Dynamical relaxation behaviors of a critical quench

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We study the universal dynamical relaxation behaviors of a quantum XY chain following a quench, paying special attention to the case in which the initial state is a critical ground state, or the postquenched Hamiltonian is at a critical point of equilibrium quantum phase transition, or both of them are critical. In such a "critical quench," we find very interesting real-time dynamical scaling behaviors and we find crossover phenomena between them. For a quench from a noncritical point to a critical point, we find that, compared to the noncritical quench, the universal power-law scaling behavior does not change; however, there may be a crossover between the exponential decaying behavior and the power-law scaling. For a quench from a critical point to a noncritical point, the power-law scaling behaviors $t^{-3/2}$ and $t^{-3/4}$ in the noncritical quenches may be changed to t^{-1} and $t^{-1/2}$, respectively. If the prequenched Hamiltonian is set to be a point that is close to but not exactly at a critical point, we find interesting crossover phenomena between different power-law scaling behaviors. We also study the quench from the vicinity of a multicritical point, we find crossover behaviors that are induced by a different mechanism, and we find another crossover exponent. All the results are related to the gap-closing properties of the energy spectrum of the critical points.

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I. INTRODUCTION

The study of the nonequilibrium properties of a many-body quantum system is a hot topic in condensed-matter physics. Of research interest are the properties of the steady state in the long-time limit and also the properties of dynamical relaxation.

A lot of important questions related to the properties of the steady state have been extensively studied, such as the question of thermalization [1], the thermodynamic relations [2], the zero and second laws [3], the heat conduction [4], the thermodynamic uncertainty relation [5], the thermodynamic force [6], the first-type of dynamical quantum phase transition (DQPT-I) [7,8], the quantum information [9,10], and the quantum response [11]. The properties of the dynamical behaviors have also been widely studied in different aspects, such as the Kibble-Zurek mechanism [12–15], the entropy production [16], the heating process [17], the aging phenomena [18], the dynamical topology [19,20], the second type of dynamical quantum phase transition (DQPT-II) [21–24], and quantum scars [25].

In the study of the aforementioned questions, the important concept of universality, which originates from the study of the equilibrium phase transition, has been applied in part of them, such as the Kibble-Zurek mechanism and the DQPTs. In DQPT-I, the critical behavior is revealed by the later-time average of the order parameter, it is the generalization of the equilibrium state phase transition to the nonequilibrim system. The concept of universality, i.e., the definition of critical exponents of such a nonequilibrium phase transition is very similar to the equilibrium phase transition. DQPT-II, which is closely related to DQPT-I, is defined by the phase-transition-like nonanalytic behaviors of the so-called dynamical free energy at a series of critical times when the overlap between the initial state and the evolving state is zero [21]. This type of DQPT has also been extensively studied in different quantum systems with different generalizations, including the mixed states [26,27], the exited states [28], the open system [29], the Floquet system [30], and the critical quench [31]. The concept of universality has also been applied to such a type of phase transition [32], although the theory is still developing [33–35].

Interestingly, besides the aforementioned DQPTs, there is a third type of DQPT (DQPT-III) [36–38] which concerns the asymptotic behavior of a physical variable approaching the steady value in the long-time limit. It is shown that the difference between the value of the correlation function at time t and the value at steady state may scale as $t^{-\mu}$, and DQPT-III is defined by the change of the scaling exponent μ, which has been found to happen at a critical frequency of a periodically driven integrable system [36–38]. Here it should be emphasized that, no matter whether there is a DQPT, the power-law decaying behavior of the correlator is also a universal property of the dynamics, i.e., the decaying exponent μ is determined by what phase the system quenchs to, not depending on the details of the system. Such type of dynamical universality has also been studied in the aperiodically driven system [39], the stochastic driven system [40], and the noise driven system [41].

Recently, such type of universal behaviors have also been found in the quench dynamics of both the noninteracting and the interacting integrable systems [42,43], with different scaling exponents. In a quantum XY chain [42], Makki and coauthors find the $t^{-1/2}$, the $t^{-3/2}$, and the $t^{-3/4}$ power laws

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of decaying, depending on whether the postquench Hamiltonian is in the commensurate phase, the incommensurate phase, or the boundary between the two phases, respectively. More importantly, it is theoretically proved by the stationary phase approximation (SPA) that the scaling law depends on the structure of the energy spectrum of the postqenched Hamiltonian, which deeply reveals the nature of such type of dynamical universality.

In the current paper, we study the universal relaxation behaviors of a quantum XY chain following a quench, paying special attention to the case in which the prequenched Hamiltonian, the postquenched Hamiltonian, or both of them are at critical points of equilibrium quantum phase transitions, which is dubbed the "critical quench." We find very interesting dynamical relaxation behaviors in such type of quench; the gap-closing of the energy spectrum may change the power-law scaling behaviors and lead to very interesting crossover phenomena. Our study also includes the multicritical points, where the crossover phenomena are induced by a different mechanism and another crossover exponent is found.

The paper is arranged as follows: In Secs. II and III, we introduce the model and the method, respectively. In Sec. IV, we give the results of the critical quench, including the quench to a critical point, the quench from a critical point, and the quench from the vicinity of a multicritical point. We conclude our paper in Sec. V.

II. MODELS

The model we studied is a quantum XY spin chain [44–46],

$$H = -\sum_{j=1}^{L}\left[\frac{1+\chi}{2}\sigma_j^x\sigma_{j+1}^x + \frac{1-\chi}{2}\sigma_j^y\sigma_{j+1}^y - h\sigma_j^z\right], \quad (1)$$

where σ^x, σ^y, and σ^z are the Pauli matrices of spin 1/2, and L is the number of sites of the chain in which the periodic boundary condition is applied.

