



## Precise low-temperature expansions for the Sachdev-Ye-Kitaev model

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(Received 26 December 2022; revised 1 June 2023; accepted 7 June 2023; published 5 July 2023)

We solve numerically the large- $N$  Dyson-Schwinger equations for the Sachdev-Ye-Kitaev (SYK) model utilizing the Legendre polynomial decomposition and reaching  $10^{-36}$  accuracy. Using this we compute the energy of the SYK model at low temperatures  $T \ll J$  and obtain its series expansion up to  $T^{7.54}$ . While it was suggested that the expansion contains terms  $T^{3.77}$  and  $T^{5.68}$ , we find that the first noninteger power of temperature is  $T^{6.54}$ , which comes from the two-point function of the fermion bilinear operator  $O_{h_1} = \chi \partial_\tau^2 \chi$  with scaling dimension  $h_1 \approx 3.77$ . The coefficient in front of  $T^{6.54}$  term agrees well with the prediction of the conformal perturbation theory. We conclude that the conformal perturbation theory appears to work even though the SYK model is not strictly conformal.

DOI: [10.1103/PhysRevB.108.035103](https://doi.org/10.1103/PhysRevB.108.035103)

### I. INTRODUCTION

The Sachdev-Ye-Kitaev (SYK) model is a quantum system of  $N$  Majorana fermions with all-to-all random interactions [1,2]. This model finds its application in different branches of science including quantum computation [3–6] and there is a variety of proposals for its realization in laboratory [7–16]. Reviews of the SYK model can be found in Refs. [17–20].

The SYK model low temperature behavior displays various interesting properties. Particularly in the large- $N$  limit it exhibits emergent conformal symmetry in the infrared. This suggests that conformal field theory methods can be used to study the model. However, the symmetry is both explicitly and spontaneously broken and therefore the term “nearly conformal quantum mechanics” or NCFT<sub>1</sub> was coined for the SYK model in Ref. [17]. The conformal symmetry still plays a crucial role and a complete framework of its applicability is not yet clear.

The SYK model is conjectured to be a dual description of the gravitational theory in two dimensions [1,17]. To construct and understand the full duality between the SYK model and the gravitational theory it is crucial to establish the low-temperature expansion of the free energy of the SYK model. In this paper we investigate such an expansion using the conformal perturbation theory approach and confirm theoretical predictions by extremely precise numerical computations. It was previously assumed that the SYK model energy at low temperatures contains noninteger powers of temperature  $T^{h_k}$ , where  $h_k$  are scaling dimensions of fermion bilinear operators  $O_{h_k}$  and the first two dimensions are  $h_1 \approx 3.77$  and  $h_2 \approx 5.68$  for the  $q = 4$  case. We show numerically that these terms are not present and the first noninteger power of temperature is  $T^{2h_1-1}$ , which comes from the two-point function of the operator  $O_{h_1}$ . We find that the coefficient in front of this power term agrees well with the prediction of the conformal perturbation theory.

The paper is organized as follows: in Sec. II, we review the SYK model. In Sec. III, we discuss the effective theory approach to the SYK model and the low-temperature expansion

of the two-point function and free energy. In Sec. IV, we discuss methods for numerical solution of the large- $N$  Dyson-Schwinger equations. Finally, in Sec. V, we present numerical results and compare them with the theoretical predictions.

### II. THE SACHDEV-YE-KITAEV MODEL

The Majorana SYK model is described by the following Hamiltonian with even integer  $q$

$$H = \frac{i^{q/2}}{q!} \sum_{i_1, \dots, i_q} J_{i_1 \dots i_q} \chi_{i_1} \cdots \chi_{i_q}, \quad (1)$$

where  $N$  Majorana fermions  $\chi_i$  with  $i = 1, \dots, N$  interact via random couplings  $J_{i_1 \dots i_q}$  drawn from a Gaussian ensemble with zero mean  $\overline{J_{i_1 \dots i_q}} = 0$  and variance  $\overline{J_{i_1 \dots i_q}^2} = (q-1)! J^2 / N^{q-1}$ . The free energy of the SYK model after disorder average is

$$-\beta F = \overline{\ln Z} = \partial_n \overline{Z^n} |_{n \rightarrow 0}, \quad (2)$$

where  $\beta = 1/T$  is the inverse temperature. In Refs. [21,22] it was explained that, for the SYK model up to  $1/N^{q-2}$  order  $\overline{Z^n} = (\overline{Z})^n$ , therefore at leading  $N$  we get  $-\beta F = \ln \overline{Z}$ . Taking disorder average of the partition function and integrating out the Majorana fermions we obtain a functional integral over two bilocal fields  $G$  and  $\Sigma$ :

$$e^{-\beta F} = \int DG(\tau_1, \tau_2) D\Sigma(\tau_1, \tau_2) e^{-I[G, \Sigma]}, \quad (3)$$

where  $G(\tau_1, \tau_2)$  has a meaning of imaginary time two-point function of the Majorana fermions

$$G(\tau_1, \tau_2) = -\frac{1}{N} \langle T \chi_i(\tau_1) \chi_i(\tau_2) \rangle \quad (4)$$

and the  $(G, \Sigma)$  action  $I$  is [17,21]

$$-\frac{I[G, \Sigma]}{N} = \ln \text{Pf}(-\sigma - \Sigma) - \frac{1}{2} \int_0^\beta d\tau_1 d\tau_2 \left[ \Sigma(\tau_1, \tau_2) G(\tau_1, \tau_2) - \frac{J^2}{q} G(\tau_1, \tau_2)^q \right], \quad (5)$$

where  $\sigma(\tau_1, \tau_2) = \delta'(\tau_1 - \tau_2)$ . In the large- $N$  limit the leading contribution to the thermodynamic free energy  $F$  comes from the saddle-point configuration  $(G_*, \Sigma_*)$  of the functional  $I$ , so  $\beta F_* = I[G_*, \Sigma_*]$ . This configuration is a solution of the Dyson-Schwinger (DS) equations

$$-\partial_{\tau_1} G(\tau_1, \tau_2) - \int_0^\beta d\tau' \Sigma(\tau_1, \tau') G(\tau', \tau_2) = \delta(\tau_{12}),$$

$$\Sigma(\tau_1, \tau_2) = J^2 G(\tau_1, \tau_2)^{q-1}, \quad (6)$$

and it has translational symmetry  $G_*(\tau_1, \tau_2) = G_*(\tau_{12})$ . It is well-known that the saddle-point solution for  $\beta, \tau_{12} \gg 1/J$  can be approximated by the power-law form [2,17,21,23]

$$G_c(\tau) = -b^\Delta \left( \frac{\pi}{\beta J |\sin \frac{\pi\tau}{\beta}|} \right)^{2\Delta} \text{sgn}(\tau), \quad (7)$$

where  $\Delta = 1/q$  and

$$b = \frac{1}{2\pi} (1 - 2\Delta) \tan(\pi\Delta). \quad (8)$$

This is a consequence of the emergent time reparametrization invariance of the DS equations in the infrared (IR) limit.

Since  $\beta F_*$  depends only on  $\beta J$ , the large- $N$  energy of the system per particle is  $\beta E = J \partial_J (\beta F_*/N)$  and thus we obtain for the energy

$$E = -\frac{J^2}{q} \int_0^\beta d\tau G_*(\tau)^q = -\frac{1}{q} \partial_\tau G_*(\tau)|_{\tau \rightarrow 0^+}. \quad (9)$$

This is an ultraviolet (UV) quantity and cannot be calculated using the conformal approximation (7). At large  $\beta J$ , the energy admits a  $1/\beta J$  expansion, so we have for the energy per coupling  $J$

$$\epsilon(\beta J) = E/J = c_0 + \frac{c_2}{(\beta J)^2} + \frac{c_3}{(\beta J)^3} + \dots, \quad (10)$$

which is a function of  $\beta J$  only. For the first three terms of the energy it was found that [17,21,24–26]

$$\epsilon(\beta J) = \epsilon_0 - \frac{\pi^2 k'(2) \alpha_0}{3qa(\beta J)^2} + \frac{2\pi^2 k'(2) \alpha_0^2}{3a(\beta J)^3} + \dots, \quad (11)$$

where  $a = [(q-1)b]^{-1}$  and  $k(h)$  is given in (13) and for  $q=4$  the ground-state energy is  $\epsilon_0 \approx -0.0406$  and  $\alpha_0 \approx 0.2648$ , which are determined numerically. The next-order terms in the energy expansion are unknown. It was suggested in Refs. [21,24] that the next-order term should have the form  $c_{h_1}/(\beta J)^{h_1}$ , where  $h_1 (\approx 3.77$  for  $q=4$ ) is the scaling dimension of the bilinear operator  $O_{h_1} = \chi_i \partial_\tau^3 \chi_i$ . On the other hand, the conformal perturbation theory and arguments based on the large- $q$  expansion in Refs. [27,28] imply that this term should be absent. The main goal of this paper is to clarify the low-temperature energy expansion of the Majorana SYK model. For this, we used a high-precision computation of the SYK model Green's function  $G_*(\tau)$  at very low temperatures (very large  $\beta J$ ). In the next section we discuss an effective theory approach to the SYK model, which uses conformal perturbation theory for computation of  $1/\beta J$  expansion of the two-point function and the free energy.

