Letter

Robustness of Kardar-Parisi-Zhang scaling in a classical integrable spin chain with broken integrability

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Recent investigations have observed superdiffusion in integrable classical and quantum spin chains. An intriguing connection between these spin chains and the Kardar-Parisi-Zhang (KPZ) universality class has emerged. Theoretical developments (e.g., generalized hydrodynamics) have highlighted the role of integrability as well as spin symmetry in KPZ behavior. However, understanding their precise role on superdiffusive transport still remains a challenging task. The widely used quantum spin chain platform comes with severe numerical limitations. To circumvent this barrier, we focus on a classical integrable spin chain which was shown to have a deep analogy with the quantum spin-$\frac{1}{2}$ Heisenberg chain. Remarkably, we find that KPZ behavior prevails even when one considers integrability-breaking but spin-symmetry preserving terms, strongly indicating that spin symmetry plays a central role even in the nonperturbative regime. On the other hand, in the nonperturbative regime, we find that energy correlations exhibit clear diffusive behavior. We also study the classical analog of the out-of-time-ordered correlator and Lyapunov exponents. We find a significant presence of chaos for the integrability-broken cases even though KPZ behavior remains robust. The robustness of KPZ behavior is demonstrated for a wide class of spin-symmetry preserving integrability-breaking terms.

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Superdiffusive spin dynamics in one-dimensional (1D) spin chains has garnered a lot of attention recently. In particular, anomalous spin transport has been observed in an *integrable* model, namely the quantum Heisenberg spin-$\frac{1}{2}$ chain with *isotropic* interactions at infinite temperature [1,2]. Subsequent numerical computations [3] have shown that the spin correlation agrees with the exact correlation function [4] known in the context of the 1D Kardar-Parisi-Zhang (KPZ) universality class [5,6]. Similar properties have been unearthed in an integrable quantum spin chain with a larger symmetry group [7]. This connection between integrability and KPZ superdiffusion has also been a topic of recent analytical and numerical studies in the context of quantum models [8–14]. Interestingly, recent experimental results have provided evidence of 1D KPZ physics in quantum spin chains as well [15,16]. Moreover, numerical studies have also revealed similar characteristics for the spin transport and correlations in integrable and isotropic *classical* models [17–20]. These developments in 1D quantum and classical spin chains suggest that both spin symmetry and integrability have pivotal implications on the existence and nature of superdiffusion. It has been argued [11] that the quantum-classical correspondence is related to the dominant role of solitons (analogous to string excitations in the quantum case) in causing superdiffusive behavior.

Naturally allied to the integrability property in the 1D spin chains is the following question: *What happens to the KPZ superdiffusion when integrability is broken?* Perturbation theory has been applied for understanding the fate of superdiffusion in the quantum Heisenberg spin-$\frac{1}{2}$ chain under the effect of weak integrability-breaking perturbations [21]. On the other hand, in the strongly chaotic regime, regular diffusion has been observed for spin transport at an infinite temperature by using conventional perturbative methods [22] where the integrable term was treated as perturbation. An extensive study of this problem is, nevertheless, still lacking in the literature. In particular, only perturbative regimes (where the weak parameter is either the integrability-breaking term or the integrable term itself) have been investigated and nonperturbative regimes are far from being understood. Needless to mention, quantum models are plagued by severe numerical limitations for such studies, thereby motivating the use of classical integrable systems (which share properties analogous to quantum chains) as one of the most promising alternative platforms.

In this Letter, we report that the KPZ superdiffusion is *robust* when a *symmetry preserving* interaction breaks integrability. This holds true even when integrability is broken strongly (nonperturbative). Our assertion is based on an extensive numerical study for a 1D classical spin chain which involves the Hamiltonian of the *integrable lattice Landau-Lifshitz* (ILLL) model [23–25] at the isotropic point and a spin-symmetry preserving, but integrability-breaking interaction (described in detail later). When the integrability-breaking term does not respect spin symmetry, we find significant deviations from KPZ behavior [26]. We consider

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both perturbative and nonperturbative regimes. Since there exists strong evidence of a quantum-classical correspondence [11,18,27,28], we expect similar behavior in the quantum case. Our numerical simulations in the classical case allow us to probe transport and correlations for spin and energy for perturbations of different magnitudes and different kinds of interactions. We find that the energy correlations exhibit ballistic or diffusive behavior depending on the strength of the perturbation.

