Letter

# Spin-statistics relation and Abelian braiding phase for anyons in the fractional quantum Hall effect

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(Received 14 September 2022; revised 28 April 2023; accepted 9 May 2023; published 22 May 2023)

Quasihole excitations in fractional quantum Hall (FQH) systems exhibit fractional statistics and fractional spin, but how the spin-statistics relation emerges from many-body physics remains poorly understood. Here we prove a spin-statistics relation using only FQH wave functions, on both the sphere and disk geometry. In particular, the proof on the disk generalizes to all quasiholes in realistic systems, which have a finite size and could be deformed into arbitrary shapes. Different components of the quasihole spins are linked to different conformal Hilbert spaces (CHSs), which are null spaces of model Hamiltonians that host the respective FQH ground states and quasihole states. Understanding how the intrinsic spin of the quasiholes is linked to different CHSs is crucial for the generalized spin-statistics relation that takes into account the effect of metric deformation. In terms of the experimental relevance, this enables us to study the effect of deformation and disorder that introduces an additional source of Berry curvature, an aspect of anyon braiding that has been largely neglected in previous literature.

DOI: 10.1103/PhysRevB.107.L201301

*Introduction*. Anyons are theoretically proposed particles in two dimensions whose adiabatic exchange leads to a phase between zero and $\pi$, interpolating the statistics of bosons and fermions [1–3]. The topological nature of fractional statistics promises applications in robust quantum computing [3–6]. One promising platform to realize anyons is a fractional quantum Hall (FQH) system, where anyons are elementary excitations [7–11]. The fractional statistics first demonstrated in the quasiholes of the Laughlin state [2,7] motivated the search for an anyonic "topological spin" that satisfies the standard spin-statistics theorem [8,11–16]. However, a complete understanding of this relation remains elusive [8,11]. Moreover, the scarcity and difficulty of accurate measurements in experiments of these fractional braiding phases [17–22] raise the question of how robust this statistical phase is in realistic experimental conditions.

The proper understanding of the anyonic spin-statistics relations is important both theoretically and experimentally. Formally, such a relationship for elementary particles requires the full machinery of the relativistic quantum field theory (r-QFT) [23–26]. In its most simplified form, Lorentz invariance (for bosons) and the additional requirement of energy being bounded from below (for fermions) are needed [23]. On the other hand, nonrelativistic explanations employ the intuitive picture that exchanging two particles involves particle self-rotation [3,27]. This argument relies on a "string attachment"[27] as shown in Fig. 1(a). While the existence of such a string cannot be physically justified for point particles, quasiparticles in condensed-matter systems are not point particles [28] [see Fig. 1(b)]. Furthermore, since Lorentz invariance is mostly irrelevant in such systems, we can in principle expect a nonrelativistic justification for the spin-statistics relation [3,8,26,29].

In the context of FQH anyons, the standard spin-statistics relation, $\gamma_{\rm exc} = 2\pi s$, fails. [Here $\gamma_{\rm exc}$ is the phase gained after exchanging two particles and $s$ is their intrinsic spin. In this paper we use the term "exchange" to refer to the process of exchanging two anyons on the two-dimensional (2D) plane, while "braiding" to refer to the process of winding one anyon around a second stationary anyon; thus for a given Abelian anyon species, the braiding phase is twice the exchange phase.] For an Abelian state at filling factor $\nu$, the intrinsic spin of a cluster of $k$ quasiholes (also referred to as "$k$ stacked"), obtained by parallel transport on the sphere [8,12,14,16,30,31], or from electron density on the disk [32,33], is shown to take the form

$$s_k = -\frac{\nu k^2}{2} + \frac{k}{2} + n\nu k, \tag{1}$$

where $n$ is the Landau-level (LL) index. It clearly contradicts the well-known exchange statistics of $\nu\pi$ [7] (e.g., for $k = 1$, $\nu = 1$, $\gamma_{\rm exc} = \pi$ but $2\pi s_k = 2\pi n$).

Experimental studies of the anyon statistics are further complicated by the coupling of the quasihole charge to the magnetic field. The Berry phase from the exchange of two anyons in QH systems is $\gamma_{\rm Berry} = \gamma_{\rm B\text{-}field} + \gamma_{\rm stats}$ where $\gamma_{\rm B\text{-}field}$ is the Aharonov-Bohm (AB) phase from the background magnetic field, and $\gamma_{\rm stats}$ encodes the exchange statistics of the anyons. Probing the latter quantities experimentally requires observing a discrete change in the total Berry curvature [19,20,34] or measuring the scattering amplitude from anyon colliders [21,22]. In both cases the quasiholes propagate along the edge, which has been shown to be robust for certain simple cases [35]. However, a microscopic understanding of the effect of disorder on anyonic statistics is lacking, especially

