


Thermal bath effects in quantum quenches within quantum critical regimesFrancesco Tarantelli *Dipartimento di Fisica dell'Università di Pisa and INFN, Largo Pontecorvo 3, I-56127 Pisa, Italy*Ettore Vicari *Dipartimento di Fisica dell'Università di Pisa, Largo Pontecorvo 3, I-56127 Pisa, Italy*

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We address the out-of-equilibrium dynamics arising from quantum-quench (QQ) protocols (instantaneous changes of the Hamiltonian parameters) in many-body systems within their quantum critical regime and in contact with (homogeneously coupled) thermal baths. We consider two classes of QQ protocols. In one of them, the thermal bath is used to prepare the initial finite-temperature Gibbs state; then, after quenching, the thermal bath is removed and the dynamics of the system is unitary. We also address a more complex QQ protocol where the thermal bath is not removed after quenching, thus the quantum evolution is also driven by the interaction with the bath, which may be described by appropriate master equations for the density matrix of the system, where a further relevant time scale, or inverse decay rate, characterizes the system-bath coupling. Under these QQ protocols, the critical system develops out-of-equilibrium scaling behaviors, which extend those for isolated critical systems, by introducing further scaling variables proportional to the temperature of the thermal bath and the decay rate of the system-bath interactions. These out-of-equilibrium scaling behaviors are checked by analyzing QQ protocols within fermionic Kitaev wires, or equivalently quantum Ising chains, supplemented with a particular modelization of thermal bath that guarantees the asymptotic thermalization within the Lindblad master equation for the dynamics of open systems.

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Thanks to the recent experimental progress in the realization and control of the dynamics of quantum many-body systems, see, e.g., Refs. [1,2], the out-of-equilibrium quantum dynamics of many-body systems has become an important theoretical issue. In particular, out-of-equilibrium phenomena have been addressed within the critical regimes of many-body systems at continuous quantum transitions (CQTs) [3–5], where collective behaviors give rise to zero-temperature singularities in the equilibrium low-energy properties of the system, and the universal critical behaviors are determined by a limited number of relevant features, such as the global symmetry, the symmetry-breaking pattern, dimensionality, etc.. Within critical regimes and in the appropriate thermodynamic or finite-size scaling (FSS) limits, one can achieve a complete characterization of the complex dynamics of many-body systems by controlling a limited number of renormalization-group (RG) perturbations. The universal scaling behaviors at CQTs extend beyond the equilibrium conditions [5]. Indeed dynamic protocols entailing out-of-equilibrium evolutions develop scaling behaviors as well, in the appropriate limits, related to the universality class of the CQT. For example, out-of-equilibrium scaling behaviors emerge when analyzing the quantum evolutions arising from a quantum quench (QQ), see, e.g., Refs. [5–11], or from slow changes of the Hamiltonian parameters across the transition point, such as the protocols associated with the so-called quantum Kibble-Zurek (KZ) problem, see, e.g., Refs. [5,12–23].

These out-of-equilibrium issues at quantum transitions have been mostly addressed within isolated many-body

systems, unitarily driven by their Hamiltonian and the Schrödinger equation. In this paper, we extend such studies to investigate how the interaction with a thermal bath, coupled homogeneously to the system, affects the out-of-equilibrium dynamics of many-body systems within the critical regime of a zero-temperature quantum transition, such as that arising from a QQ or a slow crossing of the quantum critical regime.

The role of the temperature within the equilibrium critical behavior at a CQT is generally associated with one of the relevant RG perturbations at the stable fixed point of the RG flow controlling the quantum criticality [3–5,24]. Therefore the quantum scaling behavior can be only observed in the zero-temperature limit. More precisely, the quantum scaling limit requires that the zero-temperature critical point is approached keeping the ratio T/Δ fixed, where Δ is the gap at the quantum critical point, which is generally power-law suppressed. For example, in the FSS limit the gap is suppressed as $\Delta \sim L^{-z}$ at the critical point, where L is the size of the system and $z > 0$ is the universal dynamic exponent associated with universality class of the CQT. Within the equilibrium critical regime the temperature enters the asymptotic FSS laws through a further dependence of the scaling functions on the scaling variable $\Xi \equiv TL^z \sim T/\Delta$.

The role of the temperature becomes less definite when we consider out-of-equilibrium behaviors, because the temperature of the system is an equilibrium concept. However, one may consider the effects of thermal baths in contact with the system during its out-of-equilibrium dynamics. The main feature of a thermal bath is that it eventually drives the system toward thermalization at its temperature T , in the

large-time limit of the evolution of the system in contact with the thermal bath. The thermalization process must somehow introduce a further timescale τ in the problem, characterizing the approach of the system to the thermal state when it is put in contact with the thermal bath. Such timescale is expected to play an important role in the out-of-equilibrium critical dynamics of the system in contact with the thermal bath. In this paper, we investigate these issues within the simplest dynamic protocols giving rise to out-of-equilibrium behaviors, i.e., those entailing instantaneous QCs of the Hamiltonian parameters starting from equilibrium thermal conditions.

A quench protocol is generally performed by suddenly varying a parameter within a family of Hamiltonians, such as

$$\hat{H}(w) = \hat{H}_c + w\hat{H}_p, \quad (1)$$

where \hat{H}_c and \hat{H}_p are independent of the parameter w , and $[\hat{H}_c, \hat{H}_p] \neq 0$. In a standard QC protocol for closed systems, one usually starts from the ground state $|\Phi_0, w_i\rangle$ of the Hamiltonian $\hat{H}(w_i)$ associated with an initial value w_i of the parameter w , with corresponding density matrix $\rho_i = |\Phi_0, w_i\rangle\langle\Phi_0, w_i|$. At a given time, $t = 0$ say, the Hamiltonian parameter is suddenly changed from w_i to $w \neq w_i$, and the subsequent quantum evolution is supposed to be unitarily driven by the Hamiltonian $\hat{H}(w)$, that is $|\Psi(t)\rangle = e^{-i\hat{H}(w)t}|\Phi_0, w_i\rangle$ (hereafter we set $\hbar = 1$). Several interesting issues have been investigated within QC dynamic protocols. They include the long-time relaxation and the consequent spreading of quantum correlations and entanglement, the statistics of the work, localization effects due to the mutual interplay of interactions and disorder, dynamical phase transitions, the dynamic scaling close to quantum transitions, effects of dissipation or of measurements due to interactions with an environment (see, e.g., Refs. [5,9,25–78]).

To focus on the out-of-equilibrium dynamics close to a quantum transition, we assume that the Hamiltonian \hat{H}_c in Eq. (1) is critical, thus $w = w_c = 0$ represents a quantum critical point. We recall that the critical behavior around the CQT point $w_c = 0$ is characterized by a diverging length scale $\xi \sim |w|^{-\nu}$ of the quantum critical modes, and the power-law suppression $\Delta \sim \xi^{-z}$ of the gap. The out-of-equilibrium dynamics at CQTs develops scaling behaviors controlled by the universality class of the quantum transition, for example, when the Hamiltonian parameters are slowly varied across the critical regime [5,21,23], and in the case of *soft* QC protocols when both the initial and final values of the quenching parameters are such to maintain the system within the critical regime [5,9,59]. In particular, soft QCs require that the energy scale of the QC [i.e., the difference of the energy $\langle\Psi(t)|\hat{H}(w)|\Psi(t)\rangle$ of the evolving state $|\Psi(t)\rangle$ for $t > 0$ and the ground state of $\hat{H}(w)$] is sufficiently small, i.e., comparable with the energy gap $\Delta \sim L^{-z}$ of the spectrum at the transition point in finite-size systems.

To study the effects of a thermal bath in the out-of-equilibrium behavior arising from a QC within the critical regime, we consider two protocols where the thermal baths are involved in different ways.

