Condensation of elastic pseudogauge fields and quantized response in the condensates

Andreas Sinner [1,*] and Zeinab Rashidian [2]

[1] *Institute of Physics, University of Opole, 45-052 Opole, Poland*

[2] *Department of Physics, Lorestan University, Khoramabad, 68151-44316, Iran*

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The parallels between the models of the weakly interacting Bose gas and elastic pseudogauge fields coupled to the massive Dirac fermions in two spatial dimensions reveal an intricate property of the Berry term in the former as compared to the Chern-Simons term of the latter. Guided by the Bogoliubov theory of superfluidity, we discuss the measurable consequences for the Berry term, which manifest themselves in a quantized response in any spatial dimension and distinguishes between both time directions. We argue that the same basic principles lay behind the topological *d*-wave superconductivity in two spatial dimensions. The same ideas applied to the model with elastic pseudogauge fields reveal a plethora of possible condensed phases, both with preserved and violated spatial isotropy and, consequently, different topological responses. The ground state of these phases is degenerate, hence all of them are equally probable. The spectral anisotropy raises parallels to the Lifshitz transitions, which occur due to static strain.

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I. INTRODUCTION

The model of weakly interacting nonrelativistic bosons is arguably one of the most thoroughly studied theoretical models [1–3] and one may ask why further studies are worth pursuing. A hand-waving way to answer it is to point to the case of its relativistic cousin, the ϕ^4 model, which despite several decades of the most intense research still remains the first choice model when new ideas have to be developed or examined. Here one can think of both the classical work [4,5] or much more recent studies [6,7]. As with the relativistic case, the model of the weakly interacted Bose gas still deserves attention, since it suggests numerous parallels to other physical systems and reveals quite unexpected properties, which might be helpful for better understanding of the interconnection between established theories with seemingly very different origins.

Intriguing aspects of the physics of the weakly interacting Bose gas model might be guessed from two of its fundamental properties: (1) The instability of the ground state, which leads to the emergence of the off-diagonal long-range order, usually attributed to the Bose-Einstein condensation of interacting bosons and (2) its original manifest Galilean invariance. Because of the first property, the low-energy excitations in the condensed phase are gapless sound waves with linear dispersion, propagating through the condensate. In this regard, the original Galilean invariance of the model is broken but leaves traces of its existence behind, namely, the time derivative in the Lagrangian connects the gapped with the gapless modes and forms the so-called Berry term [8]. This term is antisymmetric and intriguingly similar to the Chern-Simons term known from the 2+1-dimensional electrodynamics taken in a particular gauge.

If this similarity is noted, then the natural questions to ask are: (1) Can one measure this term and, if so, does it yield a quantized response, and what does this quantization mean in concrete terms? (2) Can one obtain a model in 2+1 dimensions, which shares several properties of the weakly interacting bosons, starting with the system of massive Dirac fermion coupled to a gauge field in a particular gauge? We know that this model is prototypical for the topological field theories [9,10] and quantum Hall effects [11]. If so, what is the nature and the features of the condensation of the respective bosonic fields?

In this paper, we are doing just that. It is indeed possible to construct the correlation function which measures only the Berry term in the model of weakly interacting Bose gas. Mainly because of the gapless spectrum in the condensed phase, the result indeed turns out to be quantized, taking only values ± 1, irrespective of the space-time dimension. We argue that this exact quantization is actually the consequence of the coexistence of collective modes propagating forwards (advanced) and backwards (retarded) in time and that it is indeed possible to distinguish both in a measurement. Then, constructing the effective interacting Chern-Simons theory from the initial fermionic model, we recognize term by term all ingredients of the bosonic model, only with a different but still gapless dispersion. The interaction term appears due to the finite bandwidth of the lattice fermions and only disappears if this is taken to infinity. The condensation of the gauge fields stabilizes the model. While it makes little sense to speculate on the condensation of the gauge fields related to the electromagnetism in the electrodynamics, it is possible to associate them with the pseudogauge fields of the elastic fields [12] or phonons [13–17]. We identify several distinct condensed phases of the pseudogauge fields and corresponding

*andreas.sinner@uni.opole.pl

symmetry-breaking patterns. The quantized response of the Chern-Simons term in analogy to the Berry term in the model of the weakly interacting Bose gas turns out to be possible if the condensation of the pseudogauge fields occur in one particular spatial direction. In this case, the spectrum of collective excitations reveals a high degree of anisotropy specific to the Lifshitz transitions observed on honeycomb lattices under constant strain [18–21]. As in the case of weakly interacting bosons, the deeper meaning of the quantized response is our ability to distinguish between both time directions.

Structurally, the paper is organized as follows: In Sec. II, we recapitulate the model of the weakly interacting Bose gases in the usual framework of the Bogoliubov approach. We introduce the correlator which detects the presence of the Berry term in the Lagrangian and carry out the explicit evaluation. In Sec. III, we start out searching for the similar physics as suggested by the model of weakly interacting Bose gas. We consider the system of massive Dirac electrons in 2D coupled to the effective or pseudogauge fields and carry out asymptotic expansions in powers of inverse quasiparticle mass. Different symmetry-breaking patterns, i.e., the condensation mechanisms for the pseudogauge fields are considered in Sec. IV. In Sec. V, we discuss the result of the Berry term correlation for each symmetry-breaking possibility and compare it with the respective results from the Kubo formula and discuss the consequences in the Conclusions, Sec. VI.

II. QUANTIZED CORRELATIONS OF THE BERRY TERM IN THE MODEL OF WEAKLY INTERACTING BOSE GAS

We start with the consideration of the action of weakly interacting Bose gas in continuum in 2+1d space-time. In the notation we adopt, it reads ($\hbar = 1$), cf. Appendix A,

$$\mathcal{S} = \bar{\varphi} \cdot \left[\partial_\tau - \frac{\nabla^2}{2m} - \mu\right]\varphi + \frac{g}{2}(\bar{\varphi}\cdot\varphi)^2, \tag{1}$$

where the three parameters of the model, the boson mass m, the chemical potential μ, and the interaction strength g are all positive quantities. The effect we are interested in is easier to understand in the real field representation. To do this, we introduce the two-component Bogoliubov bosonic spinors. Then, by a simple reordering of all terms, we get

$$\mathcal{S} = \frac{1}{2}\begin{pmatrix}\bar{\varphi}\\ \varphi\end{pmatrix}^{\mathrm{T}} \cdot \begin{pmatrix}\partial_\tau - \frac{\nabla^2}{2m} - \mu & 0\\ 0 & -\partial_\tau - \frac{\nabla^2}{2m} - \mu\end{pmatrix}\begin{pmatrix}\varphi\\ \bar{\varphi}\end{pmatrix} + \frac{g}{8}\left[\begin{pmatrix}\bar{\varphi}\\ \varphi\end{pmatrix}^{\mathrm{T}} \cdot \begin{pmatrix}\varphi\\ \bar{\varphi}\end{pmatrix}\right]^2. \tag{2}$$

The Bogoliubov spinors are introduced to capture the emergence of the off-diagonal long-range order [22] that appears in form of the anomalous $\varphi\varphi$ or $\bar{\varphi}\bar{\varphi}$ terms, which violate the particle number conservation. The sign change in front of the time derivative in the quadratic part is a consequence of the partial integration and bosonic statistics of the fields φ, which is the main difference in the fermionic counterpart, the Bogoliubov-deGennes model of superconductors [23,24]. Ultimately, this sign difference is due to the Galilean invariance of the model Eq. (1) and does not occur in this form in relativistic ϕ^4 models, as it does not for the kinetic energy $\nabla^2/2m$. Formally, both channels in the quadratic part of the action, i.e., $\bar{\varphi}\varphi$ and $\varphi\bar{\varphi}$, can be identified as channels with opposite (forward and backward) time directions. It is possible to imagine a measurement which distinguishes between both time directions. Introducing the real valued fields occurs by rotating the Bogoliubov spinors. In this way, we map the $U(1)$-invariant action Eq. (1) on the $O(2)$-invariant action of the real-valued longitudinal (A_1) and transversal (A_2) fields [13,25]:

$$\begin{pmatrix}\bar{\varphi}\\ \varphi\end{pmatrix}^{\mathrm{T}} = \begin{pmatrix}A_1\\ A_2\end{pmatrix}^{\mathrm{T}}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ i & -i\end{pmatrix},\quad \begin{pmatrix}\varphi\\ \bar{\varphi}\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -i\\ 1 & i\end{pmatrix}\begin{pmatrix}A_1\\ A_2\end{pmatrix}. \tag{3}$$

