

## Casimir effect for a stack of graphene sheets

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We consider a stack of parallel sheets composed of conducting planes with tensorial conductivities. Using the scattering matrix approach, we derive explicit formulas for the Casimir energy of two, three, and four planes as well as a recurrence relation for arbitrary planes. Specifically, for a stack of graphene, we solve the recurrence relations and obtain formulas for the Casimir energy and force acting on the planes within the stack. Moreover, we calculate the binding energy in the graphene stack with graphite interplane separation, which amounts to  $E_{ib} = 9.9$  meV/atom. Notably, the Casimir force on graphene sheets decreases rapidly for planes beyond the first one. For the second graphene layer in the stack, the force is 35 times smaller than that experienced by the first layer.

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### I. INTRODUCTION

The Casimir effect [1], which was originally considered for two perfect slabs, now plays an important role in various phenomena in physics, chemistry, and biology. For instance, books [2] and recent reviews [3] have highlighted its significance in these fields. The importance of the Casimir force, which can have different values and signs depending on the types of materials used, such as graphene [4], topological insulators [5], chiral metamaterials [6], Weyl semimetals [7], the shape of material boundaries, and the external conditions including temperature, chemical potential, and magnetic field, has been demonstrated in chemistry [8], biophysics [9], and in layered systems such as graphite.

Since the 1970s, van der Waals/Casimir energy and force have been studied for multilayered periodic systems, including a periodic stack of dielectric materials [10]. The Lifshitz formula has been generalized for layered dielectric systems using nonstandard recursion relations for Fresnel coefficients and the fluctuation-dissipative theorem [11]. The Casimir force acting on a plane in a stack perpendicular to the  $z$  axis was calculated as the difference in the  $(z, z)$  component of the regularized stress tensor of neighboring planes. In Ref. [12], a plane in a stack was considered as a piston for a five-plane system. The Casimir force for a five-layered magnetodielectric planar system was discussed in Ref. [13] as an ideal system for detection of the temperature dependence of the Casimir force, and a path-integral approach was used in Ref. [14] to calculate the force for a magnetodielectric layered planar system. The superconducting structure of  $n$ -cavity made of

$n$ -plasma sheets was analyzed in Ref. [15]. Different geometries of multilayered systems were considered in Ref. [16] in the framework of normal-mode techniques, and a modal approach was developed for layered materials in Ref. [17]. Finally, Ref. [18] explored the thermal and electrostatic manipulation of the Casimir force in graphene-dielectric multilayers and demonstrated the possibility of consistent modulation of the Casimir pressure by changing the number of graphene sheets in the stack.

The Casimir energy for a layered system with a different shape, i.e., a set of cylindrical concentric shells, has been discussed in Ref. [19]. In Ref. [20], the layered system of conductive planes was analyzed in details. Two models of conductivity were considered, namely, the constant conductivity and the Drude-Lorentz seven-oscillator model of graphene conductivity, based on the conductivity of graphite [21].

It is important to note that, in all of these references, the scalar type of materials was considered. Conversely, the conductivity of graphene has a tensorial form [4,22]. For two planes with matrix Fresnel reflection coefficients, the formula for the Casimir energy, including some variations and a correct derivation, can be found in Refs. [23,24], respectively. In a system consisting of two planes without virtual photon production, this formula can be represented as an integral over imaginary frequency  $\xi = -i\omega$ ,

$$E_2 = \Re \iint \frac{d^2k}{(2\pi)^3} \int_0^\infty d\xi \ln \det[\mathbf{I} - \exp(-2dk_E)\mathbf{r}'_1\mathbf{r}_2], \quad (1)$$

where  $\mathbf{r}'_1$  and  $\mathbf{r}_2$  are the Fresnel reflection matrices for the first and second planes,  $d$  represents the interplane distance, and  $k_E = \sqrt{\xi^2 + \mathbf{k}^2}$ . The Casimir free energy and pressure between two graphene sheets at both nonzero and zero temperature were calculated in Ref. [25] using the formalism of

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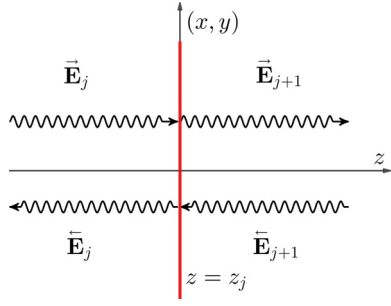


FIG. 1. The scattering on the plane  $j$  is shown. The waves moving from the left (right) to the right (left) are defined by factor  $\exp(+ik_z z)[\exp(-ik_z z)]$ .

the polarization tensor, which considers both the transversal and longitudinal conductivity of graphene, i.e., its tensorial form. The generalization of this expression for more planes is not obvious and requires careful derivation, which is the main focus of this paper.

In Sec. II, the scattering problem for a layered system containing  $n$  parallel conductive planes with tensorial conductivity is formulated. A general expression of Casimir's energy is obtained, and it is calculated for  $n = 2, 3$ , and 4 planes with different conductivities and interplane distances.

In Sec. III, recurrent relations for the energy are derived, and a relation for energies for  $n, n - 1$ , and  $n - 2$  planes is obtained. In the case of a stack of graphene with equal interplane distance, this recurrent relation can be solved in manifest form. By using eigenvalues of the reflection Fresnel matrix, the matrix form of solutions is represented as a sum of scalar forms over eigenvalues. Analogous to Casimir's energy, the Casimir force acting on a specific plane in the stack is determined.

Finally, Sec. IV is dedicated to a particular case of a stack of graphenes, where the Casimir energy and force are numerically calculated at zero temperature, considering the conductivity tensor determined within the framework of three-dimensional quantum electrodynamics [4].

Throughout this paper, the units where  $\hbar = c = 1$  are used.

## II. SCATTERING PROBLEM

Let us consider the scattering problem for the system of  $n$  conductive planes, which are perpendicular to the axis  $z$  in the points  $z_1 < z_2 < \dots < z_n$ . This system divides all space on  $n + 1$  domains ( $D_1|z_1|D_2|z_2| \dots |z_n|D_{n+1}$ ). For a plane  $j$  at position  $z = z_j$  (see Fig. 1), we have the following scattering problem:

$$\begin{pmatrix} \vec{\mathbf{E}}_j \\ \vec{\mathbf{E}}_{j+1} \end{pmatrix} = \mathcal{S}_j \begin{pmatrix} \vec{\mathbf{E}}_j \\ \vec{\mathbf{E}}_{j+1} \end{pmatrix}, \quad (2)$$

where

$$\mathcal{S}_j = \begin{pmatrix} \mathbf{r}_j & \mathbf{t}'_j \\ \mathbf{t}_j & \mathbf{r}'_j \end{pmatrix}, \quad (3)$$

is the scattering  $4 \times 4$  matrix on the plane  $j$  ( $j = 1, \dots, n$ ). Here,  $\mathbf{r}_j$  and  $\mathbf{t}_j$  are the  $2 \times 2$  reflection and transmission coefficients of plane  $j$ , correspondingly. Here,  $\vec{\mathbf{E}}_j$  ( $\vec{\mathbf{E}}_j$ ) is

the two-component vector  $(E_x, E_y)^T$  in the domain  $D_j$ . The movement to the right  $\vec{\mathbf{E}}_j$  (left  $\vec{\mathbf{E}}_j$ ) is defined by factor  $\exp(+ik_z z)[\exp(-ik_z z)]$ . To simplify notation, we do not write out the factors with a position of planes. They may be restored by replacements:

$$\begin{aligned} \mathbf{r}_j &\rightarrow \mathbf{r}_j \exp(2ik_z z_j), \\ \mathbf{r}'_j &\rightarrow \mathbf{r}'_j \exp(-2ik_z z_j), \\ \mathbf{t}_j &\rightarrow \mathbf{t}_j, \\ \mathbf{t}'_j &\rightarrow \mathbf{t}'_j. \end{aligned} \quad (4)$$

The scattering matrix of total system  $\mathcal{S}$  is defined by the relation:

$$\begin{pmatrix} \vec{\mathbf{E}}_1 \\ \vec{\mathbf{E}}_{n+1} \end{pmatrix} = \mathcal{S} \begin{pmatrix} \vec{\mathbf{E}}_1 \\ \vec{\mathbf{E}}_{n+1} \end{pmatrix} = \begin{pmatrix} \mathbf{R} & \mathbf{T}' \\ \mathbf{T} & \mathbf{R}' \end{pmatrix} \begin{pmatrix} \vec{\mathbf{E}}_1 \\ \vec{\mathbf{E}}_{n+1} \end{pmatrix}. \quad (5)$$

