Letter

Extrinsic and intrinsic nonlinear Hall effects across Berry-dipole transitions

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Three-dimensional Hopf insulators are a class of topological phases beyond the tenfold-way classification. At the critical point of the transition between two distinct Hopf insulators with rotational symmetry, the band-touching points are point Berry dipoles with two opposite Berry charges overlapping in a mirror-symmetric way, and carry unique Berry curvature structures leading to a special quantization of Berry flux. Close to such Berry-dipole transitions, we find that the extrinsic and intrinsic nonlinear Hall conductivity tensors in the weakly doped regime are characterized by two universal functions of the ratio between doping level and bulk energy gap, and are directly proportional to the change in Hopf invariant across the transition. Our work suggests that the nonlinear Hall effects display a generalized-sense quantized behavior across Berry-dipole transitions, establishing a correspondence between nonlinear Hall effects and Hopf invariants.

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Introduction. Quantum responses that can extract topological invariant information are of great interest in condensed matter physics. For topological gapped systems, since the topological protection and the existence of a bulk energy gap can tolerate perturbations, there may exist quantum responses that are quantized and directly connected to the topological invariant encoded in the band structure [1], with the quantum (anomalous) Hall insulators being the most well-known example, where the Hall conductance is quantized and connected to the Chern number [2–4]. In comparison, metallic systems are known to hardly support quantized responses since the Fermi surface deforms under perturbations and generally lacks of a topological characterization. Known exceptions include ballistic conductors where the conductance is shown to be quantized and connected to the Euler characteristic of the Fermi sea [5,6], and Weyl semimetals without inversion and mirror symmetries where a quantzied circular photogalvanic effect can emerge in the absence of disorders and interactions [7,8]. It is worth mentioning that the circular photogalvanic effect is a second-order optical response effect and its quantization in Weyl semimetals is rooted in the quantized Berry-monopole charge of the Weyl point enclosed by the Fermi surface [9].

Recently, extrinsic nonlinear Hall effect (ENHE) and intrinsic nonlinear Hall effect (INHE) in metallic systems, as another two kinds of second-order quantum responses derived by semiclassical equations of motion, have gained considerable interest since they have a quantum geometry origin and can emerge in systems without linear-order anomalous Hall effect [10–12]. It was shown that the ENHE and INHE depend on the Berry curvature dipole [13] and Berry-connection polarizability (BCP) [14], respectively, and both of them are Fermi-surface properties. The adjectives "extrinsic" and "intrinsic" applied to distinguish them reflect one essential difference between these two kinds of nonlinear Hall effects; namely, the former depends on the relaxation time associated with carrier scatterings [13], whereas the latter does not involve any time scale and only depends on the band geometry quantity [14]. Although both of them require the breaking of inversion symmetry, the ENHE is a time-reversal-even effect and can emerge in systems with time-reversal symmetry, but the INHE is a time-reversal-odd effect and can only appear in systems without time-reversal symmetry. Interestingly, when the system has neither time-reversal symmetry nor inversion symmetry, but their combination, PT symmetry, the ENHE is absent as the Berry curvature is forced to vanish [15], whereas the INHE can be significant. Two recent works have shown that the INHE could be applied to measure the Néel vector of PT-symmetric antiferromagnets [16,17].

Since various quantum geometry quantities are prominent near band crossings or avoid crossings, topological semimetals or doped small-gap topological insulators are ideal material systems to seek for strong ENHE and INHE [18–39]. For topological insulators, it is known that the change of global topology across a topological phase transition is a result of the dramatic change in quantum geometry near the band edge. Thus, although the ENHE and INHE are Fermi-surface properties that are generally not possible to determine the topological invariant associated with the whole Brillouin zone, they are possible to detect the change of band topology if the topological insulators are weakly doped. Previous works indeed found that the ENHE can manifest topological phase transitions described by low-energy Dirac Hamiltonians through the dramatic sign change of a nonlinear Hall conductivity tensor (NHCT) [22,40,41]. However, since therein, the tilt or warping of massive Dirac cones is necessary to have nonzero ENHE [13,26]; the NHCT does not have a simple and universal form revealing the change of topological invariant.

