Topological pump of $SU(Q)$ quantum chain and Diophantine equation

Yasuhiro Hatsugai *
Department of Physics, University of Tsukuba, Tsukuba 305-8571, Japan

Yoshihito Kuno
Graduate School of Engineering Science, Akita University, Akita 010-8502, Japan

(Received 29 November 2022; revised 17 March 2023; accepted 5 May 2023; published 2 June 2023)

We propose a general construction scheme of topological pump for correlated systems with local gauge symmetry. The bulk-edge correspondence is one of its natural consequences. As a concrete example, an $SU(Q)$ quantum chain associated with a local $[U(1)]^{\otimes Q}$ gauge invariance of colored fermions is discussed in detail. The $SU(Q)$ invariant dimer phases are characterized by the Z_Q Berry phases as a topological order parameter with a d-dimensional twist space ($d = Q - 1$) as a synthetic Brillouin zone. By inclusion of the symmetry breaking perturbation specified by a rational parameter $\Phi = P/Q$, the pump that encloses around the phase boundary, is characterized by the Q Chern numbers associated with the currents due to uniform infinitesimal twists. The analysis of the systems under the open/periodic/twisted boundary conditions clarifies the bulk-edge correspondence of the pump where the large gauge transformation generated by the center-of-mass (CoM) plays a central role. An explicit formula for the Chern number is given by using the Diophantine equation. Numerical demonstrations by the exact diagonalization and the DMRG for finite systems ($Q = 3, 4,$ and 5) have been presented to confirm the general discussions for low-energy spectra, edge states, CoM's, Chern numbers, and the bulk-edge correspondence. A modified Lieb-Schultz-Mattis type argument for the general $SU(Q)$ quantum chain is also mentioned.

DOI: 10.1103/PhysRevB.107.235106

I. INTRODUCTION

$U(1)$ gauge invariance is a key ingredient for the quantum Hall effects as pointed out by Laughlin [1] and is true for the Chern insulators where the conserved current associated with the gauge field is closely related with the topological numbers: the Chern number of the gapped many-body state and the Thouless—Kohmoto—Nightingale—den Nijs (TKNN) integers of the one-particle bands [2–6]. It guarantees topological stability of the phase without any further symmetry protection. Existence of the nontrivial phases with nonzero Chern number is only allowed with time-reversal symmetry breaking that suggests chiral nature of the phase.

As for most of the topological phases, the bulk is hidden, in a sense that the topological number of the bulk without boundaries is not a physical observable. What have been observed experimentally are low-energy localized modes as the chiral edge states localized near the boundaries [7–9]. The edge states reflect the topological number of the bulk as the bulk-edge correspondence [10–17] where $U(1)$ local gauge field associated with the Aharonov-Bohm flux is crucial due to the Laughlin argument. The effective theory also justifies the bulk-edge correspondence for topologically nontrivial systems [18].

Focused studies in the decades reveal various chiral modes of quantum and nonquantum phenomena in quite different phenomena have a topological origin. They are under the control of the bulk-edge correspondence where the Chern number of the bulk predicts the direction and the number of the chiral modes. The first nonquantum example is one-way propagating modes of a gyromagnetic photonic crystal [19–21], mechanical chiral modes in microtubes [22], and coupled optical resonators [23] are also governed by the bulk-edge correspondence. The concept is applied to a wide variety of phenomena in photonics [24–26] and topological circuits [27,28]. It also includes mechanical systems [29,30] and cold atoms [31]. The bulk-edge correspondence is also a key concept in the focused studies of topological insulators in the decades [32–36]. One of the recent surprises is that equatorial waves near the equator of the earth, that is well known in geophysics, are the chiral edge modes associated with the nonzero-Chern number [37]. There exist chiral edge modes in evolutionary game theory [38] and biological flows in neural progenitor cells [39]. The bulk-edge correspondence is universal. In these classical phenomena, chiral edge states are topologically stable and protected by the bulk gap, although the Chern numbers of the bulk are hardly observed but guarantee the existence of the edge states. Even in these phenomena, one can introduce $U(1)$ gauge fields to the governing equation by the minimal coupling to the spatial derivative. The $U(1)$ gauge field is fictitious and never observed but it predicts the chiral edge states associated with the Laughlin argument, which implies the bulk-edge correspondence in classical systems as well.

The local $U(1)$ gauge invariance in one dimension implies an adiabatic charge transport of a gapped quantum chain

*https://patricia.ph.tsukuba.ac.jp/~hatsugai

associated with time as an additional synthetic dimension. This is a topological pump originally proposed by Thouless [40–44] in (1+1) dimensions where the transported charge is quantized as is written by the Chern number. Similar pumping for other degrees of freedom have been also proposed [45–47]. Here we do not need time-reversal symmetry breaking in the 1D system for the nontrivial topological pump. This idea of the topological pump is old but only after more than three decades' experimental trials, the topological pump has been finally realized in cold atom experiments [48,49]. The pump is real and topological. These discoveries motivate to clarify the effects of edge states in the topological pump [50]. The topological pump is an adiabatic transfer of the charge. As for a system with boundaries, the adiabatic cycle of the pump implies everything is going back to the original state after the period assuming the gap. It implies nothing is transported in total. What occurs is that the contributions due to the bulk and edges are canceled. That is, the pumped charge due to bulk is given by a back action of the edge states. In a suitable normalization, a localization length of the edge states scales to zero in a large system-size limit. It implies a quantization of contribution due to edge states in a large system. It guarantees quantization of the pumped charge due to bulk. This is the key idea of a topological nature of the topological pump and the bulk-edge correspondence of the pump [50]. The bulk-edge correspondence is special in the topological pump, that is, the edge is hidden and the bulk gives a physical observable, the center-of-mass (CoM), that is a time integral of the current [50–53]. The pumped $U(1)$ charge due to the edge states is never experimentally observed due to the gapless nature of the edge states. It implies the breakdown of the adiabaticity. In other words, contribution of the edge states can not be measured experimentally in a finite speed pump since the adiabatic condition can not be satisfied rigorously at the moments edge states appear. What is measured is that of the bulk. In the experiment, one measures a motion of the CoM of the system. Its derivative is the current.

Recently this bulk-edge correspondence of the topological pump is also investigated for interacting fermions [52,54], quantum spins [55,56], and bosons [57–59]. This cancellation mechanism is also applied for fractional quantum Hall states [60]. A phase transition point between gapped symmetry protected topological phases (SPT) [61,62] is a source of nontrivial topology. This topological transition is characterized by the change of quantized Berry phases [63–68] where the gap closes. Breaking the symmetry constraint that supports the quantization of the Berry phase, topological pump is realized for the gapped systems. The role of the edge states of spin pumping is also discussed in a mathematically rigorous way for the AKLT Hamiltonian [69] and its modifications [70]. Note that experimental studies for the topological pump using cold atoms are rapidly developing as well [71–73].

Although the topological pump of correlated systems is one of the most active and unsolved fields of theoretical/experimental topological physics, the guiding principle of the search for the nontrivial pump is missing. Especially what kind of degrees of freedom can be pumped is totally unclear for the correlated systems where various degrees of freedom, such as charge, spins, and orbitals, are correlated and interacting each other. Based on the situation in the correlated systems as a background, we propose a general scheme to construct a topological pump for gapped correlated systems. We only require a local gauge symmetry for the system. It directly implies the bulk-edge correspondence. Since the local gauge symmetry implies associated local constraints, which can be one of the fundamental character of the correlated systems, this general proposal can be applied for various correlated systems that motivates not only theoretical but also experimental studies for a wide class of physical systems where various degrees of freedom interact with each other. As a concrete and fundamental example, we here investigate topological pump of the $SU(Q)$ quantum chain. It is surprising rich and the bulk-edge correspondence clarifies all details of the rich structures. One of the surprises is the Diophantine equation that has successfully explained the TKNN integer of the quantum Hall effect (Harper equation) on a lattice also used to explain the Chern numbers of the $SU(Q)$ pump analytically.

Since the paper is extensive, let us describe the contents of this paper and the summary of the main findings before the discussion.

After the Introduction, in Sec. II, the $SU(Q)$ quantum chain ["$SU(Q)$ spin" chain], due to Affleck [74–77] is described as a generalization of the $S = 1$ bilinear-biquadratic quantum chain. By using a colored fermion representation, symmetries of the system are described. Especially Z_Q symmetry and the large gauge transformation due to Q gauge symmetry are introduced. A gapped $SU(Q)$ symmetric dimer phase is discussed for a periodic system that is a source of the nontrivial topological pump.

In Sec. III, by using a time as a synthetic dimension and introducing the $SU(Q)$ symmetry breaking term, an $SU(Q)$ topological pump is proposed that goes around the gap closing dimer transition of the $SU(Q)$ quantum chain. The Q currents associated with the gauge symmetries are introduced and the CoM that generates the large gauge transformation is defined. The systems with open/twisted/periodic boundary conditions are carefully discussed. The $d = Q - 1$ dimensional synthetic Brillouin zone as a parameter space to define the current and Q-closed paths passing through the Z_Q symmetric point are introduced where the averaged currents along the paths are used to define the topological pump in the adiabatic approximation. As for the periodic boundary condition, the time integral of each current gives the pumped charge that is a topological number of the bulk (Chern number). The averaged CoM along the path for the open boundary condition is not continuous in time. A topological number of the edge states is defined by using this discontinuities. The bulk-edge correspondence that is an equivalence of the above two topological numbers is proposed for the Q different topological numbers of the bulk and edges.

In Sec. IV, we discuss $SU(Q)$ symmetric systems without a symmetry breaking term. Here Z_Q quantization of the Berry phases is defined by the loop in the synthetic Brillouin zone and discussed in detail. The gap closing point of this symmetric system is the origin of nontrivial topological pump. A modified Lieb-Schultz-Mattis type argument is also given associated with the (anti)translational symmetry of the uniform problem.

In Sec. V, at the beginning, Z_Q symmetry of systems with odd number of sites are discussed in relation to the edge states with numerical justifications. Also $Z_Q \times Z_Q$ emergent symmetry for systems of the open boundary condition is discussed, which appears taking an infinite-size limit with the open boundary condition. It is justified by the numerical calculations of low-energy spectra. Then systems with the symmetry breaking term is discussed perturbatively. Possible level crossings of the low-energy spectra due to the edge states are predicted associated with the emergent $Z_Q \times Z_Q$ symmetry of an infinite system with boundaries. The topological numbers of the system with edges are explicitly given due to this emergent symmetry. Based on the bulk-edge correspondence, an explicit analytical formula of the Q Chern numbers of bulk is given by using the Diophantine equation of the TKNN for the quantum Hall effect on the lattice.

In Sec. VI, numerical evaluation of the topological numbers of edges and bulk are explicitly given by using the exact diagonalization and DMRG calculation for $Q = 3, 4$, and 5 systems that justify the consistency of the discussion.

In the Appendices, supplemental discussions including technical details are given.

As in the Laughlin argument of the quantum Hall effect [1], the local gauge invariance is the key idea in the topological pump. The role is twofold. It implies a conservation law of the charge that induces level crossings of a low-energy multiplet due to the edge states. It results in the singularities of CoM that is a topological number due to the edge states. The large gauge transformation associated with labeling of the sites defines a current of the pump and the twist, which defines a bulk topological number. Then the two topological numbers of the bulk and the edges are necessarily related with each other. This is the bulk-edge correspondence of the topological pump. This scenario is general. As for a correlated system, there can be various local gauge invariances due to coupled degree of freedom, which implies variety of the bulk-edge correspondence. We demonstrate the validity of the general scheme in the present studies.

II. $SU(Q)$ QUANTUM CHAIN

A. Fermion representation

Let us start considering an $S = 1$ quantum spin chain with nearest-neighbor bilinear-biquadratic interaction

$$H_S(\omega_S) = \sum_j [\cos\omega_S(\boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1}) + \sin\omega_S(\boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1})^2],$$

where $[S_{j,\alpha}, S_{j,\beta}] = i\sum_\gamma \epsilon_{\alpha\beta\gamma} S_{j,\gamma}$, $(\alpha, \beta = 1, 2, 3)$ and $\boldsymbol{S}_j^2 = S(S+1)$, $(S = 1)$. It has a long history of study [78–82]. We discuss its nonuniform $SU(Q)$ extension, $(Q = 2, 3, \cdots)$ by a fermion representation due to Affleck [74–77] (see also Appendix A)

$$\boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1} = (H^{(1)}(\{1\}) - H^{(2)}(\{1\}))|_{Q=3} + (\text{const}), \quad (1)$$

$$(\boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1})^2 = H^{(2)}(\{1\})|_{Q=3} + (\text{const}), \quad (2)$$

where $H^{(1,2)}$ are defined for general Q as

$$H^{(1,2)}(\{J^{(1,2)}_{j,\alpha\beta}\}) = \sum_j h^{(1,2)}_{j,j+1},$$

$$h^{(1)}_{j,j+1} = \sum_{\alpha\beta}^{Q} J^{(1)}_{j,\alpha\beta} c^\dagger_{j,\alpha} c^\dagger_{j+1,\beta} c_{j+1,\alpha} c_{j,\beta}$$

$$= \sum_{\alpha\beta}^{Q} J^{(1)}_{j,\alpha\beta} h^{ex}_{j,\alpha;j+1,\beta}, \quad (3)$$

$$h^{(2)}_{j,j+1} = \sum_{\alpha\beta}^{Q} J^{(2)}_{j,\alpha\beta} c^\dagger_{j,\alpha} c^\dagger_{j+1,\alpha} c_{j+1,\beta} c_{j,\beta}$$

$$= \sum_{\alpha\beta}^{Q} J^{(2)}_{j,\alpha\beta} h^{ph}_{j,\alpha;j\beta}, \quad (4)$$

$$h^{ph}_{j,\alpha;j\beta} = \psi^\dagger_{j,\alpha;j+1,\alpha} \psi_{j,\beta;j+1,\beta}, \quad (5)$$

where $(\boldsymbol{J}^{(1,2)}_j)^\dagger = \boldsymbol{J}^{(1,2)}_j$, $(\boldsymbol{J}^{(1,2)}_j)_{\alpha,\beta} = J^{(1,2)}_{j,\alpha\beta}$, and $c_{j,\alpha}$, $(\alpha = 1, \cdots, Q)$ is a canonical fermion annihilation operator for a color $\alpha = 1, \cdots, Q$ at the site j, $\{c_{j,\alpha}, c^\dagger_{j',\beta}\} = \delta_{jj'}\delta_{\alpha\beta}$ with a constraint at each site $^\forall j$,

$$\sum_\alpha \hat{n}_{j,\alpha} = 1, \quad (6)$$

where $\hat{n}_{j,\alpha} = c^\dagger_{j,\alpha} c_{j,\alpha}$ [83]. The exchange of the colors $h^{ex}_{i,\alpha;j,\beta}$ and the pair hopping $h^{ph}_{i,\alpha;j,\beta}$ at the link i, j are defined as

$$h^{ex}_{i,\alpha;j,\beta} = c^\dagger_{i,\alpha} c^\dagger_{j,\beta} c_{j,\alpha} c_{i,\beta}, \quad (7)$$

$$h^{ph}_{i,\alpha;j,\beta} = \psi^\dagger_{i,\alpha;j,\alpha} \psi_{i,\beta;j,\beta}, \quad (8)$$

where $\psi_{i,\alpha;j,\beta} = c_{i,\alpha} c_{j,\beta}$ is a pairing amplitude. They operate as

$$h^{ex}_{i,\alpha;j,\beta} |\beta_i \alpha_j\rangle = |\alpha_i \beta_j\rangle,$$

$$h^{ph}_{i,\alpha;j,\beta} |\beta_i \beta_j\rangle = |\alpha_i \alpha_j\rangle,$$

where $|\alpha_i \beta_j\rangle = c^\dagger_{i,\alpha} c^\dagger_{j,\beta} |0\rangle$. The models described by the Hamiltonians $H^{(1)}$ and $H^{(2)}$ can be implemented by using a scattering process between different hyperfine states of cold atoms. Such an idea has been proposed in [84,85]. Especially $H^{(1)}$ can be realized experimentally by using cold atoms [86,87]. Up to constant, H_S reduces to the sum of $H^{(1)}$ and $H^{(2)}$ when $Q = 3$. The spin-1 operators at the site j is written by a generator of the $SO(3)$ spatial rotation as $S_{j,\alpha} = \sum_{\beta\gamma} c^\dagger_{j,\beta} S^\alpha_{\beta\gamma} c_{j,\gamma}$, where $S^\alpha_{\beta\gamma} = -i\epsilon_{\alpha\beta\gamma}$, $(\alpha = 1, 2, 3)$ [88] (see also Appendix A). As for the boundary condition, we discuss both of the open boundary condition and the periodic boundary condition $(c_{L+1,\alpha} \equiv c_{1,\alpha})$ assuming the lattice sites are labeled as $j = 1, \cdots, L$ unless otherwise specified.

We discuss each of the $H^{(1,2)}$ separately for general Q since the transformation properties are different. After discussing their gauge symmetries in Sec. II B, we focus on topological properties of a unique ground state of the Hamiltonian $H^{(2)}(\{J_i\})$ for general Q by introducing a dimerization that preserves $SU(Q)$ symmetries. To realize a topological pump based on this dimerized $SU(Q)$ quantum chains, we further

include a symmetry breaking term H_B for $H^{(2)}$. We demonstrate their topological properties numerically for several Q's. As for $Q=3$, $H^{(1)}$ and $H^{(2)}$ are reproduced by the bilinear-biquadratic Hamiltonian $H_S(\omega_S)$ at $\omega_S=\frac{\pi}{4}, 0$ respectively.