To obtain matrix  $\mathcal{S}$ , we represent the set of connected scattering equations in Eq. (2) into the form  $\mathbf{M} \cdot \mathbf{E} = 0$ , where

$$\mathbf{M} = \begin{bmatrix} \mathbf{A}_1 & \mathbf{C}_1 & 0 & 0 & \dots & 0 & 0 \\ 0 & \mathbf{A}_2 & \mathbf{C}_2 & 0 & \dots & 0 & 0 \\ 0 & 0 & \mathbf{A}_3 & \mathbf{C}_3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \mathbf{A}_{n+1} & \mathbf{C}_{n+1} \end{bmatrix},$$

$$\mathbf{E} = \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \mathbf{E}_3 \\ \vdots \\ \mathbf{E}_{n+1} \end{bmatrix},$$

and

$$\mathbf{A}_i = \begin{pmatrix} \mathbf{r}_i & -\mathbf{I} \\ \mathbf{t}_i & 0 \end{pmatrix}, \quad \mathbf{C}_i = \begin{pmatrix} 0 & \mathbf{t}'_i \\ -\mathbf{I} & \mathbf{r}'_i \end{pmatrix}, \quad \mathbf{E}_i = \begin{pmatrix} \vec{\mathbf{E}}_i \\ \vec{\mathbf{E}}_i \end{pmatrix}. \quad (6)$$

Here,  $\mathbf{I}$  is the identity matrix  $2 \times 2$ . To obtain a relation between  $\mathbf{E}_1$  and  $\mathbf{E}_{n+1}$ , we make elementary transformations of lines in the matrix  $\mathbf{M}$  to the form where the last line has the first and last elements only. To do this, we sequentially make zero blocks, starting with the third line.

Let us illustrate these transformations by using an example with three planes. We multiply the second line of  $\mathbf{M}$  on the left on  $-\mathbf{A}_3 \mathbf{C}_2^{-1}$  and add it to third line of the matrix:

$$\mathbf{M} = \begin{pmatrix} \mathbf{A}_1 & \mathbf{C}_1 & 0 & 0 \\ 0 & \mathbf{A}_2 & \mathbf{C}_2 & 0 \\ 0 & -\mathbf{A}_3 \mathbf{C}_2^{-1} \mathbf{A}_2 & 0 & \mathbf{C}_3 \end{pmatrix}. \quad (7)$$

Next, we multiply on the left the first line on  $\mathbf{A}_3 \mathbf{C}_2^{-1} \mathbf{A}_2 \mathbf{C}_1^{-1}$  and add it to the third line and then multiply the third line on  $\mathbf{C}_3^{-1}$ :

$$\mathbf{M} = \begin{pmatrix} \mathbf{A}_1 & \mathbf{C}_1 & 0 & 0 \\ 0 & \mathbf{A}_2 & \mathbf{C}_2 & 0 \\ \mathbf{C}_3^{-1} \mathbf{A}_3 \mathbf{C}_2^{-1} \mathbf{A}_2 \mathbf{C}_1^{-1} \mathbf{A}_1 & 0 & 0 & \mathbf{I} \end{pmatrix}. \quad (8)$$

The last line corresponds to equations:

$$\mathbf{K}_3 \begin{pmatrix} \vec{\mathbf{E}}_1 \\ \vec{\mathbf{E}}_1 \end{pmatrix} = \begin{pmatrix} \vec{\mathbf{E}}_4 \\ \vec{\mathbf{E}}_4 \end{pmatrix}, \quad (9)$$

where

$$\mathbf{K}_3 = (\mathbf{C}_3^{-1} \mathbf{A}_3)(\mathbf{C}_2^{-1} \mathbf{A}_2)(\mathbf{C}_1^{-1} \mathbf{A}_1) = \mathbf{B}_3 \mathbf{B}_2 \mathbf{B}_1, \quad (10)$$

and  $\mathbf{B}_i = \mathbf{C}_i^{-1} \mathbf{A}_i$  (the general structure of these matrices may be found in Appendix A). For a system with  $n$  planes, we obtain

$$(-1)^{n+1} \mathbf{K}_n \begin{pmatrix} \vec{\mathbf{E}}_1 \\ \vec{\mathbf{E}}_1 \end{pmatrix} = \begin{pmatrix} \vec{\mathbf{E}}_{n+1} \\ \vec{\mathbf{E}}_{n+1} \end{pmatrix}, \quad (11)$$

where

$$\mathbf{K}_n = \mathbf{B}_n \mathbf{B}_{n-1} \cdots \mathbf{B}_1 \quad (12)$$

are the matrices  $4 \times 4$ . For  $n = 0$ , there is no scattering at a whole, and therefore,  $\mathbf{K}_0 = \mathbf{I}_4$ , the identity matrix  $4 \times 4$ . By using simple algebra, we can represent this system in the form of Eq. (5) and obtain the scattering matrix of the total system in terms of the matrix  $\mathbf{K}$  (we omit index  $n$  in  $\mathbf{K}_n$  for simplicity):

$$\begin{aligned} \mathbf{R} &= -\mathbf{K}_{22}^{-1} \mathbf{K}_{21}, \\ \mathbf{R}' &= \mathbf{K}_{12} \mathbf{K}_{22}^{-1}, \\ \mathbf{T}' &= (-1)^{n+1} \mathbf{K}_{22}^{-1}, \\ \mathbf{T} &= (-1)^{n+1} (\mathbf{K}_{11} - \mathbf{K}_{12} \mathbf{K}_{22}^{-1} \mathbf{K}_{21}). \end{aligned} \quad (13)$$

For  $n = 0$ , we obtain  $\mathbf{R} = \mathbf{R}' = 0$  and  $\mathbf{T} = \mathbf{T}' = \mathbf{I}$ , as should be the case. The determinant of this block matrix reads

$$\det \mathbf{K} = \det(\mathbf{K}_{22}) \det(\mathbf{K}_{11} - \mathbf{K}_{12} \mathbf{K}_{22}^{-1} \mathbf{K}_{21}) = \frac{\det \mathbf{T}}{\det \mathbf{T}'}. \quad (14)$$

As noted in Ref. [24], due to symmetry, the scattering in opposite directions must be considered. All quantities in the opposite direction we will mark by asterisks  $\overset{*}{g} = g(-k_z)$ . In Ref. [24], it was shown that

$$\begin{aligned} \overset{*}{\mathbf{R}}' &= (\mathbf{R}' - \mathbf{T} \mathbf{R}^{-1} \mathbf{T}')^{-1}, & \overset{*}{\mathbf{R}} &= (\mathbf{R} - \mathbf{T}' \mathbf{R}'^{-1} \mathbf{T})^{-1}, \\ \overset{*}{\mathbf{T}}' &= (\mathbf{T} - \mathbf{R}' \mathbf{T}'^{-1} \mathbf{R})^{-1}, & \overset{*}{\mathbf{T}} &= (\mathbf{T}' - \mathbf{R} \mathbf{T}^{-1} \mathbf{R}')^{-1}, \end{aligned} \quad (15)$$

where the right-hand side is calculated for  $+k_z$ . Changing  $k_z \rightarrow -k_z$ , we obtain the inverse formulas:

$$\begin{aligned} \mathbf{R}' &= (\overset{*}{\mathbf{R}}' - \overset{*}{\mathbf{T}} \overset{*}{\mathbf{R}}'^{-1} \overset{*}{\mathbf{T}}')^{-1}, \\ \mathbf{R} &= (\overset{*}{\mathbf{R}} - \overset{*}{\mathbf{T}}' \overset{*}{\mathbf{R}}'^{-1} \overset{*}{\mathbf{T}})^{-1}, \\ \mathbf{T}' &= (\overset{*}{\mathbf{T}} - \overset{*}{\mathbf{R}} \overset{*}{\mathbf{T}}'^{-1} \overset{*}{\mathbf{R}})^{-1}, \\ \mathbf{T} &= (\overset{*}{\mathbf{T}}' - \overset{*}{\mathbf{R}} \overset{*}{\mathbf{T}}'^{-1} \overset{*}{\mathbf{R}})^{-1}. \end{aligned} \quad (16)$$

These relations are valid for a scattering matrix of each plane, too.

The Casimir energy is defined by the logarithm of the determinant of the scattering matrix, and it may be represented in the following form [24]:

$$\det \mathcal{S} = \frac{\det \mathbf{R}}{\det \overset{*}{\mathbf{R}}'} = \frac{\det \mathbf{T}'}{\det \overset{*}{\mathbf{T}}}. \quad (17)$$

By using the relation in Eq. (13), we obtain the following formula (see Appendix A):

$$\det \mathcal{S} = \frac{\det \mathbf{T}'}{\det \overset{*}{\mathbf{T}}'} = \frac{\det \overset{*}{\mathbf{K}}_{22}}{\det \mathbf{K}_{22}}. \quad (18)$$

The contribution to the Casimir energy,

$$\ln \det \mathcal{S} = \ln \det \overset{*}{\mathbf{K}}_{22} - \ln \det \mathbf{K}_{22}, \quad (19)$$

is the difference of scatterings in opposite directions.