By the Jordan-Winger transformation, the model can be transformed to a free-fermion model:

$$H_f = -\sum_{j=1}^{L}(c_j^\dagger c_{j+1} + \chi c_j^\dagger c_{j+1}^\dagger + \text{H.c.}) + h\sum_{j=1}^{L}(2c_j^\dagger c_j - 1). \quad (2)$$

Here, we have restricted our study in the even-fermionic-number-parity sector and adopted the antiperiodic boundary conditions.

By the Fourier transformation, the model can be transformed to the momentum space,

$$H = \sum_{k>0} H_k = \sum_{k>0}\mathbf{\Psi}_k^\dagger \mathbf{H}_k \mathbf{\Psi}_k, \quad (3)$$

where $\mathbf{\Psi}_k = (c_k, c_{-k}^\dagger)^T$ and

$$\mathbf{H}_k = \begin{pmatrix} z_k & -iy_k \\ iy_k & -z_k \end{pmatrix}. \quad (4)$$

The wave vector k belongs to $\{\pm(2n-1)\pi/L, n = 1, 2, \ldots, L/2\}$ because we have applied the antiperiodic boundary conditions. Here $z_k = 2(h - \cos k)$ and $y_k = 2\chi\sin k$.

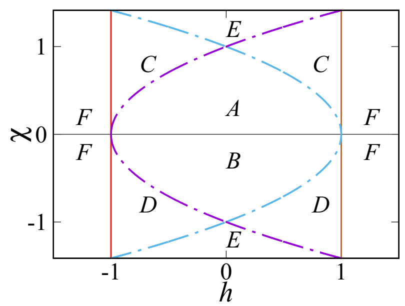

FIG. 1. Equilibrium phase diagram of the XY model (1): for $|h| < 1$, the system is in the ferromagnetic or antiferromagnetic ordered phase; for $|h| > 1$, the system is in the paramagnetic disordered phase. A, B, and E are incommensurate phases; C, D, and F are commensurate phases. The dashed lines are the boundaries between the incommensurate and commensurate phases.

The Hamiltonian H_k is already in a small Hilbert space of 2×2, and it can be easily diagonalized by the Bogoliubov transformation

$$\gamma_k = u_k^* c_k + v_k^* c_{-k}^\dagger, \quad (5)$$

$$\gamma_{-k}^\dagger = -v_k c_k + u_k c_{-k}^\dagger, \quad (6)$$

where

$$u_k = \cos\frac{\theta_k}{2}, \quad v_k = i\sin\frac{\theta_k}{2}, \quad (7)$$

with θ_k being the Bogoliubov angle defined as

$$\tan\theta_k = -y_k/z_k. \quad (8)$$

This gives the energy spectrum and the ground state of the model as

$$\varepsilon_k = \sqrt{y_k^2 + z_k^2}, \quad (9)$$

$$|\Phi\rangle = \prod_{k>0}(u_k + v_k c_k^\dagger c_{-k}^\dagger|0\rangle), \quad (10)$$

where $|0\rangle$ is the fermionic vacuum. It is obvious that $k = 0$ and π are two gap-closing points of the model,

The equilibrium phase diagram of the model is shown in Fig. 1, where $h = 1$ and $h = -1$ are the two critical lines determined by the gap-closing momentum $k = 0$ and π, respectively; the dashed lines are the boundaries between the commensurate and incommensurate phases, which are determined by $\cos k_0 = h/(1-\chi^2)$, with $k_0 = 0$ and π. The incommensurate phase is defined if there is an additional saddle point besides the saddle points $k = 0$ and π; otherwise, it is a commensurate phase.

III. METHOD

Let there be a closed quantum system at a ground state $|\Phi(0)\rangle$ of a prequenched Hamiltonian H_0, and then suddenly change the parameter of the system to make the Hamiltonian be the postquenched Hamiltonian H_1; this is called a quantum quench. Generally, the evolution of a quantum state is determined by the time-dependent Schrödinger equation; for the quantum quench, this leads to a unitary evolution of the state

$$|\Phi(t)\rangle = e^{-iH_1 t}|\Phi(0)\rangle; \quad (11)$$

for the quantum XY model, this is equivalent to solving the time-dependent Bogoliubov–de Gennes (BdG) equation [45]

$$i\partial_t \phi_k(t) = \mathbf{H}_k \phi_k(t), \tag{12}$$

where $\mathbf{H}_k$ is the one for the postquenched Hamiltonian H_1, defined in Eq. (4), and consequently

$$\phi_k(t) = e^{-i\mathbf{H}_k t}\phi_k(0), \tag{13}$$

where $\phi_k(t) = [v_k(t), u_k(t)]^T$. Noting that after the Jordan-Wigner transformation, the Fourier transformation, and the Bogoliubov transformation, the ground state $|\Phi(0)\rangle$ is already encoded by $\phi_k(0) = [v_k(0), u_k(0)]^T$; here $v_k(0) = v_k$ and $u_k(0) = u_k$, which are defined in Eq. (7), are those for the prequeched Hamiltonian H_0, and $|\Phi(0)\rangle = |\Phi\rangle$ is related to v_k and u_k by Eq. (10). Using the Taylor expansion and the fact that $\mathbf{H}_k^2 = \varepsilon_k^2\mathbf{I}$, where $\mathbf{I}$ is a 2×2 identity matrix, the calculation of Eq. (13) can be easily performed, which gives the explicit relation between $\phi_k(t)$ and $\phi_k(0)$, i.e.,

$$\begin{aligned} v_k(t) = {} & v_k(0)[\cos(\varepsilon_k t) - i\sin(\varepsilon_k t)\cos\theta_k] \\ & - u_k(0)\sin(\varepsilon_k t)\sin\theta_k, \end{aligned} \tag{14}$$

$$\begin{aligned} u_k(t) = {} & u_k(0)[\cos(\varepsilon_k t) + i\sin(\varepsilon_k t)\cos\theta_k] \\ & + v_k(0)\sin(\varepsilon_k t)\sin\theta_k. \end{aligned} \tag{15}$$