We consider a one-dimensional periodic chain of three-component classical spins $\vec{S}$ of unit length. The Hamiltonian is given by

$$H = -\sum_{n=1}^{N}[J\ln(1+\vec{S}_n\cdot\vec{S}_{n+1}) + \lambda\vec{S}_n\cdot\vec{S}_{n+1}], \tag{1}$$

where N is the length of the spin chain, J is the strength of the integrable part, and λ is the strength of the integrability-breaking perturbation. We set $N = 2048$ and $J = 1$ in our computations unless otherwise mentioned. The ILLL spin chain at the isotropic point, where the KPZ phenomenology has been observed recently [18], is recovered for $\lambda = 0$. Thus we refer to our model described by the Hamiltonian in Eq. (1) as the *isotropic perturbed* ILLL (*ip*ILLL) model. Notice that the Hamiltonian in Eq. (1) remains invariant under a global rotation of spin vectors, thereby obeying spin rotation symmetry. The spin dynamics in this system is determined by Hamilton's equations of motion

$$\frac{d\vec{S}_n}{dt} = \{\vec{S}_n, H\} = \vec{S}_n\times\vec{B}_n, \quad \vec{B}_n = -\vec{\nabla}_{\vec{S}_n}H. \tag{2}$$

In order to understand transport properties for a conserved quantity $q = \sum_{n=1}^{N} q_n$, we compute $C_q(x,t)$, the connected correlator for q, defined as

$$C_q(x,t) = \langle[q_x(t) - \langle q_0(0)\rangle_{\text{eq}}][q_0(0) - \langle q_0(0)\rangle_{\text{eq}}]\rangle_{\text{eq}}. \tag{3}$$

Here, $\langle\cdot\rangle_{\text{eq}}$ denotes average with respect to the equilibrium distribution $e^{-\beta H}/Z$, where Z is the partition function at temperature T and $\beta = 1/T$ is the inverse temperature. We are interested in the spin correlation $C_s(x,t)$ for S^z, the z component of the spin $\vec{S}$, and the energy correlation $C_e(x,t)$ associated with the local energy defined as

$$e_n = -J\ln(1+\vec{S}_n\cdot\vec{S}_{n+1}) - \lambda\vec{S}_n\cdot\vec{S}_{n+1}. \tag{4}$$

We expect that the correlation $C_q(x,t)$ satisfies

$$C_q(x,t) = \frac{1}{t^\alpha} f^q\left(\frac{x-ct}{t^\alpha}\right), \tag{5}$$

where $f^q(\cdot)$ is a scaling function and $\alpha > 0$ the scaling exponent. It is worth noting that unlike in a nonlinear fluctuating hydrodynamics description for generic nonintegrable models, where KPZ behavior is associated with sound modes [29], here we have $c = 0$. The exponent α can be directly extracted from the mean squared deviation (MSD) for q,

$$\langle\Delta x^2\rangle_q := \sum_{x=1}^{N} x^2 C_q(x,t) \propto t^{2\alpha}. \tag{6}$$

To evaluate numerically these quantities (energy and spin correlations as well as corresponding MSDs) for the *ip*ILLL spin chain, we perform numerical simulations that evolve the spin chain starting from equilibrium initial conditions at the chosen temperature. We then average over these equilibrium initial conditions to obtain our results. See Supplemental Material [26] (see also Refs. [30–32]) for more details regarding the simulation methods.