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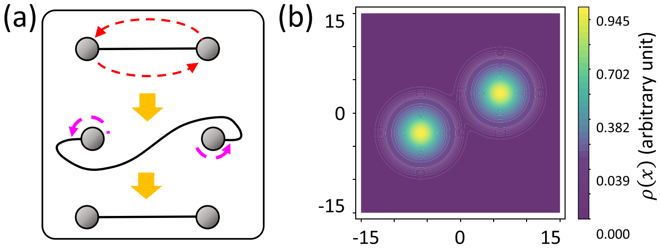


FIG. 1. (a) The "attached string" argument: exchanging the positions of two particles must be followed by self-rotation. (b) Two particles (electrons or anyons) in a single LL slightly deforms each other, allowing a well-defined "string attachment."

for quasiholes that reside completely in the bulk. A very important aspect of anyons is that they are *not point particles* but occupy a finite area with a shape that can be easily deformed. While previous studies have drawn attention to the correction to the braiding phase due to the overlapping tails of the quasiholes [36–40], even in the large-separation limit where the quasiholes are well-separated, deforming the shapes of the quasiholes themselves can lead to further correction to the Berry phase. This has important implications to the experimental braiding of anyons and the potential realization of robust universal topological quantum computers.

In this Letter, we propose an intuitive and rigorous understanding for the relationship between the intrinsic spin and statistics for all types of Abelian anyons. These include not only the special cases of fermions and bosons but also anyons from $k$-stacked quasiholes that occupy a finite area, as well as those with *an arbitrary shape* (e.g., deformed by external potential or disorder). A modification to the standard spin-statistics relation, which generalizes to the braiding of a $k$ stack around a $k'$ stack, has been proposed as [31,32,41]

$$\theta_{k,k'} = 2\pi(s_k + s_{k'} - s_{k+k'}). \tag{2}$$

Here we prove it analytically by treating the adiabatic exchange as a perturbation to some rotationally invariant Hamiltonian. Furthermore, in the case where $k = k'$ (identical particles) it is natural to define a physically relevant intrinsic spin, of which the topological spin [30,31] is the special case, so that the standard spin-statistics relation is recovered. We also discuss the effect of disorder on the Berry phase measurement on a quantum Hall (QH) droplet. While our discussion is limited to quasiholes in Abelian phases, we can easily generalize to Abelian quasi-electrons [42,43] and Abelian braiding of non-Abelions [44,45], and in principle the intrinsic spin is well-defined for such cases.

*$k$-stacked quasiholes on the sphere.* We first look at a QH fluid realized on the spherical geometry with a magnetic monopole at the center [46,47], generalizing the original arguments in Ref. [16]. Inserting a $k$-stacked quasihole into the ground state of a generic quantum Hall phase with $N_e$ electrons results in the total number of fluxes $N_\phi = \nu^{-1}N_e - s_c - s_f + k$, where $s_c$, $s_f$ are the cyclotron and guiding center topological shifts [48,49]. We can then take the difference of the adiabatic rotation phase about the $z$ axis between the two states, a $k$-stacked quasihole at the north pole ($\gamma_1$) and at the south pole ($\gamma_2$) [44], from the difference between their angular momentum:

$$\Delta\gamma_{21} = \gamma_2 - \gamma_1 = 2\pi k\nu N_\phi + 4\pi\left(\gamma_{s_1} + \gamma_{s_2} + \gamma_{s_3}\right), \tag{3}$$

$$\gamma_{s_1} = \frac{\nu k}{2}s_c, \quad \gamma_{s_2} = \frac{\nu k}{2}s_f, \quad \gamma_{s_3} = -\frac{\nu}{2}k^2. \tag{4}$$

The first term on the right-hand side (RHS) captures the coupling of the $k$-stacked anyon to the total magnetic flux. The additional terms are understood as the parallel transport on the sphere inducing a self-rotation of the quasihole due to the presence of the Gaussian curvature [30,31,48]. For the special case of the Laughlin state, $s_f = \nu^{-1} - 1$ and the three terms in Eq. (4) sum up to Eq. (1).

On the disk this quantity corresponds to the Berry phase of a $k$ stack moving in an infinitely large circle [50]. The north pole is mapped to the center of the disk while the south pole is mapped to infinity. A $2\pi$ rotation of the quasiholes about the $z$ axis on the sphere corresponds to their adiabatic dragging in a circular loop on the disk. It is important to note that only $\Delta\gamma_{21}$ is measurable as the AB phase since the curvature is zero everywhere on the disk. This is contrary to the naive view that, since a quasihole moving in a circular loop on the disk is equivalent to rotating the entire system, the associated Berry phase should be given by only $\gamma_2$ [8,16]. Subtracting from $\gamma_2$ the angular momentum of the neutral ground state [8] gives the angular momentum of the quasihole [32], but still does not give the right phase. Instead, the measurable quantity is the *excess* angular momentum from that of the quasihole placed at the rotation center.