### III. SACHDEV-YE-KITAEV EFFECTIVE THEORY

The effective theory approach to the SYK model provides a way to compute higher-power temperature corrections (higher  $1/\beta J$  terms) of the SYK free energy and the two-point function using the conformal perturbation theory. In this approach one assumes that the  $(G, \Sigma)$  action  $I$  can be viewed as a conformal field theory (CFT) action, perturbed by an infinite series of irrelevant bilinear operators  $O_h(\tau)$

$$I = I_{\text{CFT}} + \sum_h g_h \int_0^\beta d\tau O_h(\tau), \quad (12)$$

with the scaling exponents  $h$  obtained from the solution of the equation  $k(h) = 1$  with

$$k(h) = \frac{\Gamma(2\Delta - h)\Gamma(h + 2\Delta - 1)}{\Gamma(2\Delta - 2)\Gamma(2\Delta + 1)} \left( 1 - \frac{\sin(\pi h)}{\sin(2\pi\Delta)} \right). \quad (13)$$

For any  $\Delta$  the lowest scaling exponent is  $h_0 = 2$ , but the higher scaling exponents depend on  $q$  and for  $q=4$  the next three are

$$h_1 \approx 3.7735, \quad h_2 \approx 5.6795, \quad h_3 \approx 7.6320. \quad (14)$$

The bilinear operators  $O_{h_k}$  schematically are

$$O_{h_k}(\tau) = \chi_i(\tau) \partial_\tau^{2k+1} \chi_i(\tau). \quad (15)$$

Conformal SYK two-point function in (7) is obtained using the  $I_{\text{CFT}}$  action

$$G_c(\tau_{12}) = -\frac{1}{Z} \int D\chi_i \frac{1}{N} \chi_i(\tau_1) \chi_i(\tau_2) e^{-I_{\text{CFT}}}. \quad (16)$$

The  $1/\beta J$  corrections to the two-point function can be computed using conformal perturbation theory with the action in (12). For the first-order correction one uses the three-point function [29–31]

$$\frac{1}{N} \langle \chi_i(\tau_1) \chi_i(\tau_2) O_h(\tau_3) \rangle$$

$$= \frac{c_h b^\Delta \text{sgn}(\tau_{12})}{\left| \frac{\beta J}{\pi} \sin \frac{\pi\tau_{12}}{\beta} \right|^{2\Delta-h} \left| \frac{\beta J}{\pi} \sin \frac{\pi\tau_{13}}{\beta} \right|^h \left| \frac{\beta J}{\pi} \sin \frac{\pi\tau_{23}}{\beta} \right|^h}, \quad (17)$$

where the structure constants  $c_h$  can be extracted from the explicit computation of the connected part of the four-point function of the Majorana fermions [17,29]

$$c_h^2 = \frac{a}{k'(h)} \frac{(h-1/2)}{\pi \tan(\pi h/2)} \frac{\Gamma(h)^2}{\Gamma(2h)}. \quad (18)$$

The second order of the conformal perturbation theory requires the computation of the four-point function, and so on [27], this in turn uses the structure constants  $c_{hh'h''}$  of the three-point function of the operators  $O_h, O_{h'}$ , and  $O_{h''}$ . These structure constants were computed in Ref. [30] from the six-point correlation function of the Majorana fermions. Finally this leads to the following expansion of the exact two-point function

$$G_*(\tau) = G_c(\tau) \left( 1 - \sum_h \frac{\alpha_h}{(\beta J)^{h-1}} f_h(\tau) - \sum_{h,h'} \frac{a_{hh'} \alpha_h \alpha_{h'}}{(\beta J)^{h+h'-2}} f_{h,h'}(\tau) - \dots \right), \quad (19)$$

where the functions  $f_h(\tau)$  are

$$f_h(\tau) = \frac{(2\pi)^{h-1} \Gamma(h)^2 [A_h(e^{\frac{2\pi i \tau}{\beta}}) + A_h(e^{-\frac{2\pi i \tau}{\beta}})]}{2 \sin \frac{\pi h}{2} \Gamma(2h-1)}, \quad (20)$$

with  $A_h(z) = (1-z)^h \mathbf{F}(h, h, 1; z)$  and  $\mathbf{F}$  is the regularized hypergeometric function. Parameters  $\alpha_h$  are related to the couplings  $g_h$  as

$$g_h^2 = \frac{J^2 k'(h)}{a} \frac{(h-1/2)}{\pi \tan(\pi h/2)} \frac{\Gamma(h)^2}{\Gamma(2h)} \alpha_h^2, \quad (21)$$

and the coefficients  $a_{hh'}$ ,  $a_{hh'h''}$ , etc. can be computed exactly [27]. The functions  $f_{h,h'}(\tau)$  are unknown and depend on  $q$ , but scale as  $f_{h,h'}(\tau) \rightarrow (\beta/\tau)^{h+h'-2}$  for  $\beta \rightarrow \infty$ . For the Majorana SYK model the leading terms in this series expansion are

$$G_*(\tau) = G_c(\tau) \left( 1 - \frac{\alpha_0}{\beta J} f_0(\tau) - \frac{a_{00} \alpha_0^2}{(\beta J)^2} f_{00}(\tau) - \frac{\alpha_1}{(\beta J)^{h_1-1}} f_1(\tau) - \frac{a_{000} \alpha_0^3}{(\beta J)^3} f_{000}(\tau) - \dots \right). \quad (22)$$

In this approach, the coupling constants  $g_h$  (or parameters  $\alpha_h$ ) are unknown real numbers. We can find  $\alpha_h$  by fitting numerical solution for  $G_*(\tau)$  by the formula (22).

Similarly, using the conformal perturbation theory for the action (12), we can obtain the  $1/\beta J$  expansion of the SYK model free energy:

$$\beta F_* = \beta F_{\text{CFT}} + \sum_h g_h \int_0^\beta d\tau \langle O_h \rangle_\beta - \frac{1}{2} \sum_h g_h^2 \int_0^\beta d\tau_1 d\tau_2 \langle O_h(\tau_1) O_h(\tau_2) \rangle_\beta + \dots, \quad (23)$$

and  $\beta F_{\text{CFT}}/N = \beta E_0 - s_0$  is the conformal part of the free energy, where  $E_0$  is the bare energy and  $s_0$  is the zero-temperature entropy [23,32]

$$s_0 = \int_0^{1/2-\Delta} dx \frac{\pi x}{\tan(\pi x)}. \quad (24)$$

Since in one-dimension we can map a line to a circle by the transformation  $\tau \rightarrow e^{i2\pi\tau/\beta}$  we expect that correlation functions on the thermal circle are determined by the conformal symmetry. Therefore all one-point correlation functions of primary operators should vanish, except for the identity [33]. Using the two-point function of operators

$$\langle O_h(\tau_1) O_h(\tau_2) \rangle_\beta = N \left( \frac{\pi}{\beta J \sin \frac{\pi \tau_{12}}{\beta}} \right)^{2h}, \quad (25)$$

we can compute its contribution to the free energy [27,34]:

$$\frac{\beta \delta^2 F_h}{N} = - \frac{\pi^{2h-\frac{1}{2}} \Gamma(\frac{1}{2}-h)}{2\Gamma(1-h)} \frac{g_h^2/J^2}{(\beta J)^{2h-2}}. \quad (26)$$

For the SYK model this CFT approach has a big caveat. The problem is that the operator  $O_{h_0} = \chi_i \partial_\tau \chi_i$  with  $h_0 = 2$  is essentially the Hamiltonian  $H$  of the SYK model and thus correlation functions with it do not have the CFT structure. For instance as was shown in Ref. [17] the two-point function of

$O_{h_0}$  measures the energy fluctuations and is a constant rather than a power law:

$$\langle O_{h_0}(\tau_1) O_{h_0}(\tau_2) \rangle_\beta = N \frac{c}{\beta^3}, \quad (27)$$

where  $c$  is the specific heat per particle,

$$\frac{c}{2\beta} = - \frac{\pi^2 k'(2)}{3qa} \frac{\alpha_0}{\beta J}, \quad (28)$$

so  $\beta F_*/N = \beta E_0 - s_0 - c/(2\beta) + \dots$ . One can say that the operator  $O_{h_0}$  breaks conformal symmetry. On the other hand if we assume that  $\langle O_{h_k} \rangle_\beta = 0$  for  $k = 1, 2, 3, \dots$ , then we can heuristically argue that, for all  $n$ ,

$$\langle O_{h_0}(\tau_1) \dots O_{h_0}(\tau_n) O_{h_k}(\tau_{n+1}) \rangle_\beta = 0 \quad (29)$$

because, as we mentioned above,  $O_{h_0} = H$ , and thus

$$\langle H \dots H O_{h_k}(\tau_{n+1}) \rangle_\beta = \overline{Z^{-1} (-\partial_\beta)^n \text{Tr}(O_{h_k} e^{-\beta H})}. \quad (30)$$

Similarly we should conclude that, if  $\langle O_{h_k} O_{h_m} \rangle_\beta \propto \delta_{km}$  for  $k, m = 1, 2, 3, \dots$ , then

$$\langle O_{h_0} \dots O_{h_0} O_{h_k} O_{h_m} \rangle_\beta \propto \delta_{km}. \quad (31)$$

We find below that our numerical results indeed support the equations (29) and (26) and show some evidence for (31) in the case of one  $O_{h_0}$  operator and  $k = m = 1$ .