Without changing the Casimir energy, we can multiply the matrix  $\mathbf{K}_n$  on the arbitrary nondegenerate matrix  $\mathbf{W}$ , which has no dependence on the positions of planes. We use this freedom and define the matrix  $\mathcal{D}_n = \mathbf{K}_{22} \mathbf{W}$  in a way that  $\mathcal{D}_n = \mathbf{I} + \dots$ . Then

$$\det \mathcal{S} = \frac{\det \overset{*}{\mathcal{D}}_n}{\det \mathcal{D}_n}, \quad (20)$$

and the Casimir energy reads

$$E_n = \mathfrak{N} \iint \frac{d^2 k}{(2\pi)^3} \int_0^\infty d\xi \ln \det \mathcal{D}_n, \quad (21)$$

where  $\mathcal{D}_n$  is calculated at the imaginary axis  $\omega = i\xi$ .

The special cases with  $n = 1, 2, 3$ , and 4 planes are considered in Appendix B. They are

$$\begin{aligned} \mathcal{D}_1 &= \mathbf{I}, \\ \mathcal{D}_2 &= \mathcal{D}_2^{(21)} = \mathbf{I} - \mathbf{r}_2 \mathbf{r}'_1, \\ \mathcal{D}_3 &= \mathcal{D}_2^{(32)} \mathbf{t}'_2^{-1} \mathcal{D}_2^{(21)} \mathbf{t}'_2 - \mathbf{r}_3 \mathbf{t}_2 \mathbf{r}'_1 \mathbf{t}'_2, \\ \mathcal{D}_4 &= \mathcal{D}_2^{(43)} \mathbf{t}'_3^{-1} \mathcal{D}_2^{(32)} \mathbf{t}'_2^{-1} \mathcal{D}_2^{(21)} \mathbf{t}'_2 \mathbf{t}'_3 \\ &\quad - \mathcal{D}_2^{(43)} \mathbf{t}'_3^{-1} \mathbf{r}_3 \mathbf{t}_2 \mathbf{r}'_1 \mathbf{t}'_2 \mathbf{t}'_3 - \mathbf{r}_4 \mathbf{t}_3 \mathbf{r}'_2 \mathbf{t}'_2^{-1} \mathcal{D}_2^{(21)} \mathbf{t}'_2 \mathbf{t}'_3 \\ &\quad - \mathbf{r}_4 \mathbf{t}_3 \mathbf{t}_2 \mathbf{r}'_1 \mathbf{t}'_2 \mathbf{t}'_3. \end{aligned} \quad (22)$$

The positions of planes are restored by replacement in Eq. (4). Here,  $\mathcal{D}_2^{(il)} = \mathbf{I} - \mathbf{r}_i \mathbf{r}'_l$  is the matrix for two planes  $i$  and  $l$  (numeration of planes starts from one for the first plane and up to  $n$  for the last one). We observe in the above expressions that the matrices  $\mathbf{t}$  appear for internal planes only. The first terms are additive contributions to the energy; the rest of the terms describe the nonadditivity of the Casimir energy. For example, the first term for  $n = 3$  planes gives the following contribution:

$$\ln \det [\mathcal{D}_2^{(32)} \mathbf{t}'_2^{-1} \mathcal{D}_2^{(21)} \mathbf{t}'_2] = \ln \det \mathcal{D}_2^{(32)} + \ln \det \mathcal{D}_2^{(21)}, \quad (23)$$

which is the sum of the contribution of pairs of neighboring planes—first with the middle and the middle with the last one. The second term in  $\mathcal{D}_3$  contains reflection matrices of the first and last planes and a transmission matrix of the middle plane. It looks like a contribution due to the interaction of the first and last terms through the middle.

### III. RECURRENT RELATIONS

To obtain a recurrent relation for  $\mathcal{D}_n$ , we use Eq. (12). The matrix  $\mathbf{K}_n$  has the following form:

$$\mathbf{K}_{n+1} = \mathbf{B}_{n+1} \mathbf{K}_n. \quad (24)$$

By using this relation for index  $n, n + 1$  and the manifest form of matrix  $\mathbf{B}_n$  in Eq. (A3), we obtain the recurrent relation for block (2, 2), the matrix  $\mathbf{K}_n$ :

$$\mathbf{K}_{n+2}^{22} = \phi_n \mathbf{K}_{n+1}^{22} + \psi_n \mathbf{K}_n^{22}, \quad (25)$$

where

$$\begin{aligned} \phi_n &= -\mathbf{t}_{n+2}^{-1} [\mathbf{I} - \mathbf{r}_{n+2} (\mathbf{r}'_{n+1} - \mathbf{t}_{n+1} \mathbf{r}_{n+1}^{-1} \mathbf{t}'_{n+1})], \\ \psi_n &= -\mathbf{t}_{n+2}^{-1} \mathbf{r}_{n+2} \mathbf{t}_{n+1} \mathbf{r}_{n+1}^{-1}. \end{aligned} \quad (26)$$

We define matrices

$$\begin{aligned} \mathcal{D}_n &= (-1)^n \mathbf{t}'_n \mathbf{K}_n \mathbf{t}'_1 \mathbf{t}'_2 \cdots \mathbf{t}'_{n-1} = \mathbf{I} + \dots, \\ \mathcal{D}_1 &= \mathcal{D}_0 = \mathbf{I}. \end{aligned} \quad (27)$$

In terms of these matrices, we obtain the relation we need:

$$\mathcal{D}_{n+2} = \Phi_n \mathcal{D}_{n+1} \mathbf{t}'_{n+1} + \Psi_n \mathcal{D}_n \mathbf{t}'_n \mathbf{t}'_{n+1}, \quad (28)$$

where

$$\begin{aligned} \Phi_n &= (\mathbf{I} - \mathbf{r}_{n+2} \mathbf{r}'_{n+1}) \mathbf{t}'_{n+1} + \mathbf{r}_{n+2} \mathbf{t}_{n+1} \mathbf{r}_{n+1}^{-1}, \\ \Psi_n &= -\mathbf{r}_{n+2} \mathbf{t}_{n+1} \mathbf{r}_{n+1}^{-1} \mathbf{t}'_n. \end{aligned} \quad (29)$$

The scattering matrix for conductive plane  $s$ , with conductivity tensor  $\eta_s = 2\pi\sigma_s$  was found in Ref. [24]:

$$\begin{aligned} \mathbf{r}_s &= \exp(2ik_z z_s) \boldsymbol{\alpha}_s, \\ \mathbf{r}'_s &= \exp(-2ik_z z_s) \boldsymbol{\alpha}_s, \\ \mathbf{t}_s &= \mathbf{I} + \boldsymbol{\alpha}_s, \\ \mathbf{t}'_s &= \mathbf{t}_s, \\ \mathbf{r}_s^* &= -\exp(-2ik_z z_s) (\mathbf{I} + 2\boldsymbol{\alpha}_s)^{-1} \boldsymbol{\alpha}_s, \\ \mathbf{r}'_s^* &= -\exp(2ik_z z_s) (\mathbf{I} + 2\boldsymbol{\alpha}_s)^{-1} \boldsymbol{\alpha}_s, \\ \mathbf{t}_s^* &= (\mathbf{I} + 2\boldsymbol{\alpha}_s)^{-1} (\mathbf{I} + \boldsymbol{\alpha}_s), \\ \mathbf{t}'_s^* &= \mathbf{t}_s^*, \end{aligned} \quad (30)$$

where

$$\begin{aligned} \boldsymbol{\alpha}_s &= -\frac{\omega^2 \boldsymbol{\eta}_s - \mathbf{k} \otimes \mathbf{k} \boldsymbol{\eta}_s + \mathbf{I} k_z \omega \det \boldsymbol{\eta}_s}{\omega^2 \text{tr} \boldsymbol{\eta}_s - (\mathbf{k} \mathbf{k} \boldsymbol{\eta}_s) + k_z \omega (1 + \det \boldsymbol{\eta}_s)}, \\ [\boldsymbol{\alpha}_s^*]_j^i &= -\frac{\omega^2 [\boldsymbol{\eta}_s]_j^i - k^i k^l [\boldsymbol{\eta}_s]_{lj} + \delta_j^i k_z \omega \det \boldsymbol{\eta}_s}{\omega^2 \text{tr} \boldsymbol{\eta}_s - (\mathbf{k} \mathbf{k} \boldsymbol{\eta}_s) + k_z \omega (1 + \det \boldsymbol{\eta}_s)}, \end{aligned} \quad (31)$$