B. $SU(Q)$ and Z_Q symmetries

When the coupling is color independent, $J^{(1,2)}_{j,\alpha\beta}=J^{(1,2)}_{j}$, the Hamiltonian $H^{(1,2)}$ is invariant for the global $SU(Q)$ transformation respectively,

$$\mathcal{U}^{(1)}H^{(1)}(\mathcal{U}^{(1)})^\dagger = H^{(1)},$$
$$\mathcal{U}^{(2)}H^{(2)}(\mathcal{U}^{(2)})^\dagger = H^{(2)},$$

$$\mathcal{U}^{(1)} = e^{-i\sum_{j,\alpha\beta} c^\dagger_{j,\alpha}u_{\alpha\beta}c_{j,\beta}}, \tag{9}$$

$$\mathcal{U}^{(2)} = e^{-i\sum_{j,\alpha\beta}(-1)^{j-1} c^\dagger_{j,\alpha}u_{\alpha\beta}c_{j,\beta}}, \tag{10}$$

where u is a generator of a $Q\times Q$ traceless hermitian matrix ($\mathrm{Tr}\,u=0$ and $u^\dagger=u$) [89]. Note that the fermions transform as ($g=(g^*)^{-1}=e^{iu}\in SU(Q)$ [90])

$$\mathcal{U}^{(1)}c_{j,\alpha}(\mathcal{U}^{(1)})^\dagger = \sum_\beta g_{\alpha\beta}c_{j,\beta}, \tag{11}$$

$$\mathcal{U}^{(2)}c_{j,\alpha}(\mathcal{U}^{(2)})^\dagger = \begin{cases}\sum_\beta g_{\alpha\beta}c_{j,\beta} & j:\text{odd}\\ \sum_\beta g^*_{\alpha\beta}c_{j,\beta} & j:\text{even}\end{cases}. \tag{12}$$

Especially $Z_Q\subset SU(Q)$ symmetry is important for the following discussion from topological view points. Although the twists introduced later break $SU(Q)$ symmetry in general, this Z_Q still remains as a symmetry at the high-symmetric twists (denoted by G, see below), which is a generalized antiperiodic boundary condition for $Q=2$ [91].

Z_Q is given by the global cyclic shift of the fermion colors as

$$c_{j,\alpha}\to c_{j,\alpha-1,\ \mathrm{mod}\ Q} = \mathcal{U}_{Z_Q}c_{j,\alpha}\mathcal{U}^\dagger_{Z_Q}, \tag{13}$$

$$\begin{pmatrix}c_{j,1}\\ c_{j,2}\\ \vdots\\ c_{j,Q}\end{pmatrix}\to\begin{pmatrix}c_{j,Q}\\ c_{j,1}\\ \vdots\\ c_{j,Q-1}\end{pmatrix},$$
$$= Z_Q\begin{pmatrix}c_{j,1}\\ c_{j,2}\\ \vdots\\ c_{j,Q}\end{pmatrix} = \mathcal{U}_{Z_Q}\begin{pmatrix}c_{j,1}\\ c_{j,2}\\ \vdots\\ c_{j,Q}\end{pmatrix}\mathcal{U}^\dagger_{Z_Q},$$

where

$$Z_Q = \begin{pmatrix}0 & 0 & \cdots & 1\\ 1 & 0 & 0 & \vdots\\ \vdots & \ddots & \ddots & 0\\ 0 & \cdots & 1 & 0\end{pmatrix} = e^{iz_Q}\in SU(Q), \tag{14}$$

$$\mathcal{U}_{Z_Q} = \prod_j e^{-ic^\dagger_{j,\alpha}(z_Q)_{\alpha\beta}c_{j,\beta}}. \tag{15}$$

See Sec. III A.

It implies that all of the eigenstates are labeled by the eigenvalues of the unitary transformation $\mathcal{U}_{Z_Q}$ as

$$\mathcal{U}_{Z_Q}|\omega^n\rangle = |\omega^n\rangle\omega^n.$$

C. Gauge symmetry

Further the Hamiltonians have a (local) $[U(1)]^Q$ gauge invariance for $\varphi_{j,\alpha}\in\mathbb{R}$, ($\forall j$ and $^\forall\alpha$) as

$$\mathcal{U}^{(1)}_G H^{(1)}(\{J^{(1)}_{j,\alpha\beta}\})(\mathcal{U}^{(1)}_G)^\dagger = H^{(1)}(\{\bar{J}^{(1)}_{j,\alpha\beta}\}),$$
$$\mathcal{U}^{(2)}_G H^{(2)}(\{J^{(2)}_{j,\alpha\beta}\})(\mathcal{U}^{(2)}_G)^\dagger = H^{(2)}(\{\bar{J}^{(2)}_{j,\alpha\beta}\}),$$
$$\mathcal{U}^{(1)}_G c_{j,\alpha}(\mathcal{U}^{(1)}_G)^\dagger = e^{i\varphi_{j,\alpha}}c_{j,\alpha}, \tag{16}$$

$$\mathcal{U}^{(2)}_G c_{j,\alpha}(\mathcal{U}^{(2)}_G)^\dagger = e^{i(-1)^{j-1}\varphi_{j,\alpha}}c_{j,\alpha}, \tag{17}$$

where

$$\mathcal{U}^{(1)}_G = e^{-i\sum_{j,\alpha}\varphi_{j,\alpha}\hat{n}_{j,\alpha}}, \tag{18}$$

$$\mathcal{U}^{(2)}_G = e^{-i\sum_{j,\alpha}(-1)^{j-1}\varphi_{j,\alpha}\hat{n}_{j,\alpha}}. \tag{19}$$

Then, the couplings in the Hamiltonian of Eq. (2) are transformed as

$$\bar{J}^{(1)}_{j,\alpha\beta} = e^{i\Omega_{j,\alpha;j+1,\beta}}J^{(1)}_{j,\alpha\beta}, \tag{20}$$

$$\bar{J}^{(2)}_{j,\alpha\beta} = e^{i(-1)^{j-1}\Omega_{j,\alpha;j+1,\beta}}J^{(2)}_{j,\alpha\beta}, \tag{21}$$

$$\Omega_{j,\alpha;j+1,\beta} = -(\varphi_{j,\alpha}-\varphi_{j,\beta})+(\varphi_{j+1,\alpha}-\varphi_{j+1,\beta}), \tag{22}$$

where $j=1,\dots,L-1$ for the open boundary condition and $\varphi_{L+1,\alpha}\equiv\varphi_{1,\alpha}$ for the periodic boundary condition. We always assume the system size L is even for the discussion of the periodic boundary condition.

Taking all local gauge parameters constant, $\varphi_{j,\alpha}=\phi_\alpha$, one has

$$e^{-i\phi_\alpha N_\alpha}H^{(1)}e^{i\phi_\alpha N_\alpha} = H^{(1)}, \tag{23}$$

$$e^{-i\phi_\alpha \bar{N}_\alpha}H^{(2)}e^{i\phi_\alpha \bar{N}_\alpha} = H^{(2)}, \tag{24}$$

where $N_\alpha=\sum_j\hat{n}_{j,\alpha}$ and $\bar{N}_\alpha=\sum_j(-1)^{j-1}\hat{n}_{j,\alpha}$. Differentiation by ϕ_α implies Q conservation laws ($\alpha=1,\dots,Q$)

$$[N_\alpha,H^{(1)}]=0,\quad [\bar{N}_\alpha,H^{(2)}]=0, \tag{25}$$

where $[N_\alpha,N_\beta]=0$ and $[\bar{N}_\alpha,\bar{N}_\beta]=0$.

Since the Z_Q operation shifts these quantum numbers as $\bar{N}_\alpha\to\bar{N}_{\alpha-1}$, it results in degeneracy if

$$(\bar{N}_1,\dots,\bar{N}_Q)\neq(\bar{N}_Q,\bar{N}_1,\dots,\bar{N}_{Q-1}).$$

D. Large gauge transformation

Taking the gauge parameters as

$$\varphi_{j,\alpha}=x_j\varphi_\alpha,\quad (j=1,\dots,L) \tag{26}$$

$$x_j=\frac{j-j_0}{L}\in\left[-\frac{1}{2},\frac{1}{2}\right],\quad j_0=\frac{L+1}{2}, \tag{27}$$

$$\Omega_{j,\alpha;j+1\beta} = \begin{cases}\frac{1}{L}(\varphi_\alpha-\varphi_\beta) & j=1,\dots,L-1\\ \left(\frac{1}{L}-1\right)(\varphi_\alpha-\varphi_\beta) & j=L\end{cases}, \tag{28}$$

the $[U(1)]^Q$ large gauge transformation $\mathcal{U}^{(1,2)}_{LG}$ is defined by

$$\mathcal{U}^{(1)}_{LG} = e^{-i\sum_\alpha\varphi_\alpha\mathcal{P}^{(1)}_\alpha}, \tag{29}$$

$$\mathcal{U}^{(2)}_{LG} = e^{-i\sum_\alpha\varphi_\alpha\mathcal{P}^{(2)}_\alpha}, \tag{30}$$

TABLE I. Energies and eigenstates of $H^{(1)}$ $(Q=3)$ where $|\alpha_a\beta_b\rangle = c^\dagger_{i\alpha}c^\dagger_{j\beta}|0\rangle$. The sites are labeled by a (:odd) and b (:even).

State	E	(N_1, N_2, N_3)
$\|1_a1_b\rangle$	J	(2,0,0)
$\|2_a2_b\rangle$	J	(0,2,0)
$\|3_a3_b\rangle$	J	(0,0,2)
$(\|1_a2_b\rangle + \|2_a1_b\rangle)/\sqrt{2}$	J	(1,1,0)
$(\|1_a2_b\rangle - \|2_a1_b\rangle)/\sqrt{2}$	$-J$	(1,1,0)
$(\|2_a3_b\rangle + \|3_a2_b\rangle)/\sqrt{2}$	J	(0,1,1)
$(\|3_a2_b\rangle - \|2_a3_b\rangle)/\sqrt{2}$	$-J$	(0,1,1)
$(\|3_a1_b\rangle + \|1_a3_b\rangle)/\sqrt{2}$	J	(1,0,1)
$(\|3_a1_b\rangle - \|1_a3_b\rangle)/\sqrt{2}$	$-J$	(1,0,1)

where

$$\mathcal{P}^{(1)}_\alpha = \sum_j x_j \hat{n}_{j,\alpha}, \quad \mathcal{P}^{(2)}_\alpha = \sum_j (-1)^{j-1} x_j \hat{n}_{j,\alpha}, \tag{31}$$

are the CoM [92–95]. They are generators of the large gauge transformations [50,53]. They induce changes in the couplings as

$$\mathcal{U}^{(1)}_{LG} H^{(1)}(\{J^{(1)}_{j,\alpha\beta}\})(\mathcal{U}^{(1)}_{LG})^\dagger = H^{(1)}(\{\bar{J}^{(1),G}_{j,\alpha\beta}\}),$$
$$\mathcal{U}^{(2)}_{LG} H^{(2)}(\{J^{(2)}_{j,\alpha\beta}\})(\mathcal{U}^{(2)}_{LG})^\dagger = H^{(2)}(\{\bar{J}^{(2),G}_{j,\alpha\beta}\}),$$

$$\bar{J}^{(1),G}_{j,\alpha\beta} = e^{i\frac{\varphi_\alpha-\varphi_\beta}{L}} J^{(1)}_{j,\alpha\beta} \begin{cases} 1 & j=1,\ldots,L-1 \\ e^{-i(\varphi_\alpha-\varphi_\beta)} & j=L \end{cases}, \tag{32}$$

$$\bar{J}^{(2),G}_{j,\alpha\beta} = e^{i(-1)^{j-1}\frac{\varphi_\alpha-\varphi_\beta}{L}} J^{(2)}_{j,\alpha\beta} \begin{cases} 1 & j=1,\ldots,L-1 \\ e^{+i(\varphi_\alpha-\varphi_\beta)} & j=L: \text{even} \end{cases}, \tag{33}$$

where

$$\mathcal{U}^{(1)}_{LG} c_{j,\alpha} (\mathcal{U}^{(1)}_{LG})^\dagger = e^{i\varphi_\alpha x_j} c_{j,\alpha}, \tag{34}$$

$$\mathcal{U}^{(2)}_{LG} c_{j,\alpha} (\mathcal{U}^{(2)}_{LG})^\dagger = e^{i\varphi_\alpha (-1)^{j-1} x_j} c_{j,\alpha}. \tag{35}$$

Note that the constraint $\sum_\alpha \hat{n}_{j,\alpha} = 1$ implies [96]

$$\sum_{\alpha=1}^{Q} \mathcal{P}^{(1)}_\alpha = \sum_j x_j = 0, \tag{36}$$

$$\sum_{\alpha=1}^{Q} \mathcal{P}^{(2)}_\alpha = \begin{cases} -\frac{1}{2} & L: \text{even} \\ 0 & L: \text{odd} \end{cases}. \tag{37}$$

E. Periodic system: Gapped ground state of dimers

To realize a topological pump, we require a gapped unique ground state for a periodic boundary condition and also with nontrivial edge states for a system with edges. See examples [50,52,54,56,97,98].

As for the $Q=3$ case, the spectra and eigenstates of the two-site systems $H^{(1,2)}_{a,b}$ are listed in Tables I and II ($J_{\alpha\beta,1}=J$) respectively ($|0\rangle$ is a fermion vacuum). Generic Q case for $H^{(2)}_{a,b}$ is summarized in Table III. The sites are labeled by a (odd) and b (even). They are consistent with the decomposition of the representations, $3\otimes 3 = \bar{3}\oplus 6$ and $3\otimes\bar{3} = 1\oplus 8$.

TABLE II. Energy and eigenstates of $H^{(2)}$ $(Q=3)$. The sites are labeled by a (:odd) and b (:even) ($\omega = \frac{-1+i\sqrt{3}}{2}$).

State	E	$(\bar{N}_1, \bar{N}_2, \bar{N}_3)$
$\|S_{ab}\rangle = (\|1_a1_b\rangle + \|2_a2_b\rangle + \|3_a3_b\rangle)/\sqrt{3}$	$3J$	(0,0,0)
$\|\omega_{ab}\rangle = (\|1_a1_b\rangle + \omega\|2_a2_b\rangle + \omega^2\|3_a3_b\rangle)/\sqrt{3}$	0	(0,0,0)
$\|\omega^2_{ab}\rangle = (\|1_a1_b\rangle + \omega^2\|2_a2_b\rangle + \omega^4\|3_a3_b\rangle)/\sqrt{3}$	0	(0,0,0)
$\|1_a2_b\rangle$	0	$(1,-1,0)$
$\|2_a1_b\rangle$	0	$(-1,1,0)$
$\|2_a3_b\rangle$	0	$(0,1,-1)$
$\|3_a2_b\rangle$	0	$(0,-1,1)$
$\|3_a1_b\rangle$	0	$(-1,0,1)$
$\|1_a3_b\rangle$	0	$(1,0,-1)$

Since we need a unique (singlet) ground state for the two site problem as a dimer in the following, we discuss $H^{(2)}$ or H_S of $\omega_S = \frac{\pi}{4}$

$$J_j = \begin{cases} J_o & j: \text{odd} \\ J_e & j: \text{even} \end{cases}, \quad J_o, J_e \leqslant 0.$$

The extension to the $SU(Q)$ case (Table III) is straightforward and due to the decomposition $Q\otimes\bar{Q} = 1\oplus(Q^2-1)$ [99]. It is an SPT protected by Z_Q symmetry [63,100]. The two-site Hamiltonian for $J_j = J < 0$ is

$$H_{ab} = J\psi^\dagger_{a,b}\psi_{a,b}$$

where $\psi_{a,b} = Q^{-1/2}\sum_\alpha \psi_{a,\alpha;b,\alpha}$, $(a\neq b)$.

The singlet is given by $|S_{ab}\rangle = \psi^\dagger_{a,b}|0\rangle$ with its energy QJ and $\bar{N}_\alpha = 0, {}^\forall\alpha$, since $[\psi_{a,b}, \psi^\dagger_{a,b}]|0\rangle = 1$ [101]. The rest of zero energy Q^2-1 states are given by the $Q-1$ states, $|\omega^n_{ij}\rangle = Q^{-1/2}\sum_\alpha \omega^{n\alpha}|\alpha_i\alpha_j\rangle$, $n=1,\cdots,Q-1$ with $\bar{N}_\alpha = 0$ and $Q(Q-1)$ states, $|\alpha_i\beta_j\rangle$, $(\alpha\neq\beta)$ with $\bar{N}_\alpha = 1, \bar{N}_\beta = -1, \bar{N}_\gamma = 0, (\gamma\neq\alpha,\beta)$ where we assume a is odd and b is even.

Noting this two-site problem, we have two different unique gapped ground states for the periodic system ($c_{L+1,\alpha} \equiv c_{1,\alpha}$) with different dimer limits $J_o = 0$ and $J_e = 0$ as

$$|g_{\text{pe},eo}\rangle = \prod_{j=1}^{L/2} \psi^\dagger_{2j+1,2j}|0\rangle, \quad (J_o = 0), \tag{38}$$

$$|g_{\text{pe},oe}\rangle = \prod_{j=1}^{L/2} \psi^\dagger_{2j-1,2j}|0\rangle, \quad (J_e = 0). \tag{39}$$

TABLE III. Energies and eigenstates of $H^{(2)}$ where $|\omega^n_{ij}\rangle$, $n = 1,\ldots,Q-1$, $|\alpha_a\beta_b\rangle$, $\alpha\neq\beta = 1,\ldots,Q$, $(\omega = e^{i\frac{2\pi}{Q}})$. The sites are labeled by a (:odd) and b (:even). Generic case.