Although the expression (19) can be obtained using the CFT approach and assuming that  $O_{h_0}$  does not violate the CFT structure of the correlation functions, the coefficients  $a_{hh'}$ ,  $a_{hh'h''}$  and the first two terms in (11) of the energy expansion were computed using more direct method, developed in Refs. [17,21,25,26]. In the Appendix C we show how to reproduce (26) using this method. Nevertheless, this method does not explain which powers of  $1/\beta J$  can be present in the energy expansion. Therefore, in this paper we approach this question using numerical computations. Particularly, we analyze  $1/\beta J$  expansion of the energy  $\epsilon(\beta J)$  and also Green's function  $G_*(\beta/2)$  at  $\tau = \beta/2$  point.

#### IV. NUMERICAL SOLUTION OF THE DYSON-SCHWINGER EQUATIONS

Conventional approach to numerical solution of the SYK Dyson-Schwinger equations (6) at finite  $\beta J$  uses iterations of the equations, starting with the free correlation function and using the Fourier series expansion [17]

$$G(\tau) = \frac{1}{\beta} \sum_{n=-\infty}^{+\infty} G(i\omega_n) e^{-i\omega_n \tau}, \quad (32)$$

where  $\omega_n = 2\pi(n+1/2)/\beta$  are the Matsubara frequencies. In the Matsubara frequency space the first equation in (6) is diagonal

$$[i\omega_n - \Sigma(i\omega_n)] G(i\omega_n) = 1. \quad (33)$$

Numerically, one keeps only Matsubara modes with  $|n| < N_M$  and because at large frequencies the Green's function decays as  $G(i\omega_n) \sim 1/(i\omega_n)$  the error of the numerical solution scales as  $O(N_M^{-1})$ . To improve accuracy of this approach one uses the knowledge of the first  $p$  high-frequency expansion terms of the Green's function  $G(i\omega_n) \approx \sum_{k=1}^p g_k/(i\omega_n)^k$ , where

$g_1 = 1$ . This helps to reduce the error to the order  $O(N_M^{-p-1})$  [35–38]. In Appendix B we discuss this approach to the Majorana SYK model.

In this paper we use the Legendre spectral method [35,39] to solve numerically the Dyson-Schwinger equations (6) much more efficiently. Namely we decompose the Green's function and the self-energy in the Legendre polynomials (there are other orthogonal polynomials which can be used [40] as well as methods [41])

$$G(\tau) = \sum_{\ell=0}^{\infty} G_{\ell} L_{\ell}(x(\tau)), \quad \Sigma(\tau) = \sum_{\ell=0}^{\infty} \Sigma_{\ell} L_{\ell}(x(\tau)), \quad (34)$$

where  $x(\tau) = 2\tau/\beta - 1$ . The Legendre coefficients  $G_{\ell}$  and  $\Sigma_{\ell}$  decay exponentially with  $\ell$  [39] and therefore if we retain only  $N_L$  Legendre coefficients in the expansion (34) we expect numerical solution to have exponentially high accuracy of the order  $O(e^{-N_L})$ .

An inherent particle-hole symmetry of the Majorana fermions leads to the symmetry of Green's function around  $\tau = \beta/2$ , so  $G(\tau) = G(\beta - \tau)$ . This implies that all the odd coefficients of the Legendre decomposition must be zero  $G_{2k+1} = 0$  for  $k = 0, 1, 2, \dots$

Using the Legendre polynomial expansion we can find the energy of the SYK model (10) as

$$\epsilon(\beta J) = \frac{1}{2\beta J} \sum_{k=1}^{\infty} k(2k+1)G_{2k}, \quad (35)$$

and also the two-point function at  $\tau = \beta/2$  point:

$$G(\beta/2) = \sum_{k=0}^{\infty} \frac{(-1)^k (2k)!}{4^k (k!)^2} G_{2k}, \quad (36)$$

where we used that  $L'_{\ell}(-1) = (-1)^{\ell+1} \ell(\ell+1)/2$  and  $L_{2k+1}(0) = 0$  and

$$L_{2k}(0) = \frac{(-1)^k (2k)!}{4^k (k!)^2}. \quad (37)$$

Details of the numerical solution with the Legendre polynomial expansion are discussed in Appendix A. In the next section we present our numerical results and compare them with the theoretical predictions.

## V. NUMERICAL RESULTS

We find numerical solution for the Green's function  $G(\tau)$  in  $q = 4$  SYK model in the range of parameter  $\beta J \in [10, 15000]$  with accuracy around  $10^{-36}$  solving the equations (6) with the use of the Legendre polynomials decomposition. To achieve this accuracy we retain  $N_L$  Legendre coefficients, where  $N_L$  is such that the last Legendre coefficient  $G_{N_L}$  is smaller than  $10^{-36}$ . As we already mentioned above the Legendre coefficients  $G_{\ell}$  and  $\Sigma_{\ell}$  decay exponentially with  $\ell$  [39] and by fitting numerical results for different  $\beta J$  we find for  $q = 4$

$$G_{\ell} \approx -e^{-3.32\ell/\sqrt{\beta J}}, \quad \Sigma_{\ell} \approx -e^{-3.20\ell/\sqrt{\beta J}}. \quad (38)$$

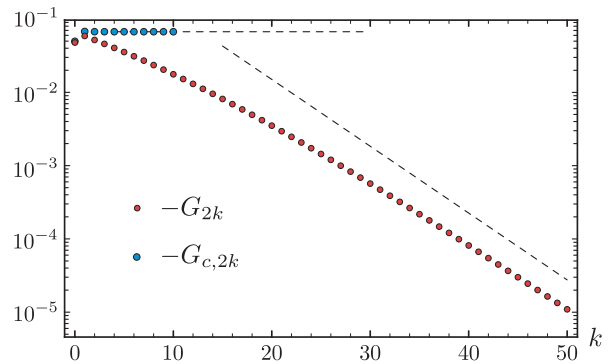


FIG. 1. Plot of the Legendre coefficients  $G_{2k}$  and  $G_{c,2k}$  of  $G_*(\tau)$  and  $G_c(\tau)$ , respectively, for  $\beta J = 1000$ . Two black dashed lines represent asymptotes (38) and (39) (only the first 11 coefficients  $G_{c,2k}$  are shown).

In contrast, the Legendre coefficients of the conformal solution  $G_c(\tau)$  in (7) for  $q = 4$  and large  $\ell = 2k$  are

$$G_{c,2k} \approx -\frac{2\sqrt{2}}{\pi^{1/4}\sqrt{\beta J}} \left( 1 + \frac{45}{2048k^4} + \dots \right), \quad (39)$$

and to derive it we used expansion in powers of  $1 - x^2$  of the function  $[\cos(\pi x/2)]^{-2\Delta}$  and the integral

$$\int_{-1}^1 \frac{dx L_{2k}(x)}{(1-x^2)^\alpha} = \frac{\Gamma(1-\alpha)\Gamma(k+\frac{1}{2})\Gamma(k+\alpha)}{\Gamma(\alpha)\Gamma(k+1)\Gamma(k+\frac{3}{2}-\alpha)}. \quad (40)$$

A plot of the Legendre coefficients  $G_{2k}$  and  $G_{c,2k}$  for  $\beta J = 1000$  is depicted in Fig. 1. We see that only first two Legendre coefficients  $G_0, G_2$  of the exact solution  $G_*(\tau)$  are close to those of the conformal solution  $G_c(\tau)$ .