and  $\mathbf{k} = (k^1, k^2) = (k_1, k_2)$ ,  $k_z = \sqrt{\omega^2 - \mathbf{k}^2}$ . All of these matrices commute with each other. At imaginary axes,  $\omega \rightarrow i\xi$  ( $k_z \rightarrow ik_E$ ), we obtain

$$\boldsymbol{\alpha}_s = -\frac{\xi^2 \boldsymbol{\eta}_s + \mathbf{k} \otimes \mathbf{k} \boldsymbol{\eta}_s + \mathbf{I} k_E \xi \det \boldsymbol{\eta}_s}{\xi^2 \text{tr} \boldsymbol{\eta}_s + (\mathbf{k} \mathbf{k} \boldsymbol{\eta}_s) + k_E \xi (1 + \det \boldsymbol{\eta}_s)}, \quad (32)$$

where  $k_E = \sqrt{\xi^2 + \mathbf{k}^2}$ . For identical planes  $\boldsymbol{\alpha}_s = \boldsymbol{\alpha}$  for all  $s$ , and

$$\begin{aligned} \mathcal{D}_{n+2} &= [\mathbf{I} + \exp(-2k_E d_{n+2, n+1}) (\mathbf{I} + 2\boldsymbol{\alpha})] \mathcal{D}_{n+1} \\ &\quad - \exp(-2k_E d_{n+2, n+1}) (\mathbf{I} + \boldsymbol{\alpha})^2 \mathcal{D}_n, \end{aligned} \quad (33)$$

where  $d_{i,j} = z_j - z_i$  is the distance between planes  $i$  and  $j$ . For identical planes  $\boldsymbol{\alpha}_s = \boldsymbol{\alpha}$  and equal interplane distances  $d_{i,j} =$

$d$ , we obtain at the imaginary axis

$$\begin{aligned} \mathcal{D}_{n+2} &= [\mathbf{I} + \exp(-2k_E d) (\mathbf{I} + 2\boldsymbol{\alpha})] \mathcal{D}_{n+1} \\ &\quad - \exp(-2k_E d) (\mathbf{I} + \boldsymbol{\alpha})^2 \mathcal{D}_n, \end{aligned} \quad (34)$$

and

$$\begin{aligned} \mathcal{D}_2 &= \mathbf{I} - \exp(-2k_E d) \boldsymbol{\alpha}^2, \\ \mathcal{D}_3 &= [\mathbf{I} - \exp(-2k_E d) \boldsymbol{\alpha}^2]^2 - \exp(-4k_E d) \boldsymbol{\alpha}^2 (\mathbf{I} + \boldsymbol{\alpha})^2, \\ \mathcal{D}_4 &= [\mathbf{I} - \exp(-2k_E d) \boldsymbol{\alpha}^2]^3 \\ &\quad - 2 \exp(-4k_E d) [\mathbf{I} - \exp(-2k_E d) \boldsymbol{\alpha}^2] \boldsymbol{\alpha}^2 (\mathbf{I} + \boldsymbol{\alpha})^2 \\ &\quad - \exp(-6k_E d) \boldsymbol{\alpha}^2 (\mathbf{I} + \boldsymbol{\alpha})^4. \end{aligned} \quad (35)$$

To solve the recurrent relations in Eq. (34), we use a generation function method extended for matrix-valued coefficients. Let us consider the matrix-valued recurrent relations:

$$\mathcal{D}_{n+2} = \mathbf{u} \mathcal{D}_{n+1} + \mathbf{v} \mathcal{D}_n, \quad (36)$$

with commutative matrices  $\mathbf{u}$  and  $\mathbf{v}$ ,  $[\mathbf{u}, \mathbf{v}] = 0$ , and a generation function:

$$\mathbf{G} = \sum_{s=0}^{\infty} \mathcal{D}_s z^s. \quad (37)$$

Here,

$$\begin{aligned} \mathbf{u} &= \mathbf{I} + \exp(-2k_E d) (\mathbf{I} + 2\boldsymbol{\alpha}), \\ \mathbf{v} &= -\exp(-2k_E d) (\mathbf{I} + \boldsymbol{\alpha})^2. \end{aligned} \quad (38)$$

Considering the recurrent relation in Eq. (36), we obtain

$$\mathbf{G} = \mathcal{D}_0 + (\mathcal{D}_1 - \mathbf{u} \mathcal{D}_0) z + \mathbf{u} \mathbf{G} z + \mathbf{v} \mathbf{G} z^2. \quad (39)$$

Therefore,

$$\mathbf{G} = \frac{\mathcal{D}_0 + (\mathcal{D}_1 - \mathbf{u} \mathcal{D}_0) z}{\mathbf{I} - \mathbf{u} z - \mathbf{v} z^2}. \quad (40)$$

Then we expand over  $z$ :

$$(\mathbf{I} - \mathbf{u} z - \mathbf{v} z^2)^{-1} = \sum_{s=0}^{\infty} \mathbf{M}_s z^s, \quad (41)$$

where

$$\mathbf{M}_s = \sum_{l=0}^{[s/2]} \mathbf{u}^{s-2l} \mathbf{v}^l C_l^{s-l}. \quad (42)$$

Here,  $[\dots]$  is the integer part, and  $C_l^s$  is the binomial coefficient:

$$C_l^s = \frac{s!}{l!(s-l)!}. \quad (43)$$

We set  $\mathbf{M}_{-1} = 0$ , and from Eq. (42), we obtain  $\mathbf{M}_0 = \mathbf{I}$ ,  $\mathbf{M}_1 = \mathbf{u}$ . Considering this expansion, we have the relation:

$$\mathbf{G} = \sum_{s=0}^{\infty} z^s [\mathcal{D}_0 \mathbf{M}_s + (\mathcal{D}_1 - \mathbf{u} \mathcal{D}_0) \mathbf{M}_{s-1}] = \sum_{s=0}^{\infty} z^s \mathcal{D}_s. \quad (44)$$

The solution of this relation reads

$$\mathcal{D}_s = \mathcal{D}_0 \mathbf{M}_s + (\mathcal{D}_1 - \mathbf{u} \mathcal{D}_0) \mathbf{M}_{s-1}. \quad (45)$$

For  $s = 0, 1$ , these relations become identities. Therefore,

$$\mathcal{D}_n = \mathbf{M}_n - \mathbf{M}_{n-1} \exp(-2k_E d)(\mathbf{I} + 2\boldsymbol{\alpha}). \quad (46)$$

Here,  $\mathcal{D}_0 = \mathcal{D}_1 = \mathbf{I}$  and  $\mathcal{D}_2, \mathcal{D}_3$  are given by Eq. (35). If the matrices  $\mathbf{u}$  and  $\mathbf{v}$  are numbers  $u, v$ , then the sum in Eq. (42) may be calculated in close form:

$$M_n = \frac{(u + \sqrt{u^2 + 4v})^{n+1} - (u - \sqrt{u^2 + 4v})^{n+1}}{2^{n+1} \sqrt{u^2 + 4v}}. \quad (47)$$

If  $\mathbf{u}$  and  $\mathbf{v}$  are diagonal matrices:

$$\mathbf{u} = \text{diag}(u_1, u_2), \quad \mathbf{v} = \text{diag}(v_1, v_2), \quad (48)$$

then the matrix  $\mathbf{M}_s$  is diagonal, too, with diagonal elements:

$$M_n^{(i)} = \frac{[u_i + \sqrt{u_i^2 + 4v_i}]^{n+1} - [u_i - \sqrt{u_i^2 + 4v_i}]^{n+1}}{2^{n+1} \sqrt{u_i^2 + 4v_i}}.$$