(i) Within the first protocol the thermal bath is used to prepare the system in a finite-temperature Gibbs state, described by the thermal density matrix (hereafter we set the Boltzmann

constant $k_B = 1$)

$$\rho_i(w_i, T) = \sum_n e^{-E_n(w_i)/T} |\Phi_n, w_i\rangle\langle\Phi_n, w_i|, \quad (2)$$

where $|\Phi_n, w_i\rangle$ are the eigenstates of $\hat{H}(w_i)$. Then the quantum evolution after the quench of the Hamiltonian parameters at $t = 0$ is unitary and driven by the Hamiltonian $\hat{H}(w)$ only, i.e., the thermal bath is removed during the quantum evolution for $t > 0$. Therefore the evolution of the density matrix is driven by the equation

$$\partial_t \rho(t) = -i[\hat{H}(w), \rho(t)], \quad \rho(t = 0) = \rho_i(w_i, T). \quad (3)$$

(ii) In the second protocol, the starting point is the same, i.e., the Gibbs state (2), but the thermal bath is not removed after quenching. Therefore the out-of-equilibrium quantum evolution for $t > 0$ is not unitary anymore, but it is also driven by the interaction with the thermal bath. Under some conditions, discussed in Refs. [5,79–84], the nonunitary evolution arising from the thermal baths can be described by a Lindblad master equation governing the time evolution of the density matrix of the system, which can be written as

$$\partial_t \rho = \mathcal{L}[\rho] \equiv -i[\hat{H}(w), \rho] + \gamma \mathbb{D}_T[\rho], \quad (4)$$

where \mathcal{L} is a Liouvillian superoperator, and \mathbb{D}_T is a dissipative driving whose strength is controlled by the homogeneous coupling γ , playing the role of the decay rate (inverse timescale) associated with the interactions between the system and the bath. The operator \mathbb{D}_T is assumed to be such that the Lindblad master equation (4) drives the system toward an equilibrium Gibbs state at temperature T in the large-time limit.

We argue that, for both types of protocols and sufficiently small temperatures of the thermal baths, the out-of-equilibrium time evolution within the critical regime develops a nontrivial out-of-equilibrium FSS (OFSS) limit, with peculiar scaling behaviors, similar to those arising for closed systems. The effects of the thermal baths can be taken into account by appropriate extensions of the out-of-equilibrium zero-temperature scaling laws describing soft quantum QCs within the critical regime of isolated systems, already put forward by earlier works [5,9]. As a theoretical laboratory to check our extended OFSS laws, we consider the quantum Ising chain [4], or the equivalent fermionic Kitaev wire [85], supplemented with a particular modelization of the thermal bath that guarantees the asymptotic thermalization within the Lindblad formulation of the dynamics of open systems with quadratic Hamiltonians [84,86], such as the fermionic Kitaev wire. In our study, we focus on fermionic Kitaev wires within their quantum critical regime, where they develop notable equilibrium and out-of-equilibrium scaling behaviors. We investigate the effects arising from the contact with thermal baths, as described by the Lindblad master equation, and identify the conditions under which they can be still casted within an out-of-equilibrium scaling framework controlled by the universality class of the CQT.

Our analyses are developed within FSS frameworks, which generally simplify the study of the universal features of critical behaviors, with respect to studies in the thermodynamic limit. In the FSS limit the general requirement of a large length scale ξ of the critical correlations is not subject to

further conditions on the system size L . It only requires that $\xi \sim L$, while critical behaviors in the thermodynamic limit requires $\xi \ll L$. Therefore much larger systems are necessary to probe analogous length scales ξ in the thermodynamic limit. Equilibrium and out-of-equilibrium FSS behaviors are often observed for systems of moderately large size, see, e.g., Refs. [5,9,57,87,88]. Thus FSS behaviors should be more easily accessed by numerical computations and experiments where the quantum dynamics can be monitored for a limited number of particles or spins, such as experiments with quantum simulators in laboratories, e.g., by means of trapped ions [89,90], ultracold atoms [91,92], or superconducting qubits [93,94].

The paper is organized as follows. In Sec. II, we present the fermionic Kitaev wire, equivalent to the quantum Ising chain, and the model of thermal bath that we use as theoretical laboratory for our study; we also outline the QQ protocols that we consider and define the observables to monitor the quantum evolution after quenching. In Sec. III, we outline the out-of-equilibrium scaling scenarios that are expected to be developed under the dynamic QQ protocols considered, and support them by numerical computations for the fermionic Kitaev wires in contact with the thermalizing bath. Finally, in Sec. IV, we summarize, draw our conclusions, and add some remarks on the extension of this study to the dynamic KZ protocols slowly crossing quantum critical regimes. The Appendix reports some details on the numerical computations for the QQ protocols within fermionic Kitaev wires in contact with a thermal bath.

II. KITAEV FERMIONIC WIRES AND THERMAL BATHS

A. The fermionic Kitaev chain

We consider fermionic Kitaev wires of L sites with open boundary conditions, whose quantum unitary dynamics is driven by the Hamiltonian [85]

$$\hat{H}_K = -J \sum_{x=1}^{L-1} (\hat{c}_x^\dagger \hat{c}_{x+1} + \hat{c}_x \hat{c}_{x+1}^\dagger + \text{H.c.}) - \mu \sum_{x=1}^L \hat{n}_x, \quad (5)$$

where \hat{c}_x is the fermionic annihilation operator associated with the site x of the chain, $\hat{n}_x \equiv \hat{c}_x^\dagger \hat{c}_x$ is the particle density operator. In the following, we assume J as the energy scale, thus we set $J = 1$.

The Hamiltonian (5) can be mapped into a quantum Ising chain, by means of the Jordan-Wigner transformation, see, e.g., Ref. [4]. The corresponding spin model is the quantum Ising chain with open boundary conditions, i.e.,

$$\hat{H}_{\text{Is}} = - \sum_{x=1}^{L-1} \hat{\sigma}_x^{(1)} \hat{\sigma}_{x+1}^{(1)} - g \sum_{x=1}^L \hat{\sigma}_x^{(3)}, \quad (6)$$

$\hat{\sigma}_x^{(k)}$ being the Pauli matrices and $g = -\mu/2$. In the following, we prefer to stick with the Kitaev quantum wire, because the thermal baths and observables that we consider are best defined within the fermionic model. However, the general scaling scenarios that will emerge apply to both models.

The Kitaev model undergoes a CQT at $\mu = \mu_c = -2$ (corresponding to $g = g_c = 1$ in the quantum Ising chain), between a disordered quantum phase for $\mu < \mu_c$ (correspond-

ing to $g > 1$) and an ordered quantum phase for $|\mu| < |\mu_c|$ (corresponding to $|g| < 1$). Thus we define

$$w = \mu - \mu_c = \mu + 2, \quad (7)$$

so that one can easily see the correspondence between the Kitaev Hamiltonian (5) and the generic one reported in Eq. (1), i.e., \hat{H}_c corresponds to the Hamiltonian (5) for $\mu = \mu_c$, and $\hat{H}_p = -\sum_{x=1}^L \hat{n}_x$. The continuous transition at $w = w_c$ belongs to the two-dimensional Ising universality class [4,5], characterized by the length-scale critical exponent $\nu = 1$, related to the RG dimension $y_w = 1/\nu = 1$ of the Hamiltonian parameter w . This implies that, approaching the critical point, the length scale ξ of the critical quantum fluctuations diverges as $\xi \sim |w|^{-\nu}$. The dynamic exponent $z = 1$ associated with the unitary quantum dynamics can be obtained from the power law $\Delta \sim \xi^{-z}$ of the vanishing gap with increasing ξ . Moreover, the RG dimension of the fermionic operators \hat{c}_j and \hat{c}_j^\dagger at the CQT is $y_c = 1/2$, and that of the particle density operator \hat{n}_x is $y_n = 1$ [4,5].

B. Modelization of the thermal bath

In our study, we consider a modelization of interaction with a thermal bath within the Lindblad master equation (4), whose asymptotic large-time behavior leads to a Gibbs density matrix at a given finite temperature T . In particular, we consider the proposal developed in Ref. [84] which applies to quantum models described by quadratic Hamiltonians, such as that of the fermionic Kitaev wires. This provides a relatively simple modelization of a thermal bath leading to thermalization in the large-time limit of the corresponding Lindblad master equation for the density matrix of the system.