We compute the energy using Eq. (35) and we can estimate the accuracy of this computation assuming that each coefficient  $G_{\ell}$  for  $\ell = 1, \dots, N_L$  is computed with accuracy  $G_{N_L}$  plus the sum in (35) from  $N_L$  to  $\infty$  with the asymptotic values of  $G_{\ell}$  from (38). This gives

$$\delta\epsilon \approx \left( \frac{N_L^3}{24\beta J} + \frac{N_L^2}{8 \times 3.32\sqrt{\beta J}} \right) e^{-3.32N_L/\sqrt{\beta J}}. \quad (41)$$

For  $\beta J \in [9600, 15000]$  we used  $N_L = 3000$  and the lowest accuracy for energy  $\epsilon(\beta J)$  is at  $\beta J = 15000$  and can be estimated as  $\delta\epsilon \approx 4 \times 10^{-31}$ . On the other hand the coefficient  $(1/\beta J)^{7.54} \approx 3 \times 10^{-32}$  at this value of  $\beta J$ , so the overall accuracy is enough to probe  $(1/\beta J)^{7.54}$  dependence of the energy. We also estimated accuracy of our computation using the second relation in Eq. (B14). This estimate shows even higher accuracy of the energy computation than Eq. (41), confirming that we can reliably probe the  $(1/\beta J)^{7.54}$  term.

We are interested in  $1/\beta J$  series expansion of the energy  $\epsilon(\beta J)$ . This is an asymptotic series and the accuracy of its approximation to  $\epsilon(\beta J)$  for large enough  $\beta J$  is proportional to  $(1/\beta J)^{p'}$ , where  $p' > p$  is the next omitted power in the series expansion up to the power  $p$ . So if we denote the series expansion up to the power  $p$  by

$$\epsilon_p(\beta J) = c_0 + c_2/(\beta J)^2 + \dots + c_p/(\beta J)^p, \quad (42)$$

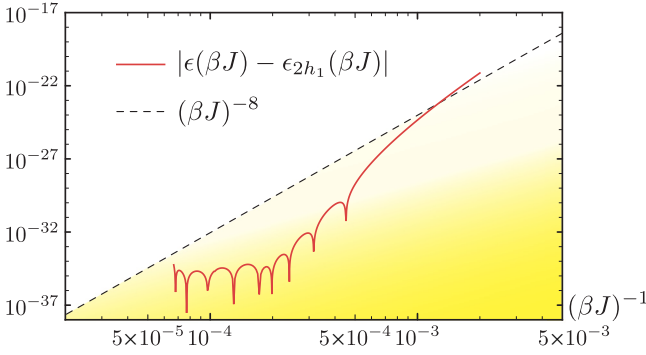


FIG. 2. Log-log plot of the difference  $|\epsilon(\beta J) - \epsilon_{2h_1}(\beta J)|$  between the numerical values  $\epsilon(\beta J)$  and their fit by the polynomial  $\epsilon_{2h_1}(\beta J)$ , which includes two non-integer powers  $2h_1 - 1$  and  $2h_1$ . We tune weights of the loss function (44) such that the difference (red line) lies below  $(\beta J)^{-p'}$ , where  $p' = 8$  is the next omitted power in the expansion (black dashed line).

then we should expect

$$|\epsilon(\beta J) - \epsilon_p(\beta J)| \leq c_{p'}/(\beta J)^{p'}. \quad (43)$$

This should be true for asymptotic series provided we take large enough  $\beta J$  such that  $c_{p'}/(\beta J)^{p'} \ll c_p/(\beta J)^p$ . We stress that powers in this expansion can be noninteger numbers.

To extract series coefficients  $c_0, c_2, \dots, c_p$  we perform linear regression fit of the numerical data for  $\epsilon(\beta J)$  by the polynomial  $\epsilon_p(\beta J)$  on the interval of couplings  $\beta J$  using the weighted least squares, i.e., minimizing the loss function

$$L_w(c_0, c_2, \dots, c_p) = \sum_i w_i [\epsilon(\beta J_i) - \epsilon_p(\beta J_i)]^2. \quad (44)$$

We tune weights  $w_i$  in such a way that the inequality (43) is fulfilled on the interval of  $\beta J$ . An example of such a fit on the interval  $\beta J \in [500, 15000]$  for  $p = 2h_1 \approx 7.54$  is depicted in Fig. 2.

On general grounds, we expect that by fitting the numerical data by the polynomial  $\epsilon_p(\beta J)$  we can find coefficients  $c_0, c_2, \dots, c_p$  with the accuracy

$$\delta c_k \propto 1/(\beta J_{\max})^{p'-k}, \quad (45)$$

where  $\beta J_{\max}$  is the maximal value of the  $\beta J$  interval and  $p' > p$  is the next omitted power in the series expansion. The validity of the estimate (45) is not entirely clear, but it should be certainly correct in the  $\beta J_{\max} \rightarrow \infty$  limit. We also analyze accuracy of our fit using the ratio  $c_3/c_2^2$ , which is independent on  $\alpha_0$  parameter and can be computed analytically from (11)

$$\frac{c_3}{c_2^2} = -\frac{384}{\pi(2 + 3\pi)} \approx -10.6988. \quad (46)$$

The accuracy of this ratio is essentially the accuracy of the coefficient  $c_3$ , since  $c_2$  should have much higher accuracy than  $c_3$  for large  $\beta J_{\max}$  according to (45). To further improve accuracy of the numerical estimate of the coefficients  $c_k$ , we fit the data for different values of  $\beta J_{\max}$  and then approximate the results to  $\beta J_{\max} \rightarrow \infty$  value. To rule out potentially possible noninteger power terms  $c_k/(\beta J)^k$  in the energy expansion, we include these terms one at a time in the polynomial  $\epsilon_p(\beta J)$

TABLE I. Fitting results for possible power expansion coefficients. We conclude that these coefficients are absent in the series expansion. We also included nonexistent coefficient  $c_1$  to check accuracy of the fit.

$c_1$	$\approx 10^{-26}$
$c_{h_1}$	$\approx 10^{-13}$
$c_{h_1+1}$	$\approx 10^{-9}$
$c_{h_1+2}$	$\approx 10^{-5}$
$c_{h_2}$	$\approx 10^{-5}$

and by fitting them we find that their coefficients  $c_k$  are tiny numbers. We listed the results of such a fit by the polynomial  $\epsilon_{2h_1}(\beta J)$  in Table I. We conclude that there are no terms with powers  $p = h_1, h_1 + 1, h_1 + 2$  or  $h_2$ , where  $h_k$  are given in (14) in the energy  $1/\beta J$  expansion. To check that non-integer power terms  $c_k/(\beta J)^k$  with  $k = 2h_1 - 1 \approx 6.54$  and  $k = 2h_1 \approx 7.54$  are present in the energy expansion we fit the numerical data for  $\epsilon(\beta J)$  by the polynomials  $\epsilon_6(\beta J)$  and  $\epsilon_7(\beta J)$  (with  $c_{2h_1-1}$  term included in  $\epsilon_7$ ) for different values of  $\beta J_{\max}$  and analyze behavior of the error  $\delta(c_3/c_2^2)$  as a function of  $\beta J_{\max}$ . According to (45) and the discussion below (46) we expect to find  $\delta(c_3/c_2^2) \propto 1/(\beta J_{\max})^{p'-3}$ , where  $p'$  is the next omitted power after the power  $p$  of the fitting polynomial  $\epsilon_p(\beta J)$ . The plot of  $\delta(c_3/c_2^2)$  as a function of  $1/\beta J_{\max}$  and its fit are depicted in Fig. 3. We find numerically  $p' \approx 6.52$  and  $p' \approx 7.44$  for  $p = 6$  and  $p = 7$  cases correspondingly.

Finally we fit the numerical data by the polynomial  $\epsilon_{2h_1}(\beta J)$  which includes  $c_{2h_1-1}$  and  $c_{2h_1}$  coefficients. Our results for the energy expansion coefficients are summarized in Table II where we included all the significant digits with the standard error in parenthesis. For these results the ratio (46) has precision  $10^{-18}$ . We will see below that the coefficient  $c_{2h_1-1}$  of the first noninteger power term in the energy expansion is in perfect agreement with the prediction of the conformal perturbation theory (26). To find numerically the parameters  $\alpha_0, \alpha_1, \alpha_2$  we compute  $G_*(\beta/2)$  on an interval of  $\beta J$  using the equation (36). We notice that  $L_{2k}(0) \approx (-1)^k/\sqrt{\pi k} + O(k^{-3/2})$  for  $k \gg 1$ , so the contribution of the Legendre coefficients  $G_{2k}$  in (36) decay with  $k$ , which makes accuracy of  $G_*(\beta/2)$  computation even higher than the accuracy of the energy computation. Similarly to the energy we fit the numerical results for  $G_*(\beta/2)$  by a polynomial of  $1/\beta J$ ,

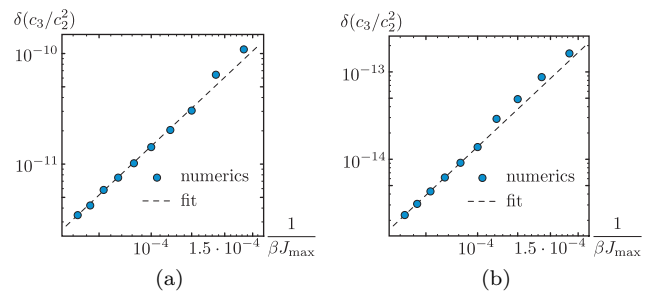


FIG. 3. Log-log plot of the error  $\delta(c_3/c_2^2)$  as a function of  $1/\beta J_{\max}$  obtained from fitting  $\epsilon(\beta J)$  by  $\epsilon_6(\beta J)$  in panel (a) and by  $\epsilon_7(\beta J)$  in panel (b). Black dashed lines are proportional to  $(\beta J_{\max})^{-3.52}$  and  $(\beta J_{\max})^{-4.44}$ .