The matrix  $\boldsymbol{\alpha}$  may be diagonalized with eigenvalues  $r_{\text{te}}, r_{\text{tm}}$ , which correspond to the reflection coefficients of TE and TM modes [26]. Therefore, the matrices  $\mathbf{u}$  and  $\mathbf{v}$  in Eq. (38) are diagonalized, too, and the matrices  $\mathcal{D}_n$  are diagonal with eigenvalues:

$$D_n^x = M_n^x - \exp(-2k_E d) M_{n-1}^x (1 + 2r_x), \quad (49)$$

where

$$M_n^x = \frac{[u_x + \sqrt{u_x^2 + 4v_x}]^{n+1} - [u_x - \sqrt{u_x^2 + 4v_x}]^{n+1}}{2^{n+1} \sqrt{u_x^2 + 4v_x}},$$

$$u_x = 1 + \exp(-2k_E d)(1 + 2r_x),$$

$$v_x = -\exp(-2k_E d)(1 + r_x)^2, \quad (50)$$

and  $x = \text{te}, \text{tm}$ . The matrix  $\boldsymbol{\alpha}$  has the following eigenvalues:

$$r_{\text{te}} = -\left(1 + \frac{k_E}{\eta\xi}\right)^{-1}, \quad r_{\text{tm}} = -\left(1 + \frac{\xi}{\eta k_E}\right)^{-1}, \quad (51)$$

for the constant conductivity case  $\boldsymbol{\eta} = 2\pi\boldsymbol{\sigma} = \eta\mathbf{I}$ , and

$$r_{\text{te}} = -\left(1 + \frac{k_E}{\eta_{\text{gr}} \tilde{k} \Phi}\right)^{-1},$$

$$r_{\text{tm}} = -\left(1 + \frac{\tilde{k}}{\eta_{\text{gr}} k_E \Phi}\right)^{-1}, \quad (52)$$

for the case of graphene at zero temperature and chemical potential:

$$\boldsymbol{\eta} = \eta_{\text{gr}} \frac{\tilde{k}}{\xi} \left( \mathbf{I} - v_F^2 \frac{\mathbf{k} \otimes \mathbf{k}}{\tilde{k}^2} \right) \Phi \left( \frac{\tilde{k}}{2m} \right), \quad (53)$$

where  $\eta_{\text{gr}} = 2\pi\sigma_{\text{gr}} = \pi e^2/2 = 0.0114$ ,  $v_F$  is the Fermi velocity,  $m$  is mass of quasiparticles in graphene (mass gap parameter), and

$$\Phi(y) = \frac{2}{\pi y} \left\{ 1 + \frac{y^2 - 1}{y} \arctan y \right\},$$

$$\tilde{k} = \sqrt{\xi^2 + v_F^2 \mathbf{k}^2}.$$

To compare with the constant conductivity case in Ref. [20], we define a new variable  $t_x$  by the relation:

$$r_x = -\frac{t_x}{1 + t_x}. \quad (54)$$

For the constant conductivity case,

$$t_{\text{te}} = \frac{\eta\xi}{k_E}, \quad t_{\text{tm}} = \frac{\eta k_E}{\xi}, \quad (55)$$

and

$$t_{\text{te}} = \frac{\eta_{\text{gr}} \tilde{k} \Phi}{k_E}, \quad t_{\text{tm}} = \frac{\eta_{\text{gr}} k_E \Phi}{\tilde{k}}, \quad (56)$$

for the graphene sheets. With this definition ( $z = k_E d$ ),

$$M_n^x = \frac{e^{-nz}}{(1 + t_x)^n} \frac{1 - f_x^{2(n+1)}}{f_x(1 - f_x^2)}, \quad (57)$$

where

$$f_x = \sqrt{(\cosh z + t_x \sinh z)^2 - 1} + \cosh z + t_x \sinh z. \quad (58)$$

Considering these relations, we obtain

$$D_n^x = \frac{e^{-nz} f_x^{1-n}}{(1 + t_x)^n} \left[ \frac{1 - f_x^{2(n+1)}}{f_x(1 - f_x^2)} - e^{-z} \frac{1 - f_x^{2n}}{1 - f_x^2} (1 - t_x) \right]. \quad (59)$$

After some algebra, we can transform this expression to the form obtained in Ref. [20] for the case of scalar constant conductivity:

$$D_n^x = \frac{\exp[-(n-1)z] f_x^{2-n}}{(1 + t_x)^n} \times \left[ \frac{1 + t_x}{f_x} \frac{1 - f_x^{2n}}{1 - f_x^2} - e^{-z} \frac{1 - f_x^{2(n-1)}}{1 - f_x^2} \right]. \quad (60)$$

Therefore, we obtain that, for a stack of  $n$  identical conductive planes with conductivity tensor  $\boldsymbol{\sigma}$  and with identical interplane distance  $d$ , the Casimir energy has the following form:

$$E_n = E_n^{\text{te}} + E_n^{\text{tm}}$$

$$= \iint \frac{d^2 k}{(2\pi)^3} \int_0^\infty d\xi (\ln D_n^{\text{te}} + \ln D_n^{\text{tm}}), \quad (61)$$

where function  $D_n$  is given by Eq. (60). For an infinite number of planes  $n \rightarrow \infty$ , the Casimir energy is divergent, but the energy per unit plane  $\bar{E}_n = E_n/n$  is finite; it is given by the same expression with replacement of  $D_n^x \rightarrow D^x$ , where

$$D^x = \lim_{n \rightarrow \infty} \sqrt[n]{D_n^x} = \frac{e^{-z} f_x}{1 + t_x}. \quad (62)$$

Let us analyze the limits of small and large separations. To make this, we change integrand variables  $k_i \rightarrow k_i/d$  and  $\xi \rightarrow \xi/d$  to make integrand dimensionless. The only place with  $d$  dependence is the argument of function  $\Phi = \Phi(\tilde{k}/2md)$ . For small separations  $d \rightarrow 0$ , we observe that  $\Phi \rightarrow 1$ , and the integrand does not depend on the interplane distance, and therefore,  $E_n \sim 1/d^3$ . In the zero order of the Fermi velocity  $v_F = 0$ , the relations in Eq. (56) are transformed to the constant conductivity case in Eq. (55), and the energy coincides with that obtained in Ref. [20] for the constant conductivity case. The Drude-Lorentz model considered in Ref. [20] gives  $1/d^{5/2}$  dependence. For large separations  $d \rightarrow \infty$ , the

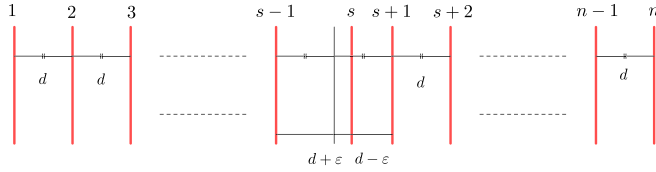


FIG. 2. The plane  $l = s$  in the stack of  $n$  planes is shifted on  $\varepsilon$  to the right.

function  $\Phi \approx 8\tilde{k}/(3\pi md) \rightarrow 0$ . Expanding the integrand in Eq. (61) over  $t_x \ll 1$ , we obtain

$$E_n = -\frac{\eta_{\text{gr}}^2(n-1)}{225d^3(md)^2} \sim \frac{1}{d^5}. \quad (63)$$

The dependence  $1/(m^2d^5)$  is observed in Ref. [4]. For the case of graphene/perfect metal, the energy is  $\sim 1/(md^4)$ . To obtain a force acting on the plane  $s$  in the stack, we consider a stack of graphene with equal distances except for plane  $s$ , which is shifted on the distance  $\varepsilon$  (see Fig. 2), and calculate the derivative:

$$\mathcal{D}'_n = \left. \frac{\partial \mathcal{D}_n}{\partial \varepsilon} \right|_{\varepsilon=0}. \quad (64)$$

For this system, the recurrent relation in Eq. (33) is valid:

$$\begin{aligned} \mathcal{D}_l &= \mathbf{u}\mathcal{D}_{l-1} + \mathbf{v}\mathcal{D}_{l-2}, & 1 \leq l \leq s-1, \\ \mathcal{D}_s &= \mathbf{u}(\varepsilon)\mathcal{D}_{s-1} + \mathbf{v}(\varepsilon)\mathcal{D}_{s-2}, & l = s, \\ \mathcal{D}_{s+1} &= \mathbf{u}(-\varepsilon)\mathcal{D}_s + \mathbf{v}(-\varepsilon)\mathcal{D}_{s-1}, & l = s+1, \\ \mathcal{D}_l &= \mathbf{u}\mathcal{D}_{l-1} + \mathbf{v}\mathcal{D}_{l-2}, & l \geq s+2, \end{aligned} \quad (65)$$

where

$$\begin{aligned} \mathbf{u} &= \mathbf{I} + \exp(-2k_E d)(\mathbf{I} + 2\boldsymbol{\alpha}), \\ \mathbf{v} &= -\exp(-2k_E d)(\mathbf{I} + \boldsymbol{\alpha})^2, \\ \mathbf{u}(\varepsilon) &= \mathbf{I} + \exp[-2k_E(d + \varepsilon)](\mathbf{I} + 2\boldsymbol{\alpha}), \\ \mathbf{v}(\varepsilon) &= -\exp[-2k_E(d + \varepsilon)](\mathbf{I} + \boldsymbol{\alpha})^2. \end{aligned}$$

The recurrent relations for derivatives have the following form:

$$\begin{aligned} \mathcal{D}'_l &= 0, & 1 \leq l \leq s-1, \\ \mathcal{D}'_s &= \mathbf{u}'\mathcal{D}_{s-1} + \mathbf{v}'\mathcal{D}_{s-2}, & l = s, \\ \mathcal{D}'_{s+1} &= -\mathbf{u}'\mathcal{D}_s + \mathbf{u}\mathcal{D}'_s - \mathbf{v}'\mathcal{D}_{s-1}, & l = s+1, \\ \mathcal{D}'_l &= \mathbf{u}\mathcal{D}'_{l-1} + \mathbf{v}\mathcal{D}'_{l-2}, & l \geq s+2, \end{aligned} \quad (66)$$

where

$$\begin{aligned} \mathbf{u}' &= -2k_E \exp(-2k_E d)(\mathbf{I} + 2\boldsymbol{\alpha}), \\ \mathbf{v}' &= 2k_E \exp(-2k_E d)(\mathbf{I} + \boldsymbol{\alpha})^2. \end{aligned} \quad (67)$$