The Kitaev Hamiltonian (5) with open boundary conditions can be diagonalized in the Nambu field space by a Bogoliubov transformation, see, e.g., Refs. [84,95,96], so that we can rewrite it as

$$\hat{H}_K = \sum_{k=1}^L \omega_k \hat{b}_k^\dagger \hat{b}_k, \quad (8)$$

where ω_k are values of the spectrum of the Bogoliubov eigenoperators \hat{b}_k (we are neglecting an irrelevant constant term). Note that both ω_k and \hat{b}_k depend on the Hamiltonian parameter μ . The relation between the fermionic operators \hat{c}_x and the Bogoliubov eigenoperators \hat{b}_k can be generally written as [84,95,96]

$$\hat{c}_x = \sum_{k=1}^L A_{xk} \hat{b}_k + B_{xk} \hat{b}_k^\dagger, \quad (9)$$

where A and B are appropriate $L \times L$ matrices depending on μ . Following Refs. [84,86], we write the dissipator $\mathbb{D}_T[\rho]$ in the Lindblad master equation (4) in terms of the Bogoliubov eigenoperators as

$$\begin{aligned} \mathbb{D}_T[\rho] = & \sum_k [1 - f(\omega_k, T)] (2 \hat{b}_k \rho \hat{b}_k^\dagger - \{\hat{b}_k^\dagger \hat{b}_k, \rho\}) \\ & + \sum_k f(\omega_k, T) (2 \hat{b}_k^\dagger \rho \hat{b}_k - \{\hat{b}_k \hat{b}_k^\dagger, \rho\}), \end{aligned} \quad (10)$$

where $\{\hat{A}, \hat{B}\}$ indicates the anticommutator, and

$$f(\omega_k, T) = (1 + e^{\omega_k/T})^{-1}. \quad (11)$$

When using this homogeneous dissipator term, the Lindblad master equation (4) ensures the asymptotic large-time thermalization [84]. Therefore

$$\lim_{t \rightarrow \infty} \rho(t) = \rho_t(w, T), \quad (12)$$

$$\rho_t(w, T) = \sum_n e^{-E_n(w)/T} |\Phi_n, w\rangle \langle \Phi_n, w|, \quad (13)$$

where $\rho_t(w, T)$ is the density matrix representing the thermal state, $E_n(w)$ and $|\Phi_n, w\rangle$ are the eigenvalues and eigenstates of $\hat{H}(w)$. The asymptotic approach to the thermal distribution is controlled by the decay-rate parameter γ [84]. Indeed the Liouvillian gap $\Delta_{\mathcal{L}}$ that controls the exponential approach to the asymptotic stationary state of the Lindblad equation is proportional to the decay rate γ , i.e.,

$$\Delta_{\mathcal{L}} \sim \gamma. \quad (14)$$

The above modelization of thermal baths provides a useful theoretical laboratory to investigate issues related to the out-of-equilibrium dynamics in the presence of thermal baths. Its derivation has been thoroughly discussed in Ref. [84]. We also mention that it has been employed in Refs. [86,97]. Some details of the computations using the Lindblad master equation (4) with the dissipator (10) are reported in the Appendix.

C. Quantum-quench protocols

As already anticipated in Sec. I, we consider two protocols, differing for the absence or presence of the contact with the thermal bath during the quantum evolution after quenching, giving respectively rise to unitary or dissipative dynamics after quenching. We call them *unitary* and *dissipative* QQ protocols, respectively.

(i) *Unitary* QQ protocol. In this simplest QQ protocol, the role of the thermal bath is limited to that of preparing the initial Gibbs state $\rho_t(w_i, T)$ at $t = 0$, reported in Eq. (2). This can be obtained by keeping the thermal bath in contact with the system for a sufficiently long time t_{th} , i.e., $t_{\text{th}} \gg \gamma^{-1}$. Then at $t = 0$ the Hamiltonian parameter is instantaneously quenched from $w_i < 0$ to $w \geq 0$ and the thermal bath is removed, so that the subsequent time evolution is that of a closed fermionic wire, i.e., it is unitary and only driven by the Hamiltonian of the system, cf. Eq. (3).

(ii) *Dissipative* QQ protocol. The quantum evolution starts from the same initial Gibbs state $\rho_t(w_i, T)$, but the thermal bath is maintained in contact with the system after the QQ from $w_i < 0$ to $w \geq 0$, at $t = 0$. Therefore the quantum evolution for $t > 0$ is driven by the Lindblad master equation (4) with the dissipator term (10). Note that this dynamic protocol entails a further timescale $\tau = \gamma^{-1}$, characterizing the asymptotic exponential approach to the large-time stationary Gibbs state associated with the Hamiltonian $\hat{H}(w)$ and temperature T .

D. Observables monitoring the time evolution

To characterize the dynamic properties of the quantum evolution after the QQ at $t = 0$, we consider the subtracted

particle-density average

$$n_s(t, L) = \frac{1}{L} \text{Tr} \left[\rho(t) \sum_{x=1}^L \hat{n}_x \right] - n_c(L), \quad (15)$$

where $n_c(L)$ is the ground-state particle density of the Kitaev wire of size L at the critical point $w_c = 0$ (in the infinite-size limit $n_c = 1/2 - 1/\pi$ [95]). Note that the particle density operator \hat{n}_x and the transverse spin component $\hat{\sigma}_x^{(3)}$ of the quantum Ising chain (6) are trivially related, indeed $\hat{\sigma}_x^{(3)} = 2\hat{n}_x$. In the definition of n_s , the subtraction of $n_c(L)$ simplifies the scaling behavior of $n_s(t, L)$ within the critical regime, canceling the leading analytical behavior [5,24]. To monitor the spatial correlations, we also consider

$$P(x, y, t) = \text{Tr}[\rho(t) (\hat{c}_x^\dagger \hat{c}_y^\dagger + \hat{c}_y \hat{c}_x)], \quad (16)$$

$$C(x, y, t) = \text{Tr}[\rho(t) (\hat{c}_x^\dagger \hat{c}_y + \hat{c}_y^\dagger \hat{c}_x)]. \quad (17)$$

Some details on the computation of the above quantities during the time evolution of the QQ protocols are reported in the Appendix. We finally mention that the above choice of observables to monitor the QQ dynamics is essentially motivated by the numerical convenience, to achieve the most accurate evidences of the validity of the out-of-equilibrium scaling behaviors allowing for the thermal baths, put forward below. Indeed, the equations for the correlation functions P and C within the Lindblad framework, see Appendix, allow us to obtain results for relatively large systems.

III. OUT-OF-EQUILIBRIUM SCALING

We now discuss the out-of-equilibrium behaviors arising from the QQ protocols outlined in Sec. II C. We show that they develop OFSS behaviors where the effects of the thermal baths are taken into account by appropriate extensions of the out-of-equilibrium zero-temperature scaling laws describing soft QQs in closed systems within their critical regime, already put forward by earlier works [5,9].

The OFSS behaviors that we put forward for QQ protocols considered are verified by numerical computations for the fermionic Kitaev wire up to relatively large sizes. See the Appendix for details on such calculations.

