Letter Editors' Suggestion

Insulating vortex cores in disordered superconductors

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We show that while an orbital magnetic field and disorder, acting individually, weaken superconductivity, acting together they produce an intriguing evolution of a two-dimensional type-II s-wave superconductor. For weak disorder, the critical field H_c at which the superfluid density collapses is coincident with the field at which the superconducting energy gap gets suppressed. However, with increasing disorder these two fields diverge from each other, creating a pseudogap region with insulating vortex cores. Our results naturally explain two outstanding puzzles: the gigantic magnetoresistance peak observed as a function of magnetic field in thin disordered superconducting films and the disappearance of the celebrated zero-bias Caroli–de Gennes–Matricon peak in the local density of states at the vortex core in disordered superconductors.

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Introduction. The response of an s-wave superconductor (sSC) individually to disorder and orbital magnetic field has by now been well established [1–4]. In a pristine Bardeen-Cooper-Schrieffer (BCS) superconductor, the two energy scales, the single-particle energy gap E_g measurable by scanning tunneling spectroscopy [5], and the superfluid stiffness D_s, related to the diamagnetic susceptibility [6], both vanish simultaneously at a critical temperature T_c.

Upon including disorder, extensive research in the last few decades has established that the pairing amplitude of a disordered superconductor (SC) becomes inhomogeneous, forming SC islands on the scale of the coherence length ξ separated by an insulating sea [7–10]. Ultimately, the superconductor is driven into an insulating state, not by the collapse of the single-particle energy gap [11–14], but rather by the vanishing of the superfluid phase stiffness [see Fig. 1(a)] due to enhanced quantum phase fluctuations [15–17]. Since E_g remains finite while D_s vanishes as a function of disorder, it is argued that the universal properties near the superconductor-insulator transition (SIT) are well described by an effective "bosonic" Hamiltonian [18–20].

Turning next to the effect of an applied magnetic field H on a clean BCS superconductor, it is well known that the magnetic field penetrates the SC by generating a periodic array of Abrikosov vortices [21] with a normal metallic core of size ξ with circulating currents around the vortex on the scale of the penetration depth λ. With increasing H, the density of vortices increases and overlaps at a critical field strength H_c [22], and beyond that the superconductor transitions into a metal.

How does the superconductor evolve in a combined presence of disorder V and magnetic field H? Our findings for s-wave superconductors suggest that the paradigm of Abrikosov vortices with metallic cores, which successfully described the behavior of conventional superconductors for more than half a century, must be modified in the presence of disorder (see Fig. 1 and discussions below).

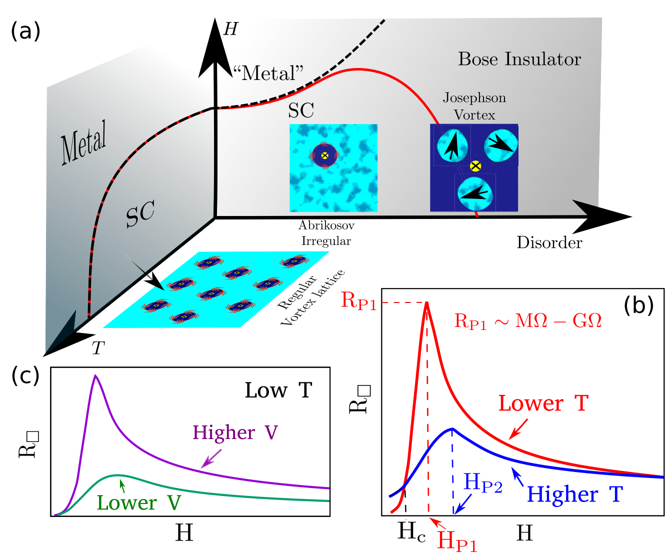

FIG. 1. (a) Schematic phase diagram of a type-II superconductor along the magnetic field (H), temperature (T), and disorder (V) axes reveals three phases: SC with single-particle gap $E_g \neq 0$ and superfluid stiffness $D_s \neq 0$; fermionic metal $E_g = 0$ and $D_s = 0$; and Bose insulator $E_g \neq 0$ and $D_s = 0$. The vortices change character from Abrikosov type with metallic cores to Josephson type with insulating cores as V is increased. (b), (c) illustrate the experimental puzzles in the magnetoresistance (MR) which rises sharply beyond the SIT at H_c by orders of magnitude [23–31], whose explanations emerge naturally from our phase diagram.

