\nu j}$ due to

$$\delta\varphi = \delta\tilde{\varphi} - \mathbf{q}\cdot\mathbf{u}_h. \tag{6}$$

We emphasize that in the present context the behavior of φ and $\mathbf{q}$ under time reversal and parity is an intrinsic property of the equilibrium phase, in contrast to externally driven systems under the influence of an external force. It is therefore only possible with respect to $\mathbf{q}$ to change the behavior under time reversal and spatial inversion simultaneously in order to preserve the symmetry properties of $\mathbf{q}$ and thus also of $\delta\varphi$. This aspect will become evident again further below when discussing the normal modes of the present spatially modulated phase of superfluid ^{3}He.

We note that φ is not fixed energetically, and is therefore the additional hydrodynamic variable. This Goldstone mode, the linear combination between a gauge transformation and a translation, one can call a relative broken translation/gauge symmetry, along the same lines of reasoning as Stern and Liu [8] for the FF state [6].

Instead of the phase variable, only its gradients can enter the energy and the hydrodynamic description

$$v^s_i = \frac{\hbar}{2m}\nabla_i\varphi = \frac{\hbar}{2m}(\nabla_i\tilde{\varphi} - q_j\nabla_i u_j) \tag{7}$$

thereby also guaranteeing translational invariance.

The two triads of unit vectors in orbit and spin space, $\mathbf{e}_\nu$ and $\mathbf{e}_j$, do not allow one to identify any preferred direction, neither in spin space nor in orbit space, just as for the ordinary B phase. Neglecting the very small dipole-dipole interaction, the relative orientation of the two triads in equilibrium, described by a rotation matrix between the two frames, $n_{\nu j}$ is not fixed energetically, just as in the bulk B phase. Therefore, deviations $\delta n_{\nu j}$ are the hydrodynamic variables (as in the B phase) and enter the energy density as

$$d\varepsilon^{PBW}_B = \Psi^n_{\nu jk}d\nabla_j n_{\nu k}. \tag{8}$$

Due to the special properties of a rotation matrix, e.g., $n_{\nu i}n_{\mu i} = \delta_{\nu\mu}$ and $n_{\nu i}n_{\nu j} = \delta_{ij}$, the $\delta n_{\nu j}$ contains three independent variables according to the three spontaneously broken rotational symmetries. They can also be parametrized by a unit vector describing the rotation direction plus the rotation angle. $n_{\nu j}$ is even under time reversal and even under spatial inversion.

The three rotation angles of spin space relative to orbit space leave the overall rotation invariance generated by the total angular momentum, $\mathbf{J}$, unbroken.

The small deviations from the equilibrium rotation matrix, $n_{\nu j}$, consistent with the properties of a rotation matrix, can be described by three rotation angles

$$\delta\Theta_\alpha = \tfrac{1}{2}\epsilon_{\alpha\mu\nu}n^0_{\mu j}\delta n_{\nu j}, \tag{9}$$

which are the expectation values of the operators

$$\delta\hat{\Theta}_\alpha(\vec{x}) = \tfrac{3}{4}\epsilon_{\alpha\mu\nu}\big[A^{*0}_{\mu j}\hat{A}_{\nu j}(\vec{x}) + A^0_{\mu j}\hat{A}^+_{\nu j}(\vec{x})\big] \tag{10}$$

with the commutator relations

$$\langle[\delta\hat{\Theta}_\alpha, \hat{L}_\beta]\rangle = -\langle[\delta\hat{\Theta}_\alpha, \hat{S}_\beta]\rangle = -i\hbar\delta_{\alpha\beta} \tag{11}$$

and where

$$\langle[\delta\hat{\Theta}_\alpha, \hat{N}]\rangle = 0 \tag{12}$$

since rotations in spin space do not break gauge invariance.

Thus we have in total four Goldstone modes, three for the relative motion of orbit and spin space and precisely one for the combination of broken gauge symmetry and broken translational symmetry as associated with the hexagonal two-dimensional density waves in the plane of the sample.

III. THE ORBITAL DYNAMICS OF THE PDW PHASE

In the preceding section we have characterized the hydrodynamic variables characteristic of the PDW phase. Here we will derive macroscopic equations for this superfluid phase making use of linear irreversible thermodynamics including the local formulation of the first and second law of thermodynamics as well as symmetry properties. The latter includes the behavior under parity, time reversal, and Galilei transformations, as well as under rotations and translations.

To derive the full set of dynamic equations is then a two-step procedure [46,55]. First one writes down the Gibbs relation, the local formulation of the first law of thermodynamics, for the hydrodynamic variables. This way one defines the thermodynamic conjugate quantities or thermodynamic forces. The thermostatic behavior is then obtained by expanding the generalized energy into the hydrodynamic variables taking into account all symmetry properties. By then taking the (variational) derivative of the generalized energy with respect to the variables one obtains the thermodynamic forces.