The generation function for this recurrent relation starts from power  $z^s$ :

$$\mathbf{G} = \sum_{l=s}^{\infty} \mathcal{D}'_l z^l = z\mathbf{u}\mathbf{G} + z^2\mathbf{v}\mathbf{G} + \mathcal{D}'_s z^s + (\mathcal{D}'_{s+1} - \mathbf{u}\mathcal{D}'_s)z^{s+1}.$$

Therefore,

$$\mathbf{G} = \frac{\mathcal{D}'_s z^s + (\mathcal{D}'_{s+1} - \mathbf{u}\mathcal{D}'_s)z^{s+1}}{\mathbf{I} - z\mathbf{u} + z^2\mathbf{v}}. \quad (68)$$

Then we use expansion for the denominator in Eq. (41) and obtain

$$\mathcal{D}'_n = \mathcal{D}'_s \mathbf{M}_{n-s} + (\mathcal{D}'_{s+1} - \mathbf{u}\mathcal{D}'_s) \mathbf{M}_{n-s-1}. \quad (69)$$

For all planes  $l \leq s$ , we must use the solution in Eq. (46). Let us now obtain an expression for force acting on graphene with number  $s$  in the stack of  $n$  graphene:

$$\begin{aligned} F_{s,n} &= -\left. \frac{\partial E_n}{\partial \varepsilon} \right|_{\varepsilon=0} = -\lim_{\varepsilon \rightarrow 0} \frac{E_n(\varepsilon) - E_n(0)}{\varepsilon} \\ &= -\Re \lim_{\varepsilon \rightarrow 0} \iint \frac{d^2 k}{(2\pi)^3} \\ &\quad \times \int_0^{\infty} d\xi \frac{\ln \det \mathcal{D}_n(\varepsilon) - \ln \det \mathcal{D}_n(0)}{\varepsilon} \\ &= -\Re \iint \frac{d^2 k}{(2\pi)^3} \int_0^{\infty} d\xi \text{tr}(\mathcal{D}_n^{-1} \mathcal{D}'_n) \\ &= -\Re \sum_{x=\text{te,tm}} \iint \frac{d^2 k}{(2\pi)^3} \int_0^{\infty} d\xi D_n^{x-1} D_n^{x'}. \end{aligned} \quad (70)$$

Direct calculation gives

$$\begin{aligned} D_n^{x-1} D_n^{x'} &\stackrel{\text{def}}{=} G_{s,n}^x \\ &= \frac{2zt_x^2 e^{-z} f[f^{2(s-1)} - f^{2(n-s)}]}{e^{-z} f[1 - f_x^{2(n-1)}] - (1+t_x)(1-f_x^{2n})}. \end{aligned} \quad (71)$$

This expression is the same as that obtained in the scalar case in Ref. [20]. For an infinite amount of graphene in a stack  $n \rightarrow \infty$  and finite  $s$ , we obtain

$$G_s^x = \frac{2zt_x^2 f^{2(1-s)}}{1 - e^z f(1+t_x)}. \quad (72)$$

Therefore, we obtain that the expression for force has the same form as that obtained for the scalar constant conductivity case but with corresponding reflection coefficients for TE and TM modes.

#### IV. STACK OF GRAPHENE

The numerical evaluation of the Casimir energy for a stack of  $n$  graphene at zero temperature is plotted in Figs. 3 and 4 as a function of parameter  $p = md$  and amount of graphene in the stack. We evaluate the function  $\mathcal{E}_n = E_n/nE_{\text{Cas}}$ —the energy per unit graphene in units of the Casimir energy  $E_{\text{Cas}} = -\pi^2/720d^3$  for two ideal planes. The bottom line in Fig. 3 relating to the case of two graphene sheets agrees with the computational results of Ref. [25]. For  $m = 0.1$  eV and interplane distance in graphite  $d_c = 0.3345$  nm, the parameter  $p_c = 1.69 \times 10^{-4}$ . With this value of mass gap, the domain of distances in Fig. 3 is  $d \in [0, 100d_c]$ . The energy  $\mathcal{E}_n$  is lowered by 16 times in this interval. For small separation, we observe the  $1/d^3$  dependence for the Casimir energy  $E_n$  as was proved above. For large separation, we observe  $1/d^5$  dependence of the energy  $E_n$ . The  $n$  dependence of  $\mathcal{E}_n$  is plotted in Fig. 4 for

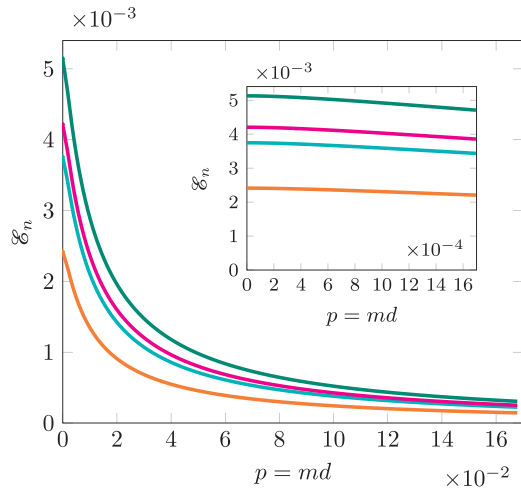


FIG. 3. The Casimir energy per unit graphene  $\mathcal{E}_n$  as a function of interplane distance. Here,  $n = 2, 4, 6, \infty$  from the bottom upwards. The insert shows the neighborhood of origin of the figure.

different interplane distances. The energy decreases in accuracy with increasing interplane distance, as should be the case (the insert in Fig. 4). It is useful to compare these calculations with the case of the constant conductivity case with universal conductivity of graphene  $\eta_{gr}$ , which was considered in detail in Ref. [20]. The constant conductivity leads to  $1/d^3$  dependence, and therefore,  $\mathcal{E}_n$  does not depend on the interplane distance. The better agreement appears for small distances, where for the graphene case,  $\mathcal{E}_n$  is approximately constant (see Fig. 3). The case of constant conductivity gives a greater value  $\sim 10\%$ . Let us estimate the binding energy due to the Casimir energy for graphite. For graphite interplane separation  $d_c = 0.3345$  nm, we obtain the following energy per unit plane in an infinite stack of graphene:  $\bar{E}_\infty = 59.23$  erg/cm<sup>2</sup>. The binding energy  $E_{ib} = \bar{E}_\infty/d_c\rho_c$ , where  $\rho_c = 2.23$  g/cm<sup>3</sup>

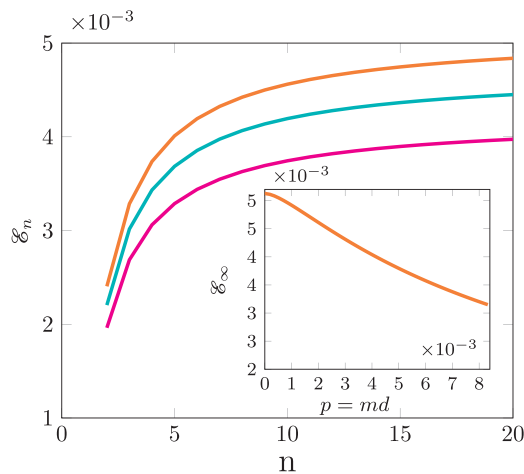


FIG. 4. The Casimir energy per unit graphene  $\mathcal{E}_n$  as a function of the number of planes in the stack. Here,  $m = 0.1$  eV, and  $d = d_c, 10d_c, 20d_c$  from the top down, where  $d_c = 0.3345$  is the interplane distance in graphite. The insert shows the energy for a stack of an infinite number of graphene as a function of interplane distance up to  $50d_c$ .