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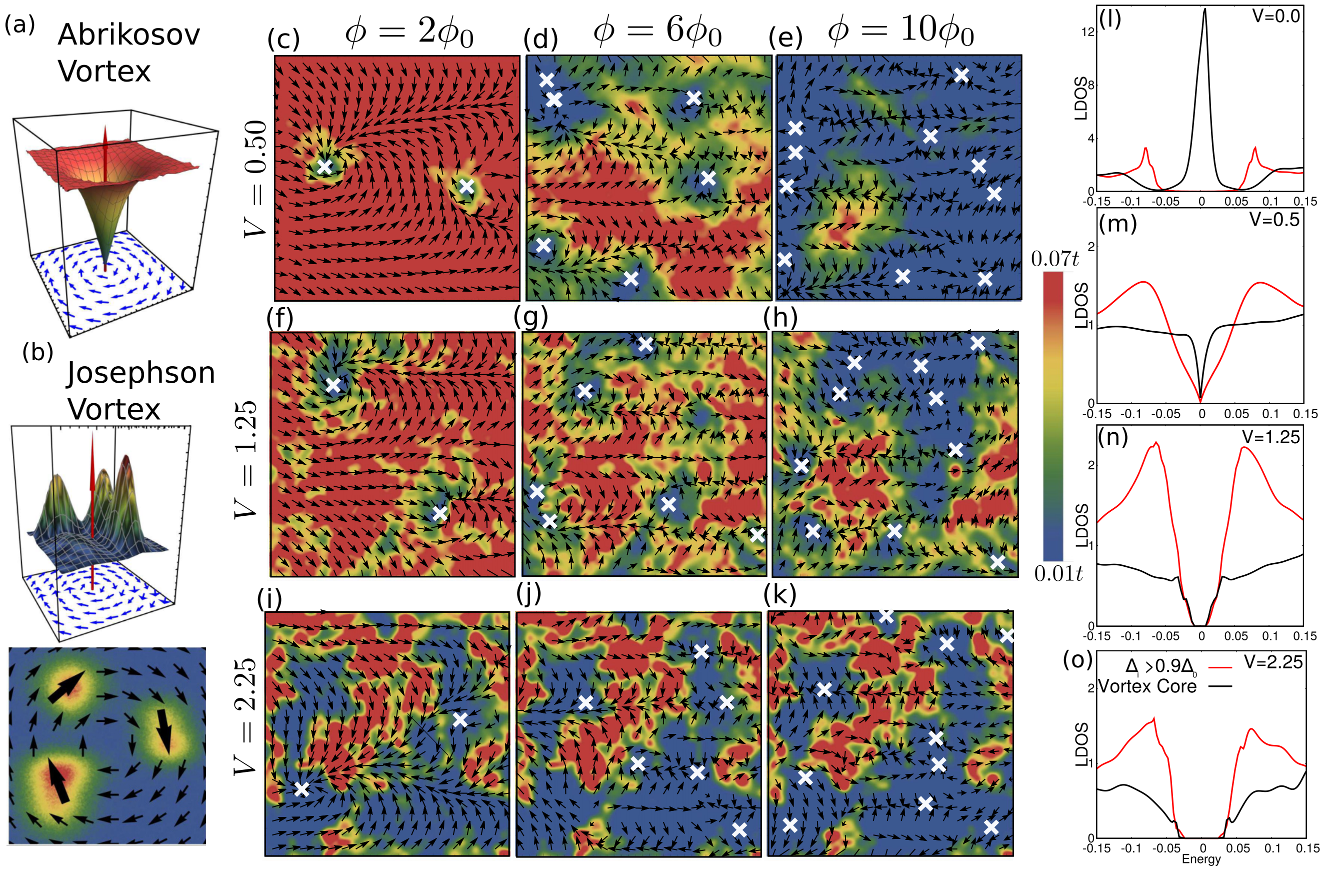