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In this work, we explore the behaviors of ENHE and INHE across the critical regime of three-dimensional Hopf insulators with C_{4z} rotational symmetry. As a class of topological insulators beyond the tenfold-way classification [42,43], the Hopf insulators follow a Z classification and do not have time-reversal and inversion symmetries [44–48]. In addition, at the critical point of the transition between two Hopf insulators with rotational symmetry, the band-touching points turn out to be neither Dirac points nor Weyl points, which are known to describe critical points of many topological phase transitions [49]. Interestingly, they resemble point dipoles with two opposite Berry charges overlapping and are thus termed Berry dipoles [50]. Across this class of Berry-dipole transitions, we find that the extrinsic and intrinsic NHCTs in the weakly doped regime are characterized by two universal functions and explicitly contain the information of the change in the Hopf invariant. Although extrinsic and intrinsic NHCTs are independent quantities of different origins, here they display similar laws and are simply related across the critical regime.

C_{4z}-rotational-invariant Hopf insulators and Berry-dipole transitions. Hopf insulators have a diversity of model realizations [44–46,51,52]. In this work we focus on the two-band realization for simplicity. The minimal models for Hopf insulators are constructed by the so-called Hopf map [44]

$$\mathcal{H}(\boldsymbol{k}) = \boldsymbol{d}(\boldsymbol{k}) \cdot \boldsymbol{\sigma} \tag{1}$$

with the three components of the $\boldsymbol{d}$-vector given by

$$d_i(\boldsymbol{k}) = \zeta^\dagger(\boldsymbol{k})\sigma_i\zeta(\boldsymbol{k}),\ \zeta(\boldsymbol{k}) = (\zeta_1(\boldsymbol{k}), \zeta_2(\boldsymbol{k}))^T,$$
$$\zeta_1(\boldsymbol{k}) = \eta_1(\boldsymbol{k}) + i\eta_2(\boldsymbol{k}),\ \zeta_2(\boldsymbol{k}) = \eta_3(\boldsymbol{k}) + i\eta_4(\boldsymbol{k}), \tag{2}$$

where σ_i are the Pauli matrices, and $\eta_i(\boldsymbol{k})$ are real functions of momentum. Mathematically, the above equations define a map from S^3 to S^2. In this paper, we focus on models with C_{4z} rotational symmetry. To be specific, we consider [50]

$$\zeta_1(\boldsymbol{k}) = \lambda^n[\sin(k_x a) + i\sin(k_y a)]^n,$$
$$\zeta_2(\boldsymbol{k}) = \lambda_z \sin(k_z c) + i\left[M + t\sum_{i=x,y,z}\cos(k_i a_i)\right], \tag{3}$$

where the lattice constants $a_x = a_y = a$ and $a_z = c$. The energy spectra of the Hamiltonian take the simple form

$$E_\pm(\boldsymbol{k}) = \pm(|\zeta_1(\boldsymbol{k})|^2 + |\zeta_2(\boldsymbol{k})|^2). \tag{4}$$

For the constructed Hamiltonian, the energy gap can only close at time-reversal invariant momenta of the Brillouin zone. Without loss of generality, we consider that the band edge is located at $\boldsymbol{\Gamma} = (0, 0, 0)$, then an expansion of the complex spinor up to the leading order in momentum gives

$$\zeta_1(\boldsymbol{k}) = v^n(k_x + ik_y)^n,$$
$$\zeta_2(\boldsymbol{k}) = v_z k_z + im, \tag{5}$$

where $v = \lambda a$, $v_z = \lambda_z c$, and $m = M + 3t$. Accordingly, the low-energy Hopf Hamiltonian has the form

$$\begin{aligned}\mathcal{H}(\boldsymbol{k}) = {}& 2v^n k_\rho^n(v_z k_z \cos n\theta + m\sin n\theta)\sigma_x \\ &+ 2v^n k_\rho^n(m\cos n\theta - v_z k_z \sin n\theta)\sigma_y \\ &+ \left(v^{2n}k_\rho^{2n} - v_z^2 k_z^2 - m^2\right)\sigma_z,\end{aligned} \tag{6}$$

where $k_\rho = \sqrt{k_x^2 + k_y^2}$, and θ is the polar angle in the k_x-k_y plane. It is worth noting that the C_{4z} rotational symmetry of the lattice Hamiltonian evolves to a continuous rotational symmetry under the low-energy approximation. The low-energy spectra are given by

$$E_\pm(\boldsymbol{k}) = \pm\left(v^{2n}k_\rho^{2n} + v_z^2 k_z^2 + m^2\right). \tag{7}$$

For the simplest case $n = 1$, one sees that the low-energy Hamiltonian at the critical point $m = 0$ is distinct to the Weyl Hamiltonian [9] and the energy dispersion is quadratic rather than the linear dispersion of usual critical points.