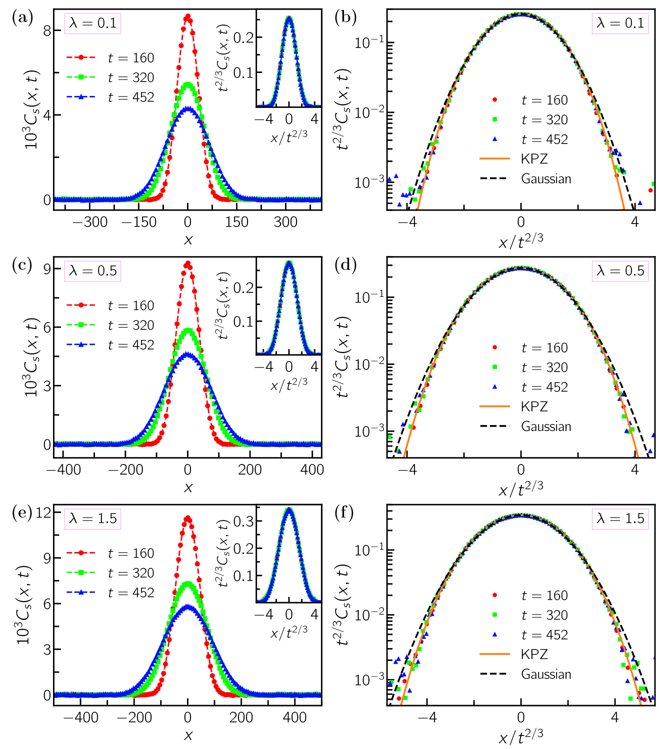

FIG. 1. Plots of the spin correlations for different values of λ. We plot $C_s(x,t)$ vs x in (a), (c), and (e) and $t^{2/3}C_s(x,t)$ vs $x/t^{2/3}$ in (b), (d), and (f). We also plot the exact KPZ correlation function [4] as well as a Gaussian function in (b), (d), and (f) for a comparison. Insets in (a), (c), and (e), show the collapse using the KPZ exponent on a normal scale. The total number of independent realizations is 2×10^5 and $N = 2048$.

We consider three cases for the *ip*ILLL model, $\lambda = 0.1, 0.5, 1.5$, which approximately fall under perturbative, intermediate, and highly nonperturbative parameter regimes, respectively. Below, we summarize our results.

Weakly perturbative regime. When $\lambda = 0.1$ the strength of the integrability-breaking term is relatively weak. Nonetheless, we find that although the system is still chaotic, even at significantly long times, the integrable part dominates over the perturbation and the KPZ superdiffusion observed in the integrable case ($\lambda = 0$) [18] survives in this case as well. This is a surprising result in itself and is consistent with similar predictions in the analogous quantum case [21]. We observe KPZ superdiffusion for spin transport and ballistic transport for energy up to time $t = 640$. We show the correlation for spin in Fig. 1(a) and its remarkable collapse when scaled with the KPZ exponent $\alpha = 2/3$ (inset). In Fig. 1(b), we plot the scaled function on a logarithmic scale to show that we see agreement with not only the KPZ exponent but also with the Prähofer-Spohn KPZ scaling function [4]. The correlation for energy has a scaling exponent $\alpha \approx 0.9$ and exhibits two