In the second step one writes down first the dynamic equations for the three types of variables in macroscopic dynamics: conservation laws for the conserved variables and balance equations for the variables associated with spontaneously broken continuous symmetries and with macroscopic variables, which relax on a long, but finite timescale. These dynamic equations contain currents and quasicurrents associated with the dynamics of the variables. To close the system of equations in the framework of linear irreversible thermodynamics one then expresses the current and quasicurrents in relations linear in the thermodynamic forces. In addition, one splits all currents and quasicurrents into reversible (no entropy generation) and into irreversible contributions [positive entropy (or heat) generation]. The irreversible contributions in the currents and quasicurrents can be derived from a dissipation function, which is an expansion quadratic in the thermodynamic forces, by taking a (variational) derivative with respect to the thermodynamic forces. When splitting the currents and quasicurrents into reversible and irreversible contributions the behavior under time reversal plays a crucial role.

In this section on the derivation of the macroscopic dynamics of the PDW phase we will comment in some detail on the points outlined above. It seems worthwhile to mention that this program has been carried out before for the three superfluid phases arising in ^{3}He in the bulk: the superfluid A phase [33,34,37–39] and the superfluid B phase [35,39], as well as for the superfluid A_1 phase [36,40,41], which arises only in a magnetic field.

Throughout the bulk of this section we focus on the orbital dynamics of the PDW phase and comment on the influence of the spin degrees of freedom and their coupling to the orbital degrees of freedom in Sec. IV.

A. Statics and thermodynamics

To obtain the static properties of our system we formulate the local first law of thermodynamics relating changes in the entropy density σ to changes in the hydrodynamic and macroscopic variables discussed above. According to the discussions in Sec. II, Eqs. (1) and (7), we get the Gibbs relation for the variables acting in orbital space

$$d\varepsilon = T\,d\sigma + \mu\,d\rho + v_i^n dg_i + \lambda_i^s dv_i^s. \tag{13}$$

The thermodynamic conjugates, temperature, T, chemical potential, μ, normal velocity, v_i^n, and λ_i^s, are defined as partial derivatives of the energy density with respect to the appropriate variables [46]. Rotational invariance of the energy requires $v_i^n g_j + \lambda_i^s v_j^s = v_j^n g_i + \lambda_j^s v_i^s$.

Let us list the symmetry properties used: Scalar quantities ε, σ, ρ, and their conjugates, are even under parity and even under time reversal. The (polar) vectors g_i, v_i^s, and their conjugates, as well as q_i, are odd under parity and odd under time reversal.

The spin degrees of freedom will be discussed in Sec. IV below.

The relations between variables and conjugates can be derived from an energy functional, $E = \int \varepsilon\, dV$, which must be invariant under time reversal as well as under parity and, in addition, must be invariant under rigid rotations and rigid translations, and be covariant under Galilei transformations.

We find in harmonic approximation

$$\begin{aligned}\varepsilon = {} & \frac{1}{2}\rho_0\left(\frac{\rho^s}{\rho^n}\right)_{ij} v_i^s v_j^s + \frac{1}{2}\left(\frac{1}{\rho^n}\right)_{ij} g_i g_j - \left(\frac{\rho^s}{\rho^n}\right)_{ij} v_i^s g_j \\ & + \frac{1}{2}c_{\rho\rho}(\delta\rho)^2 + \frac{1}{2}c_{\sigma\sigma}(\delta\sigma)^2 + c_{\sigma\rho}(\delta\sigma)(\delta\rho) \\ & + \bar{q}_i(d^\sigma\delta\sigma + d^\rho\delta\rho)\big(\rho_0 v_i^s - g_i\big)\end{aligned} \tag{14}$$

with $\bar{q}_i \equiv \frac{\hbar}{2m} q_i$

The first line has the standard two-fluid structure of superfluids, where the density material tensors take the uniaxial form $\rho_{ij}^s = \rho_\perp^s(\delta_{ij} - \hat{q}_i\hat{q}_j) + \rho_\parallel^s \hat{q}_i\hat{q}_j$ with $\hat{\mathbf{q}} = \bar{\mathbf{q}}/|\bar{\mathbf{q}}|$ the unit vector along $\bar{\mathbf{q}}$. The second line contains the scalar variables of a simple fluid and the third line shows the coupling between the two different sets of variables. Such a coupling is possible in the PDW phase, since $\bar{q}_i$ is odd under time inversion. It only involves longitudinal components of v_i^s and g_i, with the consequence that the thermodynamic coupling coefficients d^σ and d^ρ are scalars. Such a behavior has not been found before for any of the other superfluid phases of ^{3}He, but has been pointed out for the possible FF state for superfluid ^{3}He-^{4}He solutions by Stern and Liu [8].