Using the expression of $v_k(t)$, the correlation $C_{mn}(t) = \langle c_m^\dagger c_n\rangle$ can be calculated, which is

$$\begin{aligned} C_{mn}(t) &= \frac{1}{\pi}\int_0^\pi dk |v_k(t)|^2 \cos[k(m-n)] \\ &= C_{mn}(\infty) + \delta C_{mn}(t), \end{aligned} \tag{16}$$

where

$$\begin{aligned} C_{mn}(\infty) = {} & \frac{1}{\pi}\int_0^\pi dk\bigg[1 - \cos\zeta_k\cos^2\theta_k \\ & - \frac{1}{2}\sin\zeta_k \sin(2\theta_k)\bigg]\cos[k(m-n)] \end{aligned} \tag{17}$$

is the value of $C_{mn}(t)$ in the steady state. Here ζ_k is the Bogoliubov angle of the prequenched Hamiltonian, and θ_k is the Bogoliubov angle of the postquenched Hamiltonian. $\delta C_{mn}(t)$ is the difference between $C_{mn}(t)$ and $C_{mn}(\infty)$,

$$\delta C_{mn}(t) = \delta C_{mn}^{(1)}(t) + \delta C_{mn}^{(2)}(t), \tag{18}$$

where

$$\begin{aligned} & \delta C_{mn}^{(1)}(t) \\ & = -\frac{1}{\pi}\int_0^\pi dk \cos\zeta_k \sin^2\theta_k \cos(2\varepsilon_k t)\cos[k(m-n)] \\ & = \mathrm{Re}\bigg\{-\frac{1}{\pi}\int_0^\pi dk\cos\zeta_k\sin^2\theta_k e^{2i\varepsilon_k t}\cos[k(m-n)]\bigg\}, \end{aligned} \tag{19}$$

$$\begin{aligned} & \delta C_{mn}^{(2)}(t) \\ & = \frac{1}{2\pi}\int_0^\pi dk \sin\zeta_k \sin(2\theta_k)\cos(2\varepsilon_k t)\cos[k(m-n)] \\ & = \mathrm{Re}\bigg\{\frac{1}{2\pi}\int_0^\pi dk\sin\zeta_k\sin(2\theta_k) e^{2i\varepsilon_k t}\cos[k(m-n)]\bigg\}. \end{aligned} \tag{20}$$

The asymptotic behavior of $|\delta C_{mn}(t)|$ is the main topic of the current paper, which can be obtained by the SPA, where the key point is that the integral in $\delta C_{mn}^{(1)}(t)$ or $\delta C_{mn}^{(2)}(t)$ is dominated by the contributions near the extrema of ε_k, and the factor $e^{2i\varepsilon_k t}$ is replaced by a Gaussian by the Taylor expansion of ε_k at the extrema k_0, where k_0, in general, is a saddle point. Then the integrals are calculable, and the scaling behaviors can be obtained. In summary, when t is large enough, the integral (19) or (20) is approximately proportional to

$$\mathrm{Re}\bigg\{e^{i(2\varepsilon_{k_0}t+\varphi)}\int_{-\infty}^{\infty} dk(k-k_0)^q \exp[ib(k-k_0)^p t]\bigg\}, \tag{21}$$

where φ and b are trivial constants; p is determined by the asymptotic behaviors of the spectrum of the postquenched Hamiltonian in the vicinity of the saddle point, i.e., $\varepsilon_k \sim \varepsilon_{k_0} + b(k-k_0)^p$; and q is determined by the asymptotic behaviors of the term $\cos\zeta_k\sin^2\theta_k$ in Eq. (19) or the term $\sin\zeta_k\sin(2\theta_k)$ in Eq. (20) as k approaches k_0. For example, for a noncritical quench to the commensurate phase, in the vicinity of the saddle point $k=0$, $p=q=2$, which eventually leads to a power law of $t^{-3/2}$. For more details, see Appendix A of Ref. [42], where more examples are given. However, in Ref. [42] the analysis is restricted to $\delta C_{mn}^{(1)}(t)$, because the initial state is chosen as $[v_k(0), u_k(0)] = (0, 1)$. Generally, $|\delta C_{mn}^{(2)}(t)|$ follows the same scaling law as $|\delta C_{mn}^{(1)}(t)|$; however, for the critical quench studied in the current paper, we show that, in certain case, the scaling laws can be different. More importantly, the gap-closing property of ε_k may substantially change the scaling behaviors, and different crossover phenomena can also be induced.

In the current paper, in the calculations of integrals (19) and (20), m is set to be equal to n if not explicitly stated. The results do not have a qualitative difference for $m \neq n$ if the distance between the sites m and n is short. In the current paper, we only pay attention to the relaxation behavior of such a short-range correlation. For the long-range correlation, the relaxation behavior may be very different; for example, it is already known that in the quantum Ising model, the long-range correlation after quench may decay exponentially [21]. Furthermore, in the calculations of integrals (19) and (20), the system size L should be large enough to make the numerical results of the integrals accurate enough. In the current paper, if it is not clearly stated, the size of the system is taken to be $L = 10^5$.

IV. RESULTS

A. Quench to a critical point

When the prequenched Hamiltonian is noncritical and the postquenched Hamiltonian is critical, the universal property of the relaxation behavior of $|\delta C_{mn}(t)|$ is the same as the corresponding noncritical quench (where both the prequenched and postquenched Hamiltonians are noncritical).