The action Eq. (1) changes to

$$\mathcal{S} = \frac{1}{2}\begin{pmatrix}A_1\\ A_2\end{pmatrix}^{\mathrm{T}} \cdot \begin{pmatrix}-\frac{\nabla^2}{2m} - \mu & -i\partial_\tau\\ i\partial_\tau & -\frac{\nabla^2}{2m} - \mu\end{pmatrix}\begin{pmatrix}A_1\\ A_2\end{pmatrix} + \frac{g}{8}\left[\begin{pmatrix}A_1\\ A_2\end{pmatrix}^{\mathrm{T}} \cdot \begin{pmatrix}A_1\\ A_2\end{pmatrix}\right]^2. \tag{4}$$

Introducing the vector notation $\vec{A} = (A_1, A_2)^{\mathrm{T}}$ and performing a partial integration in the kinetic energy term, we further simplify this notation to

$$\mathcal{S} = -\frac{i}{2}\epsilon_{\alpha\beta}A_\alpha \cdot \partial_\tau A_\beta + \frac{1}{4m}(\partial_\alpha\vec{A})\cdot(\partial_\alpha\vec{A}) - \frac{\mu}{2}\vec{A}\cdot\vec{A} + \frac{g}{8}(\vec{A}\cdot\vec{A})^2, \tag{5}$$

where $\epsilon_{\alpha\beta}$ represents the 2D totally antisymmetric tensor. The first term is sometimes referred to as the Berry term [8] and is similar in appearance to the topological Chern-Simons term of the 2+1D electrodynamics taken in a special gauge [9,10]. The Berry term has this form in any spatial dimension, though. The potential term of the action

$$V_{\mathrm{eff}}[\vec{A}] = \frac{\mu}{2}\vec{A}\cdot\vec{A} - \frac{g}{8}(\vec{A}\cdot\vec{A})^2 \tag{6}$$

has a typical double-well form, characteristic for massive particles. For positive μ and g, the functional integral with action Eq. (5) does not converge. The convergence issue is cured if one assumes that the longitudinal field component acquires a nonvanishing vacuum expectation value $A_1 \to \rho + A_1$, cf. Appendix A. Lengthy calculations lead to the usual Bogoliubov action, with the quadratic part

$$\mathcal{S}_{\mathrm{eff}} = \frac{1}{2}\begin{pmatrix}A_1\\ A_2\end{pmatrix}^{\mathrm{T}} \cdot \begin{pmatrix}-\frac{\nabla^2}{2m} + 2\mu & -i\partial_\tau\\ i\partial_\tau & -\frac{\nabla^2}{2m}\end{pmatrix}\begin{pmatrix}A_1\\ A_2\end{pmatrix}. \tag{7}$$

The determinant of the matrix (i.e., the product of the eigenvalues) is non-negative and hence the functional integral converges. From Eq. (7), the Green's function follows:

$$G(X, X') = \int \frac{d^3Q}{(2\pi)^3}\, e^{-iQ\cdot(X-X')}\, G(Q),\quad G(Q) = \frac{2}{q_0^2 + E_{\mathrm{Bog}}^2(q)}\begin{pmatrix}\frac{q^2}{2m} & -q_0\\ q_0 & \frac{q^2}{2m} + 2\mu\end{pmatrix}, \tag{8}$$

where

$$E_{\rm Bog} = \sqrt{\frac{q^2}{2m}\left(\frac{q^2}{2m} + 2\mu\right)} \tag{9}$$

represents the Bogoliubov spectrum. The Bogoliubov spectrum is gapless, which is crucially important for the further discussions. The poles of elementary excitations to therefore reside at the points given by

$$q_0^2 + E_{\rm Bog}^2 = (-iq_0 + E_{\rm Bog})(iq_0 + E_{\rm Bog}), \tag{10}$$

which suggests the following low-energy action:

$$\mathcal{S} \approx \Phi_f \cdot [\partial_\tau + c|\nabla|]\Phi_p + \Phi_p \cdot [-\partial_\tau + c|\nabla|]\Phi_f, \tag{11}$$

which resembles the expressions for edge modes propagating along extended stringlike defects, which were proposed for gapped Dirac electron gases in graphene [26]. Microscopically, this action is obtained by a suitable Bogoliubov transformation. Here we restrict to the low-energy regime, where $E_{\rm Bog} \approx cq$ with the sound wave velocity given by $c \approx \sqrt{\mu/m}$. The first term corresponds manifestly to a particle propagating from a state Φ_p, which lies in the past, to the state $\bar{\Phi}_f$, which lies in the future, while the second term to a particle moving in the opposite direction. The obvious solutions of the extremal equations would be given by a family of radial symmetric functions of the kind $\phi(r \mp ct)$, with the radial variable constrained by $|\nabla| r = 1$.

To detect the Berry term in the action Eq. (5), we need to consider the correlator of longitudinal (A_1) and transversal (A_2) fields,

$$K_{\mu\nu}(\omega) = \frac{1}{i}\int dX\, e^{-i\omega(\tau-\tau')}\langle \partial_\tau A_{\mu,X} A_{\nu,X'}\rangle, \tag{12}$$

where the variable X contains the imaginary time τ and the spacial variable x, $X = (x, \tau)$, and $dX = d^d x d\tau$. The average operator in Eq. (12) is defined as the bosonic functional integral:

$$\langle \cdots \rangle = \frac{1}{\mathcal{Z}}\int \mathcal{D}\vec{A} \cdots e^{-\mathcal{S}_A}, \quad \mathcal{Z} = \langle 1 \rangle. \tag{13}$$

The evaluation of the Berry term correlator, cf. Appendix B, yields an expression proportional to the Bogoliubov propagator taken at zero momentum, i.e., in the large-scale limit:

$$K_{\mu\nu}(\omega) = \frac{\omega}{2} G_{\mu\nu}(\omega). \tag{14}$$

From Eq. (8) follows

$$G_{\mu\nu}(\omega) = \frac{\epsilon_{\mu\nu}}{\omega}, \tag{15}$$

which eventually yields for the Berry term correlation function Eq. (12):

$$K_{\mu\nu}(\omega) = \frac{1}{i}\int dX\, e^{-i\omega(\tau-\tau')}\langle \partial_\tau A_{\mu,X} A_{\nu,X'}\rangle = \epsilon_{\mu\nu}. \tag{16}$$

The correlator gives an exactly quantized result with the value ± 1. The actual meaning of this result is the detection of the collective modes propagating either forward (advanced) or backward (retarded) in imaginary time. This effective time direction discrimination is in some sense reminiscent of the notion of the broken time-reversal invariance, since only one particular direction is picked at a time. It is therefore clear that this should be observable in any spatial dimension. In principle, this result should be measurable in thin superfluid He_4 films or in 2D bosonic ultracold gases in optical lattices.

To obtain this result, two ingredients are crucial. First, the system has to be Galilean invariant, which in combination with the bosonic statistics of the quantum fields is the ultimate reason for the existence of the Berry term, and, second, the spectrum of elementary excitations has to be gapless. Therefore, the measurement of the quantized response of the type Eq. (12) may be considered as a tool for proving the protection of the gaplessness of the spectrum in the condensed phase. From the semirigorous renormalization group analysis of the model Eq. (1) in Refs. [27,28], we know that it is asymptotically free in the infrared in all spatial dimensions $d \geqslant 2$. Hence, in infinitely large systems the Bogoliubov approximation becomes exact and with this Eq. (16) as well.

From what was just said it follows that such effective time direction discrimination is not possible in conventional superconductors. Although they are Galilean invariant, the fermionic statistics of Grassmann fields would negate the sign change due to the integration by parts in the time derivative. Furthermore, the spectrum of elementary excitations in superconductors is gapped and the limit $\omega \to 0$ in the fermionic analog of Eq. (16) would always be zero. To the contrary, the effect discussed in this section is in many ways related to the nontrivial topological properties of the d-wave superconductivity in two spacial directions [29], which is known to capture the main phenomenology of the fractional quantum Hall effect. Here, the order parameter has nodes on the Fermi surface, which are approximated by the Dirac-like Hamiltonian. The effect of the Galilean invariance and fermionic statistics in the particle-hole channel put together then culminate in the different signs of the chemical potential for each of the spin projections, which plays the role of the effective Dirac mass. Nonetheless, even here the spectrum of quasiparticles is gapped and, hence, the appropriate version of Eq. (16) would vanish for $\omega \to 0$.