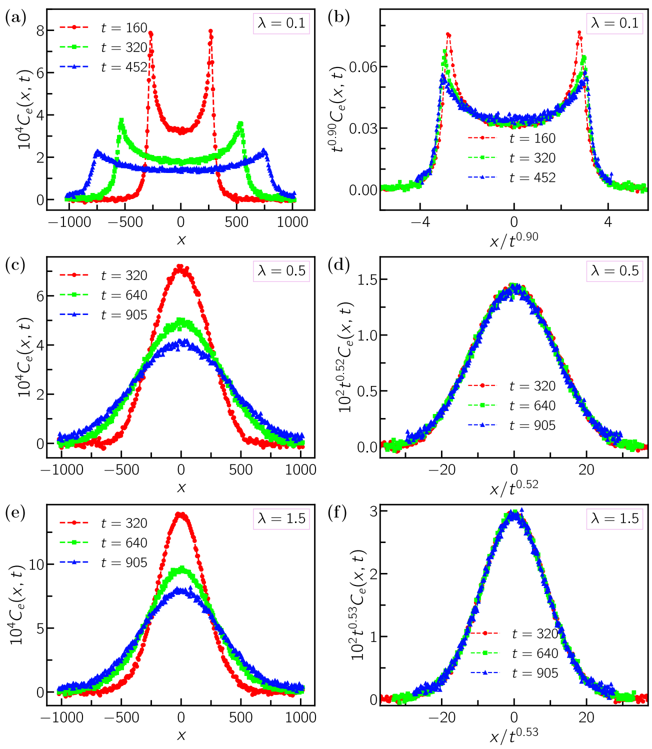

FIG. 2. Plots of the energy correlations for different values of λ. We plot $C_e(x,t)$ vs x in (a), (c), and (e). In (b), we plot $t^{0.9}C_e(x,t)$ vs $x/t^{0.9}$, in (d), we plot $t^{0.52}C_e(x,t)$ vs $x/t^{0.52}$, and in (f), we plot $t^{0.53}C_e(x,t)$ vs $x/t^{0.53}$. We consider 2×10^5 independent realizations for averaging and $N = 2048$. This figure demonstrates the unusual scenario where energy correlations can be diffusive (λ = 0.5 and λ = 1.5) although the corresponding spin correlations have KPZ behavior (Fig. 1).

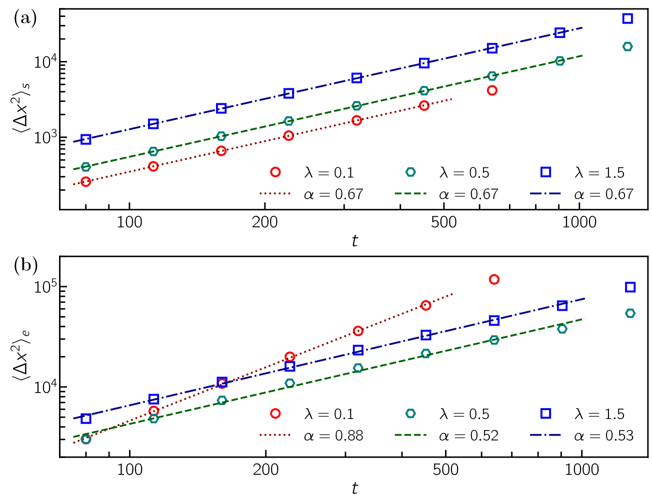

FIG. 3. Plots of the MSDs for (a) spin and (b) energy for λ = 0.1, 0.5, 1.5. We see a remarkable robustness of KPZ behavior in spin correlations even when integrability breaking is significant. The last data points are omitted during the fitting process to avoid potential boundary effects.

ballistically moving peaks [see Figs. 2(a) and 2(b)]. We plot the MSDs for spin and energy correlation in Figs. 3(a) and 3(b), respectively. Using a linear fit, we obtain $\alpha \approx 0.67$ for spin consistent with the scaling in Fig. 1(b). Similarly, we find $\alpha \approx 0.88$ for energy correlation using a linear fit in Fig. 3(b) which is close to the scaling in Fig. 2(b).

Intermediate regime. When $\lambda = 0.5$, the system is in the intermediate-coupling regime (nonpertubative) where one would expect a significant impact of integrability-breaking terms. However, remarkably in this case too, we observe that KPZ superdiffusion prevails for the spin transport. We plot the spin correlation in Figs. 1(c) and 1(d). The corresponding MSD [Fig. 3(a)] gives the exponent $\alpha \approx 0.66$ which confirms the scaling in Fig. 1(d). The energy correlation exhibits diffusive behavior for long times in this case [see Figs. 2(c) and 2(d)]. This is supported by the computation of the MSD for energy [see Fig. 3(d)] where we obtain $\alpha \approx 0.52$ for $t > 80$. It is worth noting that this is a very unusual scenario in which a model exhibits diffusive behavior in energy but KPZ superdiffusion in spin correlations.

Highly nonperturbative regime. To investigate the robustness of the KPZ behavior, we further ramp up the contribution of the integrability-breaking term. We consider $\lambda = 1.5$, where the energy contribution of the integrability-breaking term is even greater than twice that of the integrable term. To our surprise, we observe KPZ superdiffusion for spin transport in this case too. We show the KPZ scaling of the spin correlation in Fig. 1(f). As in the other cases, we also compare with the exact KPZ scaling function and find good agreement. The energy transport is diffusive in this case as well [see Figs. 2(e) and 2(f)]. The MSDs (see Fig. 3) for the spin and energy correlation give the values $\alpha \approx 0.67$ and $\alpha \approx 0.53$, respectively, consistent with the scalings, in Figs. 1(f) and 2(f).