As a preliminary example of out-of-equilibrium QQ behaviors that we want to address, in Fig. 1 we show some results for the quantum evolution of the subtracted particle density (15) along the dissipative protocol outlined in Sec. II C, after quenching a fermionic Kitaev wire of size $L = 60$, from $w_i = -0.01$ to $w = 0$, in the presence of a thermal bath at a temperature $T = 2$, and various values of the decay rate γ . The quantum evolution turns out to have a significant dependence on the decay-rate parameter γ that characterized the interactions between the system and the thermal bath. Indeed, the curves of the subtracted particle density appear to approach its equilibrium value $n_{s,\text{eq}}(w = 0, T = 2) \approx 0.0004601 \dots$ (while at $t = 0$ we have $n_{s,\text{eq}}(w = w_i, T = 2) \approx 0.126598 \dots$), faster and faster with increasing γ , actually exponentially as $\exp(-t/\tau)$ with $\tau \sim \gamma^{-1}$, confirming the role of decay rate of the parameter γ within the Lindblad master equation, cf. Eq. (14). Analogous results are obtained for other observables, such as fermionic correlation functions

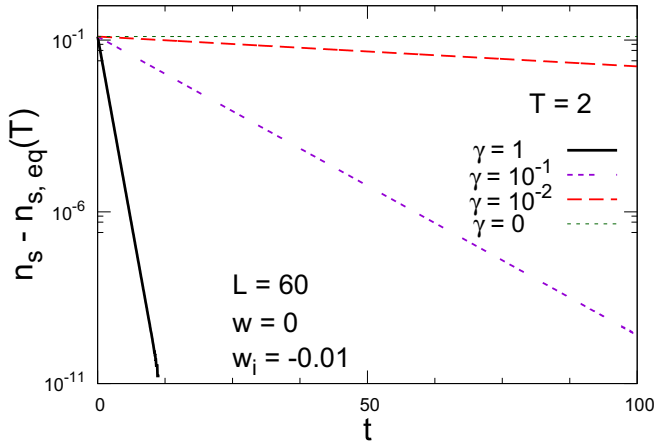


FIG. 1. The quantum evolution of the subtracted particle density $n_s(t)$, cf. Eq. (15), for the dissipative QQ protocol entailing a dissipative dynamics after the QQ at $t = 0$ of the Hamiltonian parameter w , describing the persistent interaction with the thermal bath, cf. Eqs. (4) and (10). These curves refer to a system of size $L = 60$, temperature $T = 2$ of the thermal bath, quenching from $w_i = -0.01$ to $w = 0$, and various values of the decay rate γ (the case $\gamma = 0$ corresponds to the evolution of the close system). We plot the difference $n_s(t, L, T) - n_{s,eq}(L, T)$ which is expected to vanish in the large-time limit. In this figure and in the following ones, the unity that we use are such that $\hbar = 1$, $k_B = 1$, and $J = 1$.

defined in Sec. IID. In the following we put forward an out-of-equilibrium scaling theory for these out-of-equilibrium phenomena within the quantum critical regime.

A. Zero-temperature scaling in quantum quenches

We now provide a brief summary of the out-of-equilibrium scaling theory for close systems, describing QQ protocols within the critical regime [5,9]. The initial state is the ground state associated with an initial value $w_i < 0$, and, after the instantaneous quench at $t = 0$ from w_i to w , the quantum evolution is driven by the Schrödinger equation.

Out-of-equilibrium scaling laws can be obtained by extending those valid at equilibrium, allowing for a time dependence essentially controlled by the time scaling variable $\Theta \sim t \Delta$, which is obtained by assuming that the relevant timescale of the critical modes is proportional to the inverse energy difference Δ of the lowest states. We refer to Ref. [5] for a through presentation of the scaling arguments leading to the asymptotic OFSS behaviors.

Let us consider the out-of-equilibrium evolution (after quenching) of generic observables, such as the expectation value O at time t of a local operator $\hat{O}(x)$ and its fixed-time correlations $G_O = \langle \hat{O}(x) \hat{O}(y) \rangle$. The general working hypothesis underlying out-of-equilibrium FSS frameworks is that the expectation value of $\hat{O}(x)$ and its correlation functions obey asymptotic homogeneous scaling laws [5], such as

$$O(t, \mathbf{x}, L, w_i, w) \approx b^{-y_o} \mathcal{O}(t/b^z, \mathbf{x}/b, L/b, b^{y_w} w_i, b^{y_w} w), \quad (18)$$

where b is an arbitrary (large) length scale, y_o is the RG dimension of the local operator \hat{O}_x and the RG exponents

y_w and z are determined by the universality class of the CQT (they are the RG dimensions of the Hamiltonian parameter w and the temperature T , respectively). Thus both the initial and final values of w , i.e., w_i and w , take the same RG exponent y_w , being coupled to the RG perturbation \hat{H}_p within the Hamiltonian. Note that we do not assume translation invariance, which is generally broken by the presence of boundaries, such as those arising from open boundary conditions.

OFSS can be straightforwardly derived by fixing $b = L$ in the above homogenous scaling law. Then, we expect the OFSS of the expectation value O of a generic local operator \hat{O}_x , of its spatial average $\hat{O}_a = L^{-d} \sum_x \hat{O}_x$, and its two-point correlation function G_O , develop the asymptotic OFSS behavior [5,9]

$$O(t, \mathbf{x}, L, w_i, w) \approx L^{-y_o} \mathcal{O}(\Theta, \mathbf{X}, \Phi_i, \Phi),$$

$$O_a(t, L, w_i, w) \approx L^{-y_o} \mathcal{O}_a(\Theta, \Phi_i, \Phi), \quad (19)$$

$$G_O(t, \mathbf{x}_1, \mathbf{x}_2, L, w_i, w) \approx L^{-2y_o} \mathcal{G}_O(\Theta, \mathbf{X}_1, \mathbf{X}_2, \Phi_i, \Phi),$$

where the scaling variables appearing in the scaling functions \mathcal{O} , \mathcal{O}_a , and \mathcal{G}_O are defined as

$$\Theta \equiv \frac{t}{L^z}, \quad \mathbf{X}_i \equiv \frac{\mathbf{x}_i}{L}, \quad \Phi_i \equiv L^{y_w} w_i, \quad \Phi \equiv L^{y_w} w. \quad (20)$$

The OFSS limit is obtained in the large- L and large- t limit keeping the above scaling variables fixed. These conditions ensure that the system remains within the universal critical regime during the quantum evolution. Note that in the scaling law (20) the dynamic features are essentially encoded in the time dependence of the scaling variable $\Theta \sim t \Delta$. The other features, in particular when $w_i = w$, are analogous to those arising from equilibrium FSS at CQTs [5,24], where the argument $\Phi = L^{y_w} w$ of the scaling functions is controlled by the RG dimension y_w of the relevant parameters w at the RG fixed point associated with the CQT.

The above OFSS equations can be straightforwardly applied to the observables defined in Sec. IID, after a quench from w_i to w at $t = 0$, keeping into account that the RG dimension of the subtracted particle density is $y_n = 1$, and that of the fermionic operator \hat{c}_x is $y_c = 1/2$. Note that the dominant analytical contributions to the particle density [5,24] coming from the analytical background are canceled in the difference n_s defined in Eq. (15), whose leading asymptotic behavior arises from the quantum critical modes, therefore it is analogous to that of O_a in Eq. (19), with $y_o = y_n$. Analogously one can apply the OFSS in Eq. (19) to observables and correlation functions constructed with the spin operators of the quantum spin chain (6). The OFSS functions are expected to be universal with respect to the microscopic details of the model, apart from nonuniversal multiplicative rescaling and normalizations of its arguments. Within isolated fermionic Kitaev wires and quantum Ising chains, the OFSS arising from soft QQs has been verified by numerical computations for various boundary conditions, and also along their quantum first-order transition line [5,9].

The OFSS limit is expected to be approached with power-law suppressed corrections. There are various sources of scaling corrections when approaching the OFSS. Of course, they include those that are already present at equilibrium. In particular, the irrelevant RG perturbations are sources of scaling corrections for the asymptotic behavior of the

free-energy density [5,99]. In the case of one-dimensional quantum systems undergoing CQTs belonging to the two-dimensional Ising universality class, the leading scaling corrections from irrelevant RG perturbations are suppressed as $L^{-\omega}$ with $\omega = 2$ [24,98]. However, other contributions may become more relevant [5,24,99], such as those arising from the presence of analytical backgrounds, from the presence of boundaries (which generally gives rise to $O(1/L)$ corrections), and, in the case of correlation functions, from RG mixings of the source fields [this for example happens in the case of the correlation functions of the fermionic field \hat{c}_x , for which corrections are $O(1/L)$]. These scaling corrections have been confirmed by numerical results [5,24]. Therefore we expect that the asymptotic OFSS of fermionic Kitaev wires and quantum Ising chains with open boundary conditions is generally approached with $O(1/L)$ corrections.