A Galilei transformation of Eq. (13) leads to the condition

$$\mathbf{g} = \rho\mathbf{v}^n + \boldsymbol{\lambda}^s, \tag{15}$$

which is already incorporated in the energy density ε, as can be verified by using the explicit expressions for the conjugate quantities

$$\begin{aligned} v_i^n \equiv \left(\frac{\partial\varepsilon}{\partial g_i}\right)_{\dots} = {} & \left(\frac{1}{\rho^n}\right)_{ij} g_j - \left(\frac{\rho^s}{\rho^n}\right)_{ij} v_j^s \\ & -\bar{q}_i(d^\sigma\delta\sigma + d^\rho\delta\rho),\end{aligned} \tag{16}$$

$$\begin{aligned} \lambda_i^s \equiv \left(\frac{\partial\varepsilon}{\partial v_i^s}\right)_{\dots} = {} & \rho_0\left(\frac{\rho^s}{\rho^n}\right)_{ij} v_j^s - \left(\frac{\rho^s}{\rho^n}\right)_{ij} g_j \\ & +\bar{q}_i\rho_0(d^\sigma\delta\sigma + d^\rho\delta\rho),\end{aligned} \tag{17}$$

$$\mu \equiv \left(\frac{\partial\varepsilon}{\partial\rho}\right)_{\dots} = c_{\rho\rho}\delta\rho + c_{\sigma\rho}\delta\sigma + d^\rho\bar{q}_i\big(\rho_0 v_i^s - g_i\big), \tag{18}$$

$$\delta T \equiv \left(\frac{\partial\varepsilon}{\partial\sigma}\right)_{\dots} = c_{\sigma\sigma}\delta\sigma + c_{\sigma\rho}\delta\rho + d^\sigma\bar{q}_i\big(\rho_0 v_i^s - g_i\big). \tag{19}$$

The energy density must be convex and its minimum describes the equilibrium state for the PDW phase. Generally, this imposes some restrictions on the material parameters involved. In particular, $\rho^s_\Delta < \rho_0$, $(d^\rho)^2 < (\rho^s_\Delta/\rho_0)c_{\rho\rho}$, and $(d^\sigma)^2 < (\rho^s_\Delta/\rho_0)c_{\sigma\sigma}$ for $\Delta \in \{\perp, \parallel\}$ guarantees thermodynamic stability.

B. Dynamic equations and reversible contributions

To determine the dynamics of the variables in orbit space we take into account that the first class of our set of variables, the conserved quantities, obey a local conservation law, while the dynamics of the other two classes of variables can be described by a simple balance equation, where the counter term to the temporal change of the quantity is called a quasicurrent. For the set of linearized dynamical equations we get

$$\dot\rho + \nabla_i\big(\rho_0 v^n_i + \lambda^s_i\big) = 0, \tag{20}$$

$$\dot\sigma + \sigma_0\nabla_i v^n_i + \nabla_i j^{\sigma D}_i = \frac{2R}{T}, \tag{21}$$

$$\dot g_i + \nabla_i p + \bar q_i \nabla_j \lambda^s_j + \nabla_j \sigma^D_{ij} = 0, \tag{22}$$

$$\dot{v}^s_i + \nabla_i\mu + \bar q_j \nabla_i v^n_j + \nabla_i I^D_\varphi = 0 \tag{23}$$

with $\bar q_i \equiv \frac{\hbar}{2m} q_i$.

With the help of Eq. (15) it is easy to see that the momentum density, g_i, also acts as the density current in Eq. (20), as it should. The pressure p in Eq. (22) is given by $-\partial E/\partial V$, with E the total energy (cf. Ref. [46]), and reads for our system

$$dp = \sigma dT + \rho d\mu. \tag{24}$$

The entropy production, R/T, with R the dissipation function, acts as a source term in Eq. (21). The dissipative parts of the currents (with superscript D) and R will be determined in Sec. III C.

Those parts of the currents that have the same behavior under time reversal as the time derivative of the appropriate variables are reversible. In particular, $\dot\rho$ and $\dot\sigma$ are odd, while $\dot g_i$ and $\dot v_s$ are even under time reversal. According to the second law of thermodynamics, reversible processes must not increase the entropy, meaning $R = 0$ and σ being a conserved quantity.

The reversible part of Eq. (23) follows from taking into account $\frac{\hbar}{2m}\dot\varphi + \mu = 0$ [32,53] and $\dot u_i = v^n_i$, according to translational invariance.

Using the Gibbs relation, Eq. (13), it is obvious that the contributions $\nabla_i\mu$ in Eq. (23) and λ^s_i in Eq. (20) compensate each other to give $R = 0$. Similarly, the contribution $\bar q_j\nabla_i v^n_j$ in Eq. (23) requires the contribution $\bar q_i \nabla_j\lambda^s_j$ in Eq. (22), due to $R = 0$. The latter occurrence of λ^s_i signals that the superfluid broken symmetry in the PDW phase not only comprises the phase variable but also a translation. Note that the reversible contributions $\nabla_i(\rho_0 v^n_i)$ in Eq. (20) and $\nabla_i p$ in Eq. (22) are compensated by all the other transport contributions (not shown in the linearized dynamic equations) [46].