First, we consider the quench to a *critical commensurate phase*, i.e., the postquenched Hamiltonian has the parameters $h = \pm 1$ and $|\chi| < \sqrt{2}$. A typical example is shown in Fig. 2, which is a quench from $(h, \chi) = (1.5, 2)$ to $(1, 1)$. Here the scaling relation $|\delta C_{mn}(t)| \sim t^{-3/2}$ is kept, which is the contribution of the saddle point $k = \pi$. Nearing this point,

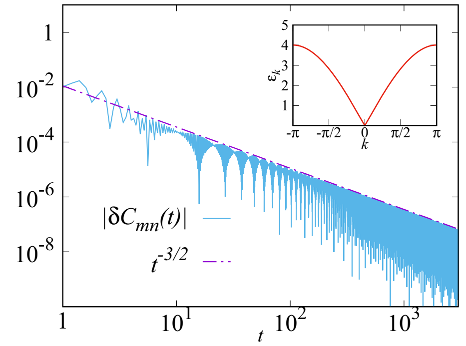

FIG. 2. Scaling behavior of a quench from $(h, \chi) = (1.5, 2)$ to (1, 1); the subset is the energy spectrum of the postquenched Hamiltonian.

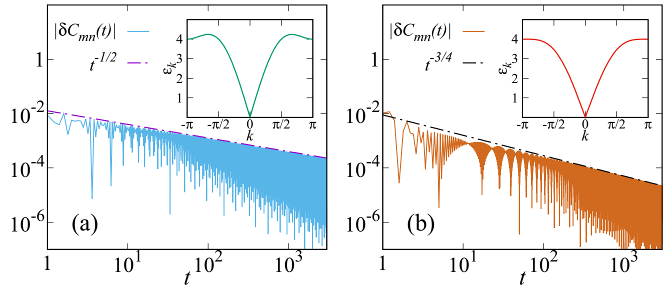

FIG. 4. (a) Scaling behavior of a quench from $(h, \chi) = (1.5, 2)$ to a critical incommensurate phase with $(h, \chi) = (1, \sqrt{3})$. (b) Scaling behavior of a quench from $(h, \chi) = (1.5, 2)$ to $(1, \sqrt{2})$. The subsets are the energy spectrum of the postquenched Hamiltonians.

$\sin\zeta_k \sim k - \pi$ and $\sin\theta_k \sim k - \pi$ too, and hence $q = 2$ in Eq. (21); furthermore, $\varepsilon_k = \varepsilon_\pi + b(k-\pi)^2$, where b is the second derivative of ε_k at π, this gives $p = 2$ in Eq. (21). The integral (21) with $q = 2$ and $p = 2$ gives a $t^{-3/2}$ scaling. It should be noted that $k = 0$ is not a saddle point, it does not contribute a power-law scaling in the dynamics but an exponential decaying, as we show in the following example.

When $|\chi| < 1$, especially when χ is small, it gives an interesting relaxation behavior; a typical example is shown in Fig. 3(a). When t is large enough, $|\delta C_{mn}(t)|$ is dominated by the $t^{-3/2}$ scaling; however, in the early time region $(0, \tau_c)$, $|\delta C_{mn}(t)|$ is dominated by an exponential decaying behavior. Such type of crossover is owing to the competition between the saddle point $k = \pi$ and the gap-closing point $k = 0$. The saddle point $k = \pi$ leads to the scaling of $t^{-3/2}$; the reason is the same as in the aforementioned case shown in Fig. 2. The exponential decaying in the early time region is owing to the contribution of the gap-closing point $k = 0$. It should be noted that, although the point $k = 0$ is not a saddle point, it is still an extreme point. The Taylor expansion of ε_k at this point should take the form

$$\varepsilon_k \sim \varepsilon_0 + bk + \cdots, \tag{22}$$

$$\text{with} \quad b = \frac{d\varepsilon_k}{dk}|_{k\to 0^+}, \tag{23}$$

because the first-order derivative b is not zero, the term bk must have certain contribution to the integrals in Eqs. (19) and (20), which is the origination of the exponential decaying behavior. Furthermore, because $\varepsilon_0 = 0$, in Eq. (21) the

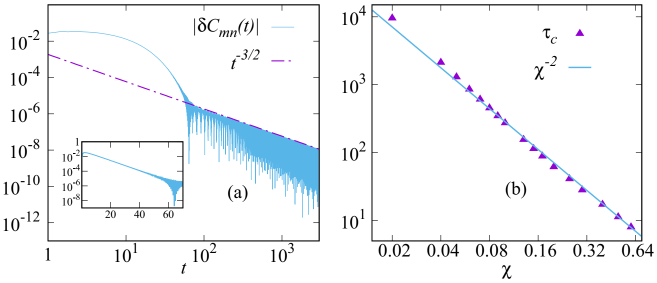

FIG. 3. Crossover of a quench from $(h, \chi) = (1.5, 2)$ to $(1, 0.2)$; the subset is a quasilogarithmic plot of the early time region.

prefactor $e^{i(2\varepsilon_0 t+\varphi)}$ does not depend on time; therefore, in such an exponential decaying region, there is no oscillation. The crossover phenomenon is obvious when the value of χ is small, because in this case the value of the $t^{-3/2}$ term is small, it can dominate the relaxation behavior only when the exponential term decays to a much smaller value. The crossover time τ_c is defined as the time that $|\delta C_{mn}(t)|$ stops the exponential decaying and begins to decay algebraically, which is numerically determined. We find that it diverges with a power law,

$$\tau_c \sim \chi^{-\kappa}, \tag{24}$$

as $\chi \to 0$, where $\kappa \approx 2.0$. This is demonstrated in Fig. 3(b).

However, for the postquenched Hamiltonian with χ exactly at 0, i.e., a quench to the Luttinger liquid (which is also a critical phase), we find that $C_{mn}(t)$ does not evolve with time; this is easy to understand, because in this case, the Hamiltonian (2) only has the hopping term and the number of particles is conserved.