TABLE II.  $1/\beta J$  expansion coefficients of the energy.

$c_0$	$-0.04063026975834491522143475022673(6)$
$c_2$	$0.19800839700257657773045(3)$
$c_3$	$-0.4194698967373753612(4)$
$c_4$	$0.664982720599833(5)$
$c_5$	$-2.57685914760(6)$
$c_6$	$9.9321852(9)$
$c_{2h_1-1}$	$-13.0232(9)$
$c_7$	$-26.8(9)$
$c_{2h_1}$	$109 \pm 5$

which includes only powers predicted theoretically in (22). More precisely, to find  $\alpha_0, \alpha_1, \alpha_2$  we fit the expression

$$\frac{\sqrt{\beta J} G_*(\beta/2)}{(\pi/4)^{1/4}} + 1 = \frac{\alpha_0 f_0(\beta/2)}{\beta J} + \frac{a_{00} \alpha_0^2 f_{00}(\beta/2)}{(\beta J)^2} + \frac{\alpha_1 f_1(\beta/2)}{(\beta J)^{h_1-1}} + \frac{a_{000} \alpha_0^3 f_{000}(\beta/2)}{(\beta J)^3} + \dots, \quad (47)$$

where we used that  $G_c(\beta/2) = -(\pi/4)^{1/4}/\sqrt{\beta J}$ . Numerical results for the coefficients of the expansion in Eq. (47) are shown in Table III. From these results we can find parameters  $\alpha_h$  for  $h_0, h_1$ , and  $h_2$ , using that  $f_0(\beta/2) = 2$ ,  $f_1(\beta/2) \approx 4.7373$ ,  $f_2(\beta/2) \approx 11.3971$  from (20). The results for  $\alpha_h$  are summarized in Table IV. We notice that numerical error of  $\alpha_2$  is large, since the two powers  $h_2 - 1 \approx 4.68$  and  $h_1 + 1 \approx 4.77$  in  $1/\beta J$  expansion are very close to each other and this lowers precision of the fit. We also obtain  $f_{00}(\beta/2) \approx 5.00969$ ,  $f_{01}(\beta/2) \approx 19.9483$  and  $f_{000}(\beta/2) \approx 2.52785$  using that  $a_{00} = 9/4$ ,  $a_{01} \approx 2.7229$  and  $a_{000} = -65/4$  [27]. As a check of our accuracy we expect from (11)

$$c_2^{\text{theory}} = \frac{1}{48} \pi (2 + 3\pi) \alpha_0. \quad (48)$$

Using the results from Tables II and IV we find

$$c_2 - c_2^{\text{theory}} \approx 10^{-17}. \quad (49)$$

This essentially shows the accuracy of  $\alpha_0$  from Table IV, since the coefficient  $c_2$  from Table II has higher accuracy, in

TABLE III.  $1/\beta J$  expansion coefficients in Eq. (47).

Power of $1/\beta J$	Coefficient
1	$0.529611247663296071(4)$
2	$0.7904019072219(3)$
$h_1 - 1$	$1.548138356(4)$
3	$-0.76275663(4)$
$h_1$	$9.40092(6)$
4	$-11.01(2)$
$h_2 - 1$	$75 \pm 20$
$h_1 + 1$	$-140 \pm 50$
5	$-10 \pm 40$

TABLE IV. Parameters  $\alpha_h$  obtained from Table III.

$\alpha_0$	$0.26480562383164805(2)$
$\alpha_1$	$0.32679563(5)$
$\alpha_2$	$6.6 \pm 1.8$

agreement with (45). Similarly, from (26), we expect

$$c_{2h_1-1}^{\text{theory}} = \frac{(h_1 - 1) \pi^{2h_1 - \frac{1}{2}} \Gamma(\frac{1}{2} - h_1) g_{h_1}^2}{\Gamma(1 - h_1) J^2} \approx -121.9375 \alpha_1^2, \quad (50)$$

where  $g_h$  is given in (21). Using the numerical result for  $\alpha_1$  from Table IV, we obtain from the conformal perturbation theory prediction

$$c_{2h_1-1}^{\text{theory}} \approx -13.0224. \quad (51)$$

We see that it coincides with our numerical estimate for  $c_{2h_1-1}$  in Table I within an error of  $8 \times 10^{-4}$ . We remark that such a good agreement with the conformal perturbation theory indirectly confirms that  $\langle O_{h_0} O_{h_2} \rangle = 0$ , otherwise it would produce the power term  $h_2 + 1 \approx 6.68$  in the energy expansion, which would strongly interfere with the term  $2h_1 - 1 \approx 6.54$  and thus lower the accuracy of the fit significantly.

## VI. DISCUSSION

In this article we show numerically that the conformal perturbation theory appears to work for the SYK model even though the SYK model is not strictly conformal. We found that there are no terms  $(\beta J)^{-h_1+1}$  and  $(\beta J)^{-h_2+1}$  in the  $1/\beta J$  expansion of the SYK free energy and the first noninteger power term is  $p = 2h_1 - 2$  ( $\approx 5.54$  for  $q = 4$ ). It comes from the two-point function of the bilinear operator  $O_{h_1} = \chi \partial_\tau^3 \chi$ . It would be interesting to investigate the low-temperature expansion of the complex SYK model [42] with nonzero chemical potential  $\mu$ , where there is a local operator  $O_h = c^\dagger \partial_\tau^2 c$  with the scaling dimension  $h \approx 2.65$  which depends slightly on  $\mu$  [32,43]. We expect that the first noninteger power in the cSYK model free energy should be  $p = 2h - 2$  ( $\approx 3.3$  for  $q = 4$ ) and is much lower than in the Majorana SYK [44]. There are variety of other random and nonrandom quantum models, where the scaling dimensions of the bilinear operators depend strongly on the models' parameters [27,28,45–48]. In some of these models the scaling dimensions of the lowest operators can be less than  $3/2$  (and for some parameters can become even relevant  $h < 1$ , thus breaking the conformal invariance completely). For these type of models the validity of the conformal perturbation theory approach is of primary importance, since it defines the leading behavior of the free energy.

## ACKNOWLEDGMENTS

We would like to thank Igor R. Klebanov and Alexey Milekhin for useful discussions and comments. We also would like to thank Michael Widom for providing us with computer cluster time.

**APPENDIX A: LEGENDRE SPECTRAL METHOD**

In this Appendix, following Ref. [35], we briefly describe the Legendre spectral method for solving the Dyson-Schwinger equations. We would like to solve numerically the Dyson-Schwinger equation

$$\partial_\tau G(\tau) + \int_0^\beta d\tau' \Sigma(\tau - \tau') G(\tau') = -\delta(\tau), \quad (\text{A1})$$

where  $\Sigma(\tau) = J^2 G^{q-1}(\tau)$ . We expand Green's function  $G(\tau)$  in the Legendre series

$$G(\tau) = \sum_{\ell=0}^{\infty} G_\ell L_\ell(x(\tau)), \quad (\text{A2})$$

where  $x = 2\tau/\beta - 1$ . Since the Legendre polynomials  $L_\ell(x(\tau))$  are defined only on the interval  $\tau \in [0, \beta]$  ( $x \in [-1, 1]$ ) the  $\delta$  function in (A1) can be omitted, but its effect must be reinstated in the boundary condition for the Green's function<sup>1</sup>  $G(0^+) + G(\beta^-) = -1$ . In terms of the Legendre coefficients this boundary condition reads

$$\sum_{\ell=0}^{\infty} ((-1)^\ell + 1) G_\ell = -1, \quad (\text{A3})$$

where we used that  $L_\ell(-x) = (-1)^\ell L_\ell(x)$  and  $L_\ell(1) = 1$ . For the derivative of the Legendre polynomial we find

$$\partial_\tau L_\ell(x(\tau)) = \frac{2}{\beta} \partial_x L_\ell(x) = \frac{2}{\beta} \sum_k D_{k\ell} L_k(x), \quad (\text{A4})$$

where  $D_{k\ell} = (2k+1)$  if  $k+\ell$  is odd and  $k < \ell$ , otherwise  $D_{k\ell} = 0$ . The derivative matrix  $D_{k\ell}$  for the first several values of  $k$  and  $\ell$  starting from 0 is

$$D_{k\ell} = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & \dots \\ 0 & 0 & 3 & 0 & 3 & \dots \\ 0 & 0 & 0 & 5 & 0 & \dots \\ 0 & 0 & 0 & 0 & 7 & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad (\text{A5})$$

so we obtain

$$\partial_\tau G(\tau) = \frac{2}{\beta} \sum_{k=0}^{\infty} \left[ \sum_{\ell=0}^{\infty} D_{k\ell} G_\ell \right] L_k(x(\tau)). \quad (\text{A6})$$