Microscopically, this can be understood by considering a realistic system where the quasihole is trapped and rotated by some potential profile. Consider a local trapping potential $\hat{H}(\theta) = \hat{H}_0 + \hat{H}_1(\theta)$ where $\hat{H}_1$ is a $\theta$-dependent perturbation to the rotationally invariant trapping potential $\hat{H}_0$. The ground state $|\psi(\theta)\rangle$ can be written as

$$|\psi(\theta)\rangle = \lambda_0|\psi_0\rangle + \lambda_1|\psi_1(\theta)\rangle, \quad \langle\psi_0|\psi_1(\theta)\rangle = 0, \tag{5}$$

where $|\psi_0\rangle$ is the ground state of $\hat{H}_0$. Tuning $\theta$ rotates the ground state, giving the Berry connection [44] $A_\theta = -[\langle\psi(\theta)|L_z|\psi(\theta)\rangle - \langle\psi_0|L_z|\psi_0\rangle]$. This depends only on the *excess* angular momentum. Intuitively, we cannot physically rotate a perfectly symmetric object on a flat surface. Only a deformed quasihole can be rotated, and the Berry phase comes from the *change* in angular momentum caused by deformation.

*Derivation of the spin-statistics theorem.* Let us now start by inserting $(k + k')$ fluxes into a FQH ground state at the north pole, creating a $(k + k')$-stacked anyon there. Next we pull $k$ of the fluxes to the south pole, leaving behind $k'$ fluxes at the north pole. The $2\pi$ rotation gives the phases $\gamma_3 = -(k + k')N_e\pi$, $\gamma_4 = (k - k')N_e\pi$, respectively. The phase difference between these two scenarios is [44]

$$\Delta\gamma_{43} = \gamma_4 - \gamma_3 = \Delta\gamma_{21} - 2\pi\nu kk' \tag{6}$$

$$= \Delta\gamma_{21} - 2\pi\left(s_k^{\text{topo}} + s_{k'}^{\text{topo}} - s_{k+k'}^{\text{topo}}\right), \tag{7}$$

where $\Delta\gamma_{21}$ is the single $k$-stacked anyon contribution from Eq. (3). The extra term is the braiding phase reexpressed in Eq. (7). Here we define a topological spin $s_k$ compatible with