We then consider the quench to a *critical incommensurate phase*, i.e., the postquenched Hamiltonian has the parameters $h = \pm 1$ and $|\chi| > \sqrt{2}$. A typical example is shown in Fig. 4(a), which is a quench from $(h, \chi) = (1.5, 2)$ to $(1, \sqrt{3})$; the postquenched Hamiltonian has an additional saddle point of $k_0 = 2\pi/3$ in the energy spectrum. It is shown that in this case the relaxation scaling law $t^{-1/2}$ is kept, falling within the prediction of the SPA.

At last, we consider a quench from $(h, \chi) = (1.5, 2)$ to $(1, \sqrt{2})$, where the postquenched Hamiltonian is at the boundary between the critical commensurate phase and the critical incommensurate phase, as shown in Fig. 4(b); it is shown that in this case the relaxation scaling law $t^{-3/4}$ is kept, consistent with the theory of the SPA.

If the postquenched Hamiltonian is close to but not exactly at the critical point, then the period becomes very large, as shown in Fig. 5(a); this is in sharp contrast to the case that is exactly at the critical point, as shown in Fig. 5(b). The reason is that the circular frequency of the oscillation of $\delta C_{mn}(t)$ is $\omega = 2\varepsilon_{k_0}$, as shown in the prefactor of the integral of Eq. (21); therefore, the circular frequency of $|\delta C_{mn}(t)|$ should be $\omega = 4\varepsilon_{k_0}$. For the quench with the postquenched Hamiltonian that is exactly at the critical point, the $k = 0$ is not a saddle point, and the circular frequency $\omega = \omega_\pi = 4\varepsilon_\pi = 16$; therefore, the period is $T = 2\pi/\omega = \pi/8$. However, for the quench with

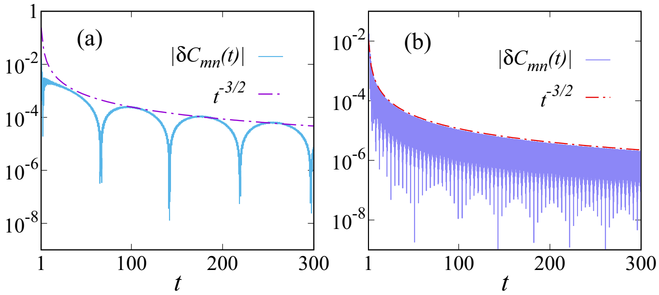

FIG. 5. (a) Dynamical relaxation of a quench from $(h, \chi) = (1.5, 2)$ to (0.99,1). (b) Dynamical relaxation of a quench from $(h, \chi) = (1.5, 2)$ to (1,1).

the postquenched Hamiltonian that is close to but not exactly at the critical point, there are two circular frequencies, in which $\omega_\pi = 15.92$ and $\omega_0 = 0.08$; correspondingly, there are two periods $T_\pi = \pi/7.96$ and $T_0 = 25\pi$. We can see that T_0 is much larger than T_π, and the oscillation behavior is very similar to a beat.

B. Quench from a critical point

If the prequenched Hamiltonian is at a critical point, the scaling behavior in the dynamics may change substantially, depending on the choosing of both the prequenched Hamiltonian and the postquenched Hamiltonian. First, we study the quench from a critical point to a commensurate phase. A typical example is shown in Fig. 6, which is a quench from $(h, \chi) = (1, 1)$ to $(0.75, 1/\sqrt{2})$. Here the scaling law of the relaxation behavior is t^{-1}, which is different from the $t^{-3/2}$ scaling of the corresponding noncritical quench. For an understanding of such a result, we pay attention to Eq. (20), in the vicinity of the saddle point $k = 0$, which is also the gap-closing point of the prequenched Hamiltonian, $\sin\zeta_k \sim 1$ and $\sin(2\theta_k) \sim k$, this means $q = 1$. Furthermore, according to the SPA, the spectrum of the postquenched Hamiltonian can be replaced by the Taylor expansion at the saddle point, and thus $\varepsilon_k = \varepsilon_0 + bk^2$, where $b = \varepsilon_0''$ is the second derivative of ε_k at $k = 0$, therefore $p = 2$. Substitute these values of p and q to the integral (21) and we get a power law of t^{-1}.

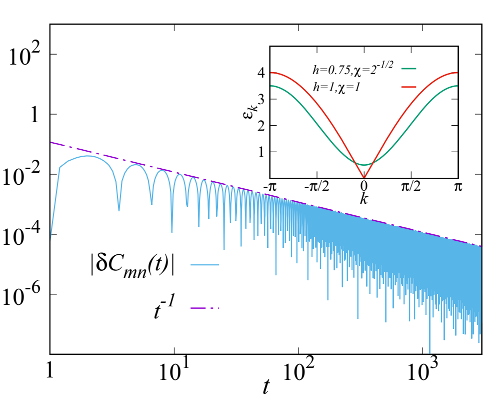

FIG. 6. Scaling behavior of a quench from $(h, \chi) = (1, 1)$ to $(0.75, 1/\sqrt{2})$. The subset shows the energy spectrum.

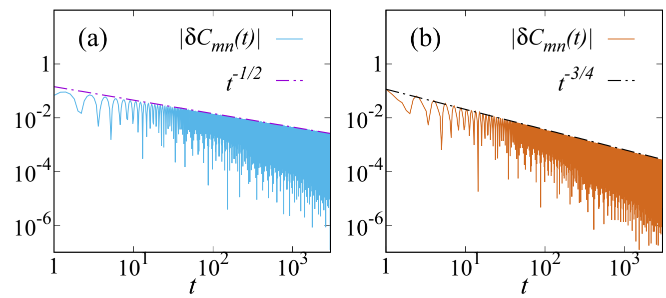

FIG. 7. (a) Scaling behavior of a quench from $(h, \chi) = (1, 1)$ to $(0.5, 1/\sqrt{2})$. (b) Scaling behavior of a quench from $(h, \chi) = (-1, 1)$ to $(0.5, 1/\sqrt{2})$.