FIG. 2. Crossover from Abrikosov to Josephson vortices with increasing disorder: (a) Schematic plot of an Abrikosov vortex showing the suppression of the pairing amplitude and the curling of the phase around the vortex core. (b) Schematic plot of a Josephson vortex centered in the non-SC region surrounded by three SC islands with finite pairing amplitude Δ. Magnetic field shown by red vertical arrows in (a) and (b). The spatial color-density map of the superconducting pairing amplitude (in one unit cell; see SM) with H increasing from left to right along each row for weak $V = 0.5$ [(c)–(e) in the top row], moderate $V = 1.25$ [(f)–(h) in the middle row], and strong $V = 2.25$ [(i)–(k) in the bottom row] disorder strengths. The phase of the order parameter is superimposed on the amplitude map by tiny arrows. (l)–(o) LDOS [47] at the vortex core (black) and away from the core (red) for weak H, with increasing disorder.

We are also motivated by experimental puzzles [23–28] that show the sheet resistance shooting up by more than eight orders of magnitude beyond a field-driven SIT. Furthermore, the line shape of magnetoresistance (MR) $\rho(H)$ is asymmetric [23,25] with a sharp rise to a peak at H_P followed by a gradual decrease to the normal-state resistance [25,29,32,33]. Interestingly, the MR peak becomes sharper and stronger with increasing disorder in the films [24,25,30,31] (see Fig. 1).

Several theoretical attempts have been made to explain this behavior ranging from a Coulomb blockade on the SC islands [34], to boson localization [35–37], a "superinsulator," and charge-vortex duality [28,38]. Suggestions have also been made that the high-field phase is a finite-temperature insulator, akin to a many-body localized state [39].

Model and methods. We consider the disordered attractive Hubbard Hamiltonian in a magnetic field:

$$\mathcal{H} = -\sum_{\langle ij\rangle,\sigma} (t e^{i\phi_{ij}} \hat{c}^\dagger_{i\sigma}\hat{c}_{j\sigma} + \text{H.c.}) - |U| \sum_i \hat{n}_{i\uparrow}\hat{n}_{i\downarrow} + \sum_{i\sigma} (V_i - \mu)\hat{n}_{i\sigma}. \tag{1}$$

This model has been used previously to study the BCS-BEC crossover in a clean system [40], the finite-temperature pseudogapped state in strong coupling [41,42], and the disorder-driven SIT [1]. In the study here we include the effects of a magnetic field in addition to disorder. Here, t and $U < 0$ denote the hopping amplitude and on-site Hubbard attraction, respectively, $c^\dagger_{i\sigma}$ ($c_{i\sigma}$) creates (annihilates) an electron on site i with spin σ on a two-dimensional (2D) square lattice, and $\hat{n}_{i\sigma}$ is the spin-resolved number operator. The magnetic field $\mathbf{H} = \nabla \times \mathbf{A}$ and the vector potential $\mathbf{A}$ is incorporated through the Peierls factor, $\phi_{ij} = \frac{\pi}{\phi_0}\int_i^j \mathbf{A}\cdot d\mathbf{l}$, where $\phi_0 = h/2e$ is the superconducting flux quantum; V is the disorder strength for uniformly distributed site energies $V_i \in [-V, V]$. We obtain the distribution of the inhomogeneous complex pairing amplitude $\Delta_i = -|U|\langle c_{i,\downarrow}c_{i,\uparrow}\rangle$ and the local density $n_i = \sum_\sigma \langle \hat{n}_{i\sigma}\rangle$ for a given disorder realization and field using the self-consistent Bogoliubov–de Gennes (BdG) method. From the longitudinal and transverse current-current correlation function we obtain the frequency-dependent conductivity $\sigma(\omega)$ and the superfluid stiffness D_s, respectively. Results for disordered cases are averaged over 15–25 disorder configurations [see Supplemental Material (SM) [43]].