Next we write the convolution as the Legendre expansion

$$\int_0^\beta d\tau' \Sigma(\tau - \tau') G(\tau') = \frac{\beta}{2} \sum_{k,\ell=0}^{\infty} [\Sigma^*]_{k\ell} G_\ell L_k(x(\tau)). \quad (\text{A7})$$

Therefore the Dyson-Schwinger equation (A1) reads

$$\sum_{\ell=0}^{\infty} \left( D_{k\ell} + \frac{\beta^2}{4} [\Sigma^*]_{k\ell} \right) G_\ell = 0, \quad (\text{A8})$$

<sup>1</sup>The  $\delta$  function implies the boundary condition  $G(0^+) - G(0^-) = -1$ . Using the KMS condition  $G(-\tau) = -G(\beta - \tau)$  we arrive to the boundary condition on the interval  $\tau \in [0, \beta]$ .

and it has to be supplemented with the boundary condition (A3). We notice that, since  $G(\tau) = G(\tau/\beta, \beta J)$ , the Legendre coefficients  $G_\ell$  depend only on combination  $\beta J$  and this is also evident from the equation (A8). To find an expression for the convolution matrix  $[\Sigma^*]_{k\ell}$  we write

$$\begin{aligned} \int_0^\beta d\tau' \Sigma(\tau - \tau') \sum_{\ell=0}^{\infty} G_\ell L_\ell(x'(\tau')) \\ = \frac{\beta}{2} \sum_{k,\ell=0}^{\infty} [\Sigma^*]_{k\ell} G_\ell L_k(x(\tau)), \end{aligned} \quad (\text{A9})$$

therefore

$$\int_{-1}^1 dx' \Sigma[\tau(x) - \tau'(x')] L_\ell(x') = \sum_{k=0}^{\infty} [\Sigma^*]_{k\ell} L_k(x). \quad (\text{A10})$$

Now multiplying both sides by the Legendre polynomial and integrating over  $x$  we obtain

$$[\Sigma^*]_{k\ell} = \frac{2k+1}{2} \int_{-1}^1 dx' dx \Sigma[\tau(x) - \tau'(x')] L_k(x) L_\ell(x'), \quad (\text{A11})$$

where we used orthogonality of the Legendre polynomials,

$$\int_{-1}^1 dx L_k(x) L_\ell(x) = \frac{2}{2k+1} \delta_{k\ell}. \quad (\text{A12})$$

It is possible to express  $[\Sigma^*]_{k\ell}$  through the Legendre coefficients  $\Sigma_\ell$  of  $\Sigma(\tau) = \sum_{\ell=0}^{\infty} \Sigma_\ell L_\ell(\tau(x))$ . The matrix elements of  $[\Sigma^*]_{k\ell}$  can be found recursively [49]:

$$\begin{aligned} [\Sigma^*]_{k,\ell+1} = -\frac{2\ell+1}{2k+3} [\Sigma^*]_{k+1,\ell} \\ + \frac{2\ell+1}{2k-1} [\Sigma^*]_{k-1,\ell} + [\Sigma^*]_{k,\ell-1}. \end{aligned} \quad (\text{A13})$$

The first two columns of the matrix  $[\Sigma^*]_{k\ell}$  are computed using the relations

$$\begin{aligned} [\Sigma^*]_{k,0} = 2 \left( \frac{\Sigma_{k-1}}{2k-1} - \frac{\Sigma_{k+1}}{2k+3} \right), \quad k \geq 1, \\ [\Sigma^*]_{k,1} = \frac{[\Sigma^*]_{k-1,0}}{2k-1} - \frac{[\Sigma^*]_{k+1,0}}{2k+3}, \quad k \geq 1, \end{aligned} \quad (\text{A14})$$

with  $[\Sigma^*]_{0,0} = -2\Sigma_1/3$  and  $[\Sigma^*]_{0,1} = -[\Sigma^*]_{1,0}/3$ . The recursion relation (A13) works only for the lower triangular part of the matrix  $[\Sigma^*]_{k,\ell}$ , i.e., for  $k \geq \ell$ . To compute elements of the upper triangular part one has to use the transpose relation

$$[\Sigma^*]_{k,\ell} = (-1)^{\ell+k} \frac{2k+1}{2\ell+1} [\Sigma^*]_{\ell,k}. \quad (\text{A15})$$

In practice we retain only  $N_L + 1$  Legendre coefficients  $G_\ell$  and  $\Sigma_\ell$  with  $\ell = 0, 1, \dots, N_L$  and to find  $G_\ell$  from given  $\Sigma_\ell$  we solve the  $(N_L + 1) \times (N_L + 1)$  linear matrix equation composed of  $k = 0, 1, \dots, N_L - 1$  equations (A8) and  $k = N_L$  equation (A3). To perform fast transform from the Legendre coefficients  $G_\ell$  to the imaginary time Green's function  $G(\tau)$  we discretize the imaginary time  $\tau = \beta(1+x)/2$  using the Legendre-Gauss-Lobatto (LGL) points  $x_i$  with  $i = 0, 1, 2, \dots, N$ , where  $x_i$  are solutions of the equation

$$(1-x^2)L'_N(x) = 0 \quad (\text{A16})$$

and they fill the interval  $x \in [-1, 1]$  with boundary values  $x_0 = -1$  and  $x_N = 1$ . The main statement for this particular choice of points  $x_i$  is that the following equation holds for any polynomial  $p(x)$  of degree less than  $2N$ :

$$\int_{-1}^{+1} dx p(x) = \sum_{i=0}^N \omega_i p(x_i), \quad (\text{A17})$$

where  $\omega_i = \frac{2}{N(N+1)} \frac{1}{[L_N(x_i)]^2}$  are called weights [50]. We assume that we expand  $G(\tau)$  in Legendre series up to the  $N_L$ th Legendre polynomial

$$G(\tau_i) = \sum_{\ell=0}^{N_L} G_\ell L_\ell(x_i) = \sum_{\ell=0}^{N_L} L_{i\ell} G_\ell, \quad (\text{A18})$$

with  $x_i = 2\tau_i/\beta - 1$ . Therefore, the transformation from the Legendre coefficients  $G_\ell$  to the discretized values of the imaginary time Green's function  $G(\tau_i)$  is a multiplication by the matrix  $L_{i\ell} = L_\ell(x_i)$ . To transform from  $G(\tau)$  to  $G_\ell$  we need to compute the integral

$$\int_0^\beta d\tau G(\tau) L_\ell(x(\tau)) = \frac{\beta}{2\ell+1} G_\ell. \quad (\text{A19})$$

On the other hand if  $\ell + N_L < 2N$  we use (A17) to get

$$\int_0^\beta d\tau G(\tau) L_\ell(x(\tau)) = \frac{\beta}{2} \sum_{i=0}^N \omega_i G(\tau_i) L_\ell(x_i), \quad (\text{A20})$$

and therefore we find

$$G_\ell = \sum_{i=0}^N S_{\ell i} G(\tau_i), \quad S_{\ell i} = \frac{2\ell+1}{2} \omega_i L_\ell(x_i). \quad (\text{A21})$$

In our numerical computations we take  $N = N_L$ . We solve the Dyson-Schwinger equation (A1) using iterations with the weighted update [17]:

$$G_\ell^{(j+1)} = (1-u)G_\ell^{(j)} + u \times \text{solution of [Eq.(A8) + Eq.(A3) with } [\Sigma^*]_{k\ell}^{(j)}], \quad (\text{A22})$$

where  $u$  is the weighting parameter and  $[\Sigma^*]_{k\ell}^{(j)}$  is constructed from  $\Sigma_\ell^{(j)} = J^2 S_{\ell i} G^{(j)}(\tau_i)^{q-1}$ . We start with the bare solution  $G_\ell^{(0)} = -\delta_{\ell,0}/2$  and weight  $u = 1/2$ . We monitor the difference  $u^{-1} \sum_i \Delta \tau_i |G^{(j+1)}(\tau_i) - G^{(j)}(\tau_i)|^2$  between successive iterations and divide  $u$  by a half if this difference increases. The iteration cycle is depicted in Fig. 4. We performed all computations in Wolfram *Mathematica* and used small computer cluster to perform computations for different values of  $\beta J$  in parallel.