B. OFSS along the unitary QQ protocol

For the simplest unitary protocol reported in Sec. II C, where the quantum evolution is that of the isolated fermionic wire, the request that the dynamics remains within the critical regime implies that the temperature of the initial Gibbs state must be appropriately suppressed in the large- L OFSS limit, to obtain a nontrivial out-of-equilibrium critical limit. This is analogous to what happens within the equilibrium FSS, where one introduces the scaling variable [3–5]

$$\Xi \equiv L^z T, \quad (21)$$

to allow for a nonzero temperature in the FSS of the observables. Therefore, like equilibrium FSS, we conjecture that the temperature of the initial Gibbs state enters the OFSS associated with the unitary QQ protocol by adding a further dependence on Ξ in the scaling functions (19). In other words, a nontrivial asymptotic OFSS limit is expected to be realized in the large- L and large- t limits keeping also Ξ fixed, beside the scaling variables already defined in Eq. (20). Therefore we expect that the OFSS of standard QQ protocols starting from ground states, cf. Eq. (19), changes into

$$O(t, \mathbf{x}, L, w_i, w, T) \approx L^{-\gamma_0} \mathcal{O}(\Theta, \mathbf{X}, \Phi_i, \Phi, \Xi), \quad (22)$$

and analogously for its spatial average O_a and the correlation function G_O .

The numerical analysis for the fermionic Kitaev wire under the unitary protocol fully support to this OFSS, obtained by extending the QQ FSS behaviors of closed systems starting from an initial ground state. This is clearly demonstrated by the curves reported in Fig. 2, associated with the quantum evolutions of the subtracted particle density $n_s(t)$ and the fermionic correlation $P(x, y, t)$ (the other fermionic correlation $C(x, y, t)$ develops an analogous OFSS).

C. OFSS along the dissipative QQ protocol

We now discuss the dynamics arising from the dissipative protocol outlined in Sec. II C, when the quantum evolution after quenching is described by the Lindblad master equation (4) with the thermal-like dissipator (10), to modelize the in-

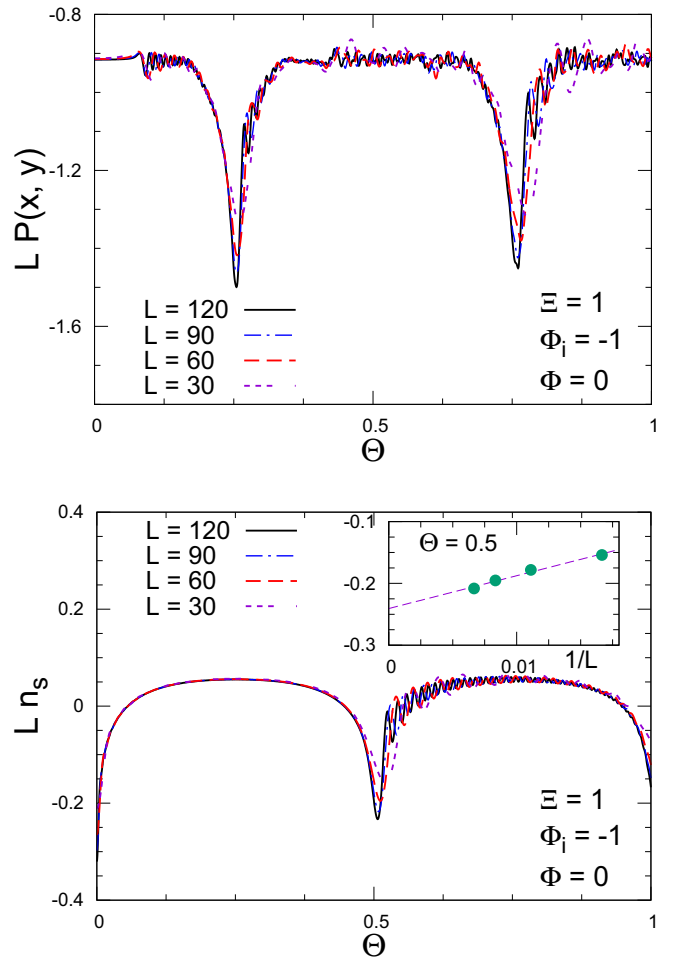


FIG. 2. OFSS behavior of the subtracted particle density (bottom) and the fermionic correlation function $P(x = L/3, y = 2L/3, t)$, cf. Eq. (16), arising from the unitary QQ protocol, for various lattice sizes L , at fixed $\Xi = L^z T = 1$, $\Phi_i = L^{y_w} w_i = -1$, and $\Phi = L^{y_w} w = 0$, vs the time scaling variable $\Theta = t/L^z$. These computations nicely support the OFSS behaviors reported in Eq. (22). The inset of the bottom figure shows that the approach to the OFSS limit is consistent with $O(1/L)$ corrections. Analogous results are obtained for other values of the scaling variables.

teraction with a thermal bath characterized by a temperature T (which does not change after quenching) and decay rate γ .

We expect that the temperature T of the thermal bath must be rescaled as in the case of the unitary QQ protocol, i.e., we must consider again the associated scaling variable Ξ already defined in Eq. (21). However, since the QQ moves the system out-of-equilibrium, also the decay rate γ , and corresponding timescale $\tau = \gamma^{-1}$, associated with the interactions with the thermal bath is expected to play a relevant role to establish a corresponding nontrivial OFSS limit. This was already noted in Ref. [97] in the analysis of dynamic protocols entailing the variation of the temperature at the critical point.

When keeping τ constant in the FSS limit where the scaling variable $\Theta = t/L^z$ is kept fixed, in the large- L limit we have eventually that

$$t = \Theta L^z \gg \tau, \quad (23)$$

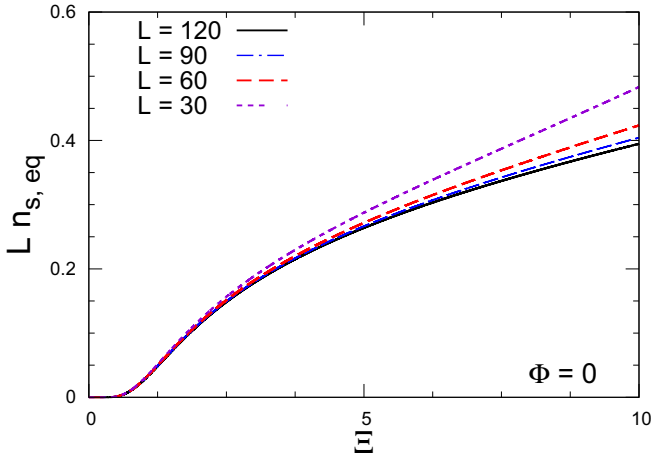


FIG. 3. Equilibrium FSS of the subtracted particle density $n_{s,\text{eq}}$ at the critical point $w = 0$ vs the rescaled temperature $\Xi = L^z T$. With increasing L , the data show the expected convergence to the equilibrium FSS reported in Eq. (24) with $y_n = 1$.

which is the condition ensuring thermalization for any finite value $\Theta > 0$. Therefore, when keeping τ fixed, the quantum evolution is not expected to develop a nontrivial OFSS limit. Indeed, in the large- L limit, the system turns out to suddenly approach an equilibrium Gibbs state (associated with the Hamiltonian parameter w and temperature T) with respect to the rescaled time Θ , without any further relevant evolution of the system for any $\Theta > 0$. Therefore, if the temperature is rescaled by keeping $\Xi = L^z T$ fixed, we must recover the equilibrium FSS behavior in the presence of a thermal bath at temperature T , such as that associated with the subtracted particle density [5,24]

$$n_{s,\text{eq}}(w, L, T) \approx L^{-y_n} \mathcal{N}(\Phi, \Xi), \quad (24)$$

where $\Phi = L^{y_w} w$, and the temperature dependence enters through the associated scaling variable $\Xi = L^z T$. In Fig. 3, we show some equilibrium data at the critical point $w = \Phi = 0$, versus Ξ , showing the approach to the asymptotic large- L equilibrium FSS (24). The realization of the equilibrium FSS within the QQ protocol at fixed γ is demonstrated by the plots reported in Fig. 4, which show the somewhat trivial convergence toward the equilibrium FSS for any finite $\Theta > 0$.