In fact, in the current case, the integral of Eq. (19) should contribute a t^{-2} scaling, because in the vicinity of the saddle point $k = 0$ the term $\cos\zeta_k \sin^2\theta_k \sim k^3$, i.e., $q = 3$ for Eq. (21). However, similar analysis shows that, in the vicinity of the saddle point $k = \pi$, the integral (19) should contribute a $t^{-3/2}$ scaling. Therefore, the decaying behavior of $|\delta C_{mn}(t)|$ satisfies a formula of the mixture of t^{-1}, $t^{-3/2}$, and t^{-2}. The $t^{-3/2}$ and t^{-2} terms decay faster than the t^{-1} term. They are overwhelmed by the t^{-1} term, and hence eventually the dynamical relaxation process is dominated by the t^{-1} scaling.

Second, we study the quench from a critical point to the boundary between the commensurate and incommensurate phases; in this case, the scaling law of the relaxation behavior may be changed or not, depending on the choosing of the prequenched (critical) Hamiltonian. As shown in Fig. 7(a), a quench from $(h, \chi) = (1, 1)$ to $(0.5, 1/\sqrt{2})$ satisfies the scaling law $t^{-1/2}$, which is different from the $t^{-3/4}$ scaling of the corresponding noncritical quench. As to the reason for such a change, it should be noted that in the current case the integral (19) still gives a scaling of $t^{-3/4}$, and the $t^{-1/2}$ scaling is given by Eq. (20). The reason is also related to the gap closing of the energy spectrum of the prequenched Hamiltonian at $k = 0$, which leads to $\sin\zeta_k \sim 1$ and subsequently $\sin\zeta_k \sin(2\theta_k) \sim k$ in Eq. (20). Therefore, in Eq. (21) $q = 1$; furthermore, because the postquenched Hamiltonian is at the boundary of the commensurate phase and the incommensurate phase, the spectrum can be expanded as $\varepsilon_k = \varepsilon_0 + \varepsilon_0'''' k^4$, and because the second and third derivatives are zero, therefore $p = 4$. Substitute these values of p and q in Eq. (21) and we get a power law of $t^{-1/2}$. However, a quench from $(h, \chi) = (-1, 1)$ to $(0.5, 1/\sqrt{2})$ still satisfies the power law of $t^{-3/4}$; this is shown in Fig. 7(b). In this case, the gap-closing point of the spectrum of the prequenched Hamiltonian is $k = \pi$. It does not interfere with the saddle point $k = 0$ of the spectrum of the postquenched Hamiltonian, and hence the power law of $t^{-3/4}$ is kept.

Last, we study the quench from a critical point to the incommensurate phase, and in this case, the scaling law is not changed. For example, in a quench from $(h, \chi) = (1, 1)$ to $(0.25, 1/\sqrt{2})$, the scaling law $t^{-1/2}$ is still kept. The gap-closing point $k = 0$ of the prequenched Hamiltonian does not change the universal relaxation behavior of such a quench. The reason can be analyzed in a similar way. If the prequenched Hamiltonian is very close to but not exactly at the critical point, crossover behavior between different scaling

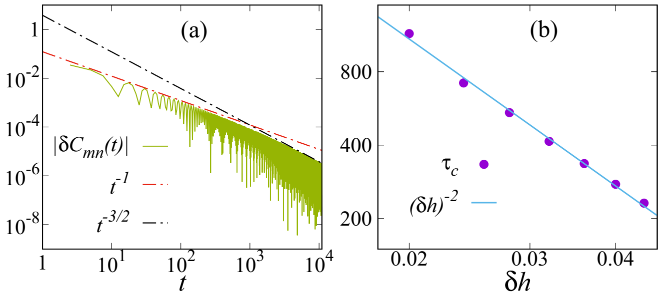

FIG. 8. (a) Crossover behavior of a quench from $(1+\delta h, \chi) = (1.02, 1)$ to $(0.75, 1/\sqrt{2})$; (b) Crossover time τ_c versus δh.

laws appears. Figure 8 shows a typical example, where the prequenched Hamiltonian is at $(1+\delta h, \chi) = (1.02, 1)$. We can see that the t^{-1} scaling behavior dominates when the time $t < \tau_c$; then it changes to the scaling $t^{-3/2}$ for $t > \tau_c$. The reason for such crossover behavior is also related to the gap closing of the prequenched Hamiltonian. The width of the exponential term $\exp[ib(k-k_0)^p t]$ in Eq. (21), i.e., the effective range of k, is $k_w \sim t^{-1/p}$; therefore, when t is not large enough, k_w is obviously larger than δh, and hence δh is negligible. In this case, the spectrum of the prequenched Hamiltonian $\varepsilon_k \sim k$, and this leads to $q = 1$ in Eq. (21); therefore, we get a t^{-1} scaling. When t is large enough, k_w is very small and subsequently δh is comparable to k. In this case, $q = 2$; therefore, we get a $t^{-3/2}$ scaling. We numerically determine the crossover time τ_c, which also satisfies a power-law scaling of $\tau_c \sim (\delta h)^{-\kappa}$. By the data fitting, we find that $\kappa \approx 2.0$; this is demonstrated in Fig. 8(b). It is a pity that currently we cannot theoretically prove why the crossover exponent takes such a value. In a similar way, we investigate the quench from the vicinity of a critical point to the boundary between the commensurate and incommensurate phases. A typical example is shown in Fig. 9(a) and the crossover time τ_c is shown in Fig. 9(b), where the crossover exponent is $\kappa \approx 2.5$.