State	E	$\bar{N}_\alpha$
$\|S_{a,b}\rangle = \psi^\dagger_{a,b}\|0\rangle$	QJ	${}^\forall\bar{N}_\alpha = 0$
$\|\omega^n_{ab}\rangle = \frac{1}{\sqrt{Q}}\sum_{\alpha=1}^{Q}\omega^{\alpha n}\|\alpha_a\alpha_b\rangle$	0	${}^\forall\bar{N}_\alpha = 0$
$\|\alpha_a\beta_b\rangle$, $\alpha\neq\beta$	0	$\bar{N}_\alpha = -\bar{N}_\beta = 1, \bar{N}_{\gamma\neq\alpha,\beta} = 0$

Note that both states are labeled by the occupations, $\bar{N}_\alpha = 0$, $\alpha = 1, \ldots, Q$.

Due to the adiabatic continuity, the ground state is gapped and unique if the interaction between the dimers ($J_{j,\alpha\beta}$, j:even) is finite but weak enough ($J_2 \neq 0$, $|J_2| \ll |J_1|$). It is a SPT phase protected by Z_Q symmetry associated with the quantized Berry phase [66,100]. See Sec. III A. As is clear in Table III, we do not require $SU(Q)$ symmetry. We may allow Z_Q invariant twists at any links as is introduced later. At $J_o = J_e$, the energy gap (of a finite system) closes as is clear from the discontinuous change of the quantized Berry phase (discussed later). This gap closing point is a source of the nontrivial topology, which we discuss in this paper.

III. TOPOLOGICAL PUMP: CURRENTS, CENTER-OF-MASS, AND BULK-EDGE CORRESPONDENCE

A. Current and synthetic Brillouin zone (twist space)

Noting the large gauge transformation, let us start considering a dimerized Hamiltonian $H^{(2)}(\{J^{(2)}_{j,\alpha\beta}\})$ by

$$J_{j,\alpha\beta} = \begin{cases} J_j & j = 1, \ldots, L-1 \\ J_j e^{-i(\varphi_\alpha - \varphi_\beta)} & \text{PBC},\ j = L : \text{even}, \\ 0 & \text{OBC},\ j = L \end{cases} \tag{40}$$

$$J_i = J_0 + \delta J (-1)^j \cos \frac{2\pi t}{T} \in \mathbb{R}, \tag{41}$$

where $J_0, \delta J \in \mathbb{R}$ and t is a time with a period T. To be explicit, we take $J_0 < 0$ and $\delta J > 0$ in the following numerical demonstration. We also include a symmetry breaking term to realize the topological pump. To be concrete, let us consider a following term for the generic $SU(Q)$ case [it reduces to the staggered potential for $SU(2)$ case]:

$$H_B(t) = \sum_{j,\alpha} \hat{n}_{j,\alpha} \Delta_\alpha(t), \tag{42}$$

$$\Delta_\alpha(t) = \Delta \sin 2\pi \left(\frac{t}{T} + \Phi\alpha \right), \tag{43}$$

where $\Phi = \frac{P}{Q}$ and Δ is a strength of a symmetry breaking [102]. The integers P and Q are mutually coprime. We omit the superscript "$^{(2)}$" unless explicitly specified and both of the periodic/open systems are discussed.

As a topological pump of the $SU(Q)$ quantum chain, we discuss a time-dependent Hamiltonian

$$H(t) = H^{(2)}\left(\left\{J^{(2)}_{j,\alpha\beta}\right\}\right) + H_B(t), \tag{44}$$

where $J_j = J_j(t)$ is also time dependent as specified later [see Eq. (138)]. It is also written by the $SU(Q)$ spins due to Affleck [76,103]. Further assuming the ground state of the periodic system is unique and gapped and the time dependence is adiabatically slow compared with the many-body energy gap, we consider a many-body $SU(Q)$-charge pump. As we have mentioned repeatedly, it is also understood as the "$SU(Q)$-spin" pump. There can be various protocols (time dependencies of the Hamiltonian), to be simple, we use the one, Eq. (138), for the numerical demonstration in Sec. VI.

Noting that Eq. (40), the Hamiltonian $H_{\rm op}(\{J_{j,\alpha\beta}\})$ for the open system is φ_α independent and $H_{\rm pe}(\{J_{j,\alpha\beta}\}) \equiv H_{\rm tw}$, for the periodic system, is with a twisted boundary condition. The difference between $H_{\rm op}$ and $H_{\rm tw}$ is only at the boundary link, $j = L$. Then the large gauge transformation by $\mathcal{U}_{LG}$ induces $\mathcal{O}(L^{-1})$ twists for each link as

$$\bar{H}_{\rm op,pe} = H_{\rm op,pe}(\{\bar{J}_{j,\alpha\beta}\}), \tag{45}$$

$$\bar{J}_{j,\alpha\beta} = e^{i(-1)^{j-1}\frac{\varphi_\alpha - \varphi_\beta}{L}} J_j \begin{cases} (j = 1, \ldots, L-1): & \text{open} \\ (j = 1, \ldots, L): & \text{periodic} \end{cases}. \tag{46}$$

Explicitly for the open/periodic cases, they are written as

$$\bar{H}_{\rm op} = \mathcal{U}_{LG} H_{\rm op} \mathcal{U}^\dagger_{LG}, \tag{47}$$

$$\bar{H}_{\rm pe} = \mathcal{U}_{LG} H_{\rm tw} \mathcal{U}^\dagger_{LG}, \tag{48}$$

$$\mathcal{U}_{LG} = e^{-i\sum_\alpha \varphi_\alpha \mathcal{P}_\alpha}. \tag{49}$$

The twists are uniform both for the periodic $\bar{H}_{\rm pe}$ and the open $\bar{H}_{\rm op}$ Hamiltonians. The periodic system is translational invariant by the period 2 with dimerization.

Without dimerization, uniformity of the $H_{\rm pe}$ is written as $J_j = J$ (j independent). This is inherited as *antitranslation* invariance of $\bar{H}_{\rm pe}$ as

$$\mathcal{A}_\mathcal{T} \bar{H}_{\rm pe} \mathcal{A}^{-1}_\mathcal{T} = \bar{H}_{\rm pe}, \tag{50}$$

$$\mathcal{A}_\mathcal{T} = \mathcal{U}_\mathcal{T} \mathcal{K}, \tag{51}$$

$$\mathcal{U}_\mathcal{T} c_{j,\alpha} \mathcal{U}^\dagger_\mathcal{T} = c_{j+1,\alpha}, \tag{52}$$

where $\mathcal{U}_\mathcal{T}$ is unitary and $\mathcal{A}_\mathcal{T}$ is antiunitary ($\mathcal{K}$ is a complex conjugate). It implies $\bar{J}_{j,\alpha\beta} = (\bar{J}_{j\pm1,\alpha\beta})^*$.

In the following, the Hamiltonian is extended by adding a symmetry breaking term H_B that is gauge invariant as $\mathcal{U}_{LG} H_B \mathcal{U}^\dagger_{LG} = H_B$. See Table IV and Fig. 1.

To define a current, let us introduce a d-dimensional twist space (d-dimensional torus), $T^d = \{(\theta_1, \theta_2, \ldots, \theta_d) | \theta_\alpha \in [0, 2\pi], \alpha = 1, \ldots, d\} (d = Q - 1)$. This is a synthetic Brillouin zone [100], which introduces twist for the Hamiltonian $H_{\rm tw}$ by

$$\begin{aligned} \varphi_1 &= \theta_1, \\ \varphi_2 &= \theta_1 + \theta_2, \\ &\vdots \\ \varphi_\alpha &= \theta_1 + \cdots + \theta_\alpha, \\ &\vdots \\ \varphi_d &= \theta_1 + \theta_2 + \cdots + \theta_d, \\ \varphi_Q &\equiv \varphi_0 = 0, \end{aligned}$$

that is,

$$\begin{aligned} \varphi_1 - \varphi_0 &= \theta_1, \\ \varphi_2 - \varphi_1 &= \theta_2, \\ &\vdots \\ \varphi_\alpha - \varphi_{\alpha-1} &= \theta_\alpha, \\ &\vdots \\ \varphi_d - \varphi_{d-1} &= \theta_d, \\ \varphi_Q - \varphi_{Q-1} &= -\theta_1 - \theta_2 \cdots - \theta_d \equiv \theta_Q, \end{aligned}$$

TABLE IV. Hamiltonians, energies and states with different boundary conditions.

	θ dependence	On $T^d(\Theta)$	Energy on $T^d(\Theta)$	Ground state	Current
$H_{\rm op}$	Independent	Independent	Independent	$\|g_{0,\rm op}\rangle$	
$\bar{H}_{\rm op}$	Uniform	Ill defined[a]	Independent	$\|g_{\rm op}\rangle = \mathcal{U}_{LG}\|g_{0,\rm op}\rangle$	$j_{\rm op} = \hbar^{-1}\langle g_{\rm op}\|\partial_\theta \bar{H}_{\rm op}\|g_{\rm op}\rangle$
$H_{\rm tw}$	At the boundaries	Well defined	Well defined[b]	$\|g_{0,\rm tw}\rangle$	
$\bar{H}_{\rm pe}$	Uniform	Ill defined[a]	Well defined[b]	$\|g_{\rm pe}\rangle = \mathcal{U}_{LG}\|g_{0,\rm tw}\rangle$	$j_{\rm pe} = \hbar^{-1}\langle g_{\rm pe}\|\partial_\theta \bar{H}_{\rm pe}\|g_{\rm pe}\rangle$

[a]Does not satisfy the periodicity in Θ.
[b]Assuming the unique ground state on Θ, $E_{\rm tw} = E_{\rm pe}$.

where θ_α is defined in modulo 2π, that is, $\theta_\alpha = 0$ and 2π are identified. It implies a formal relation $\theta_1 + \cdots + \theta_d + \theta_Q = 0$. Note that the Hamiltonian depends on

$$\varphi_\alpha - \varphi_\beta = \theta_\alpha + \cdots + \theta_{\beta+1}, \ {}^\forall\alpha, \beta, \quad \alpha \geqslant \beta + 1.$$

Since the Hamiltonian is invariant for the Z_Q shift of the fermions, $c_{j,\alpha} \to c_{j,\alpha-1}$, which induces a shift $\varphi_\alpha \to \varphi_{\alpha+1}$, and also for the constant shift of ${}^\forall\varphi_\alpha$ by subtracting θ_1 denoted by $\sim$ as

$$\begin{aligned}
&\varphi_1 \to \varphi_2 = \theta_1 + \theta_2 \sim \theta_2,\\
&\varphi_2 \to \varphi_3 = \theta_1 + \theta_2 + \theta_3 \sim \theta_2 + \theta_3,\\
&\quad\vdots\\
&\varphi_\alpha \to \varphi_{\alpha+1} = \theta_1 + \cdots + \theta_{\alpha+1} \sim \theta_2 + \cdots + \theta_{\alpha+1},\\
&\quad\vdots\\
&\varphi_{d-1} \to \varphi_d = \theta_1 + \theta_2 + \cdots + \theta_d \sim \theta_2 + \cdots + \theta_d,\\
&\varphi_d \to \varphi_{d+1} = \varphi_Q = \theta_1 + \theta_2 + \cdots + \theta_d + \theta_Q,\\
&\qquad\sim \theta_2 + \cdots + \theta_d + \theta_Q,\\
&\varphi_Q = \varphi_0 \to \varphi_1 = \theta_1 \sim 0.
\end{aligned}$$

It is given by the cyclic shift of the parameter space supplemented by θ_Q,

$$(\theta_1, \ldots, \theta_d, \theta_Q) \to (\theta_2, \ldots, \theta_Q, \theta_1). \tag{53}$$

It implies Z_Q equivalence of loops $\ell_{V_\alpha G V_{\alpha+1}}$, $\alpha = 1, \ldots, Q$ as shown later.

It is useful to express this parameter space as shown in Figs. 2 and 3. Let us start $Q = d + 1$ equivalent points V_α, $(\alpha = 1, \ldots, Q \equiv 0)$ on a $(d-1)$-dimensional sphere S^{d-1}, which is constructed recursively from the zero-dimensional sphere (two points). $G = (\frac{2\pi}{Q}, \ldots, \frac{2\pi}{Q})$ is a center-of-mass of all vertices $V_0, \ldots, V_d$, which is a center of the sphere on which all vertices lie. See Ref. [100] for the details. Its low-dimensional examples are two vertices of a line [S^0, $d = 1$, Fig. 3(a)], three vertices of a triangle on a circle [S^1, $d = 2$, Fig. 3(b)], and four vertices of a tetrahedron on a sphere in three dimension (S^2, $d = 3$). T^d is spanned by the d vectors, $\vec{e}_j = \overrightarrow{V_0 V_\alpha}/2\pi$, $\alpha = 1, \ldots, d$ as $\Theta = \theta_1\vec{e}_1 + \cdots + \theta_d\vec{e}_d$, which is abbreviated as $\Theta = (\theta_1, \ldots, \theta_d)$.

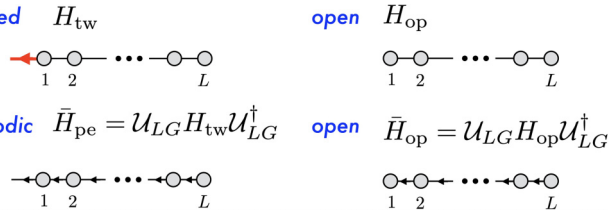


FIG. 1. Four Hamiltonians with different boundary conditions.

It defines a synthetic Brillouin zone for the twisted Hamiltonian

$$H_{\rm tw}(\Theta) = H_{\rm tw}(\Theta + 2\pi\vec{e}_\alpha), \quad (\alpha = 1, \ldots, d), \tag{54}$$

that is, all vertices ${}^\forall V_\alpha$ are identified.

Note that the Q paths

$$\ell_{V_\alpha G V_{\alpha+1}} = -\overrightarrow{GV_\alpha} + \overrightarrow{GV_{\alpha+1}}, \quad \alpha = 1, \ldots, Q, \tag{55}$$

in Fig. 2, forms loops for $H_{\rm tw}(\Theta)$. Note that $\bar{H}_{\rm pe}(\Theta)$ does not satisfies this periodicity.

By taking any path $\ell = \{\Theta | (\Theta = \Theta(\theta), \theta \in [0, 2\pi]\}$ parameterized by θ in the parameter space, let us define a current

$$\mathcal{J}^\ell \equiv \hbar^{-1}\partial_\theta \bar{H} = \hbar^{-1}\partial_\theta H(\{\bar{J}_{j,\alpha\beta}\}). \tag{56}$$

Equations (47) and (48) imply respectively

$$\mathcal{J}_{\rm op} = \hbar^{-1}\partial_\theta \bar{H}_{\rm op} = -i\hbar^{-1}\sum_\alpha \partial_\theta\varphi_\alpha[\mathcal{P}_\alpha, \bar{H}_{\rm op}], \tag{57}$$

$$\mathcal{J}_{\rm pe} = \hbar^{-1}\partial_\theta \bar{H}_{\rm pe} = -i\hbar^{-1}\sum_\alpha \partial_\theta\varphi_\alpha[\mathcal{P}_\alpha, \bar{H}_{\rm pe}] + \hbar^{-1}\mathcal{U}_{LG}\partial_\theta H_{\rm tw}\mathcal{U}_{LG}^\dagger. \tag{58}$$

Then the evaluation by the time-dependent state $|G_{\rm op}(t)\rangle$ for the open boundary condition that obeys the Schrödinger equation, $i\hbar\partial_t|G_{\rm op}(t)\rangle = \bar{H}_{\rm op}|G_{\rm op}(t)\rangle$, $(\langle G_{\rm op}|G_{\rm op}\rangle = 1)$ gives

$$j^G_{\rm op} \equiv \langle G_{\rm op}(t)|\mathcal{J}_{\rm op}|G_{\rm op}(t)\rangle = \sum_\alpha \partial_\theta\varphi_\alpha\partial_t P^G_{{\rm op},\alpha}, \tag{59}$$

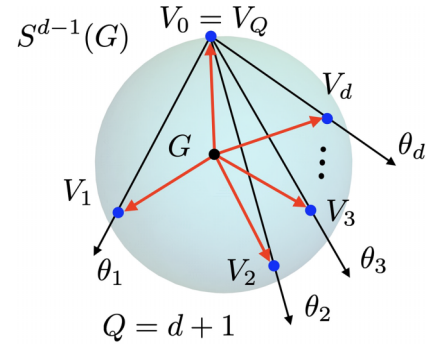


FIG. 2. d-dimensional parameter space of the twist and Q loops $\ell_{V_0GV_1}, \ell_{V_1GV_2}, \ldots, \ell_{V_dGV_0}$ on torus T^d.

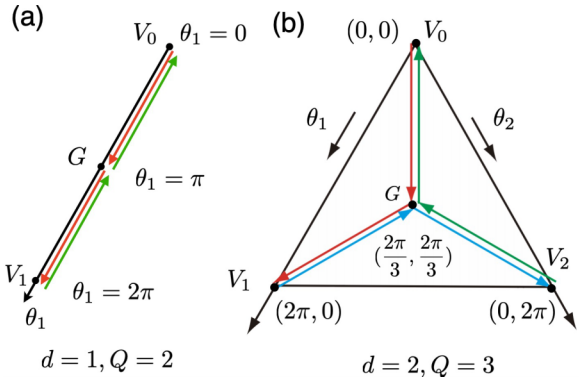


FIG. 3. Examples for the paths in the parameter spaces for (a) $d = 1, Q = 2$ and (b) $d = 2, Q = 3$ where G is a Z_Q symmetric point. The twist at G, $Q = 2$ corresponds to the antiperiodic boundary condition.

where

$$P^G_{\mathrm{op},\alpha}(t) = \langle G_{\mathrm{op}}(t)|\mathcal{P}_\alpha|G_{\mathrm{op}}(t)\rangle, \tag{60}$$

is a CoM of $|G_{\mathrm{op}}(t)\rangle$. Note that the similar discussion for the periodic boundary condition is not simple. Equation (58) has an extra term (H_{tw} depends on φ_α) and also the operator $\mathcal{P}_\alpha$ itself does not have a physical meaning since the origin of the reduced coordinate x_j is arbitrary for the periodic boundary condition.