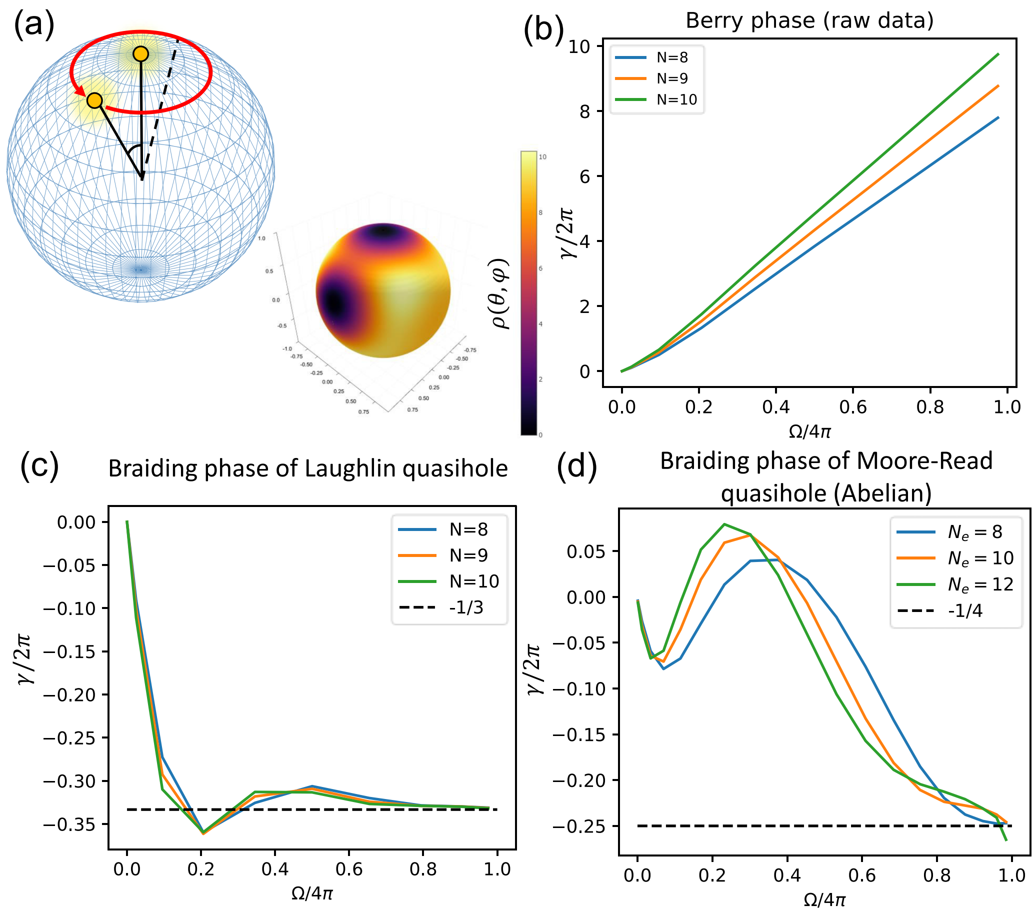


FIG. 2. (a) Schematic diagram of the braiding process on the sphere. The inset shows a sample heatmap of the electron density on the sphere calculated for a two-quasihole Laughlin state. (b) Total Berry phase for braiding two Laughlin quasiholes shows an $\Omega$-dependent part (linear as $\Omega \to 4\pi$) and an $\Omega$-independent part. The latter can be extracted as the braiding phase. (c) Braiding phase of two Laughlin quasiholes calculated for different system sizes. (d) Braiding phase of two Abelian Moore-Read quasiholes (two half-fluxes) calculated for different system sizes.

the clustering of anyons [11,30]:

$$s_k^{\mathrm{topo}} = -\frac{\nu k^2}{2}. \tag{8}$$

This topological spin originates from an *intrinsic spin* that we justify later. While Eq. (7) agrees with Eq. (2), our calculation reveals the microscopic origin of each term. For rotationally invariant quasiholes, this formula gives a braiding phase of $2\pi \nu k k'$ for an adiabatic braiding of a $k$-stacked anyon around a $k'$-stacked anyon.

The analytical derivation above is consistent with numerical calculations, which also allow us to go beyond the Laughlin and study other FQH states [see Figs. 2(a)–2(c)]. In Fig. 2(d) we show some results for the braiding of two Moore-Read quasiholes (charge $e/4$) [18,19,45]. Adding one additional magnetic flux to the Moore-Read ground state gives a quasihole of charge $e/2$, which can be split into two half-fluxes of charge $e/4$ each without any punishment by $V_3^{3bdy}$ (model Hamiltonian for the Moore-Read state). The two-quasihole Hilbert space is isomorphic to the two-quasihole Laughlin Hilbert space, and in principle, one would expect an Abelian braiding. (The non-Abelian property of the Moore-Read state only manifests when there are four or more quasiholes.) For the Moore-Read state, it is difficult to reproduce the analytical calculation as above, since one cannot have a single isolated quasihole. However, the two-quasihole states can be constructed numerically by finding the ground state of two local potentials on the sphere [44]. The braiding phase can be extracted as the solid angle-independent part of the total Berry phase when rotating two-quasihole states about the $z$ axis. Our numerical results show the intrinsic spin of the Moore-Read state is still a well-defined quantity and the spin-statistics relation still holds when the quasihole manifold is Abelian. The implication of the intrinsic spin to non-Abelian braiding remains a subject for future study.

The generalized spin-statistics relation can also be understood as a special case of Eq. (5), where we deform a rotationally invariant $\hat{H}_0$, of which the $(k + k')$-stacked anyon is the ground state. This deformation pulls a $k$-stacked anyon far away from the center of rotation, and again we rotate the entire system with $\hat{H}_1$ parametrized by $\theta$, and measuring $\Delta\gamma_{43}$ as the excess angular momentum from the perturbation [44]. Given that anyons are not point particles, the "string attachment" shown in Fig. (1) is physical even when $\hat{H}_1$ consists of two well-separated, perfectly circular confining potentials, since their deformed tail, while exponentially suppressed by the separation, allows us to "attach" the string and