For the Hamiltonian constructed by Hopf map, the Hopf invariant characterizing it can be determined by [46]

$$N_h = \frac{1}{2\pi^2}\int d^3k\,\epsilon_{abcd}\hat{\eta}_a\partial_{k_x}\hat{\eta}_b\partial_{k_y}\hat{\eta}_c\partial_{k_z}\hat{\eta}_d, \tag{8}$$

where $\hat{\eta}_i = \eta_i/\sqrt{\eta_1^2 + \eta_2^2 + \eta_3^2 + \eta_4^2}$, ϵ_{abcd} is the fourth-order Levi-Civita symbol, and a sum over repeated indices is implied. For the low-energy Hopf Hamiltonian, one has [53]

$$N_h = -\mathrm{sgn}(v_z m)\frac{n}{2}. \tag{9}$$

When m changes sign, the Hopf invariant changes by n. It is worth emphasizing that the low-energy Hopf Hamiltonian can only determine the change of Hopf invariants since it only faithfully captures the low-energy part of the bands. One needs the tight-binding Hamiltonian to determine the absolute values of Hopf invariants before and after the transition.

For the low-energy Hopf Hamiltonian (6), the three components of the Berry curvature can be determined by [54]

$$\Omega_l^{(\pm)}(\boldsymbol{k}) = \pm\epsilon_{ijl}\frac{\boldsymbol{d}(\boldsymbol{k})\cdot(\partial_{k_i}\boldsymbol{d}(\boldsymbol{k}) \times \partial_{k_j}\boldsymbol{d}(\boldsymbol{k}))}{4d^3(\boldsymbol{k})}, \tag{10}$$

where $d(\boldsymbol{k}) = |\boldsymbol{d}(\boldsymbol{k})|$ is the norm of the $\boldsymbol{d}$-vector, the superscript $+(-)$ refers to the conduction (valence) band, and ϵ_{ijl} is the third-order Levi-Civita symbol. An explicit calculation gives [53]

$$\begin{aligned}\Omega_x^{(\pm)}(\boldsymbol{k}) &= \pm\frac{2nv_z v^{2n}k_\rho^{2n-1}(m\sin\theta + v_z k_z\cos\theta)}{d^2(\boldsymbol{k})}, \\ \Omega_y^{(\pm)}(\boldsymbol{k}) &= \pm\frac{2nv_z v^{2n}k_\rho^{2n-1}(-m\cos\theta + v_z k_z\sin\theta)}{d^2(\boldsymbol{k})}, \\ \Omega_z^{(\pm)}(\boldsymbol{k}) &= \pm\frac{2n^2 v^{2n}k_\rho^{2(n-1)}\left(m^2 + v_z^2 k_z^2\right)}{d^2(\boldsymbol{k})}.\end{aligned} \tag{11}$$

At the critical point $m = 0$, it turns out that the integral of the Berry curvature over a closed surface enclosing the band-touching point at $\boldsymbol{\Gamma}$ identically vanishes; however, the integral over the upper ($k_z > 0$) or lower ($k_z < 0$) half of the surface is quantized, namely [50],

$$\frac{1}{2\pi}\int_{k_z>0}\boldsymbol{\Omega}\cdot d\mathbf{S} = n. \tag{12}$$

Band-touching points with such a quantized behavior are termed Berry dipoles [50]. Pictorially, such band-touching points correspond to point dipoles for which the Berry monopole and Berry antimonopole composing them overlap but do not annihilate due to mirror symmetry protection [50,55]. Before proceeding, it is worth emphasizing that here the Berry dipoles are different from the Berry curvature dipole defined in Eq. (14) below, which has the physical meaning as the dipole moment of Berry curvature over the occupied states.

The results in Eqs. (9) and (12) indicate that the change of Hopf invariants across the transition equals the quantized value of the Berry dipole at the critical point, so this class of topological phase transitions is also termed Berry-dipole transitions [50]. As the Berry dipole carries a unique Berry curvature structure, it is natural to expect that the ENHE would display some novel behaviors across such Berry-dipole transitions.