III. CONDENSATION OF THE ELASTIC PSEUDOGAUGE FIELDS COUPLED TO THE MASSIVE TWO-DIMENSIONAL DIRAC FERMION GAS

The proposed parallels between the Berry and Chern-Simons terms in particular dimensions suggests the existence of other analogies going in the opposite direction. We can imagine a model which comprises the massive lattice Dirac fermions [30] and the elastic lattice fields in two spatial dimensions. The latter are known to couple minimally to the lattice electrons [12], which makes it possible to consider them as the effective pseudogauge fields. The dynamics of the pseudogauge fields is extremely slow, such that the interaction between them can be ignored completely. In this regard, they are different from the phonons, where the interaction has to be fully accounted for [13–17,31]. The fermionic mass breaks the time-reversal symmetry, bears the topological charge, and is responsible for the finite Hall conductivity [11]. By coupling fermions to the gauge fields and integrating the fermions, the topological charge passes through into the effective bosonic theory, where it appears in the form of the

Chern-Simons term. Such an effective low-energy theory might have a structure similar to the model of the weakly interacting Bose gas with its intrinsic singularities. If this is the case, then one should consider the condensation of the pseudogauge as the physically reliable means to cure them. Ultimately, we can conclude on direct parallels on Berry versus Chern-Simons terms. Our starting point is the usual action

$$\mathcal{S}_F[\psi^\dagger, \psi, \vec{A}] = \psi^\dagger \cdot [G^{-1} + \vec{A} \cdot \vec{\sigma}]\psi, \tag{17}$$

where the inverse Green's function of the massive Dirac fermion is

$$G^{-1} = \partial_\tau \sigma_0 + i\vec{\nabla} \cdot \vec{\sigma} + m\sigma_3, \tag{18}$$

with the kinetic energy operator $i\vec{\nabla} \cdot \vec{\sigma} = \sum_{\alpha=1,2} i\partial_\alpha \sigma_\alpha$, and the two-component gauge field

$$\vec{A} \cdot \vec{\sigma} = A_1\sigma_1 + A_2\sigma_2, \tag{19}$$

which can be thought of as the gauge field taken in the so-called axial gauge $A_0 = 0$. The Fourier transformed for the fermionic Green's function reads

$$G^{-1}(Q) = iq_0\sigma_0 + q \cdot \sigma + m\sigma_3. \tag{20}$$

The way the gauge field couples to the fermions can be interpreted as a z component of the magnetic field, since A depends on time and space coordinates only, i.e., the derivatives with respect to the z coordinate are zero. Hence, only the z component of $\vec{\nabla} \times \vec{A}$ (i.e., effectively the magnetic field) would be finite.

To calculate the correlation function of the type of Eq. (12), we introduce the partition function

$$\mathcal{Z} = \int \mathcal{D}\vec{A} \int \mathcal{D}\psi^\dagger \mathcal{D}\psi \, e^{-\mathcal{S}_F[\psi^\dagger, \psi, \vec{A}]}, \tag{21}$$

which is the function of the two-models parameter, the fermion mass m and the finite bandwidth Λ. The calculation of the correlation function of the type of Eq. (12) requires evaluation of the functional integral:

$$\langle \partial_\tau A_{\mu,X} A_{\nu,X'} \rangle = \frac{1}{\mathcal{Z}} \int \mathcal{D}\vec{A}\, \partial_\tau A_{\mu,X} A_{\nu,X'} \int \mathcal{D}\psi^\dagger \mathcal{D}\psi \, e^{-\mathcal{S}_F[\psi^\dagger, \psi, \vec{A}]}. \tag{22}$$

The action is quadratic in fermionic fields and the integration of the fermions can be done exactly,

$$e^{-\bar{\mathcal{S}}[\vec{A}]} = \int \mathcal{D}\psi^\dagger \mathcal{D}\psi \, e^{-\mathcal{S}_F[\psi^\dagger, \psi, \vec{A}]} = \det[G^{-1} + \vec{A} \cdot \vec{\sigma}], \tag{23}$$

or, consequently,

$$\bar{\mathcal{S}}[\vec{A}] = -\mathrm{tr}\log[G^{-1} + \vec{A} \cdot \vec{\sigma}]. \tag{24}$$

Then the expression for the correlation function changes to

$$\langle \partial_\tau A_{\mu,X} A_{\nu,X'} \rangle = \frac{\int \mathcal{D}\vec{A}\, \partial_\tau A_{\mu,X} A_{\nu,X'}\, e^{-\bar{\mathcal{S}}[\vec{A}]}}{\int \mathcal{D}\vec{A}\, e^{-\bar{\mathcal{S}}[\vec{A}]}}, \tag{25}$$

which allows us to factorize out the field-independent parts of the action:

$$\bar{\mathcal{S}}[\vec{A}] = -\mathrm{tr}\log[1 + G\vec{A} \cdot \vec{\sigma}]. \tag{26}$$

The tr operator denotes the functional trace, which includes the integration over all continuous summations over all discrete degrees of freedom. Expansion to the fourth order in powers of $(G\vec{A} \cdot \vec{\sigma})$ formally reads

$$\bar{\mathcal{S}}[\vec{A}] \approx -\mathrm{tr}\big[(G\vec{A} \cdot \vec{\sigma}) - \tfrac{1}{2}(G\vec{A} \cdot \vec{\sigma})^2 + \tfrac{1}{3}(G\vec{A} \cdot \vec{\sigma})^3 - \tfrac{1}{4}(G\vec{A} \cdot \vec{\sigma})^4 + \cdots\big]. \tag{27}$$

By assuming the lattice regularization, one cannot neglect the quatric terms, which describe the interaction between the gauge fields. The correlator of the type of Eq. (12) should then be renormalizedto account for the factors appearing due to the particular regularization [32],

$$K_{\mu\nu}(\omega) = \frac{\mathcal{N}}{i} \int d^d x d\tau \, e^{-i\omega(\tau-\tau')} \langle \partial_\tau A_{\mu,X} A_{\nu,X'} \rangle, \tag{28}$$

where $\mathcal{N}$ is such a counterterm, which will be specified at a later stage.

The evaluation of each term from Eq. (27) was partially carried out in Ref. [13]. We present it with some additional information in Appendix C. In the explicit form, the action becomes

$$\bar{\mathcal{S}}[\vec{A}] = -V[\vec{A}] - \frac{s_\Lambda}{8\pi} \epsilon_{\mu\nu} A_\mu \cdot i\partial_\tau A_\nu - \frac{1}{24\pi|m|} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}^{\mathrm{T}} \cdot \begin{pmatrix} \partial_2^2 & \partial_1\partial_2 \\ \partial_1\partial_2 & \partial_1^2 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}, \tag{29}$$

with the effective potential

$$V[\vec{A}] = \frac{\Lambda}{8\pi} \vec{A} \cdot \vec{A} - \frac{1}{128\pi\Lambda} (\vec{A} \cdot \vec{A})^2, \tag{30}$$

and Λ denotes the bandwidth. The quantity s_Λ is defined in Appendix C and appears as an artifact of the lattice regularization. The linear and cubic terms do not appear in this expansion, since they are either traceless or not rotationaly invariant, but not because of the infinite cutoff.

IV. DIFFERENT PSEUDOGAUGE FIELDS CONDENSING PATTERNS

Structurally, the action obtained in Eq. (29) is similar to the model of the weakly interacting Bose gas defined in Eq. (5). Both models differ in the important point, though, that in the model of weakly interacting Bose gas only the longitudinal component can condense. For the pseudogauge fields, each of the components can condense separately or both together. In the case with only one of the components condensing, the system arrives in the state with a selected spatial direction, and the condensate breaks the spatial isotropy. If both components of the pseudogauge field condense, then the spatial isotropy is not violated and the system remains isotropic as a whole. Hence, generally we should consider both possibilities. We may estimate the stability of the phases by calculating the ground-state energy. The phase with the smaller ground-state energy would be the physical one. Introducing the Bogoliubov shift

$$\vec{A} \to \sqrt{\alpha}\hat{e} + \vec{A}, \tag{31}$$

where $\hat{e}$ is a vector, which distinguishes between the anisotropic and symmetric symmetry-breaking patterns which we specify later. The condensate α of the pseudogauge field $\vec{A}$ reflects the change in the topography of the initially flat two-dimensional sheet in which the Dirac fermions and the pseudogauge fields reside. Therefore, the Bogoliubov shift represents the zero temperature formation mechanism of the lattice defects. In the other words, α should be related to the order parameter of the rippled phase. Plugging this into the expression for the potential and retaining only terms containing only the condensate, we get the expression for the mean-field potential,

$$V[\alpha] = \Omega\frac{\Lambda}{8\pi}\hat{e}\cdot\hat{e}\left(\alpha - \frac{\alpha^2}{16\Lambda^2}\hat{e}\cdot\hat{e}\right), \tag{32}$$

where $\Omega = \int d^2x \int d\tau$ is the space-time volume of the system. For the anisotropic phase, $\hat{e} = (1, 0)$ or $\hat{e} = (0, 1)$ and $\hat{e}\cdot\hat{e} = 1$. The minimum of the potential is then obtained for $\alpha = 8\Lambda^2$, which then becomes

$$V_{\mathrm{A}}[\alpha] = \Omega\frac{\Lambda^3}{2\pi}. \tag{33}$$

For the isotropic phase $\hat{e} = (1, 1)$ and the minimum of the potential is obtained for $\alpha = 4\Lambda^2$, which leads to the very same value of the ground-state energy:

$$V_{\mathrm{S}}[\alpha] = \Omega\frac{\Lambda^3}{2\pi}. \tag{34}$$

From the point of view of the smallest ground-state energy, both symmetry-breaking patterns are equally probable. Although the value of the condensed fraction is twice as high in the anisotropic phase, this does not seem to be a sufficient argument why this phase should be less probable. If we strengthen the requirements on the vectors $\hat{e}$ by demanding them to be always normalized to the unity, then also this difference disappears and both states become energetically indistinguishable. The discovered degeneracy of the ground state mimics similar degeneracy in the ground state, which occurs when the system of 2D Dirac fermions interacts with the longitudinal monochromatic phonons reported in Refs. [15,16].