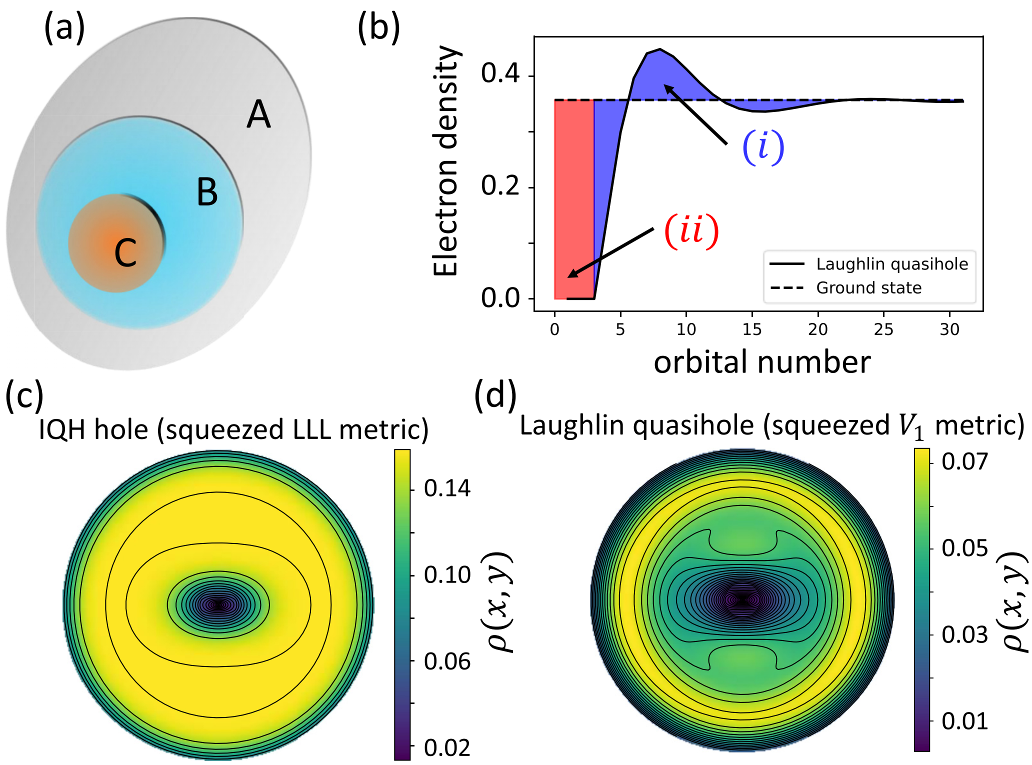


FIG. 3. (a) CHS hierarchy [43]: *A* - full many-body Hilbert space, *B* - a single LL, *C* - sub-CHS. (b) Comparing the electron densities of a Laughlin quasihole (solid line) and the neutral ground state (dashed line) shows the origin of (i) $\gamma_{s_2}$ and (ii) $\gamma_{s_3}$. Electron density of (c) the IQH hole squeezed by the guiding center metric and (d) the Laughlin quasihole squeezed by the $\hat{V}_1^{2bdy}$ null-space metric [44].

track the self-rotation during the exchange. In this picture, the relationship between spin (adiabatic self-rotation) and statistics (adiabatic exchange) can be rigorously established without r-QFT, for any types of Abelian anyons in the QH systems.

*Conformal Hilbert-space angular momentum.* To justify that $s_k^{\text{topo}}$ fully captures the Berry phase of the self-rotation of the rotationally invariant $k$-stacked quasihole, we note in Eq. (4) that the intrinsic angular momentum can be separated into three parts: $L_{\text{cy}} = \gamma_{s_1}$ is the cyclotron angular momentum associated with different LLs; $\gamma_{s_2}$ comes from the FQH topological shift, related to the dipole moment at the edge of the QH fluid [49] and vanishes for the IQHE. Here $L_{\text{LL}} = \gamma_{s_2} + \gamma_{s_3}$ is the total guiding center angular momentum within a single LL, as illustrated in Fig. 3(a).

The separation of $L_z$ into $L_{\text{cy}}$ and $L_{\text{LL}}$ is due to $L_{\text{LL}}$ being well-defined within a sub-Hilbert space (a single LL). For any physical operation within a single LL, only $L_{\text{LL}}$ is physically accessible. A single LL is an example of the conformal Hilbert space (CHS) introduced in Ref. [43]. Thus $L_{\text{LL}}$ is the angular momentum defined within this conformal Hilbert space, characterized by the guiding center metric, or the "shape" of the CHS. We can deform the shape of a hole within a single LL with a local potential, thus changing the expectation value of $L_{\text{LL}}$, while keeping $L_{\text{cy}}$ invariant, i.e., in the limit of large magnetic field so there is no LL mixing [see Fig. 3(c)]. Thus only the deformed guiding center density can be measured.

Similarly we can further separate $L_{\text{LL}}$ into $\gamma_{s_2}$ and $\gamma_{s_3}$, where the latter is the angular momentum defined within a sub-CHS, the null space of $\hat{V}_1^{\text{2bdy}}$ interaction denoted as $\mathcal{H}_1$ [43]. Only $\gamma_{s_3}$ is physically relevant if we braid or deform anyons within $\mathcal{H}_1$ with a local potential much smaller than the incompressibility gap (i.e., $\hat{V}_1^{\text{2bdy}}$ gives the dominant energy scale), so only the metric characterizing $\gamma_{s_3}$ is relevant, while that of $\gamma_{s_2}$ will be invariant. An example is given in Fig. 3(d), showing a Laughlin quasihole deformed by an elliptical potential well in the null space of $\hat{V}_1^{2bdy}$, in complete analogy to Fig. 3(c). For Abelian FQH phases, $\gamma_{s_3}$ can be explicitly computed from a unitary transformation to the composite fermion (CF) basis [44,51]. By construction the CFs are particles within $\mathcal{H}_1$, and the $k$-stacked CF holes give the CHS angular momentum of $\gamma_{s_3}$ [51].