ENHE across Berry-dipole transitions. Sodemann and Fu revealed that an a.c. electric field would result in a mixture of d.c. and second-harmonic Hall-type currents in the second-order-response regime in systems without inversion symmetry [13], i.e., $\boldsymbol{j} = \boldsymbol{j}^0 + \boldsymbol{j}^{2\omega}$ with $j_i^0 = \chi_{ijk}\mathcal{E}_j\mathcal{E}_k^*$ and $j_i^{2\omega} = \chi_{ijk}\mathcal{E}_j\mathcal{E}_k$. The extrinsic NHCT is given by [13]

$$\chi_{ijk}^{\text{ext}} = -\frac{e^3\tau}{2(1+i\omega\tau)}\epsilon_{ilk}D_{jl}, \tag{13}$$

where τ denotes the relaxation time whose value depends on the carrier scattering. D_{jl} is the Berry curvature dipole given by [13]

$$D_{jl} = \sum_\alpha \int \frac{d^3k}{(2\pi)^3} f^{(\alpha)}\big(\partial_{k_j}\Omega_l^{(\alpha)}\big) = -\sum_\alpha \int \frac{d^3k}{(2\pi)^3}\big(\partial_{k_j} f^{(\alpha)}\big)\Omega_l^{(\alpha)}, \tag{14}$$

where the sum is over all bands, and $f^{(\alpha)}$ represents the Fermi-Dirac distribution function of the αth band. The first line of this formula reveals that the Berry curvature dipole is the dipole moment of the Berry curvature over the occupied states, while the second line indicates that this quantity can also be interpreted as a Fermi-surface property.

Using the Berry curvature in Eq. (11), we find that the extrinsic NHTC only contains four nonzero components, including χ_{zxx}^{ext}, χ_{xxz}^{ext}, χ_{zyy}^{ext}, and χ_{yyz}^{ext}. However, there is in fact only one independent component. This can be inferred by noting that the rotational symmetry forces $\chi_{zxx}^{\text{ext}} = \chi_{zyy}^{\text{ext}}$, and the Hall nature forces $\chi_{zxx}^{\text{ext}} = -\chi_{xxz}^{\text{ext}}$ and $\chi_{zyy}^{\text{ext}} = -\chi_{yyz}^{\text{ext}}$, which can also be simply inferred from the antisymmetric property of the Levi-Civita symbol in Eq. (13). Focusing on the zero-temperature limit and only showing the explicit form of χ_{zxx}^{ext}, one has [53]

$$\chi_{zxx}^{\text{ext}} = N_h\chi_1\mathcal{B}_1(\mu/m^2), \tag{15}$$

where N_h is the Hopf invariant given by Eq. (9), $\chi_1 = e^3\tau/2(1+i\omega\tau)$, and $\mathcal{B}_1(x)$ is a dimensionless universal function of the form

$$\mathcal{B}_1(x) = \frac{2(x-1)^{3/2}}{3\pi^2x^2}\Theta(x-1). \tag{16}$$

Here electronic doping is assumed, i.e., $\mu > 0$. Since the energy gap of the low-energy Hamiltonian (6) is equal to $2m^2$, $\mathcal{B}_1(\mu/m^2)$ is a universal function of the ratio between chemical potential and bulk energy gap.

According to Eqs. (9) and (15), one sees that the extrinsic NHCT reverses its sign across a Berry-dipole transition, thus allowing a sensitive probe of the transition. Remarkably, if the relaxation time is assumed to be constant, the extrinsic NHCT turns out to be a universal function multiplied with the change of Hopf invariants across the transition. Particularly, the peak value of the extrinsic NHCT is quantized and directly connected to the change of Hopf invariants. It might be interesting to note that another class of systems exhibiting a similar connection between quantized peak values of response functions and topological invariants is one-dimensional topological superconductors with chiral symmetry [56,57]. Using this result, it is possible to precisely probe the change of topological invariants across Berry-dipole transitions by measuring the evolution of current with the change of doping level.

INHE across Berry-dipole transitions. Since the time-reversal and inversion symmetries are both broken in the Hopf Hamiltonian, the INHE is also allowed. However, quite different from the ENHE, the INHE introduced by Gao, Yang, and Niu depends on a band geometry quantity known as BCP [14], which does not have a direct connection with the Berry curvature. Although the ENHE and INHE have different origins, as we will show below, a simple relation exists between Berry curvature and BCP for the low-energy Hopf Hamiltonian, leading to the INHE displaying a similar behavior like the ENHE across Berry-dipole transitions.