C. Quench from a critical point to another critical point

An interesting question is, if both the prequenched and postquenched Hamiltonians are all at the critical points, then what happens? In this case, we can find a mixture of the aforementioned results. For example, Fig. 10 shows a quench from $(h, \chi) = (-1, 1)$ to (1,0.1), and we can see that the early-time

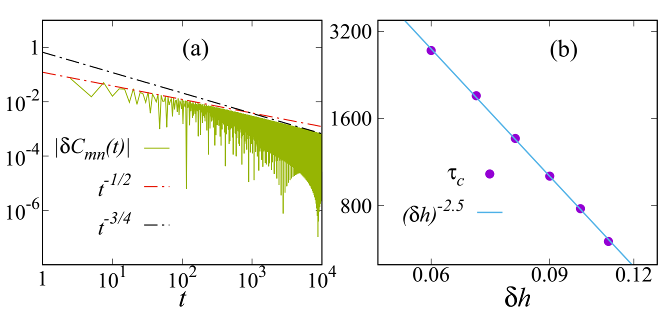

FIG. 9. (a) Crossover behavior of a quench from $(1+\delta h, \chi) = (1.1, 0.5)$ to $(0.5, 1/\sqrt{2})$. (b) Crossover time τ_c versus δh.

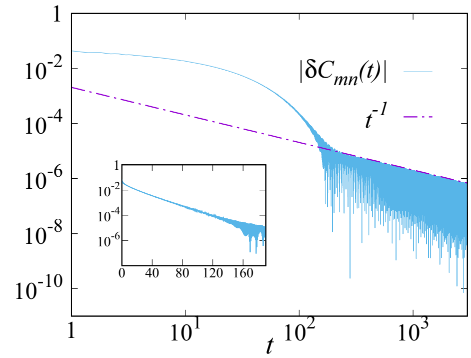

FIG. 10. Crossover behavior of a quench from $(h, \chi) = (-1, 1)$ to (1,0.1). The subset is a quasi-log plot for the early time.

dynamics in the region $(0, \tau_c)$ is dominated by an exponential decaying, and the long-time dynamics is dominated by a power-law of t^{-1}. The crossover time τ_c also satisfies the scaling law $\tau_c \sim \chi^{-\kappa}$, with $\kappa \approx 2.5$.

D. Quench from a multicritical point

In the phase diagram of Fig. 1, there are two multicritical points, with $(h, \chi) = (\pm 1, 0)$. In this section, we study the quench from such a multicritical point or from the vicinity of such a multicritical point.

If the prequenched Hamiltonian is exactly at a multicritical point, we find that the scaling of the relaxation behavior does not change. First, we investigate the quench from such a multicritical point to a commensurate phase; for example, a quench from $(h, \chi) = (1, 0)$ to the point $(0.75, 1/\sqrt{2})$ gives a scaling behavior of $^{-3/2}$. In this case, because χ is zero for the prequenched Hamiltonian, then the integral in Eq. (20) is zero, and $\delta C_{mn}(t)$ is completely determined by Eq. (19), where $\cos\zeta_k = 1$ and $\sin^2\theta_k \sim k^2$, i.e., $q = 2$ in Eq. (21). This leads to a $t^{-3/2}$ scaling behavior.

Second, we investigate the quench from the multicritical point to the incommensurate phase and the boundary between the commensurate and incommensurate phases, we find the $t^{-1/2}$ and $t^{-3/4}$ scaling behaviors, respectively. The theoretical analysis for these results can be performed similarly.

However, if we set the prequenched Hamiltonian to be a point that is very close but not exactly at the multicritical point, we can find very interesting crossover phenomena. For example, a quench from $(h, \chi) = (1, 0.0005)$ to the point $(0.75, 1/\sqrt{2})$ exhibits a crossover between the $t^{-3/2}$ and $t^{-1/2}$ scaling behaviors, as shown in Fig. 11(a). The crossover time τ_c versus different values of χ is shown in Fig. 11(b). τ_c satisfies the scaling law $\tau_c \sim \chi^{-\kappa}$, where $\kappa = -1$. The condition for such a type of crossover is that the value of χ should be very small (but not equal to zero), and then the energy spectrum of the prequenched Hamiltonian $\varepsilon_k \sim k^2$ and subsequently in Eq. (19) the term $\cos\zeta_k \sim 1$. This leads to the $t^{-3/2}$ scaling behavior. Meanwhile, also because the energy spectrum of the prequenched Hamiltonian $\varepsilon_k \sim k^2$, in Eq. (20) the term $\sin\zeta_k \sin\theta_k \sim \chi$, and this leads to a $t^{-1/2}$ scaling. In summary, the scaling behavior of $|\delta C_{mn}(t)|$ in the current case takes the form

$$|\delta C_{mn}(t)| = a t^{-3/2} + b t^{-1/2}, \tag{25}$$

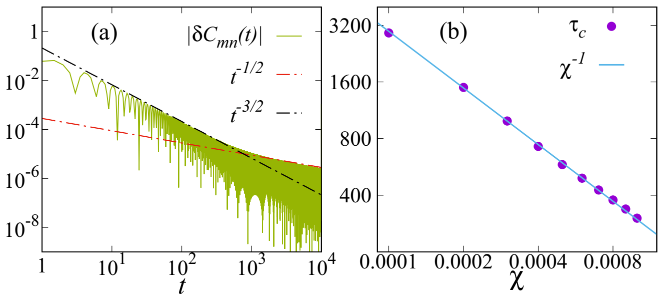

FIG. 11. (a) Crossover behavior of a quench from $(h, \chi) = (1, 0.0005)$ to $(0.75, 1/\sqrt{2})$. (b) Crossover time τ_c versus χ.