The stability investigation of each phase should be complemented by the fluctuation analysis. Lengthy calculations presented in Appendices D and E lead us to the quadratic parts of the action in each case. The considered anisotropic case assumes that A_1 condenses. The quadratic action becomes

$$\bar{\mathcal{S}}_A[\vec{A}] = \vec{A}\cdot\Pi_A^{-1}\vec{A}, \tag{35}$$

where this inverse propagator reads

$$\Pi_A^{-1}(P) = \begin{pmatrix} \Lambda + \frac{p_2^2}{12|m|} & \frac{s_\Lambda}{4}p_0 + \frac{p_1p_2}{12|m|} \\ -\frac{s_\Lambda}{4}p_0 + \frac{p_1p_2}{12|m|} & \frac{p_1^2}{12|m|} \end{pmatrix}. \tag{36}$$

The inversion of the matrix Eq. (36) reveals the structure of the spectrum to the leading order

$$E_A(P) = |p_1|\sqrt{\frac{\Lambda}{12|m|}} + O(p^3). \tag{37}$$

The spectrum Eq. (37) combines phenomenological features which are specific to two distinct classes of critical systems. First, they are gapless, which is typical for the superfluids and, in particular, for the Bogoliubov spectrum. Second, they reveal a striking anisotropy, which is typical for the so-called Lifshitz transition [19–21,33]. In terms of the effective Dirac description of graphene, the phenomenology of the Lifshitz transition is understood as follows: the in-plane strain acting on both sublattices in opposite directions changes the distance between the nearest lattice atoms within each layer and correspondingly increases the electronic hopping amplitude between them. This creates an additional term which resembles the pseudogauge field considered here [12]. The difference is that this field is static, while the ones considered here possess a full space-time dependence. The action of this field displaces the Dirac points from their original positions until the system arrives at a Lifshitz transition point at some critical value of the pseudogauge-field amplitude, characterized by a fusion of the Dirac points with resulting anisotropic spectrum of the kind of Eq. (37). Finally, such anisotropy of the spectrum is typical for directed edge modes propagating along the artificially created channels (edge domain walls) on the boundary between regions with different properties, e.g., in gapped monolayer graphene [26] or in strained and twisted bilayer graphene [34]. In the latter case, these emerging edge domain walls separate domains with different vorticity, i.e., local topology, of the lattice [35].

In the isotropic case, where both components of the pseudogauge field condense, we obtain the following inverse propagator:

$$\Pi_I^{-1}(P) = \begin{pmatrix} \Lambda + \frac{p_2^2}{3|m|} & s_\Lambda p_0 + \Lambda + \frac{p_1p_2}{3|m|} \\ -s_\Lambda p_0 + \Lambda + \frac{p_1p_2}{3|m|} & \Lambda + \frac{p_1^2}{3|m|} \end{pmatrix}. \tag{38}$$

The inversion of the propagator reveals the structure of the elementary excitations

$$E_I(P) = \sqrt{\frac{\Lambda}{3|m|}}|p_1 - p_2| + O(P^2), \tag{39}$$

which looks as a generalization of Eq. (37).

The main result of these calculations suggests that both possibilities of condensates do essentially occur due one and the same mechanism. There is no clear energetic argument which favors the isotropic case. The only difference in the ground state is the smaller value of the condensate α, but it is not clear whether this is a sufficient criterion to dismiss the anisotropic case, since it can be cured by a trivial redefinition of the variables. Perhaps the solution of this problem could be achieved by renormalization group analysis by coupling minimally both quadratic actions to each other and looking at the competition between them. We leave this for the future activities.

V. BERRY TERM VERSUS KUBO CORRELATORS IN BOTH PHASES

Although both phases do not differ much from the spectral point of view, they have quite different behaviors in the Berry term correlator. Going through the formal evaluation of Eq. (28), we get a result similar to Eq. (14),

$$K_{A;\mu\nu}(\omega) = 2\pi\mathcal{N}\frac{\omega}{2}\Pi_{A;\mu\nu}(\omega), \tag{40}$$

where the prefactor appears due to the rescaled pseudogauge fields. In the anisotropic phase, we obtain

$$\Pi_{\mu\nu}(\omega) = \epsilon_{\mu\nu}\frac{4}{\mathcal{C}_\Lambda}\frac{\text{sgn}(m)}{\omega}. \tag{41}$$

With the choice of the renormalization factor, $\mathcal{N} = \frac{\mathcal{C}_\Lambda}{4\pi}$ then follows

$$K_{A;\mu\nu}(\omega) = \epsilon_{\mu\nu}\text{sgn}(m), \tag{42}$$

which is essentially the familiar topological Chern number of the 2D massive Dirac electron gas. Things are different in the isotropic case. Here, the off-diagonal term does not vanish as the frequency and momentum are taken to zero. At zero momenta and small frequencies, we obtain a diverging result:

$$K_{I;\mu\nu}(\omega) \sim -\frac{\Lambda}{\omega}. \tag{43}$$

Since ω is a Matsubara frequency, this divergence is on the imaginary axis and there is no response on the real axis. We therefore notice that the two phases react differently to the measurement of the topological order, rendering the isotropic phase topologically trivial.

To bring this argumentation on more formal footing, we evaluate the Hall conductivity of the massive 2D Dirac electron gas from the Kubo formula [11]

$$\sigma_{\mu\nu}(\omega) = \frac{2\pi}{\omega}\int dX\, e^{-i\omega(\tau-\tau')}\langle j_\mu(X)j_\nu(X')\rangle, \tag{44}$$

with the fermionic current operators

$$j_\mu(X) = (\psi^\dagger\sigma_\mu\psi)_X, \tag{45}$$

and the averaging operator denoting the Grassmann functional integration over the fields ψ. Introducing the sources in the form of the pseudogauge fields, the Kubo formula changes to

$$\sigma_{\mu\nu}(\omega) = \frac{2\pi}{\omega}\int dX\, e^{-i\omega(\tau-\tau')}\left.\frac{\delta^{(2)}}{\delta A_{\mu,X}\delta A_{\nu,X'}}\right|_{\vec{A}=0}\langle e^{-\vec{A}\cdot\vec{j}}\rangle, \tag{46}$$

where the functional integration now runs over the action Eq. (17). To obtain the Hall conductivity, it is sufficient to expand the fermionic determinant up to the Chern-Simons term and then perform the functional derivatives. But our intention is to verify to what extend the nontrivial topology of the model persists in each of the condensed phases. No functional integration over pseudogauge fields is required at any step of the derivation up until Eqs. (36) and (38). Therefore, we can perform all the calculations from the previous sections until we arrive at

$$\sigma_{\mu\nu}(\omega) = \frac{2\pi}{\omega}\int dX\, e^{-i\omega(\tau-\tau')}\left.\frac{\delta^{(2)}}{\delta A_{\mu,X}\delta A_{\nu,X'}}\right|_{\vec{A}=0} e^{-\vec{A}\cdot\Pi^{-1}\vec{A}}, \tag{47}$$

where Π^{-1} is either Eq. (36) or Eq. (38). Performing the Fourier transform and diagonalizing the expression in the momentum space yields

$$\sigma_{\mu\nu}(\omega) = \frac{2\pi}{\omega}\int\frac{d^3Q}{(2\pi)^3}e^{-i\vec{q}\cdot\vec{x}'}e^{-i\tau'(q_0+\omega)} \times\int dX\, e^{-i\vec{q}\cdot\vec{x}}e^{-i\tau(q_0+\omega)}\Pi^{-1}_{\mu\nu}(Q). \tag{48}$$

The remaining integrals are trivial and we therefore obtain

$$\sigma_{\mu\nu}(\omega) = \frac{2\pi}{\omega}\Pi^{-1}_{\mu\nu}(\omega). \tag{49}$$

For the Hall conductivity $\mu \neq \nu$. Hence, using Eqs. (36) and (38), we obtain results almost identical with the Berry term correlator, which is especially striking for the anisotropic case.