We are now ready to explicitly write down the general spin-statistics relation:

$$\gamma_{k_1,\eta_1;k_2,\eta_2} = 2\pi\left(\bar{S}_{k_1,\eta_1} + \bar{S}_{k_2,\eta_2} - s_{k_1+k_2}^{\text{topo}}\right), \tag{9}$$

giving the phase obtained by adiabatically braiding a cluster of $k_1$ quasiholes around a cluster of $k_2$ quasiholes. Here $\eta_1$ and $\eta_2$ parametrize the deformation, or the internal structure of the quasiholes; $\bar{S}_{k_i,\eta_i}$ denotes the intrinsic spin of the cluster $k_i$ with deformation $\eta_i$. For rotationally invariant $k_1$ stack, we get $\bar{S}_{k_1,\eta_1=0} = s_{k_1}^{\text{topo}}$ in Eq. (8). However, Eq. (9) also includes the effect of the internal quasihole structure, encoded in parameters $\eta_1$ and $\eta_2$, which may present an additional contribution to the measured statistics (see subsequent section).

For identical particles ($k_1 = k_2 = k$ and $\eta_1 = \eta_2 = \eta$), Eq. (9) can be simplified to the more familiar form

$$\gamma_{k,\eta} = -4\pi\tilde{S}_{k,\eta}, \tag{10}$$

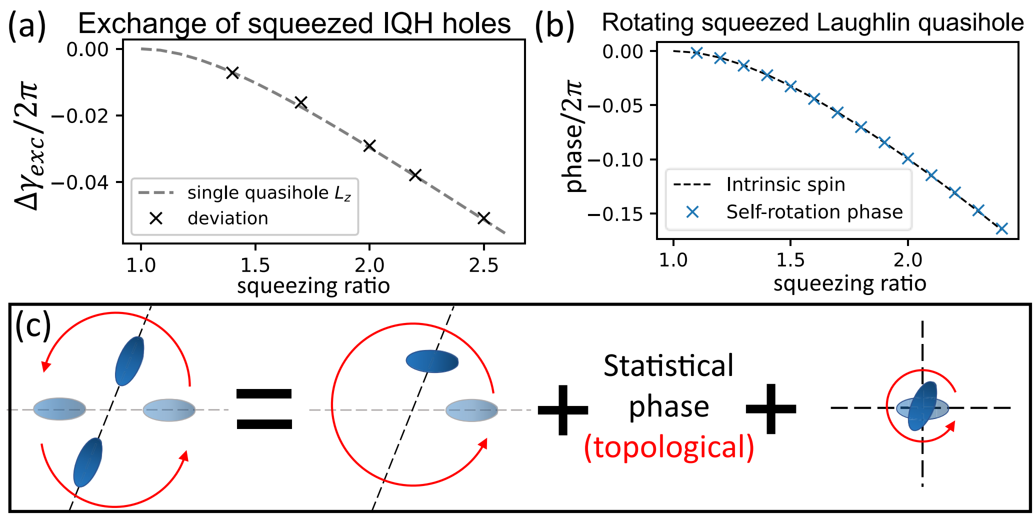


FIG. 4. (a) Deviation from fermionic statistics at different squeezing ratios (crosses) compared with the deviation of the intrinsic spin from topological spin (dashed line). (b) Self-rotation phase of a single squeezed Laughlin quasihole (crosses) compared with its intrinsic spin (dashed line) at different squeezing ratios. (c) Different contributions to the total Berry phase. The intrinsic spin captures both topological and self-rotation phases.

which recalls the standard spin-statistics relation in relativistic quantum field theory (the minus sign signifies that the quasihole is a deficiency of electrons). For a rotationally invariant $k$ stack, $\tilde{S}_{k,\eta=0} = s_k^{\text{topo}} = -\nu k^2/2$. For the more general cases (e.g., deformed quasiholes) the intrinsic spin of the quasihole is modified as $\bar{S}_{k,\eta} = s_k^{\text{topo}} + \Delta s_k$, so the braiding phase becomes $-4\pi\tilde{S}_{k,\eta} = -4\pi(s_k^{\text{topo}} - \Delta s_k)$. The additional factor $-4\pi\Delta s_k$ comes from the self-rotation of the two quasiholes, which becomes nontrivial if their intrinsic shape deviates from the rotationally invariant wave packet.