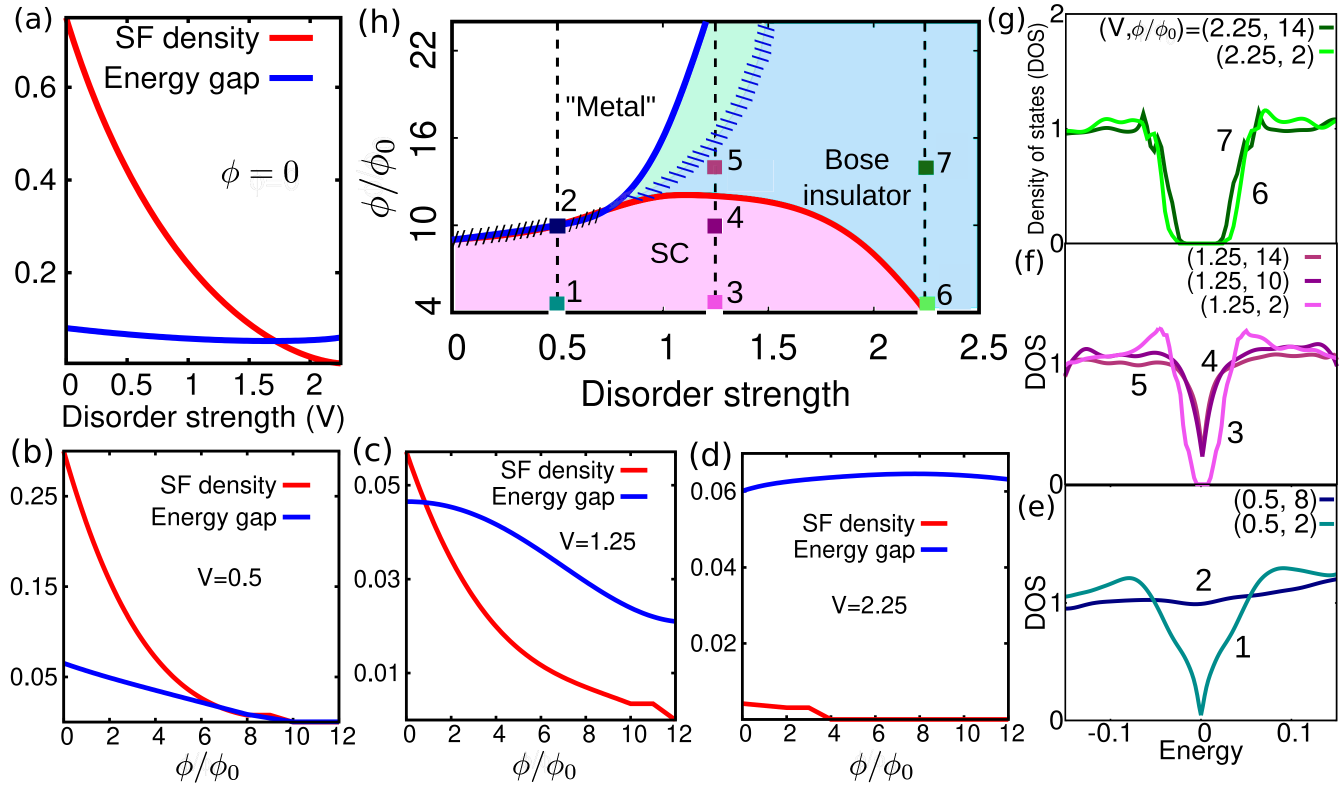

FIG. 3. Phase diagram of the 2D superconductor in the disorder V–magnetic field H plane. (a)–(d) Evolution of superfluid stiffness D_s, including harmonic quantum phase fluctuations [52–54] (see SM) and single-particle spectral gap E_g for $H = 0$ vs disorder V [weak ($V = 0.5$), moderate ($V = 1.25$), and strong ($V = 2.25$)]. (e)–(g) Average DOS, $N(\omega)$ for different H and V. The magnetic field H is in units of ϕ, the number of SC flux quanta penetrating a 36×36 magnetic unit cell. (h) Phase boundaries in the V-H plane: Critical fields obtained from vanishing of D_s (red trace) and vanishing of E_g (blue trace). The phase diagram features three distinct regions: (I) SC ($E_g \neq 0$ and $D_s \neq 0$); (II) Fermi metal or gapless fermionic Anderson insulator [55]) ($E_g = 0$ and $D_s = 0$); and (III) Bose insulator ($D_s = 0$, $E_g \neq 0$). Region III is further divided by a hashed line where the "two-particle" gap in $\sigma(\omega)$ disappears.

Evolution from Abrikosov to Josephson vortices. The field-induced spatial inhomogeneities in the pairing amplitude and the nature of vortex cores for different disorder strengths in Fig. 2 give insights into (a) the underlying mechanism of the field-driven transition of a type-II disordered SC, and (b) the nature of the resulting non-SC state.

We observe the evolution of Abrikosov vortices with a metallic core for low disorder to "core-less" Josephson vortices at higher disorder [48,49]. With increasing disorder [Figs. 2(f)–2(k)], the pairing amplitude $\Delta(\mathbf{r})$ becomes strongly inhomogeneous even in the absence of H and forms SC puddles separated from insulating regions. It is now energetically favorable for flux lines to penetrate the system in regions where Δ is already small, typically coinciding with high disorder regions. These Josephson vortices [Fig. 2(b)] cause the SC phase on the adjacent grains to wind around.