VI. CONCLUSIONS

This similarity between the Kubo and Berry term correlators gives a hint that the latter measures the topological order in the model Eq. (17). Then, by analogy it also means that the Berry term of the model of the weakly interacting Bose gas Eq. (5) measures something similar and represents some kind of space-time topological invariant, which is present in all spatial dimensions. The different sign for each of the index combinations corresponds in the original action to the direction of the imaginary time, hence what the Berry term correlator actually detects is the temporal propagation direction of either advanced and/or retarded sound mode through the condensate. The possibility to distinguish between both temporal directions in a measurement, i.e., picking of one particular time direction, may be interpreted as an effective time reversal breaking.

Importantly, this effective time direction discrimination is not measurable and therefore principally absent in superconductors. The reason is that the Berry term does not appear in the Bogolibov-de Gennes Lagrangian due to the fermionic statistics of the Grassmann fields. Because of the additional minus sign due to the commutation of Grassmann numbers, the sign of the time derivative in both advanced and retarded channels would be exactly the same, hence an analog of the Berry term correlator (if such can be formulated for Grassmann fields at all, we did not check it) would always be positive. Moreover, the spectrum of collective excitations in superconductors is gapped. Therefore, in the limit $\omega \to 0$, the result of the Berry term correlator evaluation would be exactly zero. Things are different for the case of the d-wave conductivity in two spatial dimensions, as pointed out in the main text. Here too, the combined effect of the Galilean invariance in the particle-hole channel, fermionic statistics and gapless nodal order parameter also results in nontrivial topology of the superconducting state with charge fractionalization [29]. One can attempt a speculation on a potential duality between the fermionic and bosonic theories. Even then, the spectrum of quasiparticles remains gapped and the infrared limit obscures the observation of the quantized response from the correlator of the type of Eq. (16).

The analogy to the weakly interacting bosons suggests a similar interpretation for the case of condensed pseudogauge fields as well. Here we have detected additional constrains, though. For these fields, two different symmetry-breaking mechanisms have been found, with and without the preservation of the spatial isotropy. From the point of view of the smallest ground-state energy, both symmetry-breaking patterns are equally likely, hence none of them can be neglected as such. The breaking of the spatial isotropy by the condensate seems to preserve the topological properties of the model, while the spatially isotropic condensate seems to negate the effect of the time-reversal symmetry broken by the presence of the mass gap. The spectrum of collective excitations on the condensed phase is linear and therefore represents a kind of sound wave. However, it also reveals a selected direction in the momentum space, which means that whose sound waves are directed. This phenomenological picture is specific to various Lifshitz transitions, which occur under the static strain at critical strain values also related to the bandwidth [18–21]. The spatial anisotropy of the spectrum in Eqs. (37) and (39) is typical for directed edge modes propagating along the artificially created channels [26]. These two crucial observations provide a strong phenomenological support that the condensation of the elastic pseudogauge fields reported in this paper results in the formation of quasi-one-dimensional stringlike defects of the lattice, similar to the edge domain walls, which separate domains with different lattice topologies reported recently in a different context [34,35]. It is left to future activities to study the thermodynamics and transport at the critical point and of the corresponding phases.

Talking about the spontaneous symmetry breaking of the effective model in terms of pseudogauge fields is meaningless unless we have an interaction between them. If the model Eq. (17) is taken in continuum, i.e., with an infinite ultraviolet cutoff Λ, then the interaction term in the effective potential in Eq. (A7) is totally suppressed, while the formally infinite and negative mass term is eliminated from the considerations by means of the specially invented dimensional regularization technique [36], which therefore eliminates all sources of instability in the effective model. Things are different on the lattice, though, especially if the ratio gap size m to ultraviolet cutoff Λ is not negligible anymore. This constrain produces the quatric interaction term, which then triggers the mechanism of the spontaneous symmetry breaking.

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APPENDIX A: THE WEAKLY INTERACTING BOSE GAS MODEL IN THE CONDENSED PHASE

We use the following notation in this paper: The dot product operations implies the integration of the corresponding terms over the position coordinates and the imaginary time, e.g.,

$$a\cdot b=\int d\tau\int d^dx\, a(x,\tau)b(x,\tau), \tag{A1}$$

$$(a\cdot b)^2=\int d\tau\int d^dx\, a^2(x,\tau)b^2(x,\tau). \tag{A2}$$

The condensate is split off the longitudinal field as its homogeneous part by the Bogoliubov shift:

$$A_1\to\sqrt{\rho}+A_1,\quad A_2\to A_2. \tag{A3}$$

Shifting the fields changes the first and second terms of the effective potential Eq. (6) as

$$\vec{A}\cdot\vec{A}\to\Omega\rho+2\sqrt{\rho}\hat{e}\cdot\vec{A}+\vec{A}\cdot\vec{A}, \tag{A4}$$

$$(\vec{A}\cdot\vec{A})^2\to\Omega\rho^2+4\rho^{\frac{3}{2}}\hat{e}\cdot\vec{A}+4\rho\vec{A}\cdot\mathcal{P}_\uparrow\vec{A}+2\rho\vec{A}\cdot\vec{A} + 4\sqrt{\rho}(\hat{e}\cdot\vec{A}\vec{A}\cdot\vec{A})+(\vec{A}\cdot\vec{A})^2, \tag{A5}$$

where $\hat{e}=(1,0)^{\mathrm{T}}$, $\Omega=\int d\tau\int d^dx$ being the formally infinite integral measure (space-time volume),

$$\mathcal{P}_\uparrow=\begin{pmatrix}1&0\\0&0\end{pmatrix},\quad \hat{e}\cdot\vec{A}=\int d\tau\int d^dx\,A_1,\quad\text{and}$$

$$(\hat{e}\cdot\vec{A}\vec{A}\cdot\vec{A})=\int d\tau\int d^dx\,A_1\left(A_1^2+A_2^2\right). \tag{A6}$$

Thus, as a whole, the potential term in Eq. (6) changes as

$$\begin{aligned}V_{\text{eff}}[\vec{A}]=\frac{\mu}{2}\vec{A}\cdot\vec{A}-\frac{g}{8}(\vec{A}\cdot\vec{A})^2\to&-\frac{1}{8}\Omega\rho[g\rho-4\mu]\\&-\frac{1}{2}\sqrt{\rho}[\rho g-2\mu](\hat{e}\cdot\vec{A})\\&-\frac{1}{4}[\rho g-2\mu]\vec{A}\cdot\vec{A}-\frac{g}{2}\rho\vec{A}\cdot\mathcal{P}_\uparrow\vec{A}\\&-\frac{g}{2}\sqrt{\rho}(\hat{e}\cdot\vec{A}\vec{A}\cdot\vec{A})-\frac{g}{8}(\vec{A}\cdot\vec{A})^2.\end{aligned} \tag{A7}$$

The stability criterion of the model corresponds to the minimal ground-state energy with respect to the condensate

$$0=\frac{\Omega}{8}\frac{\partial}{\partial\rho}[g\rho^2-4\mu\rho], \tag{A8}$$

which fixes the condensate density to

$$\rho=\frac{2\mu}{g}, \tag{A9}$$

and annihilates the linear term in Eq. (A7). The remaining terms of the effective potential become

$$V_{\text{eff}}[\vec{A}]=-\mu\vec{A}\cdot\mathcal{P}_\uparrow\vec{A}-\sqrt{\frac{\mu g}{2}}(\hat{e}\cdot\vec{A}\vec{A}\cdot\vec{A})-\frac{g}{8}(\vec{A}\cdot\vec{A})^2, \tag{A10}$$

truncating the action to quadratic order in A's (with support of the renormalization group argument, by which the interaction strength parameter g scales down to zero in the infrared [25,27,28]), we get the quadratic effective Bogoliubov action:

$$\mathcal{S}_{\text{eff}}[\vec{A}]=-\frac{i}{2}\epsilon_{\alpha\beta}A_\alpha\cdot\partial_\tau A_\beta+\frac{1}{4m}(\partial_\alpha\vec{A})\cdot(\partial_\alpha\vec{A})+\mu\vec{A}\cdot\mathcal{P}_\uparrow\vec{A}. \tag{A11}$$

In conventional fashion, Eq. (A11) becomes Eq. (7).