The above results suggest that also the decay rate γ of the system-bath interactions must be rescaled to observe a nontrivial OFSS limit as a function of the time scaling variable Θ , to create the conditions for a balanced competition between the critical Hamiltonian driving and the interactions with the thermal bath. As already put forward in the case of other homogeneous dissipative terms in the Lindblad equation [5,55,100–102], for example associated with particle-decay or particle-pumping dissipative mechanisms, a nontrivial OFSS limit is obtained by rescaling the decay rate of the dissipative term, so that the scaling variable

$$\Gamma \equiv L^z \gamma \sim \gamma / \Delta \quad (25)$$

is kept fixed in the OFSS limit, where Δ is the energy difference of the lowest eigenstates of $\hat{H}(w)$ at the critical point $w = w_c = 0$. Then an OFSS behavior emerges from the non-

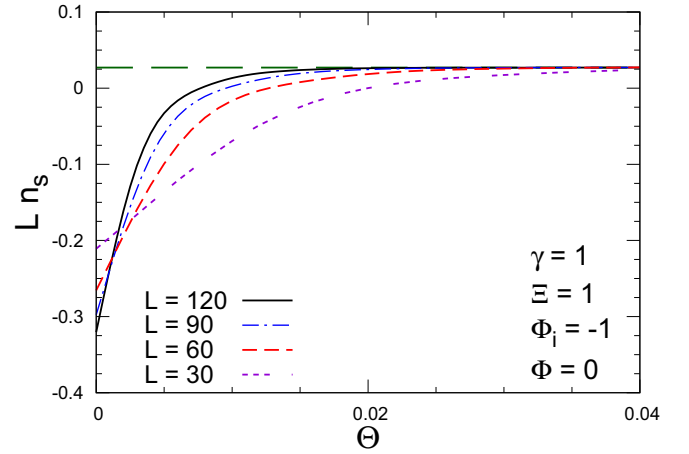


FIG. 4. Quantum evolution of the subtracted particle density arising from the dissipative QQ protocol, when rescaling all quantities involved in the quench protocol, except for the decay rate γ . With increasing L , the curves appear to approach the equilibrium FSS value at finite temperature (where the temperature dependence enters through the scaling variable $\Xi = L^z T$) faster and faster, reflecting a nonuniform convergence for any $\Theta > 0$. The dashed line shows the equilibrium value of n_s for $\Phi = 0$ and $\Xi = 1$, which is asymptotically approached by the various curves.

trivial competition between the critical unitary dynamics and the dissipative driving arising from the thermal bath.

In conclusion, on the basis of the above scaling arguments, the OFSS arising from the dissipative QQ protocols in the presence of a thermal bath is expected to be given by

$$O_a(t, L, w_i, w, T, \gamma) \approx L^{-y_o} \mathcal{O}_a(\Theta, \Phi_i, \Phi, \Xi, \Gamma) \quad (26)$$

and

$$\begin{aligned} G_O(t, \mathbf{x}_1, \mathbf{x}_2, L, w_i, w, T, \gamma) \\ \approx L^{-2y_o} \mathcal{G}_O(\Theta, \mathbf{X}_1, \mathbf{X}_2, \Phi_i, \Phi, \Xi, \Gamma). \end{aligned} \quad (27)$$

In the large- Γ limit, the above OFSS behaviors at fixed Ξ is expected to approach the corresponding equilibrium FSS, faster and faster in terms of Θ , matching the behavior at finite γ . Moreover, we also expect that the equilibrium FSS is also approached in the large- Θ limit at fixed Γ and Ξ , independently of Γ , but faster and faster with increasing Γ .

Again, the numerical results for the particle density $n_s(t)$ and correlation functions P and C fully support the above OFSS equations, i.e., Eq. (26) for $n_s(t)$ with $y_o = y_n = 1$, and Eq. (27) for P and C with $y_o = y_c = 1/2$. Some results are reported in Fig. 5. We also stress that analogous results are expected for other observables, for example the correlation functions of the spin operator of the equivalent formulation provided by the quantum Ising chains.

IV. CONCLUSIONS

We have reported a study of the effects of thermal baths to the out-of-equilibrium dynamics of many-body systems within their quantum critical regime close to a zero-temperature CQT. In particular, we analyze the out-of-equilibrium quantum evolution arising from QQs of the

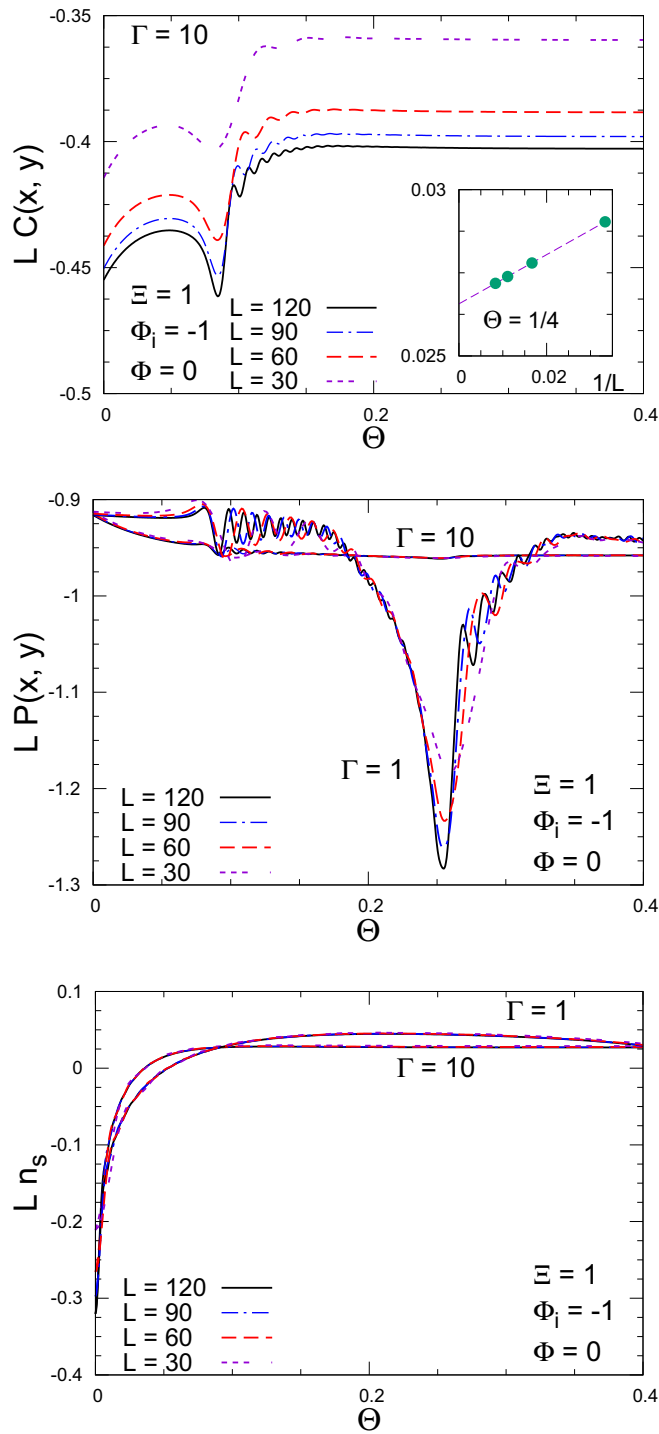


FIG. 5. Quantum evolutions along the dissipative protocol, fully supporting the OFSS reported in Eqs. (26) and (27). We report curves for $L n_s$ (bottom), $LP(x = L/3, y = 2L/3, t)$ (middle), and $C(x = L/3, y = 2L/3, t)$ (top), for various values of L , at fixed $\Phi_i = -1$, $\Phi = 0$, $\Xi = 1$, and two values of $\Gamma = L^z \gamma$, i.e., $\Gamma = 1, 10$ (except for the top figure where we only report data for $\Gamma = 10$ to ensure a good readability). The inset of the top figure shows that the OFSS is approached with $O(1/L)$ corrections. Analogous results are obtained for other values of the scaling variables.