Local density of states (LDOS). The clean SC with an Abrikosov vortex shows a peak in the LDOS near zero bias [50], called the Caroli–de Gennes–Matricon (CdGM) peak, originating from the low-lying quasiparticle bound states formed at the metallic core [see Fig. 2(l) and also in SM [43]]. With increasing disorder, the peak in the vortex core is replaced by a suppression of density of states (evolving from a V shape to a deeper U shape to a hard gap) with increasing disorder [Figs. 2(m)–2(o)]. While for weak disorder the LDOS fills up and evolves into a metallic DOS with increasing H, it remains a hard gap in the Bose insulator.

Energy scales and phase diagram. The contrasting behavior of the superfluid stiffness D_s and the spatially averaged single-particle energy gap scale E_g as a function of the strength of disorder V is shown in Fig. 3(a). In the absence of the magnetic field, the superconductor transitions into an insulating state as signaled by the collapse of D_s, though E_g remains finite beyond the SIT [7,9,10]. Here, we focus on their H dependence [Figs. 3(b)–3(d)] in different disorder regimes.

For weak disorder ($D_s \gg E_g$), both D_s and E_g decrease with increasing H and vanish linearly at the critical H_c [56], consistent with the expectations from Abrikosov theory for a clean sSC and with experimental observations [57]. For moderate disorder ($D_s \sim E_g$), E_g decreases more gradually with H compared to D_s and the two curves cross at a finite ϕ. For high disorder ($D_s \ll E_g$), E_g barely changes with increasing H, while D_s declines precipitously. The energy gaps are extracted from the average density of states (DOS) $N(\omega)$ [Figs. 3(e)–3(g)] shown for different combinations of V and H (see SM [43]). We notice that the standard BCS-type "hard" gap in $N(\omega)$ for a clean and disordered sSC [7,9] turns into a soft pseudogap in a finite field H, particularly at weaker V.

In the resulting phase diagram in Fig. 3(h) we identify three phases: the superconductor in which both D_s and E_g