APPENDIX B: EVALUATION OF THE CORRELATOR EQ. (12) IN GENERAL FORM

Integrating by parts in Eq. (12) in the time integral first gives

$$K_{\mu\nu}(\omega) = \omega \int dX\, e^{-i\omega(\tau-\tau')} \langle A_{\mu,X} A_{\nu,X'} \rangle.$$

Fourier transforming and diagonalizing in the momentum space gives

$$K_{\mu\nu}(\omega) = \omega \int dX\, e^{-i\omega(\tau-\tau')} \int \frac{d^3Q}{(2\pi)^3} \int \frac{d^3P}{(2\pi)^3} \delta(Q+P) \times e^{-iQ\cdot X} e^{-iP\cdot X'} \langle A_{\mu,Q} A_{\nu,-Q} \rangle.$$

The functional integral over the quadratic part of the action is evaluated by introducing the auxiliary sources:

$$\langle A_\nu A_\mu \rangle = \frac{\delta^2}{\delta J_\mu \delta J_\nu}\bigg|_{\vec{J}=0} \frac{1}{\mathcal{Z}} \int \mathcal{D}\vec{A}\, \exp\{-\vec{A}\cdot M\vec{A} - \vec{J}\cdot\vec{A}\}. \quad \text{(B1)}$$

Shifting the integration variables as $\vec{A} \to \vec{A} - \frac{1}{2}M^{-1}\vec{J}$, one separates $\vec{A}$ and $\vec{J}$,

$$\vec{A}\cdot M\vec{A} + \vec{J}\cdot\vec{A} \to \vec{A}\cdot M\vec{A} - \tfrac{1}{4}\vec{J}\cdot M^{-1}\vec{J}, \quad \text{(B2)}$$

which then reduces the calculation of the correlator to a simple functional derivative:

$$\langle A_\nu A_\mu \rangle = \frac{\delta^2}{\delta J_\mu \delta J_\nu}\bigg|_{\vec{J}=0} \exp\left[\frac{1}{4}\vec{J}\cdot M^{-1}\vec{J}\right] = \frac{1}{2}M^{-1}_{\mu\nu}. \quad \text{(B3)}$$

Integrating one of the momenta gives

$$\begin{aligned} K_{\mu\nu}(\omega) &= \frac{\omega}{2}\int \frac{d^3Q}{(2\pi)^3} e^{i\tau'(q_0+\omega)} e^{ix'\cdot q} \int d\tau\, e^{-i\tau(q_0+\omega)} \\ &\quad \times \int d^2x\, e^{-ix\cdot q} M^{-1}_{\mu\nu}(Q) \\ &= \frac{\omega}{2}\int \frac{d^3Q}{(2\pi)^3} \delta(q_0+\omega)\delta(q) M^{-1}_{\mu\nu}(Q) = \frac{\omega}{2} M^{-1}_{\mu\nu}(\omega), \end{aligned}$$

i.e., with the propagator matrix taken at zero momentum.

APPENDIX C: EVALUATION OF THE ACTION EQ. (27)

Below we evaluate each term in the expansion Eq. (27). The leading order term vanishes:

$$\begin{aligned} \mathrm{tr}(G\vec{A}\cdot\vec{\sigma}) &= \mathrm{Tr}\int d^3X\, G_{XX}\vec{A}_X\cdot\vec{\sigma} \\ &= \mathrm{Tr}\int d^3X \int \frac{d^3Q}{(2\pi)^3} \int \frac{d^3P}{(2\pi)^3} e^{-iQ\cdot(X-X)} e^{-iP\cdot X} G(Q)\vec{A}_P\cdot\vec{\sigma} && \text{(C1)} \\ &= \int \frac{d^3P}{(2\pi)^3} A^\mu_P \int d^3X\, e^{-iP\cdot X}\, \mathrm{Tr}\int \frac{d^3Q}{(2\pi)^3} G(Q)\sigma_\mu && \text{(C2)} \\ &= \int \frac{d^3P}{(2\pi)^3} A^\mu_P \delta(P)\, \mathrm{Tr}\int \frac{d^3Q}{(2\pi)^3} G(Q)\sigma_\mu = \mathrm{Tr}\,\vec{A}_0\cdot\vec{\sigma}\int \frac{d^3Q}{(2\pi)^3} G(Q). && \text{(C3)} \end{aligned}$$

Using the fermionic Green's function Eq. (20), we recognize

$$\begin{aligned} \mathrm{Tr}\,\vec{A}_0\cdot\vec{\sigma}\int \frac{d^3Q}{(2\pi)^3} G(Q) &= \mathrm{Tr}\,A^\mu_0\sigma_\mu \int \frac{d^3Q}{(2\pi)^3} \frac{1}{Q^2+m^2}[-iq_0\sigma_0 + q_\nu\sigma_\nu + m\sigma_3] && \text{(C4)} \\ &= 2\delta_{\mu\nu}A^\mu_0 \int \frac{d^3Q}{(2\pi)^3} \frac{q_\nu}{Q^2+m^2} = 0 && \text{(C5)} \end{aligned}$$

by angular integration. The quadratic term has to be evaluated up to second order in gradient expansion:

$$-\frac{1}{2}\mathrm{tr}(G\vec{A}\cdot\vec{\sigma})^2 = -\int \frac{d^3P}{(2\pi)^3} A^\mu_P A^\nu_{-P} \cdot \frac{1}{2}\mathrm{Tr}\int \frac{d^3Q}{(2\pi)^3} G(Q)\sigma_\mu G(Q+P)\sigma_\nu. \quad \text{(C6)}$$

Employing the Feynman-parameter trick,

$$\frac{1}{AB} = \int_0^1 dx\, \frac{1}{[(1-x)A + xB]^2}, \quad \text{(C7)}$$

we first may write

$$\begin{aligned} &-\frac{1}{2}\mathrm{Tr}\int \frac{d^3Q}{(2\pi)^3} G(Q)\sigma_\mu G(Q+P)\sigma_\nu \\ &= -\frac{1}{2}\mathrm{Tr}\int_0^1 dx \int \frac{d^3Q}{(2\pi)^3} \frac{[\not{Q} + m\sigma_3]\sigma_\mu[\not{Q}+\not{P}+m\sigma_3]\sigma_\nu}{[(1-x)(Q^2+m^2) + x((Q+P)^2+m^2)]^2}, && \text{(C8)} \end{aligned}$$

where $Q^2 = q_0^2 + q_1^2 + q_2^2$ and $\not{Q} = -iq_0\sigma_0 + q_1\sigma_1 + q_2\sigma_2$. We symmetrize the numerator by shifting $Q \to Q - xP$, which then yields

$$\to -\frac{1}{2}\mathrm{Tr}\int_0^1 dx \int \frac{d^3Q}{(2\pi)^3}\frac{[\not{Q} - x\not{P} + m\sigma_3]\sigma_\mu[\not{Q} + (1-x)\not{P} + m\sigma_3]\sigma_\nu}{[Q^2 + m^2 + x(1-x)P^2]^2}. \tag{C9}$$

To the second order in gradient expansion, we get

$$-\frac{1}{2}\mathrm{tr}(G\vec{A}\cdot\vec{\sigma})^2 \approx -\frac{1}{2}\mathrm{Tr}\int \frac{d^3Q}{(2\pi)^3}\frac{[\not{Q} + m\sigma_3]\sigma_\mu[\not{Q} + m\sigma_3]\sigma_\nu}{[Q^2+m^2]^2} \tag{C10}$$

$$-\frac{1}{2}\mathrm{Tr}\int_0^1 dx\int \frac{d^3Q}{(2\pi)^3}\frac{m\not{P}[(1-x)\sigma_\nu\sigma_3\sigma_\mu - x\sigma_\mu\sigma_3\sigma_\nu]}{[Q^2+m^2]^2} \tag{C11}$$

$$+\frac{1}{2}\mathrm{Tr}\int_0^1 dx x(1-x)\int \frac{d^3Q}{(2\pi)^3}\left[2P^2\frac{[\not{Q} + m\sigma_3]\sigma_\mu[\not{Q} + m\sigma_3]\sigma_\nu}{[Q^2+m^2]^3} + \frac{\not{P}\sigma_\mu\not{P}\sigma_\nu}{[Q^2+m^2]^2}\right]. \tag{C12}$$