Thus, our results show that as long as the integrability-breaking term in the Hamiltonian is isotropic, the spin transport shows KPZ scaling. Although our results correspond to $\beta = 1$, our observation should hold true at any temperature. In addition to these results, we study a case with $\lambda = 1$ and $J = 0.2$ such that the integrable term plays the role of perturbation. The fact that it is perturbative is established by (i) an almost unchanged Lyapunov exponent, (ii) low value of the energy of the integrable part, and (iii) almost unchanged spin alignment. Even in this case, we observe near-KPZ behavior with exponent $\alpha \approx 2/3$ [26], in stark contrast to the exponent ($\approx$0.54) in the $J = 0$ case [33]. When we consider integrability-breaking terms that do not respect spin symmetry, then we immediately find deviations from KPZ behavior [26]. We also consider different types of integrability-breaking but spin-symmetry preserving terms and our computations indicate that the robustness of KPZ behavior holds at least for a wide family of models [26].

One might wonder if the robustness in KPZ behavior even when integrability is broken ($\lambda \neq 0$) is rooted in the fact that the final system is still close to integrable (nonchaotic). To rule out this possibility, we demonstrate that the system is chaotic as soon as λ coupling is turned on. To do so, we compute the out-of-time-ordered correlator (OTOC) in the *ip*ILLL model. The OTOC has recently been studied in several classical models as a diagnostic tool to probe how initially localized perturbations spread spatially and grow (or

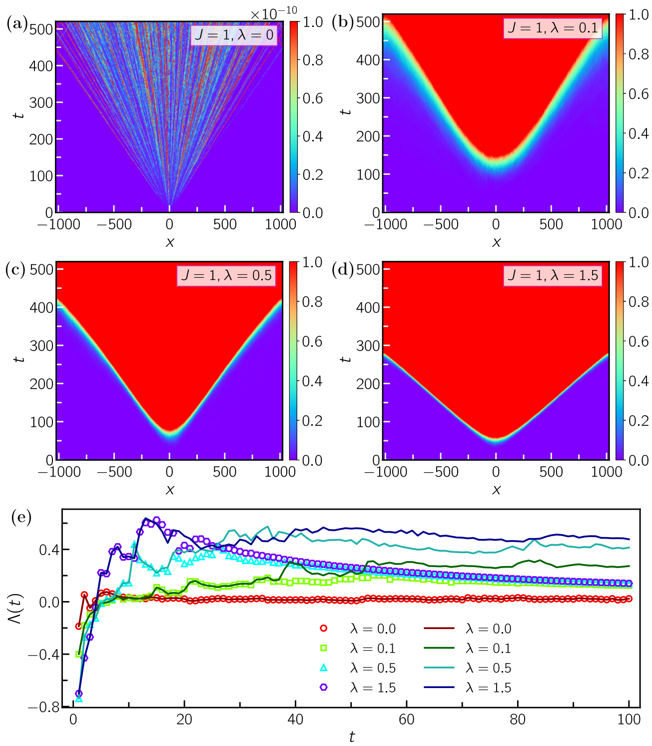

FIG. 4. Heat maps of the OTOC for different values of λ with $\epsilon = 10^{-6}$. We plot $D(x,t)$ for $\lambda = 0$ in (a), $\lambda = 0.1$ in (b), $\lambda = 0.5$ in (c), and $\lambda = 1.5$ in (d). In (e), we show the time variation of the Lyapunov exponent $\Lambda_D(t)$ and $\Lambda_L(t)$ (solid lines). We averaged over 1000 independent realizations to compute $D(x,t)$ with $N = 2048$ and $\Lambda(t)$ with $N = 512$.