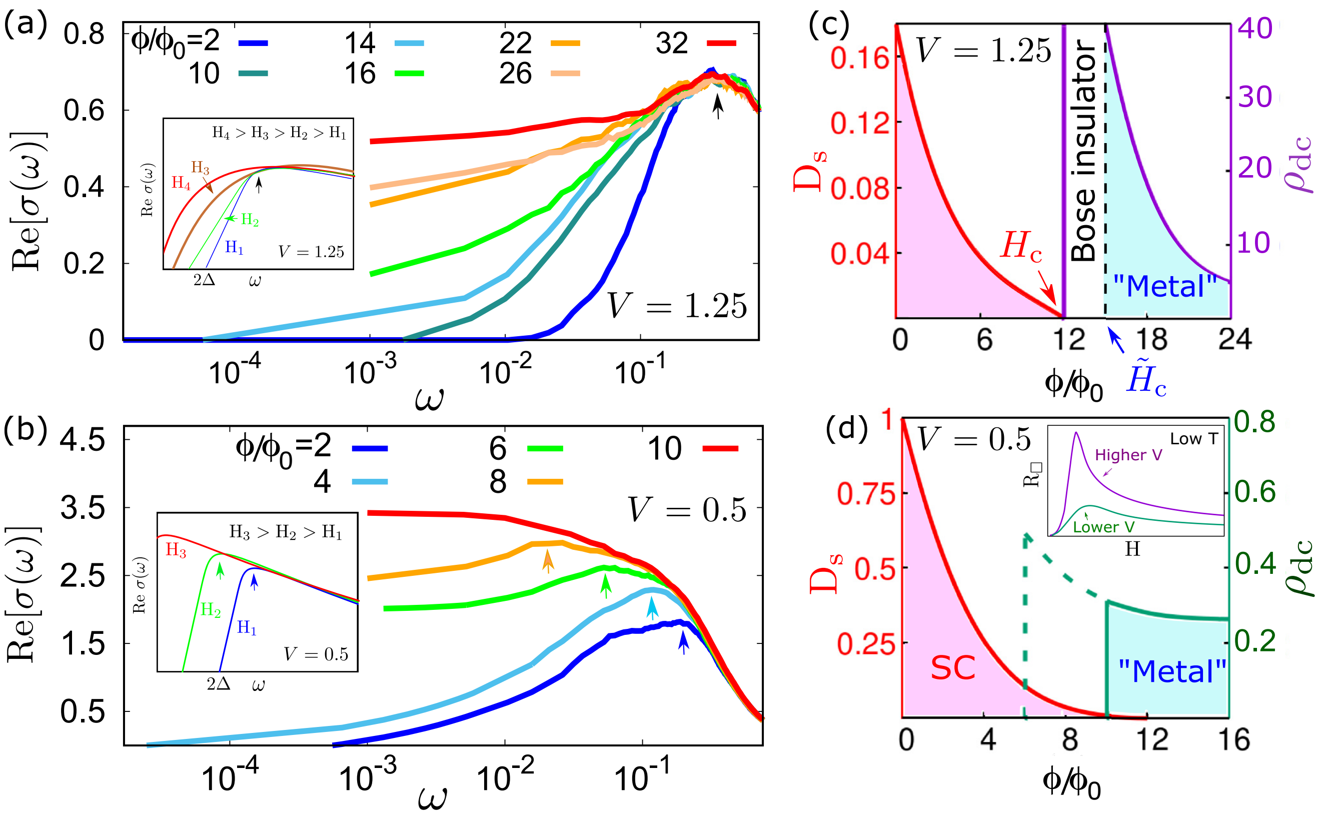

FIG. 4. Field and disorder dependence of dynamical conductivity: $\sigma(\omega)$ for (a) weak and (b) moderate disorder. The contrast in the mechanism of gap filling is highlighted in the insets on a linear scale. (c), (d) D_s (red) and sheet resistance $\rho_{\rm dc}$ (blue) for moderate $V = 1.25$ and weak $V = 0.5$ as a function of H. The pseudogapped Bose insulator is absent for weak disorder and results in a finite $\rho_{\rm dc}$ hump in the magnetoresistance. For moderate disorder, superfluidity is depleted beyond $\phi_c \approx 12\phi_0$, and the resistivity $\rho_{\rm dc}$ diverges. The "metal" region indicates a localized insulating region of electrons with $\sigma_{\rm dc} = 0$ at $T = 0$. At any finite temperature, there will be finite conduction by the variable range hopping mechanism as schematically depicted by the purple curve. The inset of (d) represents the schematic experimental observations [Fig. 1(c)] maintaining the same color coding of $\rho_{\rm dc}$ presented in the main panels of (b), (d). Note the tantalizing similarity of our results with experiments.