B. Adiabatic current

Let us start by a general discussion of the time-dependent evolution of a state by the adiabatic approximation assuming the initial state $|G(0)\rangle$ is a gapped unique ground state $|g(0)\rangle$ of the snapshot Hamiltonian $\bar{H}(t)$,

$$|G(0)\rangle = |g(0)\rangle,$$
$$\bar{H}(t)|n(t)\rangle = |n(t)\rangle E_n(t),$$

where $n = 0, 1, 2, \ldots,$ and $|g\rangle = |n\rangle$, $n = 0$, $\langle n|n'\rangle = \delta_{nn'}$. We further assume the snapshot Hamiltonian is always gapped $E_n(t) > E_g(t)$, $(n \neq g)$. When the time modulation of the Hamiltonian is slow enough, the adiabatic approximation is justified (see Appendix C) as

$$|G\rangle \approx C\left[|g\rangle + i\hbar\sum_{n\neq g}\frac{|n\rangle\langle n|\partial_t g\rangle}{E_n - E_g}\right], \tag{61}$$

where C is a time-dependent phase factor ($|C| = 1$). In the present discussion, $\bar{H}$ is $\bar{H}_{\mathrm{op}}$ or $\bar{H}_{\mathrm{pe}}$.

Under this adiabatic approximation, the observed current is given by [40] (see also Appendix C)

$$j^G = \langle G|\mathcal{J}|G\rangle \approx j, \tag{62}$$

$$j = \hbar^{-1}\langle g|\partial_\theta \bar{H}|g\rangle - iB, \tag{63}$$

$$B = \partial_\theta A_t - \partial_t A_\theta, \tag{64}$$

where A_μ $(\mu = \theta, t)$ is the Berry connection

$$A_\mu = \langle g|\partial_\mu g\rangle. \tag{65}$$

Here we assume the ground state $|g\rangle$ as a function of θ and t by $\bar{H}(\theta,t)|g(\theta,t)\rangle = |g(\theta,t)\rangle E_g(\theta,t)$. It is covariant for the phase transformation, $|g'\rangle = |g\rangle e^{i\chi}$, $[\chi = \chi(\theta,t)]$ that induces the gauge transformation for the Berry connection

$$A'_\mu = A_\mu + \partial_\mu \chi, \tag{66}$$

although the field strength B and the current j are gauge invariant.

Let us here define the average current along the path ℓ such as $\ell_{V_\alpha G V_{\alpha+1}}$ connecting two equivalent points in the parameter space (parameterized by $\theta \in [0, 2\pi]$)

$$\bar{j}^\ell(t) = \int_0^{2\pi}\frac{d\theta}{2\pi}\, j(\theta,t). \tag{67}$$

Due to the Feynman's theorem, the first term Eq. (63) is written as $\langle g|\partial_\theta \bar{H}|g\rangle = \partial_\theta E(\theta)$, $E_\theta = \langle g|\bar{H}|g\rangle$. It vanishes for the open boundary condition, since E_{op} is θ independent. As for the periodic boundary condition, again $\int_0^{2\pi} d\theta\, \partial_\theta E_{\mathrm{pe}}(\theta) = 0$ since $E_{\mathrm{pe}}(\theta) = E_{\mathrm{tw}}(\theta)$ is periodic for any closed path connecting the equivalent points in the period T. See Table IV and Fig. 1. It results in

$$\bar{j}^\ell(t) = -i\int_\ell \frac{d\theta}{2\pi}\, B. \tag{68}$$

Note that this is valid both for the open and the periodic boundary conditions. Since the current is carried by bulk, the effect of the boundaries is $\mathcal{O}(L^{-1})$. One can expect, in the infinite size limit (see Fig. 1),

$$\bar{j}^\ell_{\mathrm{pe}} = \bar{j}^\ell_{\mathrm{op}}, \quad (L \to \infty). \tag{69}$$

This is the bulk-edge correspondence for the adiabatic current.

C. Temporal gauge and discontinuity of CoM

Adiabatic pump is a periodic transfer of charge between the time period $[0, T]$ assuming that the periodicity of the Hamiltonian as $\bar{H}(t+T) = \bar{H}(t)$ under the adiabatic condition. Here $\bar{H}$ denotes $\bar{H}_{\mathrm{pe}}$ or $\bar{H}_{\mathrm{op}}$ respectively. As for the gauge fixing of the Berry connection, let us take a temporal gauge $A^{(t)}_t(\theta,t) = 0$ by

$$A^{(t)}_\theta(\theta,t) = -\int_0^t d\tau\, B(\theta,\tau). \tag{70}$$

This is apparently gauge invariant and one may check $B = \partial_\theta A^{(t)}_t - \partial_t A^{(t)}_\theta = -\partial_t A^{(t)}_\theta$. Note that $A^{(t)}_\theta$ is not periodic in time, $A^{(t)}(\theta, t+T) \neq A^{(t)}(\theta,t)$, although $\bar{H}$ is periodic in time (period T). It is further written by the Berry connection in a generic gauge as [50]

$$A^{(t)}_\theta(\theta,t) = -\int_0^t d\tau\,[\partial_\theta A_t(\theta,\tau) - \partial_\tau A_\theta(\theta,\tau)]$$
$$= A_\theta(\theta,t) - \partial_\theta\int_0^t d\tau\, A_t(\theta,\tau) - A_\theta(\theta,0). \tag{71}$$

The average current in the temporal gauge is written as

$$\bar{j}^\ell(t) = -i\int_\ell \frac{d\theta}{2\pi}B = i\partial_t \bar{A}^{(t)}_\theta(t), \tag{72}$$

$$\bar{A}^{(t),\ell}_\theta(t) = \int_\ell \frac{d\theta}{2\pi}A^{(t)}_\theta(\theta,t)$$
$$= \frac{-1}{2\pi}\int_\ell d\theta\int_0^t d\tau B(\theta,\tau). \tag{73}$$

This is valid both for open and periodic boundary condition.

As for the open boundary condition, the parameter dependence is only due to the large gauge transformation $\mathcal{U}_{LG}$ as $|g_{\rm op}\rangle = \mathcal{U}_{LG}|g_{0,\rm op}\rangle$ where $|g_{0,\rm op}\rangle$, is a ground state of $H_{\rm op}$, $(H_{\rm op}|g_{0,\rm op}\rangle = |g_{0,\rm op}\rangle E_g)$. We safely assume $|g_{0,\rm op}\rangle$ is θ independent, $\partial_\theta|g_{0,\rm op}\rangle = 0$. It implies

$$A_{t,\rm op} = \langle g_{\rm op}|\partial_t g_{\rm op}\rangle = \langle g_{0,\rm op}|\partial_t g_{0,\rm op}\rangle, \quad \partial_\theta A_{t,\rm op} = 0, \tag{74}$$

$$\begin{aligned} A_{\theta,\rm op} &= \langle g_{\rm op}|\partial_\theta g_{\rm op}\rangle = \langle g_{0,\rm op}|\mathcal{U}_{LG}^\dagger \partial_\theta \mathcal{U}_{LG}|g_{0,\rm op}\rangle \\ &= -i\sum_\alpha \partial_\theta\varphi_\alpha P_\alpha(t), \end{aligned} \tag{75}$$

where

$$P_\alpha(t) = \langle g_{0,\rm op}|\mathcal{P}_\alpha|g_{0,\rm op}\rangle = \sum_j (-1)^{j-1} x_j n_{j,\alpha}(t), \tag{76}$$

is a CoM of the α fermions without twists $[n_{j,\alpha}(t) = \langle g_{0,\rm op}(t)|\hat{n}_{j,\alpha}|g_{0,\rm op}(t)\rangle]$. It is also written by the $SU(Q)$ spins [104].

The Berry connection in the temporal gauge is written as

$$A^{(t)}_{\theta,\rm op}(\theta,t) = -i\sum_\alpha \partial_\theta\varphi_\alpha[P_\alpha(t) - P_\alpha(0)]. \tag{77}$$

The current averaged over the path ℓ is written as

$$\begin{aligned} \bar{j}^\ell &= \left[\int_\ell \frac{d\theta}{2\pi}\sum_\alpha \partial_\alpha\varphi_\alpha\right]\partial_t P_\alpha(t) \\ &= \partial_t \bar{P}^\ell(t), \end{aligned} \tag{78}$$

where

$$\bar{P}^\ell(t) = \sum_\alpha \frac{\Delta\varphi_\alpha}{2\pi} P_\alpha(t), \tag{79}$$

$$\Delta\varphi_\alpha = \varphi_\alpha(2\pi) - \varphi_\alpha(0). \tag{80}$$

Along the path $\ell_{V_0GV_1}$, $\theta_1 : 0 \to 2\pi$ and $\theta_\alpha : 0 \to 0$, $\alpha = 2,\dots,d$. It implies $\varphi_\alpha : 0 \to 2\pi$, $\Delta\varphi_\alpha = 2\pi$ for all $\alpha = 1,\dots,d$. Also $\varphi_0 = \varphi_Q = 0$. Therefore the averaged current $\bar{j}_{\rm op}^{\ell_{V_0GV_1}}$ is written as

$$\bar{j}_{\rm op}^{\ell_{V_0GV_1}} = \partial_t \bar{P}^{\ell_{V_0GV_1}}, \tag{81}$$

$$\bar{P}^{\ell_{V_0GV_1}} = -P_0, \tag{82}$$

where $P_0 \equiv -\sum_{\alpha=1}^d P_\alpha$. [Note that $P_0 = P_Q$, (L: odd) and $P_0 = P_Q + 1/2$, (L: even) due to the constraint Eq. (37).] As for $Q = 3$, $P_0 = -P_1 - P_2$ (see Fig. 3). Along the path $\ell_{V_1GV_2}$, $\theta_1 : 2\pi \to 0, \theta_2 : 0 \to 2\pi, \theta_\alpha : 0 \to 0$, $(\alpha = 3,\dots,d)$. It implies $\Delta\varphi_1 = -2\pi$, $\Delta\varphi_\alpha = 0$, $(\alpha = 2,\dots,d)$ and

$$\bar{j}_{\rm op}^{\ell_{V_1GV_2}} = \partial_t \bar{P}^{\ell_{V_1GV_2}}, \tag{83}$$

$$\bar{P}^{\ell_{V_1GV_2}} = -P_1. \tag{84}$$

Similarly along the path $\ell_{V_\alpha GV_{\alpha+1}}$, $(\alpha = 1,\dots,d-1)$, $\theta_\alpha : 2\pi \to 0$ and $\theta_{\alpha+1} : 0 \to 2\pi$. It implies $\Delta\varphi_\alpha = -2\pi$, $\Delta\varphi_\beta = 0$, $(\beta \neq \alpha)$. Then, in general,

$$\bar{j}_{\rm op}^{\ell_{V_\alpha GV_{\alpha+1}}} = \partial_t \bar{P}^{\ell_{V_\alpha GV_{\alpha+1}}}, \tag{85}$$

$$\bar{P}^{\ell_{V_\alpha GV_{\alpha+1}}} = -P_\alpha, \quad \alpha = 1,\dots,d-1,d. \tag{86}$$

Note that this is justified also for $\alpha = d$, since along the last path $\ell_{V_dGV_Q} = \ell_{V_dGV_0}$, $\theta_d : 2\pi \to 0$ and $\Delta\varphi_d = -2\pi$. Since the CoM, P_α is a physical observable of the snapshot ground state for the open boundary condition, it is periodic in time, $P_\alpha(t+T) = P_\alpha(t)$, as the Hamiltonian is periodic in time. An important observation is that $\bar{P}^\ell(t)$ is not continuous $(L \to \infty)$ and has discontinuities at $t = t_i$, $i = 1,2,\dots$ of the jumps in unit of $\pm\frac{1}{2}$ due to edge states (as shown later). Then the pumped charge $Q^\ell_{\rm op}$ in the cycle T due to the current $\bar{j}^\ell_{\rm op}$ is written as

$$\begin{aligned} Q^\ell_{\rm op} &= \int_0^T dt\ \bar{j}^\ell_{\rm op} = \sum_i \int_{t_i}^{t_{i+1}} dt\ \bar{j}^\ell_{\rm op} \\ &= \sum_i \int_{t_i+0}^{t_{i+1}-0} dt\ \partial_t \bar{P}^\ell(t) = \sum_i \bar{P}^\ell(t)\big|_{t_i+0}^{t_{i+1}-0} \\ &= -\sum_i \bar{P}^\ell(t)\big|_{t_i-0}^{t_i+0} = -\sum_i \Delta\bar{P}^\ell(t_i) = I^\ell. \end{aligned} \tag{87}$$

The discontinuities I^ℓ is defined by

$$-I^\ell = \sum_i \Delta\bar{P}^\ell(t_i), \tag{88}$$

$$\Delta\bar{P}^\ell(t_i) = \bar{P}^\ell(t)\big|_{t_i-0}^{t_i+0} = \pm\frac{1}{2}, \tag{89}$$

where $\bar{P}^\ell(t)$ is not continuous at $t = t_i$ $(i = 1,\dots)$ [105]. The sign is determined by the behavior of the edge state that causes the jump. Since the localization length (typical length scale) of the edge states is finite, it scales to zero in the rescaled coordinate x_j. It implies the contribution of the edge states localized near one of the boundaries is $\pm\frac{1}{2}$ (see Appendix G). Due to the conservation of the charge, the number of the discontinuities is even. It implies the sum of the discontinuities I is an integer. This is the quantization of the pumped charge.

The physical current $\bar{j}^\ell_{\rm op}$ is carried by the bulk even with the open boundary condition and is determined by the discontinuities due to the edge states by the back action based on the periodicity of $\bar{P}^\ell$ in time.

D. Bulk-edge correspondence

As for the periodic boundary condition, $\bar{A}^{(t),\ell}_\theta$ is smooth and the pumped charge averaged along the path ℓ is given by

$$\begin{aligned} Q^\ell_{\rm pe} &\equiv \int_0^T dt\ \bar{j}^\ell_{\rm pe} \\ &= \frac{1}{2\pi i}\int_0^T dt \int_\ell d\theta\, B_{\rm pe}(\theta,t), \end{aligned} \tag{90}$$

$$B_{\rm pe}(\theta,t) = \partial_\theta A_{t,\rm pe} - \partial_t A_{\theta,\rm pe}, \tag{91}$$

$$A_{\mu,\rm pe} = \langle g_{\rm pe}|\partial_\mu g_{\rm pe}\rangle, \tag{92}$$

where $\bar{H}_{\rm pe}|g_{\rm pe}\rangle = |g_{\rm pe}\rangle E_g$. As for the periodic boundary condition, the Berry connection $A_{\mu,\rm pe}$ and thus $B_{\rm pe}$ is also defined by the Hamiltonian $\bar{H}_{\rm pe}$, Eq. (48), that is not periodic/invariant by the shift $\Theta \to \Theta + 2\pi\vec{e}_\alpha$. See Table IV. It implies that the periodic Hamiltonian $\bar{H}_{\rm pe}$, Eq. (48), is not defined on the torus T^d. On the other hand, the ground state $|g_{0,\rm tw}\rangle$ of the twisted Hamiltonian $H_{\rm tw}$, Eq. (47), is periodic by the shift and well

defined on the torus T^d. Noting that $|g_{\rm pe}\rangle = \mathcal{U}_{LG}|g_{0,\rm tw}\rangle$ and $H_{\rm tw}|g_{0,\rm tw}\rangle = |g_{0,\rm tw}\rangle E_g^{\rm tw}$, one has

$$A_{\theta,\rm pe} = -i\sum_\alpha \partial_\theta \varphi_\alpha P_{\alpha,\rm tw} + A_{\theta,\rm tw}, \tag{93}$$

$$A_{t,\rm pe} = A_{t,\rm tw}, \tag{94}$$

where $P_{\alpha,\rm tw} = \langle g_{0,\rm tw}|\mathcal{P}_\alpha|g_{0,\rm tw}\rangle$.

It results in

$$B_{\rm pe} = B_{\rm tw} + i\sum_\alpha \partial_\theta\varphi_\alpha \partial_t P_{\alpha,\rm tw}. \tag{95}$$

Since $P_{\alpha,\rm tw}$ is smooth and periodic in time, the last term does not contribute to the total pumped charge. Then

$$Q^\ell_{\rm pe} = \frac{1}{2\pi i}\int_\ell d\theta \int_0^T dt\, B_{\rm tw}(\theta,t) \equiv C^\ell. \tag{96}$$

This integral is over a torus $T^2 = \{(\theta,t)|\theta \in [0,2\pi], t \in [0,T]\}$ without boundaries. It gives the Chern number C that is integer. Now due to Eq. (87), we have

$$Q^\ell_{\rm pe} = Q^\ell_{\rm tw} = Q^\ell_{\rm op} = C^\ell, \tag{97}$$

$$I^\ell = C^\ell. \tag{98}$$

This is the bulk-edge correspondence of the topological pump.

As for the canonical path $\ell_{V_\alpha G V_{\alpha+1}}$, $\alpha = 0,\dots d = Q-1$, it is given by

$$I^\alpha = C^\alpha = Q^\alpha, \quad \alpha = 0,\dots,Q-1, \tag{99}$$

where

$$I^\alpha \equiv I^{\ell_{V_\alpha G V_{\alpha+1}}} = \sum_i \Delta P_\alpha(t_i), \tag{100}$$

is a sum of the discontinuities of the α particle in the cycle and $C^\alpha \equiv C^{\ell_{V_\alpha G V_{\alpha+1}}}$ is the Chern number defined on a torus $T^2 = [0,T]\times V_\alpha G V_{\alpha+1}$. It implies the Chern number C^α measures the pumped charge of α fermion Q^α [95] due to the bulk-edge correspondence.