We first discuss the structure of the propagating modes, first and second sound $\sim\exp(i\omega t)\exp(i\mathbf{k}\cdot\mathbf{r})$ to lowest order $\omega \sim k$, disregarding their damping with $\omega \sim ik^2$ contained in $j^{\sigma D}_i$, σ^D_{ij}, and I^D_φ. Dealing with second sound it is customary to switch from the entropy density σ to the entropy per mass density $s \equiv \sigma/\rho$ as variable, leading to the linearized dynamic equation

$$\rho_0\dot s - s_0\nabla_i\lambda^s_i = 0. \tag{25}$$

Starting with first sound we take the time derivative of Eq. (20) and together with Eqs. (22) and(25) we get

$$\ddot\rho = \Delta p + \frac{s_0}{\rho_0}\bar q \nabla_\parallel \dot s \tag{26}$$

with $\bar q \equiv |\bar{\mathbf{q}}|$ and $\nabla_\parallel = \hat q_i\nabla_i$. For $\bar q = 0$, the well-known first sound velocity $c^2_{10} = (\partial p/\partial\rho)_s$ is found, when, as usual, the conventional thermodynamic static cross coupling $(\partial p/\partial s)_\rho$ is neglected. The existence of the $\bar q$ term signals the (anisotropic) coupling to second sound due to the broken relative gauge/translational symmetry.

For second sound we take the time derivative of Eq. (25) and express $\dot\lambda^s_i$ via Eq. (17) by the dynamic equations for g_i and v^s_i. For the moment we neglect the static couplings due to the existence of $\bar q$ [by taking $d^\rho = 0 = d^\sigma$ in Eqs. (16)–(19)] with the result

$$\ddot s = \left(\frac{\rho^s}{\rho^n}\right)_{ij} \nabla_i\nabla_j\left(\frac{s_0^2}{\rho_0^2}T - s_0\bar q\, v^n_\parallel\right) - \frac{\rho^s_\parallel}{\rho^n_\parallel}\bar q\,\nabla_\parallel \dot s. \tag{27}$$

Without the direction $\bar q_i$ the well-known (isotropic) second sound velocity $c^2_{20} = (\rho^s/\rho^n)(s_0^2/\rho_0^2)(\partial T/\partial s)_\rho$ is found, when, as usual, $(\partial T/\partial\rho)_\sigma$ is neglected. The existence of $\bar q_i$ renders second sound to be anisotropic due to the anisotropy of the superfluid densities. In addition, there is a contribution $\sim\dot s$ and a coupling to $v^n_\parallel$. The former is responsible for the fact that the second sound velocity along the direction of $\bar q_i$ is different from that of the opposite direction (along $-\bar q_i$). This situation resembles externally driven systems, although in the present case the velocity $\bar q_i$ is intrinsic.

The coupling to $v^n_\parallel$ requires the use of an additional dynamic equation for $v^n_\parallel$, which is obtained by taking the time derivative of Eq. (16) and employing Eqs. (22), (23), and (25) with the result

$$\rho^n_\parallel \dot v^n_\parallel = -\frac{\rho^n_\parallel}{\rho_0}\nabla_\parallel p - \frac{s_0\rho^s_\parallel}{\rho_0^2}\nabla_\parallel T - \frac{\rho_0}{s_0}\bar q\,\dot s + \rho^s_\parallel \bar q\,\nabla_\parallel v^n_\parallel. \tag{28}$$

The variable $v^n_\parallel$ is coupled to ρ (via p) and to s (via T) and directly to $\dot s$. Thus, all three variables are coupled, although there is no coupling to ρ in Eq. (27), nor to $v^n_\parallel$ in Eq. (26). As a result, first and second sound and $\nabla_\parallel v^n_\parallel$ are generally coupled excitations on the $\omega \sim k$ level.

For waves traveling perpendicular to q_i, there is $\nabla_\parallel = 0$ and first sound is decoupled and unaffected by the broken relative gauge/translational symmetry. Second sound velocity acquires a $\bar q^2$ contribution

$$\omega^2 = \frac{\rho^s_\perp}{\rho^n_\perp}\left(\frac{s_0^2}{\rho_0^2}(\partial T/\partial s)_\rho + \frac{\rho_0}{\rho_\parallel}\bar q^2\right)k^2_\perp \tag{29}$$

and the variable $v^n_\parallel$ is not independent, since $s_0\rho^n_\parallel \dot v^n_\parallel = -\rho_0\bar q\dot s$.

The solution of the dispersion relation for general wave directions cannot be given in closed form. An approximate formula for small $\bar q$ shows that first sound is affected by the dynamic couplings due to the broken gauge/translational symmetry to order $\bar q^2$, while second sound acquires contributions linear in $\bar q$.