decay) temporally [34–43]. In order to compute the OTOC for the spin chains, we consider the following scheme. From an equilibrium initial configuration, which we denote by A, we generate a *perturbed* copy B by replacing the $N/2$-th spin with $\vec{S}'_{N/2} = (\vec{S}_{N/2} + \vec{p}_\epsilon)/|\vec{S}_{N/2} + \vec{p}_\epsilon|$, where $\vec{p}_\epsilon = (0, 0, \epsilon)$, $\epsilon > 0$. We evolve the two copies A and B and compute the OTOC defined as [34]

$$D(x,t) = 2\big[1 - \big\langle \vec{S}^A_{N/2+x}(t) \cdot \vec{S}^B_{N/2+x}(t) \big\rangle\big], \tag{7}$$

where $\vec{S}^A_n(t)$ [$\vec{S}^B_n(t)$] is the spin at site n in the copy A (copy B) of the spin chain. In connection with the OTOC we define the finite-time Lyapunov exponent as $\Lambda_D(t) = \ln|D(0,t)/\epsilon^2|/2t$. We also find a linearized equation for $\delta\vec{S}_n = \vec{S}^B_n(t) - \vec{S}^A_n(t)$ with $\epsilon \to 0$ [26]. In terms of $\delta\vec{S}_n$, the Lyapunov exponent is $\Lambda_L(t) = \ln|\langle \delta\vec{S}^2_{N/2} \rangle/\epsilon^2|/2t$. We show the OTOC for the integrable case ($\lambda = 0$) as well as the three cases mentioned above in Figs. 4(a)–4(d) in the form of heat maps. These heat maps show nontrivial behavior as soon as λ is turned on. The OTOC in Figs. 4(b)–4(d) indicate the presence of chaotic behavior, as expected when integrability is broken. The behavior of the OTOC in the *ip*ILLL model resembles that for the classical Heisenberg model [34]. Even for small λ, the system becomes significantly chaotic. We note that the butterfly velocity (slope of the cone) increases as we increase the strength of the integrability-breaking term. In Fig. 4(e) we show the finite-time Lyapunov exponent $\Lambda(t)$ [both $\Lambda_D(t)$ and $\Lambda_L(t)$] as a function of time. It is clear that $\Lambda(t \to \infty) \approx 0$ when $\lambda = 0$. As soon as we turn on the integrability-breaking term ($\lambda \neq 0$), we see that $\Lambda(t \to \infty)$ is positive and increases with λ, thereby indicating chaos. Note that $\Lambda_D(t)$ and $\Lambda_L(t)$ agree at early times, while at late times $\Lambda_D(t)$ shows a decay ($\sim 1/t$), as expected for any finite ϵ.

In conclusion, we have studied transport properties in the presence of integrability-breaking perturbation in a classical spin chain, namely the *ip*LLL model. Our numerical investigation establishes the robustness of the KPZ physics for spin correlations under spin-symmetry preserving but integrability-breaking perturbations of the integrable Hamiltonian. The robustness of KPZ behavior remains even in the highly nonperturbative regime. In the limit $\lambda/J \gg 1$, however, we expect that the features of the classical Heisenberg spin chain will take over for the spin transport at long times, destroying the KPZ superdiffusion. In this limit we expect diffusive behavior with possible logarithmic corrections [34,44–54]. However, surprisingly, we do not observe such a crossover in time from the KPZ superdiffusion to diffusion even in the limit of large λ/J, thereby not ruling out the possibility of a different scenario. Moreover, surprising KPZ behavior has recently been reported at low temperatures for the classical Heisenberg spin chain [33]. This temperature is well below the one considered here. For integrability-breaking perturbations which do not respect spin symmetry, the KPZ superdiffusion is immediately lost [26]. Our findings on classical spin chains strongly support the corresponding results for quantum systems [21] and predict the possible robustness of KPZ physics in quantum models even deep in the nonperturbative regime. Despite this robustness to integrability-breaking terms, one cannot rule out the crossover to features finally dominated by nonintegrable terms (such as conventional diffusion [7,22,54]) at extremely long times and large system sizes inaccessible in present state-of-the-art computations.

The KPZ scaling in nonintegrable but spin-symmetry preserving systems could be rooted in a possible robustness of solitons of the ILLL in the presence of integrability-breaking but spin-symmetry preserving terms and this will be explored in the future. The anisotropic but integrable generalization of ILLL and the effect of breaking its integrability is an interesting question that is expected to yield a plethora of possibilities.

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