The twist is a gauge field and may not be easy to measure directly. It implies the Berry connection itself is not a physical observable. However, the Chern number as the topological invariant associated with the Berry connection is a pumped charge of the bulk and directly measurable by experiments. This is compensated by the discontinuities of the CoM of the edge states due to the bulk-edge correspondence. These are also physical observables but can not be accessed by a usual experimental setup since the adiabatic condition of the gapless edge states is never satisfied by a finite speed pump.

IV. Z_Q BERRY PHASE, SYMMETRY, AND GAP CLOSING

A. Z_Q quantization

In this section, let us discuss the Hamiltonian without symmetry breaking term ($\Delta = 0$). Using the Hamiltonian $H_{\rm tw}$, the Berry phase γ_ℓ is defined since the path ℓ forms a loop for $H_{\rm tw}$ as

$$i\gamma_\ell = \int_\ell d\theta\, A_\theta. \tag{101}$$

The Z_Q shift $\mathcal{U}_{Z_Q}$ as a shift in the parameter space as shown in Sec. II B induces a map of the Hamiltonians in the parameter space $\varphi_\alpha \to \varphi_{\alpha+1}$ [see Eq. (53)] and the canonical loop $\ell_{V_\alpha G V_{\alpha+1}}$,

$$\mathcal{U}_{Z_Q}\bar{H}_{\rm tw}(\boldsymbol{\theta})\mathcal{U}^\dagger_{Z_Q} = \bar{H}_{\rm tw}(\boldsymbol{\theta}'), \tag{102}$$

where two points $\boldsymbol{\theta}$ and $\boldsymbol{\theta}'$ in Θ are parameterized by the same θ as

$$\boldsymbol{\theta} \in \ell_{V_{\alpha-1}GV_\alpha}(\theta),$$
$$\boldsymbol{\theta}' \in \ell_{V_\alpha G V_{\alpha+1}}(\theta).$$

It implies that we may take

$$|g_{0,\rm tw}(\boldsymbol{\theta}')\rangle = \mathcal{U}_{Z_Q}|g_{0,\rm tw}(\boldsymbol{\theta})\rangle, \tag{103}$$

and

$$\gamma_{\ell_{V_{\alpha-1}GV_\alpha}} = \gamma_{\ell_{V_\alpha G V_{\alpha+1}}} \equiv \gamma_Q,\ \alpha = 1,\cdots Q, \tag{104}$$

since $\mathcal{U}_{Z_Q}$ does not depend on the parameter and thus $\langle g_{0,\rm tw}(\boldsymbol{\theta}')|\partial_\theta g_{0,\rm tw}(\boldsymbol{\theta}')\rangle = \langle g_{0,\rm tw}(\boldsymbol{\theta})|\partial_\theta g_{0,\rm tw}(\boldsymbol{\theta})\rangle$. Then using the fact,

$$\sum_{\alpha=1}^{Q} \ell_{V_\alpha G V_{\alpha+1}} = 0, \tag{105}$$

it results $Q\gamma_Q = 0$, (mod 2π). It implies Z_Q quantization

$$\gamma_Q = \frac{2\pi n}{Q},\quad n \in \mathbb{Z}. \tag{106}$$

This Z_Q Berry phase characterizes a symmetry protected topological phase [63,66,100,106]. It is a generalized Z_2 Berry phase that characterizes a singlet pair or a covalent bond [63,107] on the twisted link. Similarly γ_Q characterizes the $SU(Q)$ dimer phase. The dimer limit is characterized by this Z_Q Berry phase and the symmetry protection and adiabatic continuity guarantee the quantization. Unless the gap closes by the deformation to the dimer limit $J_i = 0$ (i:odd), the Berry phase is given by (see Fig. 4 and Appendix E)

$$\gamma_Q = +\frac{2\pi}{Q},\quad \text{mod } 2\pi. \tag{107}$$

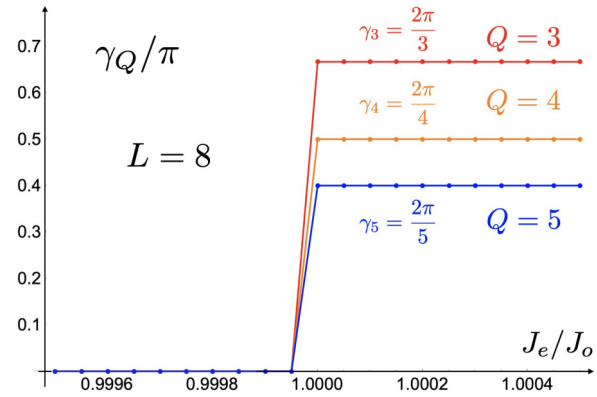


FIG. 4. Z_Q Berry phase γ_Q for $Q = 3, 4$, and 5 as a function of J_e/J_o ($L = 8$). We have confirmed that the results are reproduced by the symmetry indicators Eq. (118).

It is topologically stable unless the gap closes against finite coupling $J_p < J_e < 0$ and $J_e < J_o < 0$. It also gives

$$\gamma_{V_\alpha G V_{\beta'}} = +\frac{2\pi}{Q}(\alpha - \beta), \quad \mod 2\pi. \tag{108}$$

B. Symmetry indicators

Let us first note that the Berry phase needs to be evaluated by a single gauge fixing [63,106,108,109] (see Appendix D). Here we assume that $|g_\phi\rangle$ is gauge fixed over the loop $\ell_{V_{\alpha-1}GV_\alpha} = \ell_{V_{\alpha-1}G} - \ell_{V_\alpha G}$ by a single gauge fixing by $|\phi\rangle$ as

$$|g_\phi\rangle = P|\phi\rangle/\sqrt{N_\phi}, \tag{109}$$

where $P = |g_{0,\mathrm{tw}}\rangle\langle g_{0,\mathrm{tw}}|$ and $N_\phi = |\langle\phi|g_{0,\mathrm{tw}}\rangle|^2 \neq 0$ [110].

Noting that

$$H_{\mathrm{tw}}(\boldsymbol{\theta}') = \mathcal{U}_{Z_Q} H_{\mathrm{tw}}(\boldsymbol{\theta}) \mathcal{U}_{Z_Q}^\dagger, \tag{110}$$

where $\boldsymbol{\theta} \in \ell_{V_{\alpha-1}G}(\theta)$, and $\boldsymbol{\theta}' \in \ell_{V_\alpha G}(\theta)$, one has

$$|g_\phi(\boldsymbol{\theta}')\rangle = \mathcal{U}_{Z_Q}|g_\phi(\boldsymbol{\theta})\rangle e^{-i\Omega(\boldsymbol{\theta})}. \tag{111}$$

This extra phase factor is due to the fact that the phase convention of the state $|g_\phi(\boldsymbol{\theta}')\rangle$ by $|\phi\rangle$ at $\boldsymbol{\theta}'$ is, in general, different from that of $\mathcal{U}_{Z_Q}|g_\phi(\boldsymbol{\theta})\rangle$.

Since $\mathcal{U}_{Z_Q}$ is independent of the parameter, the Berry phase is written as

$$\begin{aligned}\gamma_Q = \gamma_{\ell_{V_{\alpha-1}GV_\alpha}} &= -i\int_{\ell_{V_{\alpha-1}GV_\alpha}} d\theta\, \langle g_\phi|\partial_\theta g_\phi\rangle \\ &= -i\int_{\ell_{V_{\alpha-1}G}} d\theta\, \langle g_\phi|\partial_\theta g_\phi\rangle + i\int_{\ell_{V_\alpha G}} d\theta\, \langle g_\phi|\partial_\theta g_\phi\rangle \\ &= +\int_{\ell_{V_{\alpha-1}G}} d\theta\, \partial_\theta \Omega = \Omega(G) - \Omega(O),\end{aligned} \tag{112}$$

where all vertices V_α are identified to the origin O in modulo 2π. Supplementing $\theta_Q = -\sum_{\alpha=1}^d \theta_\alpha$, at the vertices O and G, the parameters are

$$\boldsymbol{\theta} = O : (\theta_1, \ldots, \theta_Q) = (0, \ldots, 0), \quad \mod 2\pi, \tag{113}$$

$$\boldsymbol{\theta} = G : (\theta_1, \ldots, \theta_Q) = \left(\frac{2\pi}{Q}, \ldots, \frac{2\pi}{Q}\right), \quad \mod 2\pi. \tag{114}$$

It implies the Hamiltonian is invariant by the shift of the fermions by $\mathcal{U}_{Z_Q}$ at O and G as

$$[H_{\mathrm{tw}}(\boldsymbol{\theta}), \mathcal{U}_{Z_Q}] = 0, \quad \boldsymbol{\theta} = O, G. \tag{115}$$

Therefore $\mathcal{U}_{Z_Q}$ is a symmetry of the Hamiltonian and $e^{i\Omega}$ is an eigenvalue of the symmetry operation (symmetry indicator) as

$$\mathcal{U}_{Z_Q}|g_\phi(O)\rangle = |g_\phi(O)\rangle e^{+i\Omega(O)}, \tag{116}$$

$$\mathcal{U}_{Z_Q}|g_\phi(G)\rangle = |g_\phi(G)\rangle e^{+i\Omega(G)}. \tag{117}$$

Since $\mathcal{U}_{Z_Q}$ is trivial at O and $\Omega(O) = 0$, it results in

$$\gamma_Q = \Omega(G) = \mathrm{Arg}\,\langle g_\phi(G)|\mathcal{U}_{Z_Q}|g_\phi(G)\rangle. \tag{118}$$

Physical meaning of these quantities is clear by the adiabatic deformation to the dimer limit by the Z_Q Berry phase. It also implies the gap closing at G associated with a topological transition due to the discrete change of the Z_Q Berry phase.

It is also directly observed by the discretized formula of the Berry phase (discretizing the path $V_0 G$ into M segments $\theta_m = \frac{2\pi}{Q}\frac{m}{M}$, $(m = 0, \ldots, M)$ as

$$\begin{aligned}\gamma_Q = \lim_{M\to\infty} \mathrm{Arg}\,(&\langle\theta_0|\theta_1\rangle\langle\theta_1|\theta_2\rangle\cdots\langle\theta_{M-1}|\theta_M\rangle \\ &\times\langle\theta_M|\theta'_{M-1}\rangle\cdots\langle\theta'_2|\theta'_1\rangle\langle\theta'_1|\theta_0\rangle),\end{aligned}$$

where $|\theta'_m\rangle = \mathcal{U}_{Z_Q}|\theta_m\rangle$, $|\theta_m\rangle = |g_\phi(\theta_m)\rangle$. Since $\mathcal{U}_{Z_Q}|\theta_M\rangle = |\theta_M\rangle e^{i\Omega(G)}$ and $\mathcal{U}_{Z_Q}|\theta_0\rangle = |\theta_0\rangle e^{i\Omega(O)}$, it is written as

$$\begin{aligned}\gamma_Q &= \lim_{M\to\infty} \mathrm{Arg}\,(\langle\theta_0|\theta_1\rangle\langle\theta_1|\theta_2\rangle\cdots\langle\theta_{M-1}|\theta_M\rangle \\ &\quad\times\langle\theta_M|\mathcal{U}_{Z_Q}|\theta_{M-1}\rangle\cdots\langle\theta_2|\theta_1\rangle\langle\theta_1|\mathcal{U}_{Z_Q}^\dagger|\theta_0\rangle) \\ &= \Omega(G) - \Omega(O),\end{aligned}$$

where $\mathcal{U}_{Z_Q}^\dagger|\theta_0\rangle = |\theta_0\rangle e^{-i\Omega(O)}$ and $\langle\theta_M|\mathcal{U}_{Z_Q} = e^{i\Omega(G)}\langle\theta_M|$ due to Eqs. (116) and (117).

C. Modified Lieb-Schultz-Mattis (LSM) argument

As is clear, the system of the dimer limit is gapped. This gap is stable for inclusion of finite coupling between the dimers, at least, for a finite-size system. One may naturally expect this gap converges to some finite values by taking an infinite size limit $L \to \infty$ assuming the ground state is adiabatically connected to a set of disconnected dimers. As for a uniform system, existence of the gap is unclear and the problem has a long history of studies. Some of the recent studies are topological, especially in relation with the Haldane conjecture for the $S = 1$ Heisenberg model [111]. Since $H^{(2)}$ for $Q = 2$ is equivalent to the standard $S = 1/2$ Heisenberg model, $H^{(1)}$ (Appendix B), well-known Lieb-Schultz-Mattis (LSM) theorem [112,113] is applied and the energy gap of the finite system with the periodic boundary condition vanishes when $L \to \infty$. This is consistent with the existence of the gapless excitation as the des Cloizeaux and Pearson mode of the $Q = 2$ case [114]. Note that the LSM theorem also allows existence of the finite-size gap between the states, which become degenerate in the thermodynamic limit associated with the symmetry breaking. On the other hand, for the $Q = 3$ case, a series of studies [115–117] has clarified that the uniform system has doubly degenerate dimerized ground states in the $L \to \infty$ limit. The case, $Q > 3$, is also discussed by Affleck suggesting a similar conclusion (double degeneracy due to dimerization) [118]. In this subsection, we give a topological argument for the gap closing for even $Q \geqslant 2$ of the finite-size system.

The gap of the finite system under the twist is strongly constrained by considering the Berry phase, which works as a topological order parameter responding to the local twist as an external perturbation [60,63,100,119–121]. In Ref. [122], the standard $S = 1/2$ Heisenberg model, $H^{(1)}$ ($Q = 2$) was considered. If the gap remains open for all values of the twist, one can prove that the Berry phase pattern, associated with the local twist at the link, needs to be alternating in this $S = 1/2$ case. This clearly contradicts the uniformity of the system. It results in that the Berry phase can not be defined, that is, the gap closing of the system at some twist [122]. The argument

can be extended to the present system $Q > 2$ as shown here. The claim is that, as for a finite system of even $Q \geqslant 2$, the energy gap between the ground state and the next one under the twisted boundary condition vanishes at some twist.

Up to this point, we have discussed Berry phases associated with the twist at the boundary link L and 1. Let us write it as γ^L_ℓ. In a similar way, one may also define the Berry phase γ^{L-1} associated with the twist at $L-1$ and L. Let us write the Hamiltonians with the twists as

$$H^L_{\rm tw} = J_e \sum_{\alpha,\beta} e^{-i(\varphi_\alpha-\varphi_\beta)} c^\dagger_{L,\alpha} c^\dagger_{1,\alpha} c_{1,\beta} c_{L,\beta} + J_o \sum_{\alpha,\beta} c^\dagger_{L-1,\alpha} c^\dagger_{L,\alpha} c_{L,\beta} c_{L-1,\beta} + \cdots, \tag{119}$$

$$H^{L-1}_{\rm tw} = J_e \sum_{\alpha,\beta} c^\dagger_{L,\alpha} c^\dagger_{1,\alpha} c_{1,\beta} c_{L,\beta} + J_o \sum_{\alpha,\beta} e^{+i(\varphi_\alpha-\varphi_\beta)} c^\dagger_{L-1,\alpha} c^\dagger_{L,\alpha} c_{L,\beta} c_{L-1,\beta} + \cdots, \tag{120}$$

where $\cdots$ does not include $c_{L,\alpha}$. *Note that the sign of the twist is reversed.* They are related with each other by the the gauge transformation $\mathcal{U}_L = e^{-i\sum_\alpha \varphi_\alpha \hat{n}_{L,\alpha}}$, $\mathcal{U}^\dagger_L c_{L,\alpha} \mathcal{U}_L = e^{+i\varphi_\alpha \hat{n}_{L,\alpha}} c_{L,\alpha}$ (see also Appendix E) as

$$H^L_{\rm tw} = \mathcal{U}_L H^{L-1}_{\rm tw} \mathcal{U}^\dagger_L. \tag{121}$$

The Berry phases are defined by $|g^L\rangle$ and $|g^{L-1}\rangle$, which are the ground states of $H^L_{\rm tw}$ and $H^{L-1}_{\rm tw}$ respectively as

$$i\gamma^L = \int_\ell d\theta A^L, \quad A^L = \langle g^L|\partial_\theta g^L\rangle, \tag{122}$$

$$i\gamma^{L-1} = \int_\ell d\theta A^{L-1}, \quad A^{L-1} = \langle g^{L-1}|\partial_\theta g^{L-1}\rangle, \tag{123}$$

where $H^L|g^L\rangle = |g^L\rangle E$ and $H^{L-1}|g^{L-1}\rangle = |g^{L-1}\rangle E$. Noting that $|g^L\rangle = \mathcal{U}_L|g^{L-1}\rangle$ [123], it induces

$$A^L = A^{L-1} - i\sum_\alpha (\partial_\theta \varphi_\alpha)\langle g^{L-1}|\hat{n}_{L,\alpha}|g^{L-1}\rangle. \tag{124}$$

Generically, with the twist, $SU(Q)$ symmetry is (slightly) broken even without explicit symmetry breaking term H_B, that is, the fermions with different colors are not equivalent and $\langle g^{L-1}|\hat{n}_{L,\alpha}|g^{L-1}\rangle \neq \frac{1}{Q}$. However, this symmetry breaking effect due to the twist is not localized at the twisted link. The large gauge transformation Eqs. (30) and (49), $\mathcal{U}_{LG}$, maps the periodic system with $\mathcal{O}(1/L)$ twist to the system with twisted boundary condition preserving the local charge density remains unchanged because ($[\mathcal{U}_{LG}, \hat{n}_{j,\alpha}] = 0$). Then $\langle g^{L-1}|\hat{n}_{j,\alpha}|g^{L-1}\rangle$ is j independent both for the periodic/twisted system. It implies the effects are of the order of L^{-1} as $\langle g^{L-1}|\hat{n}_{L,\alpha}|g^{L-1}\rangle = \frac{1}{Q} + \mathcal{O}(L^{-1})$. Then integrating Eq. (124) over the loop $\ell_{V_\alpha G V_{\alpha+1}}$, we have for a sufficiently large system

$$\gamma^L_{\ell_{V_\alpha G V_{\alpha+1}}} = \gamma^{L-1}_{\ell_{V_\alpha G V_{\alpha+1}}} + \frac{2\pi}{Q}, \tag{125}$$

since $\Delta\varphi_\beta = -2\pi(\beta=\alpha), 0(\beta\neq\alpha)$. Note that the possible $\mathcal{O}(L^{-1})$ extra term vanishes after the integration to be consistent with the Z_Q quantization of the Berry phases. This constraint needs to be satisfied for any systems even with site-dependent J_i's. The two dimer limits, $\gamma^L_{\ell_{V_\alpha G V_{\alpha+1}}} = 0$, $\gamma^{L-1}_{\ell_{V_\alpha G V_{\alpha+1}}} = -\frac{2\pi}{Q}$ and $\gamma^L_{\ell_{V_\alpha G V_{\alpha+1}}} = +\frac{2\pi}{Q}$, $\gamma^{L-1}_{\ell_{V_\alpha G V_{\alpha+1}}} = 0$, are consistent with Eq. (125).