Up to this point we have neglected the static couplings d^ρ and d^σ in Eqs. (16)–(19). The main effect of these contributions is a considerable increase of complexity of the dispersion relation, but generally no novel effects are introduced compared to the dynamic couplings. An exception is fourth sound.

Fourth sound is found in superfluid systems, when the normal velocity is clamped ($v_i^n = 0$), e.g., in narrow pores or very thin capillaries. There is $g_i = \lambda_i^s$ and momentum conservation, Eq. (22), does not hold. As a result, we get in our case the dynamic equations $\dot\rho + \nabla_i \lambda_i^s = 0$, $\dot\sigma = 0$, and $\dot v_i^s + \nabla_i \mu = 0$ that do not show any influence of the broken gauge/translational symmetry. Of the static Eqs. (16)–(18) that now read $\lambda_i^s = \rho_{ij}^s v_j^s$ and $\delta\mu = c_{\rho\rho}\delta\rho + d^\rho \bar q \rho_\parallel^n v_\parallel^s$, only the second one contains the static coupling d^ρ.

Fourth sound, given by

$$\ddot\rho - \rho_{ij}^s \nabla_i \nabla_j \mu = 0, \tag{30}$$

is the sole propagating excitation with $\omega \sim k$ for $v_i^n = 0$. Without d^ρ the well-known anisotropic fourth sound velocity is found:

$$c_f^2 = c_{\rho\rho}(\rho_\perp^s k_\perp^2 + \rho_\parallel^s k_\parallel^2)/k^2 \tag{31}$$

with $c_{\rho\rho} = (\partial\mu/\partial\rho)_\rho$. It describes a coupled excitation of density fluctuations and $\nabla_i v_i^s$. The full dispersion relation reads

$$\omega^2 = c_f^2 k^2 + \frac{c_{\rho\rho} d^\rho \bar q \rho_\parallel^n k_\parallel}{\omega + d^\rho \bar q \rho_\parallel^n k_\parallel} \tag{32}$$

and involves density fluctuations, $\nabla_i v_i^s$ and $\nabla_\parallel v_\parallel^s$. Fourth sound allows one to detect the static cross couplings, since there are no dynamic ones.

C. Irreversible dynamics and entropy production

In Eqs. (20)–(23) we have already incorporated the reversible contributions within the framework of linearized macroscopic dynamics, which leaves only the dissipative currents (superscript D) and the entropy production R to be determined. The dissipative parts of the currents have the opposite sign under time reversal as the time derivatives of the variables. According to the second law of thermodynamics for irreversible processes, $R > 0$ is required. With the help of the full set of dynamic equations the Gibbs relation, Eq. (13), leads to an expression bilinear in the currents and thermodynamic conjugates,

$$2R = -j_i^{\sigma D}\nabla_i T - \sigma_{ij}^D A_{ij} - I_\phi^D \nabla_i \lambda_i^s > 0, \tag{33}$$

where $A_{ij} = (1/2)(\nabla_i v_j^n + \nabla_j v_i^n)$.

We can use the dissipation function R as a Lyapunov functional to derive the irreversible currents and quasicurrents. This automatically includes the famous reciprocity rules for dissipative cross couplings [55]. One can expand the function R into the thermodynamic forces using the same symmetry arguments as in the case of the energy density. We obtain

$$\begin{aligned} R = {} & \tfrac12 \kappa_{ij}(\nabla_i T)(\nabla_j T) + \tfrac12 \nu_{ijkl} A_{ij} A_{kl} + \zeta(\nabla_i \lambda_i^s)(\nabla_j \lambda_j^s) \\ & + \zeta_{ij}^n A_{ij}(\nabla_k \lambda_k^s) + \Sigma_{ijk}^D A_{jk}(\nabla_i T) + \hat q_k \Psi^D (\nabla_j \lambda_j^s)(\nabla_k T). \end{aligned} \tag{34}$$

The second rank tensors κ_{ij} and ζ_{ij}^n take the form

$$\alpha_{ij} = \alpha_\parallel \hat q_i \hat q_j + \alpha_\perp \delta_{ij}^\perp \quad \text{with} \quad \delta_{ij}^\perp = \delta_{ij} - \hat q_i \hat q_j \tag{35}$$

and the viscosity tensor ν_{ijkl} has the standard uniaxial form containing five viscosities, like in a nematic liquid crystal [46] with $\hat q_i$ replacing the nematic director. The second contribution in the second line of Eq. (34) requires an odd number of $\hat q_i$ factors resulting in

$$\Sigma_{ijk}^D = \Sigma_1^D(\hat q_k \delta_{ij}^\perp + \hat q_j \delta_{ik}^\perp) + \Sigma_2^D \hat q_i \delta_{jk}^\perp + \Sigma_3^D \hat q_i \hat q_j \hat q_k. \tag{36}$$

The condition $R > 0$ requires $\kappa_\parallel \zeta > (\Psi^D)^2$, $\kappa_\parallel \nu_3 > (\Sigma_3^D)^2$, $\kappa_\parallel \nu_1 > (\Sigma_2^D)^2$, and $\kappa_\perp \nu_5 > (\Sigma_1^D + \Sigma_2^D)^2$ with the viscosity components defined as the elastic moduli in Ref. [56].