*Deformation of anyons*. Since each CHS comes with an intrinsic geometric degree of freedom [43], the quasihole can be arbitrarily deformed while still remaining inside the null space of the model Hamiltonian. In practice, this can be done by implementing a trapping potential *within the null space* of a given model Hamiltonian, and the metric is determined by the potential profile [44]. We consider here an elliptical well that "squeezes" the quasiholes into elliptical shapes [see Figs. 3(c) and 3(d)].

We first illustrate our results with the squeezed holes at filling factor $\nu = 1$. Here there exist two possible exchange schemes. When the two holes are exchanged along a circle by pure translation, we observe a fermionic statistics expected of the $\nu = 1$ phase [44]. However, when each hole self-rotates along the exchange path, there is a deviation as shown in Fig. 4(a), the amount of deviation from fermionic statistics exactly matches the deviation of the intrinsic spin from the topological spin, multiplied by $2\pi$. When self-rotation is involved in the exchange procedure, the exchange phase contains both a topological component that depends only on the topological indices of the FQH phase, and a self-rotation component that depends on the internal shape of the quasiholes [see Fig. 4(c)]. To observe only the topological phase in real experiments, one must ensure that the quasiholes are not rotated by any additional potentials in the system, such as disorder. This is actually difficult to avoid as we will show later.

For FQH states, the physics is analogous once we replace the guiding center angular momentum and the corresponding deformation in the lowest Landau level (LLL) with that of the respective null spaces. For example, with the Laughlin $\nu = 1/3$ quasiholes, the intrinsic spin is taken to be the $\hat{V}_1^{2bdy}$ null-space angular momentum, exploiting the isometry between the LLL and the $\hat{V}_1^{2bdy}$ null space [43]. This intrinsic spin can be computed from the fermionization process [43,44]. Figure 4(b) shows the Berry phase from the self-rotation of a squeezed Laughlin quasihole agrees very well with its intrinsic spin. In particular, Fig. 4(c) readily generalizes to all Abelian FQH phases.

*Effect of disorder*. The microscopic calculations in this work also allow us to study the effect of disorder on the Berry phase measurement in the bulk. Qualitatively, any disorder in the system can deform the shape of the quasiholes, therefore changing its intrinsic spin. When a quasihole moves past a region with disorder, a self-rotation is induced *in general*, leading to an additional contribution to the Berry phase as described above. The effect of disorder on Berry phase anywhere on the QH droplet can be studied by calculating the local Berry curvature. We emphasize that our analysis applies to both quasiholes in the bulk and in the edge, as opposed to previous studies which focus quasiholes carried by the edge currents. In general, the Berry curvature contains three distinct distribution: the Aharonov-Bohm term from the background magnetic field, the presence of any extra quasiholes, and the presence of disorder [see Fig. 5(a)].

While the Berry curvature is uniform in a clean system far from other quasiholes, local disorder induces fluctuation on the Berry curvature. If a quasihole travels along the edge, far away from the disorder in the bulk, the total Berry curvature fluctuation sums to zero as a result of Gauss-Bonnet theorem [44], consistent with the analysis in Ref. [35]. Any disorder at the edge can deform the quasihole shapes and thus modifies the Berry phase of those propagating along the edge, although the *phase difference* when an addition or removal