Hamiltonian parameters within two different protocols involving a thermal bath coupled homogeneously to the system.

Within the first protocol, named unitary QQ protocol, the thermal bath is used to prepare the system at $t = 0$ in a finite-temperature Gibbs state, then the dynamics after quenching of the Hamiltonian parameters is assumed unitary, i.e., the thermal bath is removed during the quantum evolution for $t > 0$. The second protocol, named dissipative QQ protocol, starts from the same initial condition, but the thermal bath is not removed after quenching, and the quantum evolution for $t > 0$ is assumed to be described by the Lindblad master equation (4). The dissipative term of the Lindblad equation is supposed to simulate a thermal bath, such that the many-body system is driven to a large-time finite-temperature Gibbs state. This dissipative protocol is characterized by a further timescale $\tau = \gamma^{-1}$, related to the decay rate of the interactions between the system and the bath.

Within OFSS frameworks, we argue that, when the thermal baths are associated with a sufficiently small temperature, their effects can be taken into account by appropriate extensions of the zero-temperature out-of-equilibrium scaling laws describing soft QQs of isolated systems within the critical regime. For the unitary QQ protocol, where the thermal bath only determines the initial Gibbs state and the evolution is unitary, a nontrivial OFSS limit is simply obtained by rescaling the temperature as $T \sim L^{-z}$, similarly to equilibrium FSS. Along the dissipative QQ protocol, where the thermal bath is not removed after quenching, the dynamics is more complicated, and the decay rate γ plays a relevant role. Indeed, in addition to the rescaling of the temperature T associated with thermal bath, one also needs to rescale γ as $\gamma \sim L^{-z}$ to obtain a nontrivial OFSS. Otherwise, when keeping γ fixed, the dynamics converges toward the equilibrium FSS at finite temperature, which happens suddenly after quenching with respect to the timescale $t_c \sim L^z$ of the critical regime. Therefore the scaling behavior when keeping γ fixed becomes somehow trivial, reproducing the equilibrium FSS for any rescaled time $\Theta = L^{-z}t > 0$ in the large- L limit.

Our scaling arguments are supported by numerical results with the paradigmatic fermionic Kitaev model, or equivalently quantum Ising chain, at its CQT separating quantum disordered and ordered phases. We consider a particular modelization of the thermal bath that guarantees the asymptotic thermalization within the Lindblad formulation of the dynamics of open systems. However, we note that the scaling arguments used to arrive at the OFSS laws for critical QQs are general, and therefore we expect that the emerging out-of-equilibrium scenarios also apply to many-body systems at generic CQTs in contact with homogenous thermal baths, in any dimension.

We finally remark that the out-of-equilibrium scaling arguments we put forward, leading to the OFSS of QQs in the presence of a thermal bath, can be extended to other protocols giving rise to out-of-equilibrium dynamics. Another interesting class of dynamic protocols entails slow variations of the Hamiltonian parameters across the critical regime of a quantum transition, such as those associated with the quantum KZ problem (see, e.g., Refs. [5, 12–23]). In standard KZ protocols starting from the ground state for an initial parameter $w_i < 0$, the out-of-equilibrium quantum evolution arises from the linear time dependence of one Hamiltonian parameter, $w(t) = t/t_s$ in Eq. (1), where t_s is the timescale of the KZ

protocol. Since $w(t)$ crosses the critical point at $t = 0$, the system passes through the quantum critical regime, moving it away from equilibrium even in the large- t_s limit, and developing a peculiar out-of-equilibrium scaling behaviors. In particular, the interplay between the size L of the system and the timescale t_s of the protocol develops OFSS behaviors [5,23] when $t_s \rightarrow \infty$ and $L \rightarrow \infty$, keeping the scaling variables $\Omega_t \equiv t/t_s^\kappa = t/t_s^{z/(y_w+z)}$ and $\Upsilon \equiv t_s/L^{y_w+z}$ (thus $\Omega_t = t/t_s^{1/2}$ and $\Upsilon = t_s/L^2$ for the fermionic Kitaev wire or quantum Ising chain) fixed. In particular, in isolated fermionic wires subject to the above KZ protocol with $w(t) = t/t_s$, the subtracted particle-density average (15) is expected to asymptotically behave as [5]

$$n_s(t, t_s, L) \approx L^{-y_n} F(\Omega_t, \Upsilon), \quad (28)$$

in the limit of large t_s and L keeping Ω_t and Υ fixed. This is the standard KZ OFSS of isolated systems, from which one may also derive the scaling behavior of the residual defects for $w(t) > 0$ [5,14–21].

KZ-like protocols can be also extended to systems interacting with a thermal bath, such as that outlined in Sec. II B, starting from a Gibbs state for an initial $w_i < 0$ and the temperature T of the thermal bath. Then we may consider a time evolution driven by the Lindblad master equation (4), with a time-dependent Hamiltonian $\hat{H}[w(t)]$ and the dissipator term (10), where also the Bogoliubov operators are assumed to be time dependent to adapt themselves to the time dependence of w . Analogously to the OFSS of QCs in contact with thermal baths, to define a nontrivial OFSS limit in KZ protocols, we expect that both the temperature T and the decay rate γ associated with the bath must be rescaled, as $T \sim L^{-z}$ and $\gamma \sim L^{-z}$. Therefore the OFSS of the subtracted particle-density average (15) in the presence of a thermal bath is expected to turn into

$$n_s(t, t_s, L, T, \gamma) \approx L^{-y_n} F_t(\Omega_t, \Upsilon, \Xi, \Gamma), \quad (29)$$

with $\Xi \equiv TL^z$ and $\Gamma \equiv \gamma L^z$, and keeping the initial value w_i fixed. Note that if only the temperature of the thermal bath is rescaled as $T \sim L^{-z}$, while $\gamma > 0$ is kept fixed, the time interval associated with a variation of Ω_t in the KZ scaling limit, i.e., $\Delta_{\Omega} t \sim t_s^\kappa \Delta \Omega_t$, becomes eventually much larger than the timescale $\tau \sim \gamma^{-1}$ of the interaction with the thermal bath. Since $\tau/\Delta_{\Omega} t \rightarrow 0$ in the KZ limit, the system effectively thermalizes at each rescaled time Ω_t . Therefore, in the KZ limit, the quantum evolution is expected to pass through equilibrium finite-temperature states, thus effectively resulting into equilibrium evolutions reproducing the equilibrium finite-temperature FSS as a function of $L^{y_w} w(t)$. Therefore, like dissipative QQ protocols, the observation of a nontrivial OFSS in KZ protocols requires the simultaneous rescaling of the timescale τ associated with the interaction with the thermal bath. The necessary rescaling of the decay rate γ of the dissipative term in the Lindblad master equation has been also put forward for KZ protocols in the presence of other dissipative mechanisms, such as those related to particle decay or pumping [100].

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APPENDIX: DETAILS ON THE COMPUTATIONS

In this section, we provide some details of the computations for the fermionic Kitaev wire in the presence of a thermal bath.

1. Asymptotic thermal states

The dynamics of the system in contact with the thermal bath described by the Lindblad master equation (4) with the dissipator term (10) leads to thermal states, such as those described by the density matrix reported in Eq. (13). To compute the correlation functions of the fermionic operators \hat{c}_x in thermal states of the Hamiltonian $\hat{H}(w)$, one can use the relation with the Bogoliubov eigenoperators \hat{b}_k , cf. Eq. (9), and the thermal correlations of the Bogoliubov operators \hat{b}_k , i.e.

$$\langle b_k^\dagger b_q \rangle \equiv \text{Tr}[\rho_t(w, T) b_k^\dagger b_q] = \frac{\delta_{kq}}{1 + e^{\omega_k/T}}, \quad (A1)$$

corresponding to the standard Fermi-Dirac distribution function. Note also that the other correlations $\langle b_k b_q \rangle$ and $\langle b_k^\dagger b_q^\dagger \rangle$ vanish. Then the correlation functions of the original fermionic field \hat{c}_x can be straightforwardly obtained from Eq. (9).