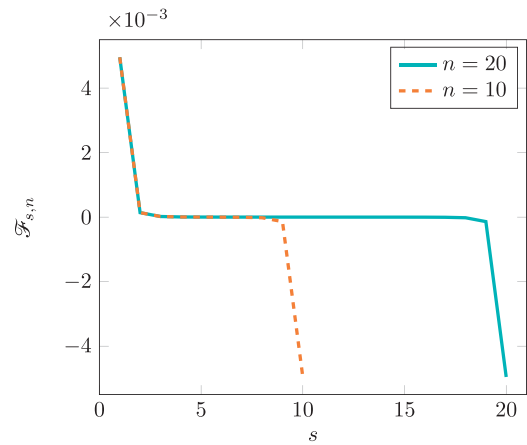


FIG. 5. The Casimir force  $\mathcal{F}_{s,n}$  acting on the graphene  $s$  in the stack of 10 and 20 planes.

is the graphite mass density. For these values, we obtain  $E_{ib} = 9.9$  meV/atom, which is 10% smaller than for the constant conductivity case [20]. From first principles, the cohesion energies are 24–26 meV/atom [27] and 24 meV/atom [28]. The experimental data give cohesion  $35 \pm 10, 15$  meV/atom [29], and  $61 \pm 5$  meV/atom [30]. Most likely, the Casimir energy gives an essential contribution to the binding energy. The numerical evaluations of force acting on the graphene  $s$  in the stack of  $n$  graphene are plotted in Figs. 5 and 6. We evaluate the function  $\mathcal{F}_{s,n} = F_{s,n}/F_{Cas}$ , where  $F_{Cas} = -\pi^2/240d^4$  is the Casimir force for two ideal planes. From Fig. 5, we observe that the value of the force falls very quickly, beginning at the first graphene sheet. The force acting on the second graphene is 35 times smaller than the force acting on the first one. Figure 6 shows that the force very quickly becomes the force for an infinite amount of graphene—already the stack of 10 graphene sheets gives the same force as for an infinite stack.

V. CONCLUSIONS

We considered here the Casimir energy for a stack of conductive planes and a force acting on the plane on this stack.

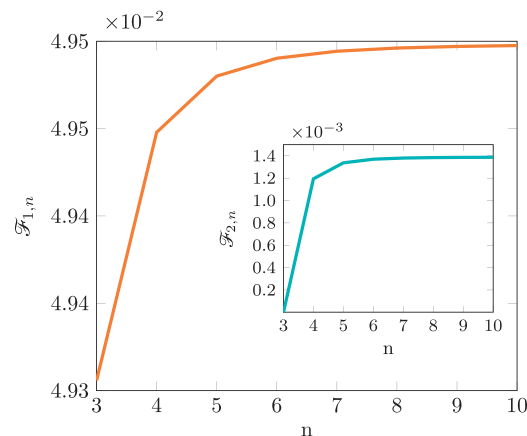


FIG. 6. The Casimir force  $\mathcal{F}_{s,n}$  acting on the first and second (insert figure) graphene in the stack of  $n$  graphene.

To calculate the Casimir force, the scattering theory is used. Starting from the scattering on the single plane, we found the scattering matrix for the set of planes and found the recurrent relation in Eq. (28) for the Casimir energy in Eq. (21). In the particular case of a stack of graphene with equal interplane distance, the recurrent relation in Eq. (35) may be solved in the manifest form in Eq. (46). The reflection Fresnel matrix may be diagonalized with eigenvalues corresponding to the TE and TM electromagnetic modes. The resulting expression for the Casimir energy is the sum of contributions of these modes in Eq. (61). The function  $D_n$  has the same form in Eq. (60) which was found in Ref. [20] for the constant conductivity case. We must use only the corresponding reflection coefficients for graphene in Eq. (52). The explicit form of these coefficients was obtained in the framework of the Dirac model, which is applicable at energies  $< \sim 3$  eV [31]. The characteristic energy of the Casimir interaction is equal to 1 eV at the separation of 100 nm between the planes and decreases with increasing separation. Because of this, at  $a > 100$  nm, the above computational results are fully justified physically. At  $a < 100$  nm, they should be considered as an estimation obtained, strictly speaking, using the Dirac model outside the region of its applicability. For small interplane separations  $md \ll 1$ , the Casimir energy is divergent as  $1/d^3$  (without mass gap parameter  $m$ ). In the opposite limit of large distances, the energy falls as  $1/(m^2 d^5)$ , in accordance to Ref. [4]. The binding energy of an infinite stack of graphene with interplane separation of graphite  $d_c$  is  $E_{ib} = 9.9$  meV/atom, which is 10% smaller than for the constant conductivity case considered in Ref. [20]. The obtained value of binding energy depends only weakly on the mass gap parameter because the case of graphite interplane separation corresponds to the case  $md_c \ll 1$  (see the insert in Fig. 3). The Casimir force acting on the graphene  $s$  in the stack of  $n$  graphene is a sum of two contributions in Eq. (70), where function  $G_{s,n}$  has the same form in Eq. (71) as for constant conductivity case found in Ref. [20], with corresponding reflection coefficients for TE and TM modes. The force acting on graphene falls very quickly—for the second plane, it is 35 times smaller than for the first graphene (see Fig. 5). The extremes of graphene squeeze the stack. The force acting on graphene in a stack is quick  $n \sim 10$  (see Fig. 6) and becomes the same as for an infinite stack. The influence of nonzero temperature and chemical potential can be considered by employing the corresponding conductivity obtained in Ref. [22]. We will analyze this influence in forthcoming papers.

#### ACKNOWLEDGMENTS

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#### APPENDIX A: SOME RELATIONS

(I) First, we prove the formula in Eq. (18). By using the relations in Eqs. (15), (16), and (13), one has

$$\det \mathcal{S} = \frac{\det \mathbf{R}}{\det \mathbf{R}'} = \frac{\det \mathbf{R}}{\det(\mathbf{R}' - \mathbf{R}\mathbf{R}^{-1}\mathbf{T}')^{-1}}$$

$$= \frac{\det[-\mathbf{K}_{22}^{-1}\mathbf{K}_{21}]}{\det[\mathbf{K}_{11}\mathbf{K}_{21}^{-1}]^{-1}} = \frac{\det \mathbf{K}_{11}}{\det \mathbf{K}_{22}}. \quad (\text{A1})$$

(II) The different forms of the matrices  $\mathbf{B}_i$ . By directly solving the relation  $\mathbf{C}_i^{-1}\mathbf{C}_i = \mathbf{I}$ , we obtain

$$\mathbf{C}_i^{-1} = \begin{pmatrix} \mathbf{r}'_i \mathbf{t}'_i{}^{-1} & -\mathbf{I} \\ \mathbf{t}'_i{}^{-1} & 0 \end{pmatrix}. \quad (\text{A2})$$

Therefore, we can represent the matrix  $\mathbf{B}_i$  in different forms:

$$\mathbf{B}_i = \begin{pmatrix} -\mathbf{t}'_i{}^{-1} & -\mathbf{r}'_i \mathbf{t}'_i{}^{-1} \\ \mathbf{t}'_i{}^{-1} \mathbf{r}_i & -\mathbf{t}'_i{}^{-1} \end{pmatrix} = \begin{pmatrix} -\mathbf{t}'_i{}^{-1} & \mathbf{t}'_i{}^{-1} \mathbf{r}_i \\ \mathbf{r}'_i \mathbf{t}'_i{}^{-1} & -\mathbf{t}'_i{}^{-1} \end{pmatrix}, \quad (\text{A3})$$

considering the relations in Eqs. (15) and (16). Therefore, the matrices have the following structure:

$$\mathbf{B}_i = \begin{pmatrix} \mathbf{b}_{11}^i & \mathbf{b}_{12}^i \\ \mathbf{b}_{12}^{*i} & \mathbf{b}_{11}^{*i} \end{pmatrix} = \begin{pmatrix} \mathbf{b}_{22}^{*i} & \mathbf{b}_{21}^{*i} \\ \mathbf{b}_{21}^i & \mathbf{b}_{22}^i \end{pmatrix}. \quad (\text{A4})$$

With replacement of Eq. (4), the dependence of the position appears:

$$\mathbf{B}_i = \begin{bmatrix} \mathbf{b}_{11}^i & \mathbf{b}_{12}^i \exp(-2ik_z z_i) \\ \mathbf{b}_{21}^i \exp(2ik_z z_i) & \mathbf{b}_{22}^i \end{bmatrix}. \quad (\text{A5})$$

Some properties:

$$\mathbf{B}_i^{-1} = \begin{pmatrix} -\mathbf{t}_i^{-1} & \mathbf{t}_i^{-1} \mathbf{r}'_i \\ -\mathbf{r}_i \mathbf{t}_i^{-1} & -\mathbf{t}_i^{-1} \end{pmatrix}, \quad \det \mathbf{B}_i = \frac{\det \mathbf{t}_i}{\det \mathbf{t}'_i}. \quad (\text{A6})$$

Therefore, due to the relations in Eq. (A4), the matrix  $\mathbf{K}$  has the following structure:

$$\mathbf{K} = \begin{pmatrix} \mathbf{K}_{11} & \mathbf{K}_{12} \\ \mathbf{K}_{12}^* & \mathbf{K}_{11} \end{pmatrix} = \begin{pmatrix} \mathbf{K}_{22}^* & \mathbf{K}_{21}^* \\ \mathbf{K}_{21} & \mathbf{K}_{22} \end{pmatrix}. \quad (\text{A7})$$