The mass term becomes

$$-\frac{1}{2}\mathrm{Tr}\int \frac{d^3Q}{(2\pi)^3}\frac{[\not{Q} + m\sigma_3]\sigma_\mu[\not{Q} + m\sigma_3]\sigma_\nu}{[Q^2+m^2]^2} = -\frac{1}{2}\mathrm{Tr}\int \frac{d^3Q}{(2\pi)^3}\frac{\not{Q}\sigma_\mu\not{Q}\sigma_\nu + m^2\sigma_3\sigma_\mu\sigma_3\sigma_\nu}{[Q^2+m^2]^2}$$

$$= \frac{1}{2}\mathrm{Tr}\int \frac{d^3Q}{(2\pi)^3}\frac{\delta_{\mu\nu}(q_0^2+m^2)\sigma_0 - q\cdot\sigma\sigma_\mu q\cdot\sigma\sigma_\nu}{[Q^2+m^2]^2} = \delta_{\mu\nu}\int \frac{d^3Q}{(2\pi)^3}\frac{q_0^2+m^2}{[Q^2+m^2]^2},$$

where we subsequently dropped terms vanishing by the angular integration. Integrating over q_0 from $-\infty$ to $+\infty$ (zero temperature) and over q up to a cutoff Λ, we further get

$$\delta_{\mu\nu}\int \frac{d^3Q}{(2\pi)^3}\frac{q_0^2+m^2}{[Q^2+m^2]^2} = \delta_{\mu\nu}\frac{1}{8\pi}\frac{\Lambda^2}{\sqrt{\Lambda^2+m^2}} \approx \delta_{\mu\nu}\frac{\Lambda}{8\pi}, \tag{C13}$$

and the corresponding term in the action becomes

$$\approx \frac{\Lambda}{8\pi}\vec{A}\cdot\vec{A}. \tag{C14}$$

The Chern-Simons term is evaluated as

$$-\frac{1}{2}\mathrm{Tr}\int_0^1 dx\int \frac{d^3Q}{(2\pi)^3}\frac{m\not{P}[(1-x)\sigma_\nu\sigma_3\sigma_\mu - x\sigma_\mu\sigma_3\sigma_\nu]}{[Q^2+m^2]^2}$$

$$= \frac{i}{2}mp_0\mathrm{Tr}\{\sigma_3\sigma_\mu\sigma_\nu\}\int \frac{d^3Q}{(2\pi)^3}\frac{1}{[Q^2+m^2]^2} \tag{C15}$$

$$= -\frac{p_0}{8\pi}\epsilon_{\mu\nu}\left[\frac{m}{|m|} - \frac{m}{\sqrt{\Lambda^2+m^2}}\right] = -\epsilon_{\mu\nu}\mathrm{sgn}(m)\frac{p_0}{8\pi}\mathcal{C}_\Lambda, \tag{C16}$$

where

$$\mathcal{C}_\Lambda = \left[1 - \frac{1}{\sqrt{1+(\Lambda/m)^2}}\right], \tag{C17}$$

i.e., the correction to the Chern number due the lattice regularization, which then becomes not an exact integer $\sim\mathrm{sgn}(m)$ but rather a nonuniversal number [32]. We finally acquire the Chern-Simons term in the action

$$\approx \frac{s_\Lambda}{8\pi}\epsilon_{\mu\nu}A_\mu i\partial_\tau A_\nu, \tag{C18}$$

where $s_\Lambda = \mathcal{C}_\Lambda\mathrm{sgn}(m)$. For the evaluation of the term to second order in gradient expansion, we notice

$$\frac{1}{2}\int_0^1 dx\, x(1-x) = \frac{1}{12}, \tag{C19}$$

such that the term becomes

$$\bar{\mathcal{S}}[\vec{A}] \approx -\frac{1}{24\pi|m|}\begin{pmatrix}A_1\\A_2\end{pmatrix}_P^{\mathrm{T}}\cdot\begin{pmatrix}p_2^2 & p_1p_2\\ p_1p_2 & p_1^2\end{pmatrix}\begin{pmatrix}A_1\\A_2\end{pmatrix}_{-P} \tag{C20}$$

$$= \frac{1}{24\pi|m|}\begin{pmatrix}A_1\\A_2\end{pmatrix}^{\mathrm{T}}\cdot\begin{pmatrix}\partial_2^2 & \partial_1\partial_2\\ \partial_1\partial_2 & \partial_1^2\end{pmatrix}\begin{pmatrix}A_1\\A_2\end{pmatrix} = \vec{A}\cdot\mathbb{K}\vec{A}, \tag{C21}$$

where we introduced the kinetic energy matrix in the last expression. Therefore, we have obtained the quadratic part of the action in the form

$$\bar{\mathcal{S}}_2[\vec{A}] = -\frac{\Lambda}{8\pi}\vec{A}\cdot\vec{A} - \frac{\mathrm{sgn}(m)}{8\pi}\mathcal{C}_\Lambda\epsilon_{\mu\nu}A_\mu i\partial_\tau A_\nu - \vec{A}\cdot\mathbb{K}\vec{A}. \tag{C22}$$

It is sufficient to determine the qubic and quartic terms only to the momentum-independent order of the gradient expansion. There is no qubic term to local order, i.e.,

$$\tfrac{1}{3}\mathrm{tr}(G\vec{A}\cdot\vec{\sigma})^3 \approx 0. \tag{C23}$$

The quartic term reads to local order

$$\bar{\mathcal{S}}_4[\vec{A}] = -\frac{1}{4}\mathrm{tr}(G\vec{A}\cdot\vec{\sigma})^4 \approx -\frac{A_1^4 + 2A_1^2A_2^2 + A_2^4}{128\pi\Lambda}$$

$$= -\frac{1}{128\pi\Lambda}(\vec{A}\cdot\vec{A})^2. \tag{C24}$$

To the leading order in gradients, we obtain Eq. (29).

APPENDIX D: PROPAGATOR OF THE PSEUDOGAUGE FIELD IN THE ANISOTROPIC CONDENSED PHASE

Since the pseudogauge fields A are bosonic, the corresponding functional integration converges if the real part of the action is positive. The Matsubara frequency appears in the free propagator with an imaginary unit i in front of it, i.e., it represents the oscillating part in the integrand and cannot influence the convergence. The condition for the functional integral to converge implies both the real part of the quadratic term and the amplitude of the interaction term to be positive numbers. But this condition is not given for momenta smaller than the cutoff Λ. This is fixed by the Bogoliubov shift $\vec{A} \to \sqrt{\alpha}\hat{e} + \vec{A}$. As pointed out in the main text, there are two possibilities to introduce the order parameter: with and without preservation of the spatial isotropy. Here we consider the latter case and the former in the next paragraph. Then $\hat{e} = (1, 0)$ or $\hat{e} = (0, 1)$, and $\hat{e} \cdot \hat{e} = 1$ and the local quadratic part transforms as

$$\frac{\Lambda}{8\pi}[\sqrt{\alpha}\hat{e} + \vec{A}] \cdot [\sqrt{\alpha}\hat{e} + \vec{A}] = \frac{\Lambda}{8\pi}[\Omega\alpha + 2\sqrt{\alpha}(\hat{e} \cdot \vec{A}) + \vec{A} \cdot \vec{A}], \tag{D1}$$

while the quartic part as

$$\frac{1}{128\pi\Lambda}([\sqrt{\alpha}\hat{e} + \vec{A}] \cdot [\sqrt{\alpha}\hat{e} + \vec{A}])^2$$
$$= \frac{1}{128\pi\Lambda}[\alpha(\hat{e} \cdot \hat{e}) + 2\sqrt{\alpha}(\hat{e} \cdot \vec{A}) + \vec{A} \cdot \vec{A}]^2 \tag{D2}$$
$$= \frac{1}{128\pi\Lambda}[\Omega\alpha^2 + 4\alpha(\hat{e} \cdot \vec{A})^2 + (\vec{A} \cdot \vec{A})^2 + 4\alpha^{3/2}(\hat{e} \cdot \vec{A}) + 2\alpha\vec{A} \cdot \vec{A} + 4\sqrt{\alpha}(\hat{e} \cdot \vec{A}\vec{A} \cdot \vec{A})], \tag{D3}$$

which leads us to the zeroth term of the effective potential

$$V_0(\alpha) = \Omega\frac{\alpha\Lambda}{8\pi}\left(1 - \frac{\alpha^2}{16\Lambda^2}\right), \tag{D4}$$

the linear term

$$V_1(A) = \sqrt{\alpha}\frac{\Lambda}{4\pi}\left(1 - \frac{\alpha}{8\Lambda^2}\right)(\hat{e} \cdot \vec{A}), \tag{D5}$$

the quadratic term

$$V_2(A) = \frac{\Lambda}{8\pi}\vec{A} \cdot \vec{A} - \frac{2\alpha}{128\pi\Lambda}\vec{A} \cdot \vec{A} - \frac{4\alpha}{128\pi\Lambda}(\hat{e} \cdot \vec{A})^2, \tag{D6}$$

and the higher terms

$$V_{3/4}(A) = -\frac{1}{128\pi\Lambda}[4\sqrt{\alpha}(\hat{e} \cdot \vec{A}\vec{A} \cdot \vec{A}) + (A \cdot A)^2]. \tag{D7}$$

The stability criterion condition requires the linear term to vanish, which is analogous to the minimum of the ground-state energy considered in the main text in Eq. (33) and leads to

$$\alpha = 8\Lambda^2. \tag{D8}$$

Then the quadratic term becomes

$$V_2(A) = -\frac{\Lambda}{2\pi}(\hat{e} \cdot \vec{A})^2, \tag{D9}$$

i.e., only one component of the pseudogauge field acquires a gap. The higher order terms

$$V_{3/4}(A) = -\frac{\sqrt{2}}{16\pi}(\hat{e} \cdot \vec{A}\vec{A} \cdot \vec{A}) - \frac{1}{128\pi\Lambda}(\vec{A} \cdot \vec{A})^2. \tag{D10}$$