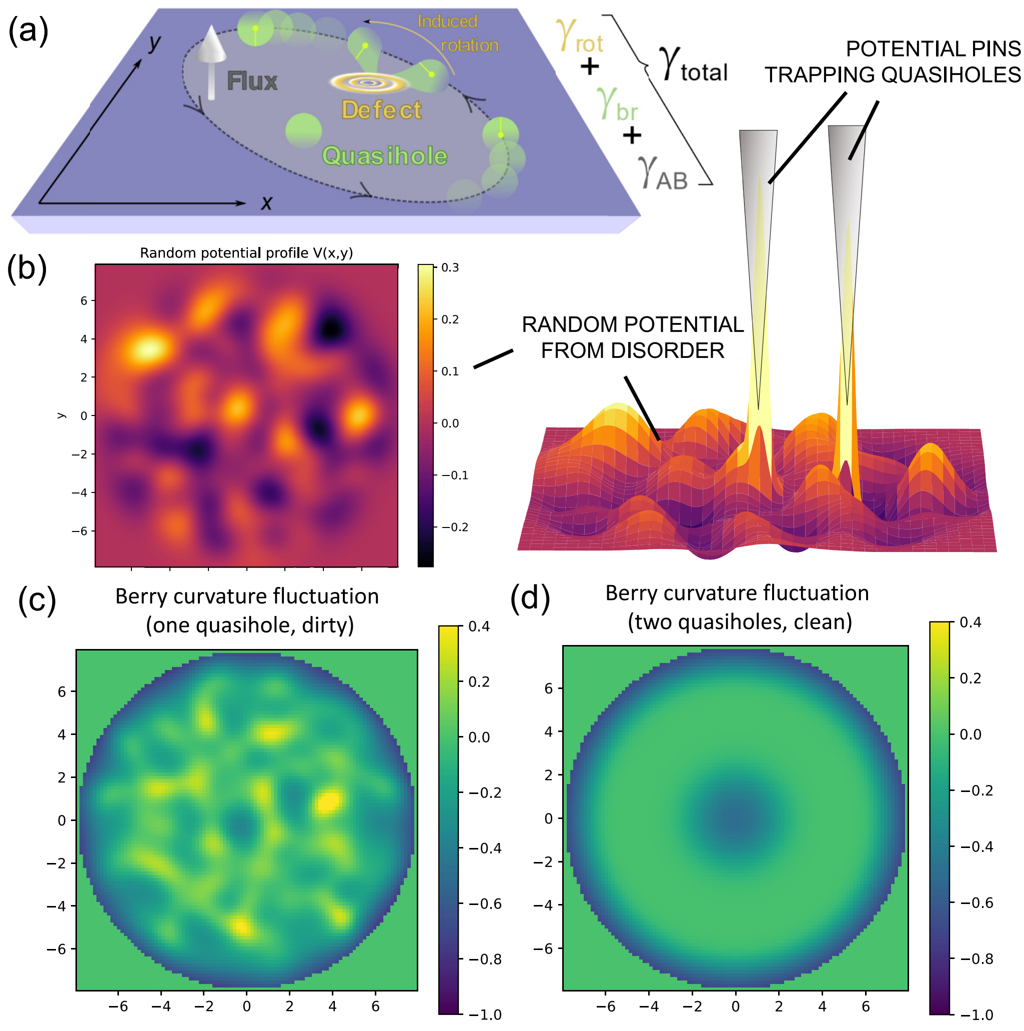


FIG. 5. (a) The total Berry phase consists of the AB phase, the statistical phase, and the self-rotation phase. The last term coming from a self-rotation induced by disorder near the path of the quasihole (b) Random potential profile $V(x, y)$ modeling a disordered system [44]. The quasiholes are trapped and manipulated by two additional potential pins (e.g., an AFM tip) illustrated on the right panel (c) Deviation of local Berry curvature from average for a dirty system (d) Deviation of local Berry curvature from average for a clean system with an extra quasihole pinned at the center. Calculations done on a $\nu = 1$ system with 30 electrons [44].

of another quasihole in the bulk will *not be affected* by such disorder.

However, procedures within the bulk may result in path-dependent Berry phases *even if the loops encircle the same area and the same number of quasiholes*. This effect is attributed to the deformed internal structure of the quasihole and can be seen in the Berry phase fluctuation around the disorder points in Fig. 5(c). This could be one of the main sources of noise in quantum computation from anyon braiding. Our analysis brings attention to the influence of disorder to the internal structures of quasiholes, and our method provides a tool to quantify such effects.

*Conclusion and outlook*. In this paper, we have proposed the microscopic mechanism for the generalized spin-statistics for Abelian anyons in quantum Hall fluids. We have defined the intrinsic spin for quasiholes of the FQH phase from the algebra of the conformal Hilbert spaces (CHS), with the special case of identical quasihole clusters giving the standard spin-statistics relation. This intrinsic spin also applies to deformed quasiholes and can be detected from the proper Berry phase measurements. It should be noted while our discussion was limited to quasiholes in Abelian phases, the CHS algebra generalizes to Abelian quasi-electrons and charged excitations of non-Abelian phases [42,43] and, in principle, the intrinsic spin is well-defined for such cases. The hierarchy of CHS reveals the relationship between different FQH phases with interesting physical consequences; the implication of this on quasihole statistics remains to be studied, especially regarding non-Abelian anyons which are of interest to quantum computing. Our analysis poses questions about whether truly robust braiding in bulk is possible in a realistic system because quasiholes are not point particles but objects with internal structures that can be influenced by disorder. The methods we proposed provide tools to quantify such effects, which we explicitly illustrated with a new source of Berry phase from the local disorder deforming the shape of the quasiholes. In dirty systems where deviation from topological behavior is detected, our calculation can help to identify potential sources of errors and potentially devise methods to mitigate such effects.

*Acknowledgments*. We are grateful to L. Mazza and T. Comparin for helpful discussions. This work is supported by the NTU grant for Nanyang Assistant Professorship and the National Research Foundation, Singapore under the NRF fellowship award (NRF-NRFF12-2020-005), and a Nanyang Technological University start-up grant (NTU-SUG).

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