The Hall current originating from INHE has the form $j_\alpha^{\text{int}} = \chi_{\alpha\beta\gamma}^{\text{int}}\mathcal{E}_\beta\mathcal{E}_\gamma$, where the intrinsic NHCT is given by [14,16,17]

$$\chi_{\alpha\beta\gamma}^{\text{int}} = -e^3\sum_n\int\frac{d^3k}{(2\pi)^3}v_\alpha^{(n)}G_{\beta\gamma}^{(n)}\frac{\partial f(E_n)}{\partial E_n} - (\alpha\leftrightarrow\beta), \tag{17}$$

where E_n, $v_\alpha^{(n)} = \partial E_n/\partial k_\alpha$ and $G_{\beta\gamma}^{(n)}$ denote the nth band's dispersion, velocity in the α direction, and BCP, respectively. Their $\boldsymbol{k}$-dependence are made implicit in Eq. (17). The gauge-independent BCP is given by [14,16,17]

$$G_{\beta\gamma}^{(n)}(\boldsymbol{k}) = 2\text{Re}\sum_{m\neq n}\frac{A_{nm,\beta}(\boldsymbol{k})A_{mn,\gamma}(\boldsymbol{k})}{E_n(\boldsymbol{k})-E_m(\boldsymbol{k})}, \tag{18}$$

where $A_{nm,\beta}(\boldsymbol{k}) = i\langle u_n(\boldsymbol{k})|\partial_{k_\beta}u_m(\boldsymbol{k})\rangle$ is the interband Berry connection. From Eq. (17), it is apparent that $\chi_{\alpha\beta\gamma}^{\text{int}} = -\chi_{\beta\alpha\gamma}^{\text{int}}$. For the Hamiltonian considered, one has [14]

$$G_{\beta\gamma}^{(+)}(\boldsymbol{k}) = -\frac{\partial_{k_\beta}\hat{\boldsymbol{d}}(\boldsymbol{k})\cdot\partial_{k_\gamma}\hat{\boldsymbol{d}}(\boldsymbol{k})}{4d(\boldsymbol{k})} = -G_{\beta\gamma}^{(-)}(\boldsymbol{k}), \tag{19}$$

where $\hat{\boldsymbol{d}}(\boldsymbol{k}) = \boldsymbol{d}(\boldsymbol{k})/d(\boldsymbol{k})$ is the normalized $\boldsymbol{d}$ vector. Interestingly, the numerator in Eq. (19) suggests that the BCP is related to the quantum metric for a two-band Hamiltonian. Since the BCP is a symmetric tensor, i.e., $G_{\beta\gamma}^{(\pm)}(\boldsymbol{k}) = G_{\gamma\beta}^{(\pm)}(\boldsymbol{k})$, there are six independent components. By straightforward calculations, one has [53]

$$G_{xx}^{(\pm)}(\boldsymbol{k}) = G_{yy}^{(\pm)}(\boldsymbol{k}) = \mp\frac{n^2v^{2n}k_\rho^{2n-2}\big(m^2+v_z^2k_z^2\big)}{d^3(\boldsymbol{k})},$$

$$G_{zz}^{(\pm)}(\boldsymbol{k}) = \mp\frac{v_z^2v^{2n}k_\rho^{2n}}{d^3(\boldsymbol{k})},\quad G_{xy}^{(\pm)}(\boldsymbol{k}) = 0,$$

$$G_{xz}^{(\pm)}(\boldsymbol{k}) = \pm\frac{nv^{2n}v_z k_\rho^{2n-1}(m\sin\theta + v_z k_z\cos\theta)}{d^3(\boldsymbol{k})},$$
$$G_{yz}^{(\pm)}(\boldsymbol{k}) = \pm\frac{nv^{2n}v_z k_\rho^{2n-1}(-m\cos\theta + v_z k_z\sin\theta)}{d^3(\boldsymbol{k})}. \quad (20)$$

A close look at Eqs. (11) and (20) finds that four of the six independent components of the BCP tensor have a simple relation with the three components of the Berry curvature, i.e.,

$$\Omega_x^{(\pm)} = 2dG_{xz}^{(\pm)}, \quad \Omega_y^{(\pm)} = 2dG_{yz}^{(\pm)}, \quad \Omega_z^{(\pm)} = -2dG_{xx(yy)}^{(\pm)}. \quad (21)$$

Bringing Eq. (20) into Eq. (17), one finds that $\chi_{\alpha\beta\gamma}^{\rm int}$ is nonzero only when $\alpha \neq \beta \neq \gamma$, and the nonzero independent components have the simple relation [53]

$$\chi_{xyz}^{\rm int} = -2\chi_{yzx}^{\rm int} = -2\chi_{zxy}^{\rm int}. \quad (22)$$