## APPENDIX B: MATSUBARA SPECTRAL METHOD

In this Appendix we discuss improved numerical method for solving the Dyson-Schwinger equations (6) using the standard Matsubara frequency approach. Since in the Majorana SYK model  $G(\beta - \tau) = G(\tau)$  Green's function in the frequency space is antisymmetric,  $G(-i\omega_n) = -G(i\omega_n)$ , and therefore the high-frequency expansions of it and of the

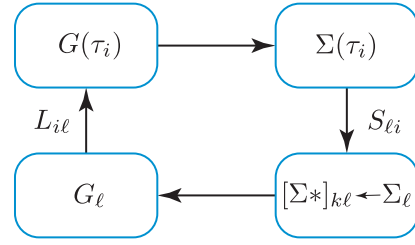


FIG. 4. Graphic representation of the iteration cycle for the Legendre spectral method. The upper horizontal arrow uses the equation  $\Sigma(\tau) = J^2 G(\tau)^{q-1}$  and the lower horizontal arrow uses the equation (A22).

self-energy have only odd powers of frequency with real coefficients

$$G(i\omega_n) = \sum_{k=1}^{\infty} \frac{g_{2k-1}}{(i\omega_n)^{2k-1}}, \quad \Sigma(i\omega_n) = \sum_{k=1}^{\infty} \frac{s_{2k-1}}{(i\omega_n)^{2k-1}}, \quad (\text{B1})$$

and  $g_1 = 1$ . Expanding the right-hand side of the DS equation

$$G(i\omega_n) = [i\omega_n - \Sigma(i\omega_n)]^{-1} \quad (\text{B2})$$

for large  $i\omega_n$  we find relations between the coefficients  $g_{2k-1}$  and  $s_{2k-1}$ :

$$g_3 = s_1, \quad g_5 = s_1^2 + s_3, \quad g_7 = s_1^3 + 2s_1 s_3 + s_5, \dots \quad (\text{B3})$$

Denoting an inverse Fourier transform of a given frequency power as

$$Q_p(\tau) = \frac{1}{\beta} \sum_{n=-\infty}^{+\infty} \frac{e^{-i\omega_n \tau}}{(i\omega_n)^p} = \text{res}_{z=0} \left( \frac{-e^{z(\beta/2-\tau)}}{2z^p \cosh\left(\frac{\beta z}{2}\right)} \right), \quad (\text{B4})$$

where the first three functions are  $Q_1(\tau) = -1/2$ ,  $Q_2(\tau) = (2\tau - \beta)/4$ ,  $Q_3(\tau) = \tau(\beta - \tau)/4$ , and using the equation  $\Sigma(\tau) = J^2 G(\tau)^{q-1}$  we find

$$s_1 = J^2/2^{q-2}, \quad (\text{B5})$$

and thus  $g_3 = J^2/2^{q-2}$ . This allows us to write expansion of the Green's function for  $J\tau \ll 1$ :

$$G(\tau) = -\frac{1}{2} - q\epsilon J\tau - \frac{(J\tau)^2}{2^q} + \dots, \quad (\text{B6})$$

where we used Eqs. (9) and (10) to write the linear  $\tau$  term. Consequently this fixes a small  $J\tau$  expansion of the self-energy

$$\Sigma(\tau) = -J^2 \left[ \frac{1}{2^{q-1}} + \frac{q(q-1)}{2^{q-2}} \epsilon J\tau + \left( \frac{q-1}{2^{2q-2}} + \frac{q^2(q-1)(q-2)}{2^{q-2}} \epsilon^2 \right) (J\tau)^2 + \dots \right]. \quad (\text{B7})$$

Since we can write  $\Sigma(\tau)$  as

$$\Sigma(\tau) = \sum_{k=1}^{\infty} s_{2k-1} Q_{2k-1}(\tau), \quad (\text{B8})$$

and only  $Q_3(\tau)$  contains  $\tau^2$  term we find

$$s_3 = J^4 \left( \frac{q-1}{2^{2q-4}} + \frac{q^2(q-1)(q-2)}{2^{q-4}} \epsilon^2 \right) \quad (\text{B9})$$



and therefore from (B3) we obtain

$$g_5 = J^4 \left( \frac{q}{2^{2q-4}} + \frac{q^2(q-1)(q-2)}{2^{q-4}} \epsilon^2 \right). \quad (\text{B10})$$

We see that only  $g_1$ ,  $g_3$ , and  $s_1$  coefficients are known explicitly, whereas already  $g_5$  and  $s_3$  contain the energy  $\epsilon$ , which itself can be computed only from the exact solution of the DS equations. Finally, we note that the Matsubara coefficients  $G(i\omega_n)$  are related to the Legendre coefficients  $G_\ell$  as [39]

$$G(i\omega_n) = \sum_{\ell=0}^{\infty} T_{n\ell} G_\ell, \quad (\text{B11})$$

where the matrix  $T_{n\ell}$  is

$$T_{n\ell} = \beta(-1)^n i^{\ell+1} j_\ell(\beta\omega_n/2) \quad (\text{B12})$$

and  $j_\ell(z)$  are the spherical Bessel functions. The matrix  $T_{n\ell}$  satisfies  $\sum_n T_{n\ell}^* T_{n\ell'} = \beta^2 \delta_{\ell\ell'} / (2\ell + 1)$ . Using the properties of the spherical Bessel functions one can relate the coefficients  $g_k$  to the Legendre coefficients  $G_\ell$ :

$$g_k = \frac{2(-1)^k}{\beta^{k-1}} \sum_{\ell=0}^{\infty} \frac{(\ell+k-1)!}{(k-1)!(\ell-k+1)!} \delta_{\ell+k,\text{odd}} G_\ell. \quad (\text{B13})$$

This relation imposes constraints on the Legendre coefficients of the Majorana SYK two-point function

$$\begin{aligned} \sum_{\ell=0:2}^{\infty} G_\ell &= -\frac{1}{2}, \\ \sum_{\ell=0:2}^{\infty} (\ell-1)\ell(\ell+1)(\ell+2)G_\ell &= -\frac{(\beta J)^2}{2^{q-2}}, \end{aligned} \quad (\text{B14})$$

where we used expressions for the  $g_1$  and  $g_3$  coefficients. The first equation in (B14) is essentially the boundary condition (A3) and our numerical solution for  $G_\ell$  automatically satisfies it. The second equation in (B14) can be used to estimate the accuracy of the numerical results for  $G_\ell$  coefficients. To increase the numerical accuracy of the Matsubara spectral method we rewrite the DS equations in terms of the reduced functions

$$\begin{aligned} \tilde{G}(i\omega_n) &= G(i\omega_n) - \frac{1}{i\omega_n} - \frac{J^2}{2^{q-2}(i\omega_n)^3}, \\ \tilde{\Sigma}(\tau) &= \Sigma(\tau) + \frac{J^2}{2^{q-1}}, \end{aligned} \quad (\text{B15})$$

and  $\Sigma(\tau)$  is computed as

$$\Sigma(\tau) = J^2 \left( \tilde{G}(\tau) + Q_1(\tau) + \frac{J^2}{2^{q-2}} Q_3(\tau) \right)^{q-1}. \quad (\text{B16})$$

Therefore, the fast Fourier transform (FFT) is performed with the functions  $\tilde{G}(i\omega_n)$  and  $\tilde{\Sigma}(i\omega_n)$ , which decay faster than  $G(i\omega_n)$  and  $\Sigma(i\omega_n)$  in the frequency space. The iteration cycle is depicted in Fig. 5. Finally, we discuss the computation of  $\ln \text{Pf}(\partial_\tau - \Sigma)$ , which contains the zero-temperature entropy

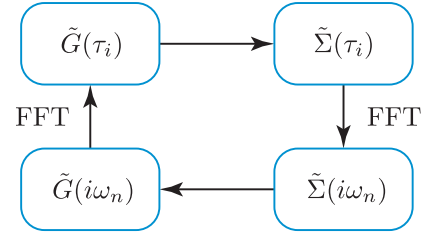


FIG. 5. Graphic representation of the iteration cycle for the improved Matsubara spectral method.