Rescaling the pseudogauge fields, the frequency and the momenta are as follows:

$$\vec{A} \to \sqrt{2\pi}\Lambda^{-\frac{3}{2}}\vec{A}, \tag{D11}$$

$$p_0 \to \Lambda p_0, \tag{D12}$$

$$p_i \to \Lambda^{\frac{1}{2}} p_i, \tag{D13}$$

and, correspondingly, for the frequency-momentum integrals, we obtain

$$\mathcal{S}_A[\vec{A}] = (\hat{e} \cdot A)^2 - \frac{s_\Lambda}{4}\epsilon_{\mu\nu}A_\mu i\partial_\tau A_\nu - \vec{A} \cdot \mathbb{K}\vec{A} + \frac{\sqrt{\pi}}{4\Lambda^{\frac{5}{2}}}(\hat{e} \cdot \vec{A}\vec{A} \cdot \vec{A}) + \frac{\pi}{32\Lambda^5}(\vec{A} \cdot \vec{A})^2. \tag{D14}$$

As the dimensional analysis shows, the remnants of the interaction term are indeed negligible in comparison to the scale-invariant quadratic part and can be omitted from further considerations. The kinetic energy matrix changes to

$$\mathbb{K} = \frac{1}{12|m|}\begin{pmatrix} \partial_2^2 & \partial_1\partial_2 \\ \partial_1\partial_2 & \partial_1^2 \end{pmatrix} \to -\frac{1}{12|m|}\begin{pmatrix} p_2^2 & p_1p_2 \\ p_1p_2 & p_1^2 \end{pmatrix}, \tag{D15}$$

the last relation being the result of the Fourier transformation. In its explicit form, the quadratic part of the action is shown in Eq. (36).

The field rescaling procedure does also affect the correlation functions Eq. (22) as

$$\langle\partial_\tau A_{\mu,X}A_{\nu,X'}\rangle \to 2\pi\langle\partial_\tau A_{\mu,X}A_{\nu,X'}\rangle, \tag{D16}$$

as only the integration kernel are not compensated by the normalization. The inversion of the matrix Eq. (36) gives

$$\Pi_A(P) = \frac{1}{\det\left[\Pi_A^{-1}(P)\right]}\begin{pmatrix} \frac{p_1^2}{12|m|} & -\frac{s_\Lambda}{4}p_0 - \frac{p_1p_2}{12|m|} \\ \frac{s_\Lambda}{4}p_0 - \frac{p_1p_2}{24\pi|m|} & \Lambda + \frac{p_2^2}{12|m|} \end{pmatrix}, \tag{D17}$$

and the $\det[\Pi^{-1}(P)]$ is factorized in terms of the elementary excitation poles

$$\det\left[\Pi_A^{-1}(P)\right] = -\left[\frac{\mathcal{C}_\Lambda}{4}ip_0 + E(P)\right]\left[\frac{\mathcal{C}_\Lambda}{4}ip_0 - E(P)\right], \tag{D18}$$

with the gapless anisotropic spectrum defined in Eq. (37). The factor $\mathrm{sgn}(m)$ is lost in the determinant calculation and, correspondingly, in its factorization.

APPENDIX E: PROPAGATOR OF THE PSEUDOGAUGE FIELD IN THE ISOTROPIC CONDENSED PHASE

In the isotropic condensed phase, both field components contribute equally to the condensate:

$$\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \to \sqrt{\alpha}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \sqrt{\alpha}\hat{e} + \vec{A}, \tag{E1}$$

i.e., $\hat{e} = (1, 1)^{\mathrm{T}}$ and $\hat{e}\cdot\hat{e} = 2$. The local quadratic part transforms as

$$\frac{\Lambda}{8\pi}[\sqrt{\alpha}\hat{e} + \vec{A}]\cdot[\sqrt{\alpha}\hat{e} + \vec{A}] = \frac{\Lambda}{8\pi}[2\Omega\alpha + 2\sqrt{\alpha}(\hat{e}\cdot\vec{A}) + \vec{A}\cdot\vec{A}], \tag{E2}$$

while the quartic part as

$$\frac{1}{128\pi\Lambda}([\sqrt{\alpha}\hat{e} + \vec{A}]\cdot[\sqrt{\alpha}\hat{e} + \vec{A}])^2 = \frac{1}{128\pi\Lambda}[\alpha(\hat{e}\cdot\hat{e}) + 2\sqrt{\alpha}(\hat{e}\cdot\vec{A}) + \vec{A}\cdot\vec{A}]^2 \tag{E3}$$

$$= \frac{1}{128\pi\Lambda}[4\Omega\alpha^2 + 4\alpha(\hat{e}\cdot\vec{A})^2 + (\vec{A}\cdot\vec{A})^2 + 8\alpha^{3/2}(\hat{e}\cdot\vec{A}) + 4\alpha\vec{A}\cdot\vec{A} + 4\sqrt{\alpha}(\hat{e}\cdot\vec{A}\vec{A}\cdot\vec{A})], \tag{E4}$$

which leads us to the zeroth term of the effective potential

$$V_0(\alpha) = \Omega\frac{\alpha\Lambda}{4\pi}\left(1 - \frac{\alpha}{8\Lambda}\right), \tag{E5}$$

the linear term

$$V_1(\vec{A}) = \frac{\Lambda\sqrt{\alpha}}{4\pi}\left(1 - \frac{\alpha}{4\Lambda^2}\right)(\hat{e}\cdot\vec{A}), \tag{E6}$$

which vanishes at $\alpha = 4\Lambda^2$, the quadratic term

$$V_2(\vec{A}) = \frac{\Lambda}{8\pi}\vec{A}\cdot\vec{A} - \frac{4\alpha}{128\pi\Lambda}(\hat{e}\cdot\vec{A})^2 - \frac{4\alpha}{128\pi\Lambda}\vec{A}\cdot\vec{A} \tag{E7}$$

$$\to -\frac{\Lambda}{8\pi}(\hat{e}\cdot\vec{A})^2 = -\frac{\Lambda}{8\pi}\begin{pmatrix} A_1 \\ A_2 \end{pmatrix}^{\mathrm{T}}\cdot\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} A_1 \\ A_2 \end{pmatrix}, \tag{E8}$$

and the higher terms:

$$V_{3/4}(\vec{A}) = -\frac{1}{16\pi}(\hat{e}\cdot\vec{A}\vec{A}\cdot\vec{A}) - \frac{1}{128\pi\Lambda}(\vec{A}\cdot\vec{A})^2. \tag{E9}$$

Combining all terms to the action and rescaling the fields $\vec{A} \to \sqrt{8\pi}\vec{A}$, we get

$$\mathcal{S}_I[\vec{A}] = -s_\Lambda\epsilon_{\mu\nu}A_\mu i\partial_\tau A_\nu - \vec{A}\cdot\mathbb{K}\vec{A} + \Lambda(\hat{e}\cdot\vec{A})^2 + \sqrt{2\pi}(\hat{e}\cdot\vec{A}\vec{A}\cdot\vec{A}) + \frac{\pi}{2\Lambda}(\vec{A}\cdot\vec{A})^2, \tag{E10}$$

the rescaled kinetic energy term being

$$\mathbb{K} = \frac{1}{3|m|}\begin{pmatrix} \partial_2^2 & \partial_1\partial_2 \\ \partial_1\partial_2 & \partial_1^2 \end{pmatrix}. \tag{E11}$$

The quadratic part of the action then becomes

$$\bar{\mathcal{S}}_I[\vec{A}] = \vec{A}\cdot\Pi_I^{-1}\vec{A}. \tag{E12}$$

The propagator in the isotropic phase reads

$$\Pi_I(P) = \frac{1}{\det[\Pi_I^{-1}(P)]} \times\begin{pmatrix} \Lambda + \frac{p_1^2}{3|m|} & -s_\Lambda p_0 - \Lambda - \frac{p_1p_2}{3|m|} \\ s_\Lambda p_0 - \Lambda - \frac{p_1p_2}{3|m|} & \Lambda + \frac{p_2^2}{3|m|} \end{pmatrix}. \tag{E13}$$

The determinant of the inverse propagator matrix is

$$\det\left[\Pi_I^{-1}(P)\right] = (\mathcal{C}_\Lambda p_0)^2 + \frac{\Lambda}{3|m|}(p_1 - p_2)^2, \tag{E14}$$

which can be factorized in terms of the poles of the propagator:

$$\det\left[\Pi_I^{-1}(P)\right] = -[\mathcal{C}_\Lambda ip_0 - E_I(P)][\mathcal{C}_\Lambda ip_0 + E_I(P)]. \tag{E15}$$

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