Also focusing on the zero-temperature limit and only showing the explicit form of $\chi_{xyz}^{\rm int}$, one has [53]

$$\chi_{xyz}^{\rm int} = -N_h\chi_2\mathcal{B}_2(\mu/m^2), \quad (23)$$

where $\chi_2 = e^3/m^2$ and

$$\mathcal{B}_2(x) = \frac{2(x-1)^{3/2}}{3\pi^2x^3}\Theta(x-1) = \frac{\mathcal{B}_1(x)}{x}. \quad (24)$$

Compared to the ENHE, one sees that the INHE displays a similar universal behavior, but a big difference is that the factor χ_2 goes divergent as $m^2 \to 0$, suggesting that the peak of the intrinsic NHCT goes divergent as the bulk energy gap decreases to infinitely small but nonzero (note that the low-energy Hopf Hamiltonian at the critical point has emergent inversion symmetry so the INHE is forced to vanish when $m = 0$). This property implies that the INHE can provide an even more sensitive probe of the Berry-dipole transition. Furthermore, one can find $\chi^{\rm ext}/\chi^{\rm int} \sim m^2\tau$, thus a simultaneous study of ENHE and INHE can also provide a probe of the relaxation time.

Discussions and conclusions. It is worth emphasizing that because the low-energy Hopf Hamiltonian does not have time-reversal symmetry, the linear Hall conductivity tensor σ_{ij} does not identically vanish. By analyzing the Berry curvature in Eq. (11), it is easy to find that σ_{xz} and σ_{yz} identically vanish, and only σ_{xy} is finite in the doped regime. This can also be figured out by noting that, despite the absence of global time-reversal symmetry, the Hamiltonian has spinless time-reversal symmetry at any k_y or k_x plane. This result suggests that the linear anomalous Hall effect only appears in the xy plane. For the ENHE and INHE, their nonzero components indicate that the Hall current will flow in the z direction if the electric vector lies in the xy plane, thus they can be easily distinguished from the linear-order Hall current according to the current direction. Of course, they can also be distinguished by using lock-in method since the linear-order and second-order Hall signals have different dependence on the frequency of the ac electric field. The ENHE and INHE can also be easily distinguished from each other by controlling the electric vector to lie in the xz or yz plane. For the former, the current will flow in parallel with the electric component in the xy plane, while the latter will flow perpendicular to the electric vector plane.

Now we discuss potential systems to observe our predictions. In theory, the considered two-band Hopf insulators provide the simplest realization of Berry-dipole transitions, however, two-band Hopf insulators remain elusive in experiment except in the context of electric circuit systems [58]. Since the Berry dipole is equivalent to the overlap of two mirror-symmetry-related Weyl points [55], a potential route to realize the Berry-dipole transition is to consider magnetic Weyl semimetals with a mirror plane and then break appropriate symmetries to gap out the nodal points [50]. The experimental implementation of two-band Hopf insulators has also been explored in the context of cold-atom systems, and it was suggested that cold-atom systems with long-range dipolar interaction and Floquet modulations may realize the Hopf insulators [59–61]. On the other hand, a recent work revealed that Bloch oscillations provide a counterpart realization of ENHE in cold-atom systems [62]. Therefore, the cold-atom systems are a potential platform to observe our predictions. Furthermore, it is worth pointing out that Berry-dipole transitions are not restricted to two-band realizations. N-fold ($N \geqslant 3$) Berry-dipole critical points also exist with linear dispersion [52]. Compared to the two-band realizations of Berry-dipole transitions, an important advantage of the N-band ($N \geqslant 3$) realizations is that their corresponding tight-binding Hamiltonians can only involve nearest-neighbor hoppings, and are thus more feasible in experiments. By investigating the $N = 3$ case, we find that the ENHE and INHE do also display the expected generalized-sense quantized behaviors across Berry-dipole transitions [53]. The only difference is that the associated universal functions have a different form due to the difference in dispersion. This suggests that this class of systems can also be applied to test our predictions on the ENHE and INHE across Berry-dipole transitions.

In conclusion, we have shown that the nonlinear Hall effects display generalized-sense quantized behaviors across Berry-dipole transitions, building up a correspondence between nonlinear Hall effects and Hopf invariants. A direction for future study is to consider semimetals with nodal points or insulators with critical points associated with more exotic Berry curvature structures and explore what kinds of quantum responses can uniquely reflect them.

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