where a and b are nonuniversal parameters. Note that $b \sim \chi$ is very small. Therefore, when the time is not long enough, the first term dominates; however, the first term decays faster than the second term, and when the value of the first term is smaller than that of the second term, the second term begins to show the importance and eventually dominates scaling behavior. It is obvious that the scaling for the first term to reach the second term is proportional to $t^{-3/2}/t^{-1/2} = t^{-1}$; therefore, the crossover time τ_c satisfies $\tau_c^{-1} = \chi$, i.e., $\tau_c = \chi^{-1}$. We can see that the mechanism of such a type of crossover behavior is very different from the aforementioned crossover behaviors in Secs. IV B and IV C. In a similar way, we study the quench from the vicinity of the multicritical point to the boundary between the commensurate and incommensurate phases, such as the quench from $(1, \chi)$ to $(0.5, 1/\sqrt{2})$, with χ being very small, and we find a crossover from the $t^{-3/4}$ scaling to the $t^{-1/4}$ scaling. Here the $t^{-1/4}$ scaling comes from the integral (20), which gives $q = 0$ and $p = 4$ for Eq. (21). Figure 12(a) shows a typical example of such a type of crossover, we can see that in the early time, the relaxation is dominated by the $t^{-3/4}$ scaling, as we expected; however, in the later time that we have reached, it is dominated by a scaling of $t^{-0.36}$ but not $t^{-1/4}$. The reason is that in this region the term of $t^{-3/4}$ is not small enough, and hence the relaxation is a mixture of $t^{-3/4}$ and $t^{-1/4}$, i.e.,

$$|\delta C_{mn}(t)| = at^{-3/4} + bt^{-1/4}. \quad (26)$$

In the numerical sense, if the difference between a and b is not significant, the data may be fitted by a uniform scaling $t^{-\mu}$, with $-3/4 < \mu < -1/4$. The $t^{-1/4}$ scaling can only dominate in very late time; this makes it very difficult to numerically determine the crossover time, because when the time is too late the revival phenomenon [47] may appear for a finite-size system. In Fig. 12(a), we have used a very large system size of $L = 10^6$, and the latest time shown is $t = 3.8 \times 10^5$, above which the revival phenomenon appears. For a further understanding of the question, we calculate the integrals of Eqs. (19) and (20) respectively, and the results are presented in Fig. 12(b), where the $t^{-3/4}$ scaling of $|\delta C_{mn}^{(1)}(t)|$ and the $t^{-1/4}$ scaling of $|\delta C_{mn}^{(2)}(t)|$ are shown very clearly. The scaling for the $t^{-3/4}$ term to reach the $t^{-1/4}$ term is proportional to $t^{-3/4}/t^{-1/4} = t^{-1/2}$, and b is proportional to χ; therefore, the crossover time τ_c satisfies $\tau_c^{-1/2} = \chi$, i.e., $\tau_c = \chi^{-2}$.

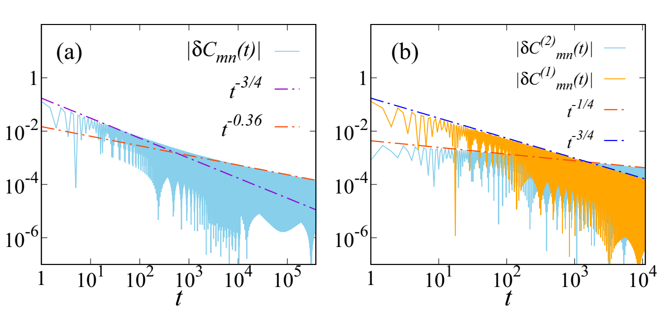

FIG. 12. (a) Scaling behavior of $|\delta C_{mn}(t)|$ of a quench from $(h, \chi) = (1, 0.01)$ to $(0.5, 1/\sqrt{2})$. (b) Scaling behaviors of $|\delta C_{mn}^{(1)}(t)|$ and $|\delta C_{mn}^{(2)}(t)|$ of the quench from $(h, \chi) = (1, 0.01)$ to $(0.5, 1/\sqrt{2})$.

V. SUMMARY AND DISCUSSION

In summary, we have studied the universal real-time dynamical relaxation behaviors of a quantum XY chain after a critical quench, where the initial state is a critical ground state, or the postquenched Hamiltonian is at a critical point of equilibrium quantum phase transition, or both of them are critical. Generally, a quench to a critical point does not change the universal power-law scaling behavior but may lead to a crossover between the exponential decaying behavior and the power-law scaling behavior. A quench from a critical point may change the power-law scaling behaviors $t^{-3/2}$ and $t^{-3/4}$ in the noncritical quenches to t^{-1} and $t^{-1/2}$, respectively. Furthermore, if the prequenched Hamiltonian is set to be in the vicinity of a critical point, we can find an interesting crossover between different scaling behaviors. Similar questions are also studied in the quench from a multicritical point, and we find crossover behaviors originating from a different mechanism, where another crossover exponent is found.

All our reported results in the critical quench are related to the gap-closing properties of the critical points; more specifically, at the critical point, $\varepsilon_k \sim k$, this shelters the asymptotic behavior of $\sin\zeta_k$ or $\sin\theta_k$ in the integrals of Eqs. (19) and (20) as k approaches zero, and subsequently leads to different scaling behaviors. It is obvious that, if the dispersion is different, it will lead to different scaling behaviors in the critical quench; the quench from the vicinity of the multicritical point, which has a dispersion of $\varepsilon_k \sim k^2$, is a typical example. It is an interesting question to study the relaxation behaviors after a critical quench in other integrable and nonintegrable systems, such as the periodically driven system [38], the aperiodically driven system [39], the stochastic driven system [40], the noise driven systems [41], and so forth. In fact, nonequilibrium dynamics involving critical states also lead to other interesting physics [31,48–50].

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Correction: Equations (7) and (8) contained errors and have been fixed.