are finite, the "metal" [55] in which both of these energy scales vanish, and the Bose insulator in which D_s is zero but E_g remains finite. The familiar Abrikosov scenario, a direct transition from a superconductor to a metal observed in a clean SC [21] with increasing magnetic field, is not found for larger disorder strengths. Instead, an insulating state of Cooper pairs intervenes in which the resistance diverges as the temperature approaches zero. We thus provide another paradigm for the fate of superconductivity as a function of magnetic field for films with stronger disorder.

Origin of gigantic magnetoresistance. In a clean sSC at $T = 0$ as expected there is no absorption of electromagnetic radiation for $H = 0$ for $\omega \leqslant 2\Delta_0$, where Δ_0 is the single-particle gap, if one ignores the weak absorption by collective modes. Disorder generates states within the gap. For weak disorder, the peak in $\sigma(\omega)$ [59] shifts to lower frequencies with increasing disorder, whereas for larger V there is no such downshift, leading to a qualitatively different gap filling for weak and strong disorder.

Based on D_s and $\sigma_{\rm dc} = \sigma(\omega \to 0)$ we identify two qualitatively different transport behaviors: For moderate disorder [Fig. 4(a)], D_s decreases with increasing field and vanishes at H_c. However, the gap in $\sigma(\omega)$ closes only for $H > \tilde{H}_c$ where $\tilde{H}_c > H_c$. In the region $\tilde{H}_c < H < H_c$, between these two critical fields the system remains a Bose insulator with $D_s = 0$ and $\sigma_{\rm dc} = 0$.

On the other hand, for weaker disorder [Fig. 4(b)] we find that D_s remains finite at $\phi \approx 6\phi_0$, whereas $\sigma_{\rm dc}$ jumps up to a finite value from zero. The dissipative response is thus short circuited by a finite D_s, ensuring dissipationless (superfluid) transport in the system. Resistive transport with finite $\sigma_{\rm dc}$ sets in for $H \geqslant H_c$ only after the collapse of D_s (E_g also vanishes at the same H_c) and leads to only a weak hump in MR.

Our analysis explains the existence of the sharp peak in the MR $\rho_{\rm dc}(H) = 1/\sigma_{\rm dc}(H)$ [Figs. 4(c) and 4(d)], its asymmetric shape, and its disorder dependence, in agreement with experiments [23–25,30]. In a recent experiment [29] it is claimed that the MR peak position $H_P \to H_c$ as $T \to 0$, an observation consistent with our $\rho_{\rm dc}$ in Fig. 4(c). Consistent with this picture, we see in Fig. 4(d) that for weakly disordered samples, the energy gap to transport collapses at the same critical field where D_s vanishes.

Conclusion. Upon increasing the magnetic field in a clean superconductor, the metallic cores of Abrikosov vortices overlap and drive the system into a normal fermionic metal. We present a another paradigm for moderately disordered superconductors in which Josephson vortices with insulating cores drive the system into a Bose insulator with a two-particle gap E_σ that is smaller than the one-particle gap E_g. With a further increase in field we find a "Bose-metal"-type phase with zero superfluid stiffness, a vanishing two-particle gap $E_\sigma = 0$, but a finite E_g [hashed line in Fig. 3(h)] [60]. Open questions remain about the nature and mechanism of transport by pairs or collective modes in the pseudogap phase. These questions, together with the role of Coulomb repulsion in the observed anomalous metallic region in a gate-tuned transition in a 2D semiconductor-superconductor heterostructure [33,61], as well as the subtleties of the Bose-metal phase [62–66] will continue to be discussed in future investigations.

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[56] While we introduced in the text the two critical fields $H_c^{D_s}$ and $H_c^{E_g}$ for clarity, in a true sense the "critical field" is identified by $H_c^{D_s}$ that marks the destruction of type-II superconductivity. For all our discussions, the simplified notation H_c always represents $H_c^{D_s}$.

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