The matrices  $\mathbf{B}_i$  may be represented in the following forms:

$$\mathbf{B}_i = \boldsymbol{\tau}_i \boldsymbol{\rho}_i = \boldsymbol{\rho}'_i \boldsymbol{\tau}'_i, \quad (\text{A8})$$

where

$$\boldsymbol{\rho}_i = \begin{pmatrix} -\mathbf{I} & \mathbf{r}'_i \\ \mathbf{r}_i & -\mathbf{I} \end{pmatrix}, \quad \boldsymbol{\rho}'_i = \begin{pmatrix} -\mathbf{I} & -\mathbf{r}'_i \\ -\mathbf{r}'_i & -\mathbf{I} \end{pmatrix},$$

$$\boldsymbol{\tau}_i = \begin{pmatrix} \mathbf{t}'_i{}^{-1} & 0 \\ 0 & \mathbf{t}'_i{}^{-1} \end{pmatrix}.$$

It easy to see that

$$\boldsymbol{\rho}_{i+1} \boldsymbol{\rho}'_i = \begin{pmatrix} \mathbf{I} - \mathbf{r}'_{i+1} \mathbf{r}'_i & \mathbf{r}'_i - \mathbf{r}'_{i+1} \\ \mathbf{r}'_i - \mathbf{r}_{i+1} & \mathbf{I} - \mathbf{r}_{i+1} \mathbf{r}'_i \end{pmatrix}. \quad (\text{A9})$$

#### APPENDIX B: DIFFERENT NUMBERS OF PLANES

##### 1. The case without planes, $n = 0$

Without planes, there is no scattering, and  $\mathbf{K}_0 = \mathbf{I}_4$  is diagonal matrix  $4 \times 4$ . We define  $\mathcal{D}_0 = \mathbf{I}$ .



## 2. Single plane, $n = 1$

The matrix  $\mathbf{K}_1 = \mathbf{B}_1$  has the following form [see Eq. (A3)]:

$$\mathbf{K}_1 = \begin{pmatrix} -\mathbf{t}'_1{}^{*-1} & -\mathbf{r}'_1\mathbf{t}'_1{}^{*-1} \\ \mathbf{t}'_1{}^{*-1}\mathbf{r}_1 & -\mathbf{t}'_1{}^{*-1} \end{pmatrix}. \quad (\text{B1})$$

Therefore,

$$\mathcal{D}_1 = -\mathbf{K}_1{}^{22}\mathbf{t}'_1 = \mathbf{I}. \quad (\text{B2})$$

This expression does not depend on the position of the plane and gives no contribution to the Casimir energy.

## 3. Two planes, $n = 2$

In this case,

$$\mathbf{K}_2 = \mathbf{B}_2\mathbf{B}_1 = \tau_2 \begin{pmatrix} \mathbf{I} - \mathbf{r}'_2\mathbf{r}'_1 & \mathbf{r}'_1 - \mathbf{r}'_2 \\ \mathbf{r}'_1 - \mathbf{r}_2 & \mathbf{I} - \mathbf{r}_2\mathbf{r}'_1 \end{pmatrix} \tau_1. \quad (\text{B3})$$

Therefore,

$$\mathcal{D}_2 = \mathbf{I} - \mathbf{r}_2\mathbf{r}'_1. \quad (\text{B4})$$

## 4. Three planes, $n = 3$

For three planes, one has

$$\begin{aligned} \mathbf{K}_3 &= \mathbf{B}_3\mathbf{B}_2\mathbf{B}_1 \\ &= -\tau_3 \begin{pmatrix} \mathbf{I} - \mathbf{r}'_3\mathbf{r}'_2 & \mathbf{r}'_2 - \mathbf{r}'_3 \\ \mathbf{r}'_2 - \mathbf{r}_3 & \mathbf{I} - \mathbf{r}_3\mathbf{r}'_2 \end{pmatrix} \begin{pmatrix} \mathbf{t}'_2{}^{*-1} & \mathbf{t}'_2{}^{*-1}\mathbf{r}'_1 \\ \mathbf{t}'_2{}^{*-1}\mathbf{r}'_1 & \mathbf{t}'_2{}^{*-1} \end{pmatrix} \tau_1. \end{aligned} \quad (\text{B5})$$

Then by using the relations:

$$\mathbf{r}'_i\mathbf{t}'_i{}^{*-1} = -\mathbf{t}'_i{}^{*-1}\mathbf{r}_i, \quad \mathbf{t}'_i{}^{*-1}\mathbf{r}'_i = -\mathbf{r}'_i\mathbf{t}'_i{}^{*-1}, \quad (\text{B6})$$

and Eqs. (15) and (16), we obtain

$$\mathcal{D}_3 = \mathcal{D}_2^{(32)}\mathbf{t}'_2{}^{*-1}\mathcal{D}_2^{(21)}\mathbf{t}'_2 - \mathbf{r}_3\mathbf{t}_2\mathbf{r}'_1\mathbf{t}'_2, \quad (\text{B7})$$

where  $\mathcal{D}_2^{(ij)} = \mathbf{I} - \mathbf{r}_i\mathbf{r}'_j$  for two planes  $i$  and  $j$ . In manifest form:

$$\mathcal{D}_3 = \mathbf{I} - \mathbf{r}_3\mathbf{r}'_2 - \mathbf{t}'_2{}^{*-1}\mathbf{r}_2\mathbf{r}'_1\mathbf{t}'_2 + \mathbf{r}_3\mathbf{r}'_2\mathbf{t}'_2{}^{*-1}\mathbf{r}_2\mathbf{r}'_1\mathbf{t}'_2 - \mathbf{r}_3\mathbf{t}_2\mathbf{r}'_1\mathbf{t}'_2.$$

By using replacement of Eq. (4), we observe that the second term  $\sim \exp(2ik_z d_{3,2})$ , the third term  $\sim \exp(2ik_z d_{2,1})$ , and the last two terms  $\sim \exp(2ik_z d_{3,1})$ , where  $d_{i,k}$  is the distance between planes  $i$  and  $k$ . A comment is in order. The first term in the expression obtained in Eq. (B7) directly gives the sum of energies of pars. Indeed,

$$\ln \det [\mathcal{D}_2^{(32)}\mathbf{t}'_2{}^{*-1}\mathcal{D}_2^{(21)}\mathbf{t}'_2] = \ln \det \mathcal{D}_2^{(32)} + \ln \det \mathcal{D}_2^{(21)}. \quad (\text{B8})$$

The second term breaks the additivity of the Casimir energy. It contains refraction coefficients for the first and last planes without the middle plane.

## 5. Four planes

We have

$$\begin{aligned} \mathbf{K}_4 &= \tau_4 \begin{pmatrix} \mathbf{I} - \mathbf{r}'_4\mathbf{r}'_3 & \mathbf{r}'_3 - \mathbf{r}'_4 \\ \mathbf{r}'_3 - \mathbf{r}_4 & \mathbf{I} - \mathbf{r}_4\mathbf{r}'_3 \end{pmatrix} \\ &\times \tau_3\tau_2 \begin{pmatrix} \mathbf{I} - \mathbf{r}'_2\mathbf{r}'_1 & \mathbf{r}'_1 - \mathbf{r}'_2 \\ \mathbf{r}'_1 - \mathbf{r}_2 & \mathbf{I} - \mathbf{r}_2\mathbf{r}'_1 \end{pmatrix} \tau_1. \end{aligned}$$

Let us express now this expression without asterisks. We use the relations in Eq. (B6). Then

$$\begin{aligned} \mathcal{D}_4 &= \mathcal{D}_2^{(43)}\mathbf{t}'_3{}^{*-1}\mathcal{D}_2^{(32)}\mathbf{t}'_2{}^{*-1}\mathcal{D}_2^{(21)}\mathbf{t}'_2\mathbf{t}'_3 \\ &- \mathcal{D}_2^{(43)}\mathbf{t}'_3{}^{*-1}\mathbf{r}_3\mathbf{t}_2\mathbf{r}'_1\mathbf{t}'_2\mathbf{t}'_3 - \mathbf{r}_4\mathbf{t}_3\mathbf{r}'_2\mathbf{t}'_2{}^{*-1}\mathcal{D}_2^{(21)}\mathbf{t}'_2\mathbf{t}'_3 \\ &- \mathbf{r}_4\mathbf{t}_3\mathbf{t}_2\mathbf{r}'_1\mathbf{t}'_2\mathbf{t}'_3. \end{aligned} \quad (\text{B9})$$

The first term describes the additive part of the Casimir energy, and the rest of the terms give nonadditivity of the energy.

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