To obtain the dissipative parts of the currents and quasicurrents we take the partial derivatives with respect to the appropriate thermodynamic force

$$j_i^{\sigma D} = -\left(\frac{\partial R}{\partial(\nabla_i T)}\right)_{\dots} = -\kappa_{ij}\nabla_j T - \Sigma_{ijk}^D A_{jk} - \hat q_i \Psi^D \nabla_k \lambda_k^s, \tag{37}$$

$$\sigma_{ij}^D = -\left(\frac{\partial R}{\partial A_{ij}}\right)_{\dots} = -\nu_{ijkl} A_{kl} - \zeta_{ij}^n \nabla_k \lambda_k^s - \Sigma_{kij}^D \nabla_k T, \tag{38}$$

$$I_\varphi^D = -\left(\frac{\partial R}{\partial(\nabla_k \lambda_k^s)}\right)_{\dots} = -\zeta \nabla_k \lambda_k^s - \zeta_{ij}^n A_{ij} - \hat q_k \Psi^D \nabla_k T. \tag{39}$$

The broken gauge/translation symmetry allows the existence of dissipative cross couplings of the heat current with the stress tensor as well as with the phase current.

Inspecting the time-reversal behavior of the dissipative currents we can verify that all contributions have the opposite sign under time reversal as the corresponding time derivative of the associated variable. Looking at the heat conduction term $\sim\kappa_{ij}\nabla_j T$ and the viscous term $\sim\nu_{ijkl}A_{kl}$ this behavior for the heat current and the stress tensor is already familiar from the hydrodynamics of a simple fluid.

IV. MACROSCOPIC SPIN DYNAMICS OF THE PDW PHASE

According to the discussion in Sec. II there are additional degrees of freedom: The magnetization s_ν, and the rotation matrix $n_{\nu j}$ describing (three) relative rotations between spin and orbit space. Neglecting the small spin-orbit coupling, the rotations are Goldstone modes. Spin-orbit coupling will be treated perturbatively [35], at the end of this section. The spin dynamics of the PDW phase is somewhat similar to that of the PDB phase [42]. Therefore we can be brief here.

Therefore, the appropriate Gibbs relations, Eqs. (2) and (8), read

$$d\varepsilon = \Psi_{\nu jk}^n d\nabla_j n_{\nu k} + h_\nu ds_\nu \tag{40}$$

with the conjugate quantities

$$h_\nu = (1/\chi)_{\nu\mu} s_\mu, \tag{41}$$

$$\Psi_{\nu ik}^n = M_{\nu\mu ijkl}\nabla_j n_{\mu l} \tag{42}$$

that do not show any cross coupling, in particular none with orbit space variables.

The preferred direction in orbit space $\bar{q}_i$ is manifest in spin space as $\bar{q}_\nu = n_{\nu i}\bar{q}_i$ and leads to the uniaxial form of second-rank tensors

$$(1/\chi)_{\nu\mu} = \chi_\perp^{-1}(\delta_{\nu\mu} - \hat{q}_\nu\hat{q}_\mu) + \chi_\parallel^{-1}\hat{q}_\nu\hat{q}_\mu. \tag{43}$$

The material tensor $M_{\nu\mu ijkl}$ contains five moduli $M_1 \dots M_5$. It is restricted by the fact that only $\nabla_j \Psi_{\nu jk}$ enters the dynamics and that $n_{\mu l}$ behaves as a rotation matrix (as does $\Psi_{\nu jk}$ with respect to ν and k). The final form is given in the Appendix, Eq. (A1).

The dynamic equations

$$\dot{s}_\nu + \nabla_k J_{\nu k} = 0, \tag{44}$$

$$\dot{n}_{\nu i} + Z_{\nu i} = 0 \tag{45}$$

have the form of a conservation law and a balance equation, respectively. According to general spin dynamics the reversible parts of the currents are [35]

$$J^R_{\nu i} = \gamma\epsilon_{\nu\mu\lambda}n^0_{\lambda j}\Psi^n_{\mu ij}, \tag{46}$$

$$Z^R_{\nu i} = \gamma\epsilon_{\nu\mu\lambda}n^0_{\mu i}h_\lambda \tag{47}$$

and describe a cross coupling between the two degrees of freedom due to the gyromagnetic ratio γ.

Dissipative dynamics enters only by higher order gradient terms, which we will not consider here in detail.