If the system is uniform, $J_o = J_e$, the antitranslation invariance of the system with the twist, $H^L_{\rm tw} = \mathcal{A}_\mathcal{T} H^{L-1}_{\rm tw} \mathcal{A}^{-1}_\mathcal{T}$, implies

$$\gamma^L_{\ell_{V_\alpha G V_{\alpha+1}}} = -\gamma^{L-1}_{\ell_{V_\alpha G V_{\alpha+1}}}, \quad \mod 2\pi. \tag{126}$$

By Eqs. (125) and (126), we have constraints for the Berry phase for a uniform system as

$$\gamma^L_{\ell_{V_\alpha G V_{\alpha+1}}} = -\gamma^{L-1}_{\ell_{V_\alpha G V_{\alpha+1}}} = +\frac{\pi}{Q}, \quad \mod 2\pi \tag{127}$$

or

$$\gamma^L_{\ell_{V_\alpha G V_{\alpha+1}}} = \pi\frac{Q+1}{Q}, \quad \gamma^{L-1}_{\ell_{V_\alpha G V_{\alpha+1}}} = \pi\frac{Q-1}{Q}, \quad \mod 2\pi. \tag{128}$$

As for the even Q, these constraints, Eqs. (127) and (128), contradict the Z_Q quantization of the Berry phases $\gamma = 2\pi\frac{n}{Q}$, $n \in \mathbb{Z}$. This contradiction implies that the Berry phase can not be well defined. It is only possible when the gap of the (finite size) system closes. A level crossing between the ground state and the next one occurs at some twist parameter. Assuming the degenerate dimer states for the infinite size system, the gap between the linear combinations of the dimer states of the finite size system closes at the twisted parameters. As for the odd Q, the second case, Eq. (128), is compatible with the Z_Q quantization ($Q \pm 1$ is even), although these quantized values are different from that of the dimer limit [124]. In principle, it allows a unique gapped ground state of the uniform system for any value of the twist. Although it does not occur in the present numerical calculations shown in Fig. 4, inclusion of long range couplings and additional terms, which respect Z_Q symmetry may realize such a ground state.

V. EMERGENT $Z_Q \times Z_Q$ SYMMETRY AND EXPLICIT CHERN NUMBERS

A. Open system: Edge states and low-energy spectrum

Although most of the discussion in the paper is for even L systems, let us consider, in this section, even/odd L systems separately, especially near the dimer limits $|J_o| \ll |J_e|$ and $|J_e| \ll |J_o|$.

L: odd. When the system size L is odd, the ground states are given for each dimer limits $J_o = 0$ and $J_e = 0$ as (see Fig. 5)

$$\left|g^{L:\rm odd}_{1,eo}, \alpha\right\rangle = c^\dagger_{1,\alpha} \prod_{j=1}^{(L-1)/2} \psi^\dagger_{2j+1,2j}|0\rangle, \quad (J_o = 0), \tag{129}$$

$$\left|g^{L:\rm odd}_{L,eo}, \alpha\right\rangle = c^\dagger_{L,\alpha} \prod_{j=1}^{(L-1)/2} \psi^\dagger_{2j-1,2j}|0\rangle, \quad (J_e = 0), \tag{130}$$

where $\bar{N}_\beta = 1$ for $\beta = \alpha$ and 0 otherwise. It implies Q-fold degeneracy of the ground states. Their charge distributions are

$$\left\langle g^{L:\rm odd}_{1,eo}, \alpha\middle|n_{j,\beta}\middle|g^{L:\rm odd}_{1,eo}, \alpha\right\rangle = \begin{cases} \frac{1}{Q} & j \neq 1 \\ 0 & j = 1, \beta \neq \alpha, \\ 1 & j = 1, \beta = \alpha \end{cases} \tag{131}$$

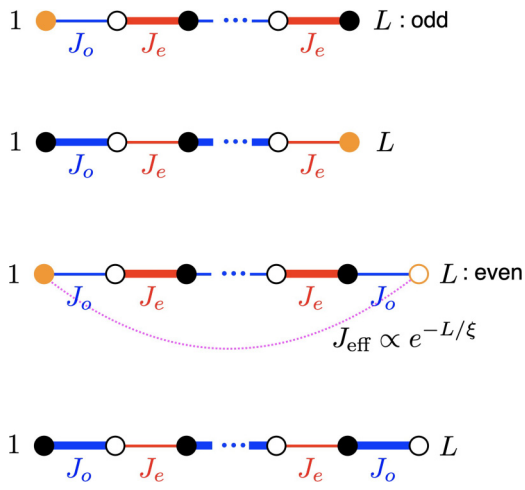


FIG. 5. Dimer configurations of open systems for even/odd sites.

$$\left\langle g^{L:\mathrm{odd}}_{L,eo}, \alpha \middle| n_{j,\beta} \middle| g^{L:\mathrm{odd}}_{L,eo}, \alpha \right\rangle = \begin{cases} \frac{1}{Q} & j \neq L \\ 0 & j = 1, \beta \neq \alpha, \\ 1 & j = 1, \beta = \alpha \end{cases} \tag{132}$$

where $|g^{L:\mathrm{odd}}_{1,eo}, \alpha\rangle$ is a product of the bulk and completely localized state at $j = 1$ with the color α. Similarly $|g^{L:\mathrm{odd}}_{L,eo}, \alpha\rangle$ is a product of the bulk and completely localized state at $j = L$ with the color α. This degeneracy is stable for inclusion of a finite coupling J_o and J_e since finite matrix elements with different quantum numbers $\bar{N}_\alpha$'s are prohibited due to the symmetry. It implies the charge distributions are modified continuously for a finite coupling. The numerical results for $L = 9$ systems with dimerization $\bar{N}_\alpha : (1, 0, 0, 0, 0)$ are obtained by the exact diagonalization and shown in Fig. 6. They are consistent with the present picture. The ground state is given by the gapped bulk and boundary states (edge states at both ends) localized near the boundaries.

L: even. As for the system with L: even, the ground states of the dimer limit are again given by

$$\left| g^{L:\mathrm{even}}_{\mathrm{op},eo}(\alpha, \beta) \right\rangle = c^\dagger_{1,\alpha} c^\dagger_{L,\beta} \prod_{j=1}^{L/2-1} \psi^\dagger_{2j+1,2j} |0\rangle, \quad (J_o = 0) \tag{133}$$

$$\left| g^{L:\mathrm{even}}_{\mathrm{op},oe} \right\rangle = \prod_{j=1}^{L/2} \psi^\dagger_{2j-1,2j} |0\rangle, \quad (J_e = 0) \tag{134}$$

where $\bar{N}_\gamma = 0, {}^\forall\gamma$ for both cases. It implies Q^2-fold degeneracy for $J_o = 0$ and gapped unique ground state for $J_e = 0$. If $|J_o| > |J_e|$, the unique gapped ground state $|g^{L:\mathrm{even}}_{eo}\rangle$ is stable for inclusion of the finite coupling J_e. However, as for $|J_o| < |J_e|$, in contrast to the L: odd case, the Q^2-fold degeneracy of the ground states is unstable for the finite size systems. The degeneracy is lifted for the finite coupling due to the residual interaction between the edge states at both ends. As for a chain of the finite length, we expect an effective coupling between the two boundary states at both ends. It is a generalization of the Kennedy's discussion [125–128]. This effective coupling $J_{\rm eff}$ is expected to behave as $e^{-L/\xi}$ for L where ξ is a correlation length between the edge states, which

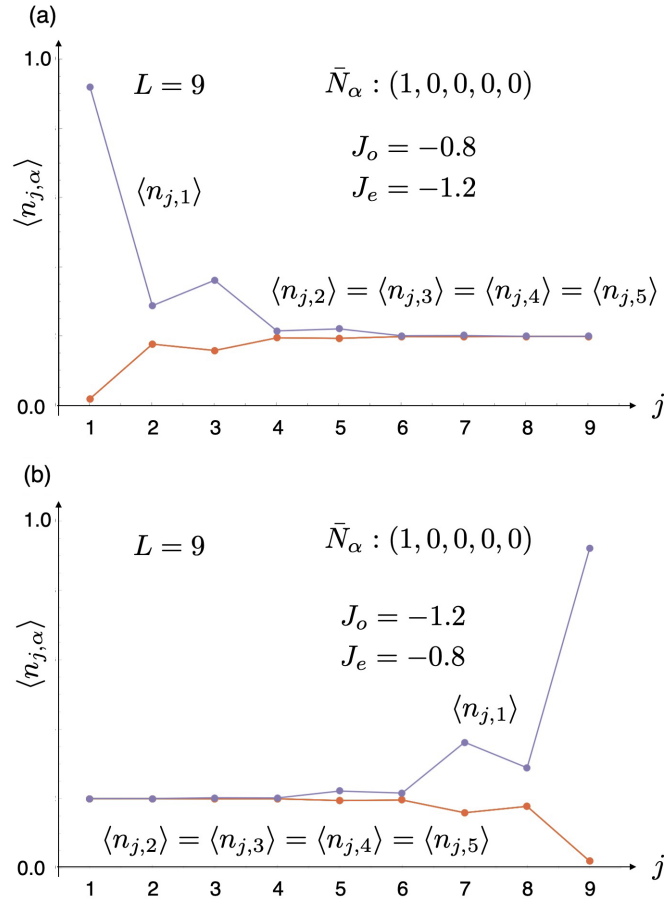


FIG. 6. Charge distribution $\langle n_{j,\alpha} \rangle$ of the unique ground state of the odd L system (L=9) with Q=5, $\bar{N}_\alpha : (1, 0, 0, 0, 0)$ (a) J_o= − 0.8, $J_e = -1.2$ and (b) $J_o = -1.2$, $J_e = -0.8$.

can be proportional to the inverse of the bulk energy gap. This is confirmed numerically for $Q = 3$ and 4 in Figs. 7 and 8.

That is, we expect an exponentially small coupling between the both ends, which operates for the Q^2-fold low-energy multiplet of the edge states. It is described by the effective Hamiltonian of the effective boundary fermions (assuming that they live at $j = 1$ and L) $c_{1,\alpha}$ and $c_{L,\alpha}$ ($\alpha = 1, \ldots, N$) as

$$h_{\rm eff} = J_{\rm eff} c^\dagger_{1,\alpha} c_{1,\beta} c^\dagger_{L,\alpha} c_{L,\beta} = Q J_{\rm eff} \psi^\dagger_{1,L} \psi_{1,L}. \tag{135}$$

The energy spectrum of the low-energy multiplet is given by the decomposition of the tensor product of $SU(Q)$ representation as $Q \otimes \bar{Q} = 1 \oplus (Q^2 - 1)$ as a generalization of singlet-triplet decomposition for the Kennedy's case. The unique ground state among the multiple is a singlet approximately given by

$$|S_{1,L}\rangle = \psi^\dagger_{1,L} \otimes \prod_{j}^{L/2-1} \psi^\dagger_{2j+1,2j} |0\rangle, \tag{136}$$

with its energy $QJ_{\rm eff}$. This is a generalization of the Kenndey's singlet and triplet for the $S = 1$ Haldane chain [125–128]. Note that the state $|S_{1,L}\rangle$ is interpreted as a tensor product of a gapped bulk and edge states. Due to the uniqueness and the $SU(Q)$ invariance, it implies that the one-point function is

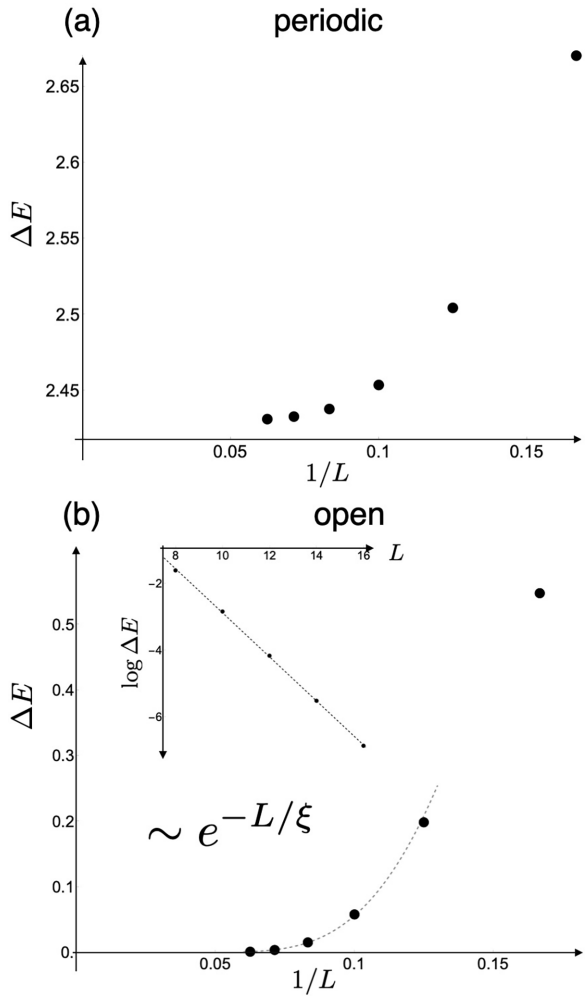


FIG. 7. Energy gap of the $SU(3)$ symmetric Hamiltonian $H^{(2)}$, (a) periodic boundary condition and (b) open boundary condition. The system sizes are $L = 6, 8, 10, 12, 14, 16$, and $J_j = -0.8(j$: odd) and $J_j = -1.2(j$: even). The data of the open boundary condition for $L \geqslant 10$ are fitted by the localization length $\xi = 1.52$.

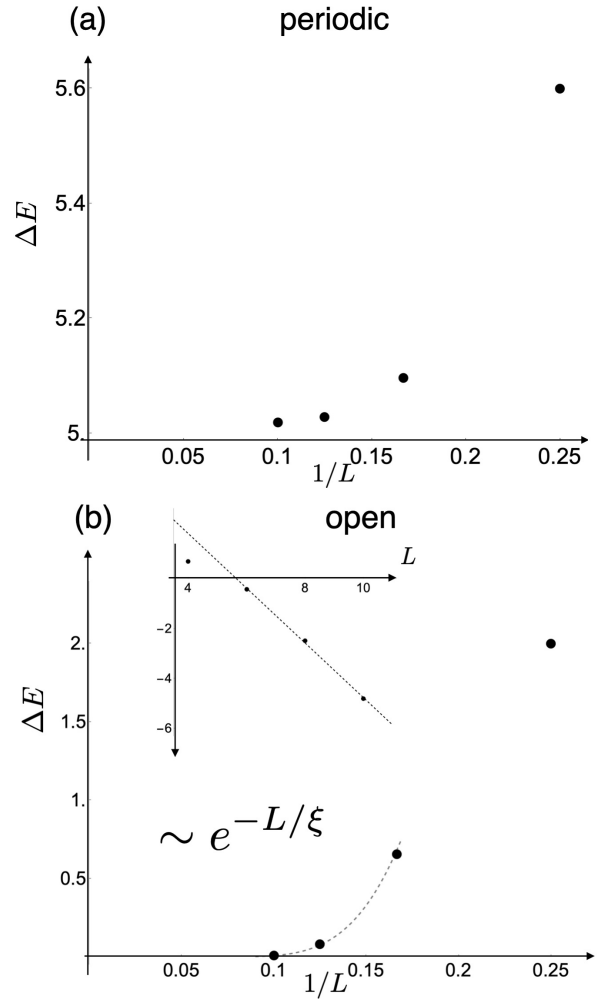


FIG. 8. Energy gap of the $SU(5)$ symmetric Hamiltonian $H^{(2)}$, (a) periodic boundary condition and (b) open boundary condition. The system sizes are $L = 4, 6, 8, 10$, and $J_j = -0.8(j$: odd) and $J_j = -1.2(j$: even). The data of the open boundary condition for $L \geqslant 6$ are fitted by the localization length $\xi = 0.91$.

constant as $\langle S_{1,L}|n_{j\alpha}|S_{1,L}\rangle = 1/Q$ for all α when the average is defined by the trace over the degenerate states. Anything localized is not observed in the charge distribution. This is to be compared with the results shown in Fig. 6 for the odd L case. The other $Q^2 - 1 (= 8, N = 3)$ are at the zero energy. The lowest 15 energies of $Q = 3$ are listed in Table V. System size dependencies of the gap for the $Q = 3$ and $Q = 5$ are shown in the insets of Figs. 7 and 8. They show $J_{eff} \propto e^{-L/\xi}$, which implies that the low-energy multiplets are described by the edge states. Assuming this behavior, we may assume exact Q^2 degeneracy for the infinite system with boundaries, that is, taking an infinite size limit for the open system. This Q^2-fold degeneracy is exact only in the $L \to \infty$ limit. In this sense, the Q^2-fold degeneracy implies that emergence of $Z_Q^{\rm left} \times Z_Q^{\rm right}$ symmetry in the infinite chain, which was originally mentioned in a chiral symmetric fermion system [129]. This corresponds to the 2^2-fold degeneracy and $Z_2 \times Z_2$ symmetry of the Haldane chain [130] and dimer phases of $S = 1/2$ quantum spin chain [131]. See Table V as well.