2. Computations for the unitary protocol

In the unitary QQ protocol, one starts from a Gibbs state associated with the Hamiltonian parameter w_i and the temperature T , then at $t = 0$ one instantaneously changes $w_i \rightarrow w$ and removes the contact with the thermal bath. Therefore the quantum evolution is unitary, described by the Schrödinger equation (3). One may easily obtain closed equations for the evolution of the correlation functions C and P defined in Eqs. (16) and (17).

We introduce the correlations

$$\mathcal{C}_{x,y} = \text{Tr}[\rho(t) \hat{c}_x^\dagger \hat{c}_y], \quad \mathcal{P}_{x,y} = \text{Tr}[\rho(t) \hat{c}_x^\dagger \hat{c}_y^\dagger], \quad (A2)$$

whose quantum evolution can be written as

$$\begin{aligned} \frac{d\mathcal{C}_{x,y}}{dt} &= i[\mathcal{C}_{x,y+1} - \mathcal{C}_{x-1,y} + \mathcal{C}_{x,y-1} - \mathcal{C}_{x+1,y}] \\ &\quad - i(\mathcal{P}_{y,x-1}^\dagger - \mathcal{P}_{y,x+1}^\dagger) + i(\mathcal{P}_{x,y-1} - \mathcal{P}_{x,y+1}), \end{aligned} \quad (A3)$$

$$\begin{aligned} \frac{d\mathcal{P}_{x,y}}{dt} &= -i[\mathcal{P}_{x,y+1} + \mathcal{P}_{x+1,y} + \mathcal{P}_{x,y-1} + \mathcal{P}_{x-1,y}] \\ &\quad - 2i\mu \mathcal{P}_{x,y} - i(\delta_{x-1,y} - \delta_{x+1,y}) \\ &\quad - i(\mathcal{C}_{x,y-1} - \mathcal{C}_{y,x-1} - \mathcal{C}_{x,y+1} + \mathcal{C}_{y,x+1}). \end{aligned} \quad (A4)$$

The initial conditions are easily obtained by the relations with the thermal correlations of the Bogoliubov operators associated with the initial Gibbs state. Then the fermionic correlation function are obtained by

$$C(x, y, t) = 2 \text{Re} \mathcal{C}_{x,y}(t), \quad P(x, y, t) = 2 \text{Re} \mathcal{P}_{x,y}(t). \quad (A5)$$

The above differential equations are solved using the four-order Runge-Kutta method. The particle density is obtained from the data of $\mathcal{E}_{x,x} = \text{Tr}[\rho(t)\hat{c}_x^\dagger\hat{c}_x]$.

3. Computations for the dissipative protocol

For the dissipative QQ protocol, where the thermal bath is kept in contact with the system, the evolution is driven by the Lindblad master equation (4), which can be equivalently written in terms of the time dependence of Heisenberg operators $\hat{O}_H(t)$, i.e., [84,86]:

$$\partial_t \hat{O}_H(t) = i[\hat{H}(w), \hat{O}_H(t)] + \gamma \widehat{\mathbb{D}}_T[\hat{O}_H(t)], \quad (\text{A6})$$

where

$$\begin{aligned} \widehat{\mathbb{D}}_T[\hat{O}_H(t)] = & \sum_k f(\omega_k) [2\hat{b}_k^\dagger \hat{O}_H(t) \hat{b}_k - \{\hat{O}_H(t), \hat{b}_k \hat{b}_k^\dagger\}] \\ & + \sum_k (1 - f(\omega_k)) [2\hat{b}_k \hat{O}_H(t) \hat{b}_k^\dagger \\ & - \{\hat{O}_H(t), \hat{b}_k^\dagger \hat{b}_k\}], \end{aligned} \quad (\text{A7})$$

where \hat{b}_k are the Bogoliubov operators associated with the Hamiltonian $\hat{H}(w)$.

The initial state at $t = 0$ is the Gibbs state for the Hamiltonian parameter w_i . This state corresponds to the steady state solution of the Eq. (A6) with $\hat{H}(w_i)$. Then, the change of the Hamiltonian parameter to $w \neq w_i$ leads to a change of the Bogoliubov operators diagonalizing the Hamiltonian. We call $\{b'_k\}$ the operators which diagonalizes $\hat{H}(w)$,

$$\hat{H}(w) = \sum_{k=1}^L \omega'_k \hat{b}'_k{}^\dagger \hat{b}'_k, \quad (\text{A8})$$

where $\{\omega'_k\}$ is the Bogoliubov spectrum associated with $\hat{H}(w)$. To evaluate the correlations of the Bogoliubov operators $\{b'_k\}$,

one can solve the Eq. (A6) for couples of operators $\{b'_k\}$, obtaining [84]

$$\begin{aligned} \langle b'_k{}^\dagger b'_k \rangle &= (1 - e^{-2\gamma t}) f(\omega'_k) + e^{-2\gamma t} \langle b'_k{}^\dagger b'_k \rangle_0, \\ \langle b'_k{}^\dagger b'_q \rangle &= e^{i(\omega'_k - \omega'_q)t - 2\gamma t} \langle b'_k{}^\dagger b'_q \rangle_0, \\ \langle b'_k{}^\dagger b'_q{}^\dagger \rangle &= e^{i(\omega'_k + \omega'_q)t - 2\gamma t} \langle b'_k{}^\dagger b'_q{}^\dagger \rangle_0, \\ \langle b'_k b'_q \rangle &= e^{-i(\omega'_k + \omega'_q)t - 2\gamma t} \langle b'_k b'_q \rangle_0. \end{aligned} \quad (\text{A9})$$

The initial values $\langle b'_k{}^\dagger b'_q \rangle_0$ of the correlations is computed on the initial Gibbs state associated with w_i , and it can be obtained using the relations between $\{b_k\}$ to $\{b'_k\}$. This relation can be formally derived as follows [84]. Introducing the fermionic Nambu field $\mathbb{C}^\dagger = (\hat{c}_1^\dagger, \dots, \hat{c}_L^\dagger, \hat{c}_1, \dots, \hat{c}_L)$, their relations with the Bogoliubov operators $\mathbb{B}(w)^\dagger = (\hat{b}'_1{}^\dagger, \dots, \hat{b}'_L{}^\dagger, \hat{b}_1, \dots, \hat{b}_L)$ corresponding to the Hamiltonian $\hat{H}_K(w)$ are obtained by a unitary transformation, $\mathbb{C} = \mathbb{T}(w)\mathbb{B}(w)$. See, e.g., Ref. [84] for more details. Therefore one can formally derive the relation between the Bogoliubov operators \hat{b}'_k and \hat{b}_k , corresponding to the Hamiltonian parameters w_i and w respectively, from the general relation

$$\mathbb{B}(w_2) = \mathbb{T}(w_2)^\dagger \mathbb{T}(w_1) \mathbb{B}(w_1). \quad (\text{A10})$$

Finally, to compute the time-dependent observables defined in Sec. IID, one can use the relations between the fermionic correlation functions associated with \hat{c}_x and those of the Bogoliubov operators \hat{b}_k , such as

$$\begin{aligned} C(x, y) = & \sum_{k,q=1}^L [A_{xk}^* A_{yq} \langle b'_k{}^\dagger b'_q \rangle + B_{xk}^* B_{yq} \langle b_k b'_q{}^\dagger \rangle \\ & + A_{xk}^* B_{yq} \langle b'_k b'_q{}^\dagger \rangle + B_{xk}^* A_{yq} \langle b_k b_q \rangle] \end{aligned} \quad (\text{A11})$$

where A and B are the matrices entering Eq. (9).

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