The resulting phonon-type modes are anisotropic, which becomes apparent by the occurrence of $k_\parallel^2 = \hat{q}_\nu\hat{q}_\mu\nabla_\nu\nabla_\mu$. In particular, the mode involving transverse spin excitations $(\bar{q}_\nu\epsilon_{\nu\mu\lambda}\nabla_\mu s_\lambda)$ has the dispersion relation

$$\omega_1^2 = \frac{\gamma^2}{\chi_\perp}\left(E_1k_\perp^2 + E_2k_\parallel^2\right), \tag{48}$$

where $E_{1,2}$ are given in the Appendix, Eqs. (A2) and (A3).

In addition, the other two modes, the longitudinal one (involving $\nabla_\nu s_\nu$) and a second transverse one (involving $\hat{q}_\nu s_\nu$), are coupled. The dispersion relations are found as the solutions of the quadratic algebraic equation

$$\left(\omega_{2,3}^2 - \frac{\gamma^2}{\chi_\parallel}A\right)\left(\omega_{2,3}^2 - \frac{\gamma^2}{\chi_\perp}F\right) + \frac{\gamma^4}{\chi_\perp^2}Ck_\parallel^2 = 0, \tag{49}$$

where the functions A, F, C are all of the form $A_1k_\perp^2 + A_2k_\parallel^2$ [for details cf. Eqs. (A4)–(A6)]. Obviously, the coupling of these two modes is due to the anisotropy and vanishes in the isotropic case, e.g., in the superfluid B phase of ^{3}He.

Taking into account spin-orbit coupling the relative orientation of spin and orbit spaces is no longer arbitrary, but fixed in equilibrium. Therefore, a distinction between greek and latin subscripts is no longer necessary. A general rotation matrix

$$n_{ij} = \cos\vartheta(\delta_{ij} - d_id_j) + d_id_j + \sin\vartheta\,\epsilon_{ijk}d_k \tag{50}$$

contains the rotation angle ϑ and the direction of the rotation axis d_i. First, the spin-orbit coupling fixes the equilibrium value $\vartheta_0 = \cos^{-1}(1/4)$ [19]. The energy for deviations from equilibrium

$$\varepsilon_{so} = \frac{B}{2}(\vartheta - \vartheta_0)^2 = \frac{B}{8\sin^2\vartheta_0}\left(n_{ii} - n^0_{ii}\right)^2 \tag{51}$$

is rather small. It is customary to derive spin hydrodynamics under the full threefold broken symmetry, and add ε_{so} at the end [35].

Second, the preferred direction in orbit space, $\bar{q}_i$, has to be the rotation axis and in equilibrium $d^0_i = \bar{q}_i$. Thus, there is a finite energy for deviations $\delta d_i = d_i - d^0_i$,

$$\varepsilon_{\text{sod}} = \frac{D}{2}(\delta d_i)^2 = \frac{D}{8\sin^2\vartheta_0}[\bar{q}_j(n_{ji} - n_{ij})]^2 \tag{52}$$

with $D > 0$. The fact that $\bar{q}_i$ and, thus, d_i are time-reversal negative, has no effect for the static considerations here.

The spin-orbit energy ε_{so}, Eq. (50), results in a gap in the longitudinal spin-wave branch

$$\omega_2^2(k \to 0) = \frac{\gamma^2}{\chi_\parallel}B \equiv \tilde{B} \tag{53}$$

that is manifest in NMR as the longitudinal shift. This mode is coupled to one of the transverse modes (ω_3) even without spin-orbit coupling.

Finally, the energy ε_{sod}, Eq. (52), gives rise to gaps in the transverse spin modes ($\epsilon_{ijk}\hat{q}_js_k$) of the form

$$\omega_{1,3}^2(k \to 0) = \frac{\gamma^2}{\chi_\perp}\frac{1 + \cos\vartheta_0}{2\sin^2\vartheta_0}D \equiv \tilde{D}. \tag{54}$$

There is no spontaneously broken rotational symmetry left and all modes acquire a gap. In addition, all three spin-wave modes are coupled.

Such a situation is found in the (isotropic) B phase of ^{3}He only after having applied an external magnetic field. In the PDW phase an external field, H_i, will not induce qualitatively new features to the spin-wave modes. In particular, the gaps of the modes are then given, for a field parallel to $\bar{q}_i$, by $\omega_1^2 = \tilde{D}$ and $\omega_3^2 = \tilde{D} + \omega_L^2$ and, for a transverse field by $\omega_2^2 = \tilde{B} + \omega_L^2$, where $\omega_L = \gamma H$ is the Larmor frequency.