B. Low-energy multiplet of edge states with symmetry breaking

Let us consider a dimerized system $|J_e| \neq |J_o|$, $(J_i < 0)$ with open boundary condition. We assume the system size L is even assuming it is sufficiently large (compared with the gap). When $|J_o| < |J_e|$, the low-energy sector of the system is composed of a Q-fold degenerate multiplet with edge states localized near $j = 1$ and $j = L$, which is spanned by degenerate Q^2 low-energy states Eq. (133), $\{|g^{L:\text{even}}_{eo;\alpha,\beta}\rangle|\alpha, \beta = 1, \ldots, Q\}$. This multiplet is separated from the other states by the bulk gap. The interaction between the both ends are negligibly small since we assume the system size is large. Then the low-energy multiplet is Q^2-fold degenerate and the Z_Q symmetry breaking term H_B, Eq. (43), operates within this multiplet perturbatively assuming that the symmetry breaking is small compared with the bulk gap. This perturbative discussion is exact as for the level crossing (selection rule) within the multiplet assuming the gap between the multiplet and the others (global spectral structure) is finite where the energy scale of the splitting is governed by the gap of the

TABLE V. Lowest 15 energies of $H^{(2)}$ with $Q = 3$ ($L = 6$ and 8).

	Periodic		Open		Open	
	$(-0.8, -1.2), (-1.2, -0.8)$		$(-1.2, -0.8)$		$(-0.8, -1.2)$	
(J_1, J_2)	$L = 6$	$L = 8$	$L = 6$	$L = 8$	$L = 6$	$L = 8$
1	−12.00000000	−15.91335471	−11.54504463	−15.51898367	−9.19845500	−12.87602132
2	−9.32998146	−13.40936964	−8.73554793	−12.84032668	−8.65117764	−12.67776804
3	−9.32998146	−13.40936964	−8.73554793	−12.84032668	−8.65117764	−12.67776804
4	−9.32998146	−13.40936964	−8.73554793	−12.84032668	−8.65117764	−12.67776804
5	−9.32998146	−13.40936964	−8.73554793	−12.84032668	−8.65117764	−12.67776804
6	−9.32998146	−13.40936964	−8.73554793	−12.84032668	−8.65117764	−12.67776804
7	−9.32998146	−13.40936964	−8.73554793	−12.84032668	−8.65117764	−12.67776804
8	−9.32998146	−13.40936964	−8.73554793	−12.84032668	−8.65117764	−12.67776804
9	−9.32998146	−13.40936964	−8.73554793	−12.84032668	−8.65117764	−12.67776804
10	−8.40000000	−12.45492049	−8.01759677	−12.29865018	−7.32249080	−11.45644523
11	−8.04273842	−12.45492049	−8.01759677	−12.29865018	−6.72949546	−10.84747170
12	−8.04273842	−12.45492049	−8.01759677	−12.29865018	−6.72949546	−10.84747170
13	−8.04273842	−12.45492049	−8.01759677	−12.29865018	−6.72949546	−10.84747170
14	−8.04273842	−12.45492049	−8.01759677	−12.29865018	−6.72949546	−10.84747170
15	−8.04273842	−12.45492049	−8.01759677	−12.29865018	−6.72949546	−10.84747170

bulk. When the dimerization pattern is reversed, $|J_o| > |J_e|$, the system is gapped even for the open boundary condition and the ground state is adiabatically connected to the unique gapped one, Eq. (134).

Let us extend the time dependence of the pump by shifting the timing of the dimerization as follows:

$$J_i = J_0 + (-1)^j \delta J \cos\left(2\pi\left(\frac{t - t_0}{T}\right)\right), \quad J_0 < 0, \delta J > 0, \tag{137}$$

$$\Delta_\alpha = \Delta \sin\left(2\pi\left(\Phi\alpha + \frac{t}{T}\right)\right). \tag{138}$$

This is the pumping protocol of the topological pump. As for the open boundary condition, the edge states only appear when the coupling is weak at the both boundaries (L: even), that is, $|J_o| < |J_e|$ ($J_0 < 0, \delta J > 0$). This period is specified by

$$\frac{t_0}{T} + \frac{1}{4} + n < \frac{t}{T} < \frac{t_0}{T} + \frac{3}{4} + n, \quad {}^\exists n \in \mathbb{Z}. \tag{139}$$

Let us first discuss energies of the symmetry breaking Hamiltonian H_B, Eq. (43), within the Q-fold degenerate multiplet $|\alpha, \beta\rangle = |g^{L:\text{even}}_{\text{op},eo}(\alpha, \beta)\rangle$, $\alpha, \beta = 1, \ldots, Q$,

$$H_B(t)\left|g^{L:\text{even}}_{\text{op},eo}(\alpha, \beta)\right\rangle = \left|g^{L:\text{even}}_{\text{op},eo}(\alpha, \beta)\right\rangle E_{\alpha\beta}(t),$$

$$E_{\alpha\beta}(t) = \Delta_\alpha(t) + \Delta_\beta(t).$$

See Table VI and Fig. 9. Assuming the system size is sufficiently large $L \to \infty$, the CoM, P_α, and the quantum numbers $\bar{N}_\alpha$ are also shown. For example, as for the state, $|1_1, 1_L\rangle = |g^{L:\text{even}}_{\text{op},eo}(1, 1)\rangle$, the CoM's are $P_1 = x_1 \cdot (+1) + x_L \cdot (-1) \to -1$, $P_2 = 0$, and $P_3 = 0$. As for the state, $|1_1, 2_L\rangle = |g^{L:\text{even}}_{\text{op},eo}(1, 2)\rangle$, $P_1 = x_1 \cdot (+1) \to -\frac{1}{2}$, $P_2 = x_L \cdot (-1) \to -\frac{1}{2}$ and $P_3 = 0$. Generically, as for the state, $|\alpha_1, \alpha_L\rangle$, $\bar{N}_\alpha = (0, \cdots, 0)$. Its CoM's are $P_\alpha = -1$ and $P_\gamma = 0$ for any $\gamma \neq \alpha$. As for the state, $|\alpha, \beta\rangle$, $(\alpha \neq \beta)$, $\bar{N}_\alpha = +1, \bar{N}_\beta = -1, \bar{N}_\gamma = 0$ for any $\gamma \neq \alpha, \beta$. Its CoM's are $P_\alpha = P_\beta = -\frac{1}{2}, P_\gamma = 0$ ($\gamma \neq \alpha, \beta$). Explicit examples for $Q = 3$ are shown in Table VI.

Examples of the time dependence of $E_{\alpha\beta}(t)$ for $Q = 3$, $Q = 4$, and $Q = 5$ are shown in Fig. 9. Generically the lowest energies are always given by $E_{\alpha\alpha}(t)$ if $\Delta > 0$. Then the jth level crossing between $E_{\alpha\alpha}(t)$ and $E_{\beta\beta}(t)$, $(\alpha \neq \beta)$, at $t = t_j$, $(j = 1, \ldots, Q)$ occurs when

$$\sin 2\pi\left(\frac{P}{Q}\alpha + \frac{t_j}{T}\right) = \sin 2\pi\left(\frac{P}{Q}\beta + \frac{t_j}{T}\right), \tag{140}$$

$$P\alpha \equiv j \pmod{Q}, \tag{141}$$

$$P\beta \equiv j + 1 \pmod{Q}. \tag{142}$$

Let us write as $\alpha = \tau_j$ and $\beta = \tau_{j+1}$ where the Diophantine equation due to TKNN [2,132] is

$$j = \tau_j P + s_j Q, \tag{143}$$

where $\tau_j, s_j \in \mathbb{Z}$ is used in modulo Q.

TABLE VI. Energies $E_{\alpha\beta}(t) = \Delta_\alpha(t) + \Delta_\beta(t)$, quantum numbers $\bar{N}_\alpha$, and CoM's P_α of the symmetry breaking term H_B for the multiplet of the boundary spins, $\Delta_\alpha = \Delta \sin 2\pi(\Phi\alpha + \frac{t}{T})$, $\alpha = 1, 2, 3$ where $\Phi = P/Q, P = 1, Q = 3$. The generic Q case is discussed in a straightforward way.

Multiplet	$E_\alpha + E_\beta$	$\bar{N}_\alpha$	$P_\alpha, (L \to \infty)$
$\|1_1, 1_L\rangle$	$E_{11} = 2\Delta_1$	(0, 0, 0)	(−1, 0, 0)
$\|2_1, 2_L\rangle$	$E_{22} = 2\Delta_2$	(0,0,0)	(0, −1, 0)
$\|3_1, 3_L\rangle$	$E_{33} = 2\Delta_3$	(0,0,0)	(0, 0, −1))
$\|1_1, 2_L\rangle$	$E_{12} = \Delta_1 + \Delta_2$	(+1, −1, 0)	$(-\frac{1}{2}, -\frac{1}{2}, 0)$
$\|2_1, 1_L\rangle$	$E_{21} = \Delta_2 + \Delta_1$	(−1, +1, 0)	$(-\frac{1}{2}, -\frac{1}{2}, 0)$
$\|1_1, 3_L\rangle$	$E_{13} = \Delta_1 + \Delta_2$	(+1, 0, −1)	$(-\frac{1}{2}, -\frac{1}{2}, 0)$
$\|3_1, 1_L\rangle$	$E_{31} = \Delta_3 + \Delta_1$	(−1, 0, +1)	$(-\frac{1}{2}, 0, -\frac{1}{2})$
$\|2_1, 3_L\rangle$	$E_{23} = \Delta_2 + \Delta_3$	(0, +1, −1)	$(0, -\frac{1}{2}, -\frac{1}{2})$
$\|3_1, 2_L\rangle$	$E_{32} = \Delta_3 + \Delta_2$	(0, −1, +1)	$(0, -\frac{1}{2}, -\frac{1}{2})$

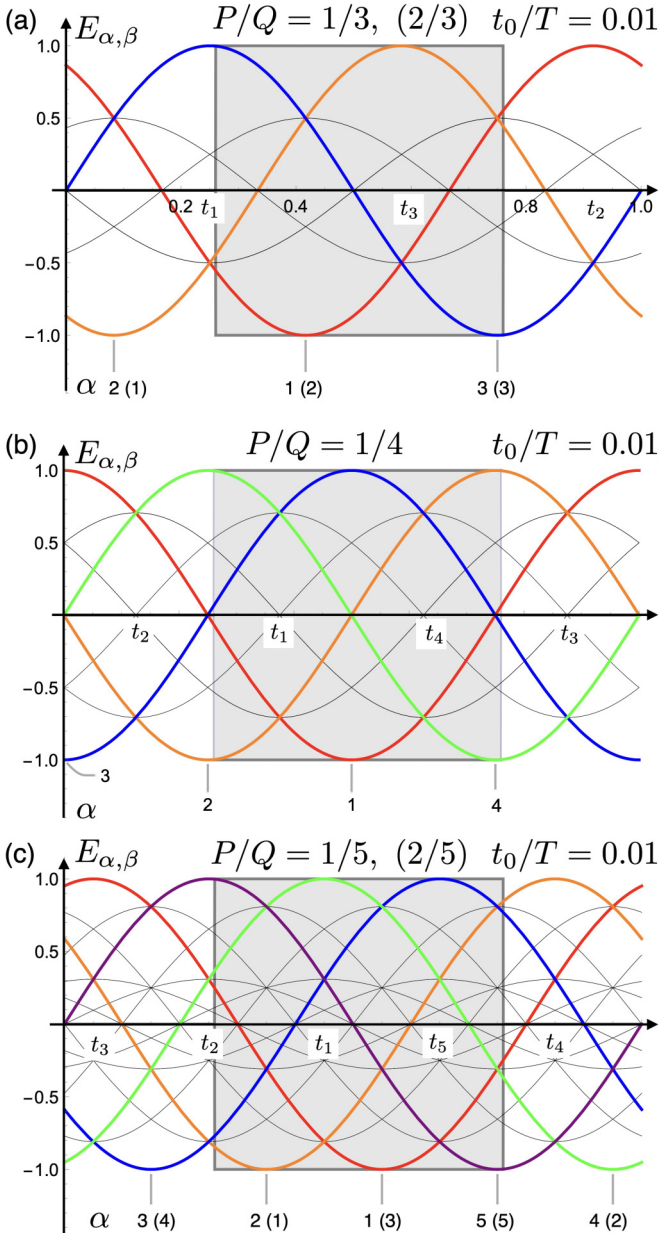


FIG. 9. Normalized low-energy spectra of the symmetry breaking term $H_B(t)$ for the multiplet $|\alpha_L\beta_R\rangle$ where L and R are effective spins at both ends, $E_{\alpha\beta}(t) = \Delta_\alpha(t) + \Delta_\beta(t)$ as a function of t. The gray rectangles show the region $|J_o| < |J_e|$ that is specified by Eq. (139) for $t_0/T = 0.01$. Colored lines are $E_{\alpha\alpha}$ and the numbers shown in the bottom denote α. They are for $P = 1$, $\Phi = 1/Q$ and the ones in the parenthesis are for $P = 2$, $\Phi = 2/Q$. (a) $Q = 3$ (b) $Q = 4$, and (c) $Q = 5$.

It implies that the level crossing from the energy E_{τ_j,τ_j} to $E_{\tau_{j+1},\tau_{j+1}}$ occurs at t_j (see Fig. 9) [133]

$$\frac{t_j}{T} = \frac{3}{4} - \frac{j+\frac{1}{2}}{Q}, \quad \mathrm{mod}\,1. \tag{144}$$

Let us discuss the shift t_0 dependence of the low-energy spectrum of the Hamiltonians with open boundary condition. The low-energy spectrum without symmetry breaking perturbation for $Q = 3$ is shown in Fig. 10. It shows (approximate)

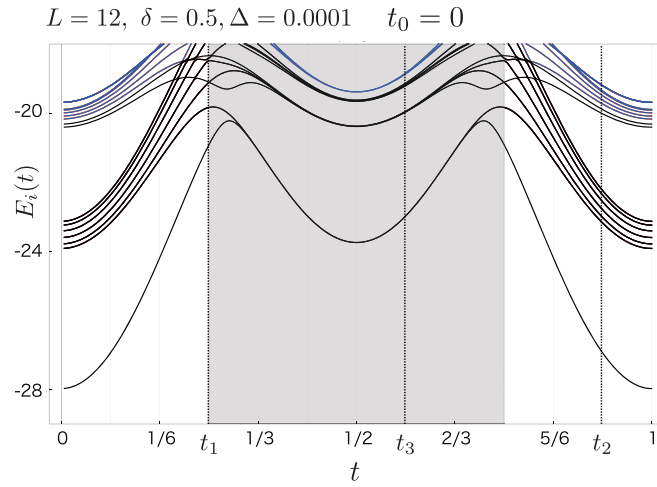
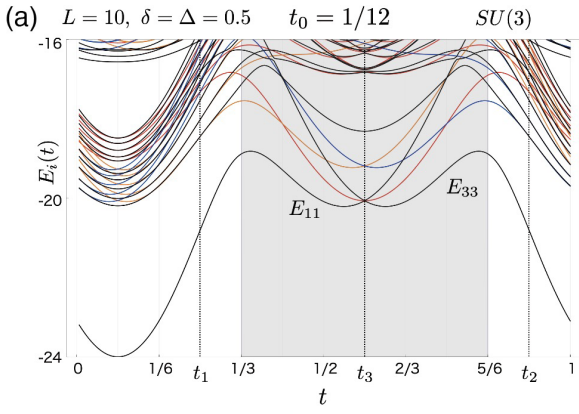


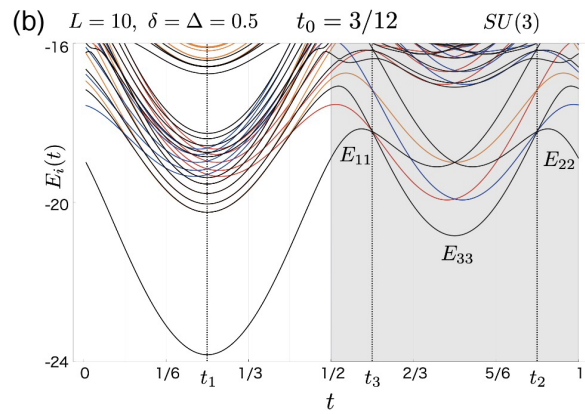
FIG. 10. The lowest 15 energy levels of the $SU(3)$ symmetric Hamiltonian of the 12-site system with open boundary condition within $\bar{N}_\alpha$ sectors, where $\delta = 0.5$, $\Delta = 0.00001$. To avoid numerical instability of the diagonalization, $\Delta = 0.0$ is not used. The black lines are for $\bar{N}_\alpha : (0, 0, 0)$, the red ones are for $(1, -1, 0)$ and $(-1, 1, 0)$, the orange ones are for $(0, 1, -1)$ and $(0, -1, 1)$, and the blue ones are for $(-1, 0, 1)$ and $(1, 0, -1)$. The colored lines are degenerate with the black ones and hidden within the width of the lines. The gray region is for $|J_0(t)| < |J_e(t)|$ where the low-energy spectrum is composed of approximately degenerate 3^2 edge states of the multiplet. This justifies that the calculation for the sector $\bar{N}_\alpha = 0$ $(^\forall\alpha)$ can be enough as for the low-energy spectra.

$Q^2 = 9$ fold degeneracy due to edge states when the coupling at the boundary is weak as specified in the period by Eq. (139). Generically the degeneracy is Q^2. Assuming the emergent $Q \times Q$ symmetry of the infinite chain with boundaries, this degeneracy is lifted by the symmetry breaking perturbation H_B. The low-energy spectra for $Q = 3$ and $Q = 5$ are shown in Figs. 11 and 12.