(24) and has a  $1/\beta J$  expansion of the form

$$\begin{aligned} 2 \ln \text{Pf}(\partial_\tau - \Sigma) &= -(q+1)\beta J \epsilon_0 + 2s_0 \\ &\quad - (q-3) \frac{c_2}{\beta J} - (q-2) \frac{c_3}{(\beta J)^2} - \dots, \end{aligned} \quad (\text{B17})$$

where  $c_k$  are the expansion coefficients of the energy (10). A naive computation with the use of the Legendre coefficients

$$2 \ln \text{Pf}(\partial_\tau - \Sigma) = \ln 2 \det \left( 1 + \frac{\beta^2}{4} D^{-1} [\Sigma^*] \right) \quad (\text{B18})$$

has very low accuracy of the order  $O(N_L^{-1})$ . Notice that in the matrix  $D_{kl}$  the last  $k = N_L$  row is  $(-1)^\ell + 1$  and represents the boundary condition (A3). To improve the accuracy one can compute Matsubara coefficients from the Legendre coefficients using (B11) and then compute

$$\begin{aligned} \ln \text{Pf}(\partial_\tau - \Sigma) &= \frac{1}{2} \ln \left[ 2 \cosh^2 \left( \frac{\beta J}{2^{q/2}} \right) \right] \\ &\quad + \sum_{n=0}^{N_M} \ln \left( 1 + \frac{i\omega_n \tilde{\Sigma}(i\omega_n)}{\omega_n^2 + J^2/2^{q-2}} \right), \end{aligned} \quad (\text{B19})$$

where  $\tilde{\Sigma}(i\omega_n)$  is the reduced self-energy (B15). We remark that, whereas the accuracy of  $g_k$  computed from  $G_\ell$  using (B13) decay exponentially with  $k$  because of the exponentially growing factors, the accuracy of  $G(i\omega_n)$  computed using (B11) remains exponential.

### APPENDIX C: COMPUTATION OF THE FREE ENERGY USING KITAEV-SUH RESONANCE THEORY

We can rewrite the SYK ( $G, \Sigma$ ) action (5) in the following form:

$$\begin{aligned} -\frac{I}{N} &= \ln \text{Pf}(-\hat{\Sigma}) - \frac{1}{2} \int_0^\beta d\tau_1 d\tau_2 \left[ \hat{\Sigma}(\tau_1, \tau_2) G(\tau_1, \tau_2) \right. \\ &\quad \left. - \frac{J^2}{q} G(\tau_1, \tau_2)^q \right] + \frac{1}{2} \int d\tau_1 d\tau_2 \sigma(\tau_1, \tau_2) G(\tau_1, \tau_2), \end{aligned} \quad (\text{C1})$$

where we made a replacement  $\hat{\Sigma} = \sigma + \Sigma$ . The first part of this action without the last term is invariant under reparametrizations of time [21] and can be viewed as  $-I_{\text{CFT}}/N$  introduced in (12). The saddle-point configuration of  $I_{\text{CFT}}$  is  $(G_c(\tau), \Sigma_c(\tau))$ . The last term in (C1) is considered as a perturbation. The leading correction to the free energy is obtained

from first-order perturbation theory:

$$\begin{aligned} -\frac{\beta\delta F}{N} &= \frac{1}{2} \int_0^\beta d\tau_1 d\tau_2 \sigma(\tau_1, \tau_2) \langle G(\tau_1, \tau_2) \rangle \\ &= \frac{\beta}{2} \int_0^\beta d\tau \sigma(\tau) G_c(\tau). \end{aligned} \quad (C2)$$

We cannot use direct expression for the source function  $\sigma(\tau) = \delta'(\tau)$  in this formula, as it leads to uncontrolled divergence. It was proposed in Ref. [21] that the source function  $\sigma$  can be replaced by the expression

$$\sigma(\tau) = - \sum_h \sigma_h \frac{J^2 \text{sgn}(\tau)}{b^\Delta a^{1/2} |J\tau|^{h+1-2\Delta}} u(\xi), \quad (C3)$$

where  $a = [(q-1)b]^{-1}$  and the sum goes over the scaling exponents of the bilinear operators in (15) and  $u(\xi)$  with  $\xi = \ln|J\tau|$  is the window function, which serves as a regulator with the normalization  $\int d\xi u(\xi) = 1$ . From the computation of the corrections to the two-point function, one derives that the coefficients  $\alpha_h$  in (19) are related to  $\sigma_h$  as [21]

$$\alpha_h = \frac{\sigma_h a^{1/2}}{-k'(h)}. \quad (C4)$$

Then the contribution to  $\delta F$  comes from the  $h_0 = 2$  scaling exponent and the second-order term of  $\tau/\beta$  expansion of  $G_c$ :

$$\begin{aligned} -\frac{\beta\delta F}{N} &= -\frac{\beta\sigma_0 J^2}{2b^\Delta a^{1/2}} \\ &\times 2 \int_0^{\beta/2} d\tau \frac{u(\xi)}{|J\tau|^{3-2\Delta}} \left( -\frac{b^\Delta}{|J\tau|^{2\Delta}} \frac{\pi^2 \Delta}{3} \frac{\tau^2}{\beta^2} \right), \end{aligned} \quad (C5)$$

where the factor of two in front of the integral is due to the other half of the thermal circle. Using normalization of the window function and (C4) we find the leading correction to the free energy:

$$\frac{\beta\delta F}{N} = \frac{\pi^2 k'(2)}{3qa} \frac{\alpha_0}{\beta J}, \quad (C6)$$

which agrees with (28) and (11). It is not clear why a similar calculation with the higher-scaling exponents  $h$  would not produce  $1/(\beta J)^{h-1}$  contribution. The next  $\delta^2 F$  contribution to the free energy comes from second-order perturbation theory,

$$\begin{aligned} -\frac{\beta\delta^2 F}{N} &= \frac{N}{8} \int_0^\beta d\tau_1 \cdots d\tau_4 \sigma(\tau_1, \tau_2) \sigma(\tau_3, \tau_4) \\ &\times \langle G(\tau_1, \tau_2) G(\tau_3, \tau_4) \rangle_{\text{conn}}. \end{aligned} \quad (C7)$$

The connected correlation function of two  $G(\tau_1, \tau_2)$  fields is equal to the connected part of the four-point function

$$\langle G(\tau_1, \tau_2) G(\tau_3, \tau_4) \rangle_{\text{conn}} = \frac{1}{N} \mathcal{F}(\tau_1, \tau_2; \tau_3, \tau_4). \quad (C8)$$

The  $h_0 = 2$  contribution to the four-point function is special and we only consider  $h \neq 2$  sector here. It is well known that in this sector the four-point function can be written as [17,21,29]

$$\begin{aligned} \mathcal{F}_{h \neq 2}(\tau_1, \tau_2; \tau_3, \tau_4) &= G_c(\tau_{12}) G_c(\tau_{34}) \\ &\times \sum_{k=1}^{\infty} c_h^2 \chi_2^{hk} F_1(h_k, h_k, 2h_k, \chi), \end{aligned} \quad (C9)$$

where  $c_h$  are given in (18) and the cross-ratio  $\chi$  at finite temperature is

$$\chi = \frac{\sin \frac{\pi \tau_{12}}{\beta} \sin \frac{\pi \tau_{34}}{\beta}}{\sin \frac{\pi \tau_{13}}{\beta} \sin \frac{\pi \tau_{24}}{\beta}}. \quad (C10)$$

Taking only  $\sigma_h$  terms from  $\sigma$  functions and picking term with the scaling dimension  $h$  in (C9) we find the contribution to the free energy:

$$\begin{aligned} -\frac{\beta\delta^2 F_h}{N} &= \frac{J^4 c_h^2 \sigma_h^2}{8a} \int_0^\beta d\tau_1 \cdots d\tau_4 \\ &\times \frac{u(\xi_{12}) u(\xi_{34})}{|J\tau_{12}|^{h+1} |J\tau_{34}|^{h+1}} \chi_2^h F_1(h, h, 2h, \chi). \end{aligned} \quad (C11)$$

Since the window function decays quickly when  $\xi = \ln|J\tau|$  is not small, we assume that the integrals acquire the main contribution for  $\tau_{12} \rightarrow 0$  and  $\tau_{34} \rightarrow 0$ , therefore we use that  ${}_2F_1(h, h, 2h, \chi) \approx 1 + \cdots$  and get

$$\begin{aligned} -\frac{\beta\delta^2 F_h}{N} &= \frac{J^4 c_h^2 \sigma_h^2}{8a} 4 \int_0^{\beta/2} dx dx' \\ &\times \int_0^\beta dy dy' \frac{u(\xi) u(\xi')}{|Jx|^{h+1} |Jx'|^{h+1}} \left( \frac{\frac{\pi x}{\beta} \frac{\pi x'}{\beta}}{\sin^2 \frac{\pi(y-y')}{\beta}} \right)^h, \end{aligned} \quad (C12)$$

where we introduced  $x = \tau_{12}$ ,  $x' = \tau_{34}$  and  $y = (\tau_1 + \tau_2)/2$ ,  $y' = (\tau_3 + \tau_4)/2$ . Using normalization of the window function  $u(\xi)$ , the relation between  $\sigma_h$  and  $\alpha_h$  in (C4) as well as the one between  $\alpha_h$  and  $g_h$  in (21), and the expression for  $c_h$  in (18), we finally obtain

$$\frac{\beta\delta^2 F_h}{N} = -\frac{1}{2} g_h^2 \int_0^\beta dy dy' \left( \frac{\pi}{\beta J \sin \frac{\pi(y-y')}{\beta}} \right)^{2h}. \quad (C13)$$

This is exactly equal to the result of the conformal perturbation theory (26).

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