V. SUMMARY AND PERSPECTIVE

In this paper we have studied the macroscopic behavior of the spatially modulated PDW state of superfluid ^{3}He observed experimentally in thin layers. Making use of the available experimental results and microscopic models, we have used the spatially modulated phase of the macroscopic wave function as an order parameter. This leads to the remarkable result that only one Goldstone mode in orbit space exists, coupling the superfluid aspects to the spatial modulations of the order parameter in the plane of the sample. We find that this type of Goldstone mode leads to a propagating mode sharing aspects of second sound and acoustic waves arising from the spatial variations of the order parameter.

In contrast to all other superfluid systems studied so far, we observe that even to lowest order in the wave vector first and second sound no longer decouple from the component of the momentum density parallel to the preferred direction in the plane, **q**. In addition, due to the coupling of the phase of the order parameter to in-plane spatial modulations, even the velocity of first sound becomes anisotropic. We also find that the velocities of first and second sound contain both, the static and dynamic coupling terms to the order parameter. For clamped normal fluid, we find that for fourth sound only one static cross coupling associated with density variations enters the velocity. Therefore we predict that measurements of the fourth sound velocity could be used to measure the influence of this static cross-coupling term.

The analysis of the structure of sound modes, in particular of the velocity of first, second, and fourth sound, might very well serve as a key instrument to identify experimentally the presence of a FF phase. This question has been a central issue in the field for a number of years (cf. Ref. [57] for a review of this challenge). Along these lines the investigation of collective modes and the speed of sound has been studied in detail about a decade ago using more microscopic techniques. As an example we mention Ref. [58] where the Kadanoff and Baym method using field-theoretical Green's functions has been applied to study the FFLO state. It was shown in Ref. [58] that the velocities of sound waves in two spatial dimensions become anisotropic for the FFLO state. Here we have demonstrated in the truly hydrodynamic limit in three dimensions, using an approach combining symmetry arguments and irreversible thermodynamics, that the velocities of first, second, and fourth sound become anisotropic due to the order parameter of the FF state selecting a preferred direction. Thus the analysis presented here can serve as a powerful tool to identify experimentally the presence of the long sought after FF state. Finally we emphasize the unique feature of fourth sound velocity becoming anisotropic exclusively due to a static cross coupling associated with density variations.

We have also discussed an important difference to the supersolid phase anticipated for solid ^{4}He. In the latter case the order parameter is rather different from the one studied here for the superfluid phase of ^{3}He for a spatially modulated pair density wave. In particular, for the superfluid solid phase in ^{4}He a displacement field is arising as an independent variable.

As for spin space we obtain, among other results, three pairs of propagating spin waves provided the magnetic dipole interaction and external fields are neglected. These results resemble quite closely those for the PDB phase.

Our results thus indicate a combination of certain aspects of a FF-type state with two-dimensional in-plane order parameter variations of square or hexagonal symmetry with various types of spin waves showing some similarities to those found in the distorted superfluid ^{3}He-B.

As a perspective it seems important to achieve experimentally large enough monodomains of the new superfluid phase to identify the symmetry of the order parameter structure in the planes of the samples. This would also facilitate study on the influence of the static cross coupling on the velocity of fourth sound.

Although the preferred direction of the aerogel enters the spin space dynamics, when the spin-orbit coupling is taken into account, the elastic deformations of the aerogel do not.

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APPENDIX: SPIN-WAVE VELOCITIES

The material tensor $M_{\nu\mu ijkl}$ introduced in Eq. (42) reads [42]

$$\begin{aligned} M_{\nu\mu ijkl} = & M_1\delta_{\nu\mu}\delta_{ij}\delta_{kl} + M_3\hat{q}_\nu\hat{q}_\mu\delta_{ij}\delta_{kl} \\ & + M_4\delta_{\nu\mu}\hat{q}_i\hat{q}_j\delta_{kl} + M_5\hat{q}_\nu\hat{q}_\mu\hat{q}_i\hat{q}_j\delta_{kl} \\ & + M_2(\epsilon_{kip}\epsilon_{jlt} + \epsilon_{kjp}\epsilon_{ilt})n^0_{\nu p}n^0_{\mu t}. \end{aligned} \tag{A1}$$

The abbreviations introduced in the dispersion relations for the spin waves, Eqs. (48) and (49), are given by [42]

$$E_1 = 2M_1 + M_3, \tag{A2}$$

$$E_2 = 2M_4 + M_5 \tag{A3}$$

and

$$A = 2\left(M_1k^2 + M_4k_\parallel^2\right) + 8\left(1 - \frac{\chi_\parallel}{\chi_\perp}\right)M_2k^2, \tag{A4}$$

$$F = (2M_1 + M_3 + 8M_2)k^2 + (2M_4 + M_5)k_\parallel^2, \tag{A5}$$

$$\begin{aligned} C = & 8M_2\left([8M_2 + M_3]k^2 + M_5k_\parallel^2 + 2\left(\frac{\chi_\parallel}{\chi_\perp} - 1\right)\right. \\ & \left.\times\left(M_1k^2 - M_4k_\parallel^2\right)\right). \end{aligned} \tag{A6}$$

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