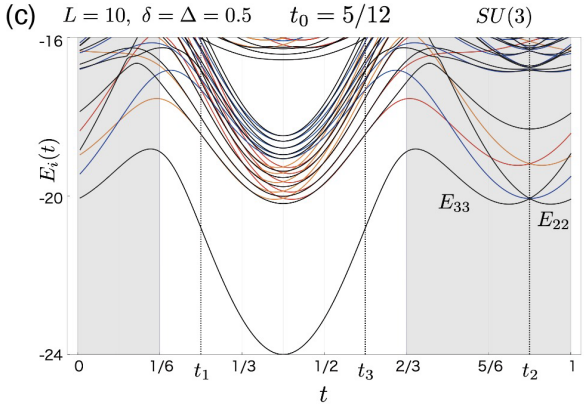
As the results indicate, the hybridization of the edge states at both boundaries is negligibly small. Then approximate level crossings in the figures are identified only by the spectrum of the symmetry breaking Hamiltonian H_B between the edge states, which are explicitly shown in Figs. 11 and 12. For example, in Fig. 11(a), there is a threefold (approximate) level crossing at $t = t_3$. This should be compared with the level crossing in Fig. 9(a) at $t = t_3$. As for the spectrum of H_B, the threefold degeneracy is given by the change of the ground state from the state with the energy E_{11} (red) to that with E_{33} (blue). At the level crossing, some other states with energies $E_{\alpha\beta}$ $(\alpha \neq \beta)$ also pass through the level crossing. Correspondingly, in Fig. 11(a), the ground state is given by the state of the sector $N_\alpha : (0, 0, 0)$ and the other $(-1, 1, 0)$ is passing through. The behavior of the ground-state energy is cusp like. However, it should not be a rigorous level crossing due to the mixture of the edge states at both ends. It induces tiny (exponentially small as a function of the system size, L) level repulsion. This level repulsion vanishes by taking $L \to \infty$. This is negligibly small for the present parameter in Fig. 11. The emergent symmetry $Z_Q^{\mathrm{left}} \times Z_Q^{\mathrm{right}}$ $(L \to \infty)$ protects this (asymptotic) level crossing. Just before the level



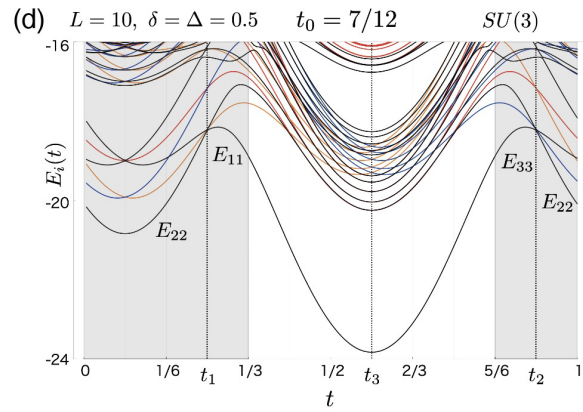
$\Delta P_1(t_3) = +1,\ \Delta P_3(t_3) = -1$ and $I^\alpha : (+1, 0, -1)$.



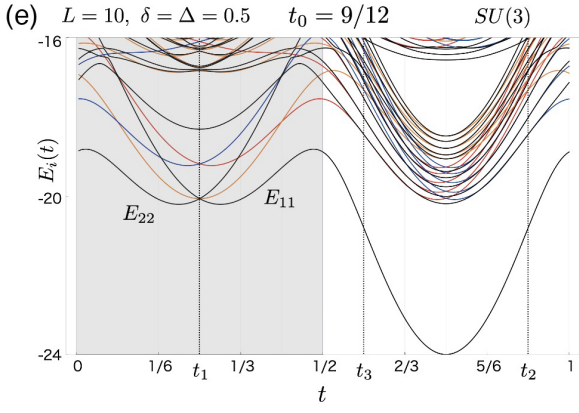
$\Delta P_1(t_3) = +1,\ \Delta P_3(t_3) = -\Delta P_3(t_2) = -1,$
$\Delta P_2(t_2) = -1$ and $I^\alpha : (+1, -1, 0)$.



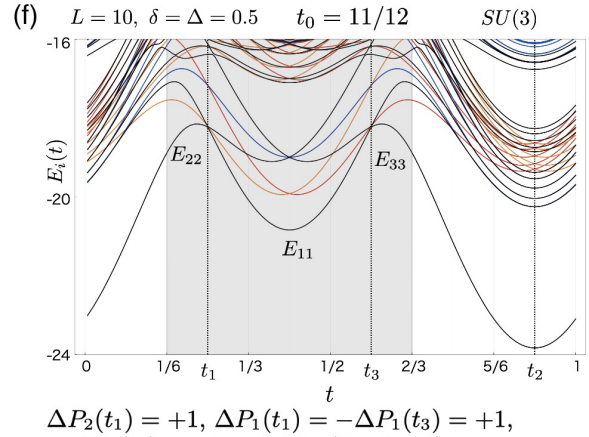
$\Delta P_3(t_2) = +1,\ \Delta P_2(t_2) = -1$ and $I^\alpha : (0, -1, +1)$.



$\Delta P_3(t_2) = +1,\ \Delta P_2(t_2) = -\Delta P_2(t_1) = -1,$
$\Delta P_1(t_1) = -1$ and $I^\alpha : (-1, 0, +1)$.



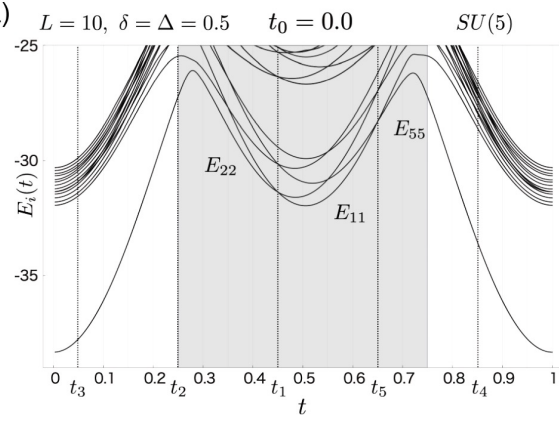
$\Delta P_2(t_1) = +1,\ \Delta P_1(t_1) = -1$ and $I^\alpha : (-1, +1, 0)$.



$\Delta P_2(t_1) = +1,\ \Delta P_1(t_1) = -\Delta P_1(t_3) = +1,$
$\Delta P_3(t_3) = -1$ and $I^\alpha : (0, +1, -1)$.

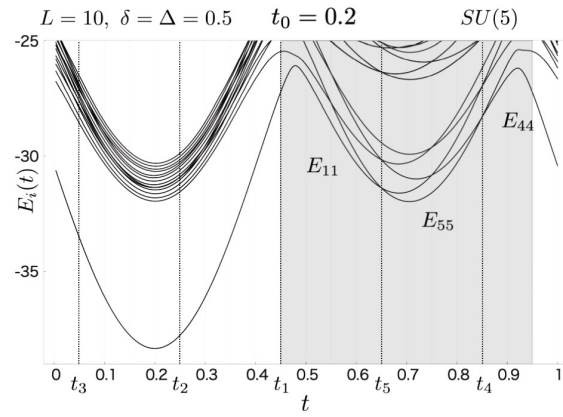
FIG. 11. t_0 dependence of the lowest 15 energy levels of the $SU(3)$ symmetric Hamiltonian of the 12-site system with open boundary condition within $\bar{N}_\alpha$ sectors, where $\delta = 0.5$, $\Delta = 0.5$, and $\Phi = 1/3$ ($Q = 3$). The black lines are for $\bar{N}_\alpha : (0, 0, 0)$, the red ones are for $(1, -1, 0)$ and $(-1, 1, 0)$, the orange ones are for $(0, 1, -1)$ and $(0, -1, 1)$, and the blue ones are for $(-1, 0, 1)$ and $(1, 0, -1)$. The gray region is for $|J_0(t)| < |J_e(t)|$ where the low-energy spectrum is composed of the multiplet of the edge states of the dimension Q^2. One expects $Q \times Q$ emergent symmetry, in the $L \to \infty$ limit, that is responsible for the level crossings within the multiplet. The lowest eigenstate is identified by $E_{\alpha\alpha}$ assuming the emergent $Q \times Q$ symmetry for the infinite system. The dimensions of the Hilbert spaces are 35169 for (0,0,0) and 27888 for the others (Compare with the Fig. 9). Due to Eq. (100), $I^\alpha = \sum_i \Delta P^\alpha(t_i)$.

(a)



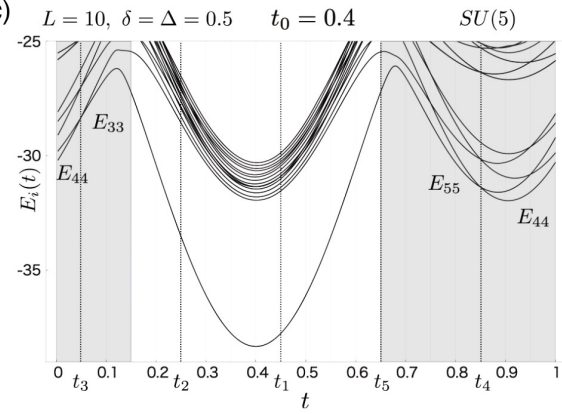
$\Delta P_2(t_1) = +1,\ \Delta P_1(t_1) = -\Delta P_1(t_5) = -1,$
$\Delta P_5(t_5) = -1$ and $I^\alpha : (0, +1, 0, 0, -1)$.

(b)



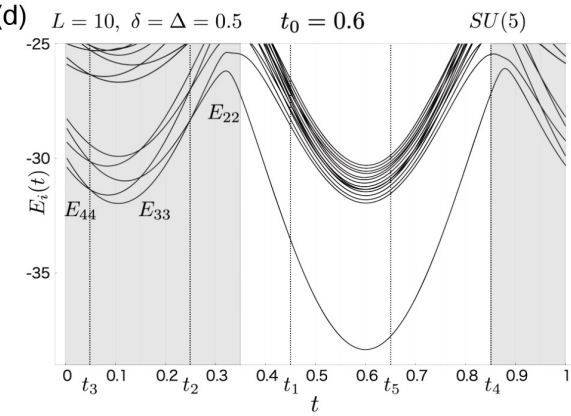
$\Delta P_1(t_5) = +1,\ \Delta P_5(t_5) = -\Delta P_5(t_4) = -1,$
$\Delta P_4(t_4) = -1$ and $I^\alpha : (+1, 0, 0, -1, 0)$.

(c)



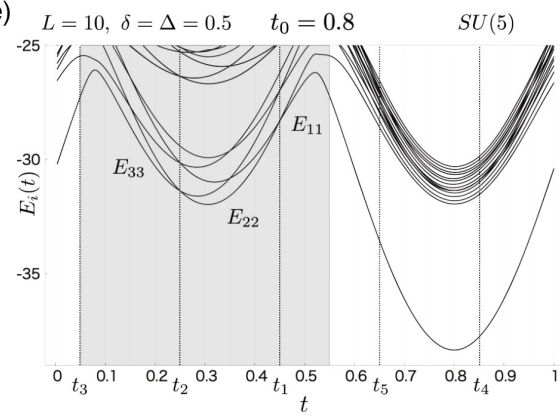
$\Delta P_5(t_4) = +1,\ \Delta P_4(t_4) = -\Delta P_4(t_3) = -1,$
$\Delta P_3(t_3) = -1$, and $I^\alpha : (0, 0, -1, 0, +1)$.

(d)



$\Delta P_4(t_3) = +1,\ \Delta P_3(t_3) = -\Delta P_3(t_2) = -1,$
$\Delta P_2(t_2) = -1$ and $I^\alpha : (0, -1, 0, +1, 0)$.

(e)



$\Delta P_3(t_2) = +1,\ \Delta P_2(t_2) = -\Delta P_2(t_1) = -1,$
$\Delta P_1(t_1) = -1$ and $I^\alpha : (-1, 0, +1, 0, 0)$.

FIG. 12. t_0 dependence of the lowest 15 energy levels of the $SU(5)$ symmetric Hamiltonian of the 10-site system with open boundary condition within $\bar{N}_\alpha$ sectors, where $\delta = 0.5$, $\Delta = 0.5$, and $\Phi = 1/5$ ($Q = 5$). Only the date for $\bar{N}_\alpha : (0, 0, 0, 0, 0)$ are shown, which give ground-state energies. The gray region is for $|J_o(t)| < |J_e(t)|$ where the low-energy spectrum is composed of the multiplet of the edge states of the dimension Q^2. One expects $Q \times Q$ emergent symmetry, in the $L \to \infty$ limit, that is responsible for the level crossings within the multiplet. The lowest eigenstate is identified by $E_{\alpha\alpha}$ assuming the emergent $Q \times Q$ symmetry for the infinite system. The dimensions of the Hilbert spaces are 127905 (compare with the Fig. 9). Due to Eq. (100), $I^\alpha = \sum_i \Delta P^\alpha(t_i)$.

crossing ($t = t_3-$), the edge state due to $\alpha = 1$ near $j = 1$ and $\alpha = 1$ near $j = L$ are the ground state. The edge states at both ends contribute to P_1 by -1 in pair (see Table VI). Note that there is another contribution due to bulk as well. After the crossing ($t = t_3+$), the edge state changes to the one due to $\alpha = 3$ near $j = 1$ and $\alpha = 3$ near $j = L$. As for the CoM P_1, contribution from the edge state with energy E_{11} vanishes at $t = t_3$. The bulk contribution remains the same (since it is continuous in time t) at $t = t_3$. Then it implies $\Delta P_1 = +1$. The similar consideration implies $\Delta P_3 = -1$. In the present case, there is no further level crossing in Fig. 11(a). Then according to Eq. (100), $I^1 = +1$, $I^2 = 0$, and $I^3 = -1$. Assuming this emergent symmetry and the level crossings, each jump of the CoM ΔP_α is identified for each level crossing as shown in the caption of Figs. 11 and 12 supplemented with the sum of the discontinuities I^α. In the next section, CoM's for $Q = 3$ case is directly calculated by using the DMRG calculation. Also direct calculation of the Chern number C^α are compared in the following section. It enables us to confirm the bulk-edge correspondence Eqs. (87) and (98).

C. Explicit Chern numbers and Diophantine equation

To be simple let us first consider a system at $t_0 = +0$ (see Fig. 9). In the pump cycle, one may see a series of the jumps in P_α. As for the sum of the discontinuities I^α, a pair of the jumps, except the first and the last ones, is canceled (see Figs. 11 and 12). The last jump is due to the level crossing from some state with energy $E_{\alpha\alpha}$ to the state with E_{QQ}. It results in $I^Q = -1$. Similarly the first jump is a level crossing from the state with energy $E_{\alpha\alpha}$ ($\alpha = \tau_{\frac{Q-1}{2}}$) to the state with energy $E_{\beta\beta}$ ($\beta = \tau_{\frac{Q-1}{2}-1}$) for Q: odd and from $E_{\alpha\alpha}$ ($\alpha = \tau_{\frac{Q}{2}}$) to $E_{\beta\beta}$ ($\beta = \tau_{\frac{Q}{2}-1}$) for Q: even. It results in $I^{\tau_{\frac{Q-1}{2}}} = +1$ for Q (odd) and $I^{\tau_{\frac{Q}{2}}} = +1$ for Q (even).

In a similar way as for the time dependence Eq. (138), the edge states appear at $t/T = t_{\rm ini}/T \equiv t_0/T + 1/4$ and vanish at $t/T = t_{\rm fin}/T \equiv t_0/T + 3/4$ for each t_0. Within the period $[t_{\rm ini}, t_{\rm fin}]$, the level crossings due to the edge state $\beta = \tau_{j+1}$ to $\alpha = \tau_j$ occur at $t = t_j$, Eq. (144), that cause the jumps in $\Delta\tilde{P}^{\tau_j} = -1$ and $\Delta\tilde{P}^{\tau_{j+1}} = +1$ for all j's that satisfy $t_{\rm ini} < t_j < t_{\rm fin}$ (see Fig. 9). Since the paired jumps inside the period cancel with each other, the first one $t_{j_{\rm ini}}$, $t_{\rm ini} < t_{j_{\rm ini}} < t_{\rm ini} + 1/Q$ gives the sum of the discontinuity, $I^{\tau_{j_{\rm ini}+1}} = +1$. Similarly, the last one $t_{j_{\rm fin}}$, $t_{\rm fin} - 1/Q < t_{j_{\rm fin}} < t_{\rm fin}$ gives to the sum of the discontinuity $I^{\tau_{j_{\rm fin}}} = -1$. Otherwise $I^\alpha = 0$ ($\alpha \notin \{\tau_{j_{\rm ini}+1}, \tau_{j_{\rm fin}}\}$). The conditions are written as [134]

$$\frac{t_0}{T} + \frac{1}{4} < \frac{3}{4} - \frac{j_{\rm ini} + \frac{1}{2}}{Q} < \frac{t_0}{T} + \frac{1}{4} + \frac{1}{Q}, \tag{145}$$

$$\frac{t_0}{T} + \frac{3}{4} - \frac{1}{Q} < \frac{3}{4} - \frac{j_{\rm fin} + \frac{1}{2}}{Q} < \frac{t_0}{T} + \frac{3}{4}. \tag{146}$$

It implies

$$j_{\rm ini} + 1 = \left\lfloor Q\left(\frac{1}{2} - \frac{t_0}{T}\right) + \frac{1}{2} \right\rfloor, \tag{147}$$

$$j_{\rm fin} = \left\lfloor -Q\frac{t_0}{T} + \frac{1}{2} \right\rfloor, \tag{148}$$

where $\lfloor x \rfloor$ is the largest integer less than x. Finally we have with using the bulk-edge correspondence

$$I^\alpha(t_0) = C^\alpha(t_0) = \begin{cases} -1 & \alpha \equiv \tau_{\lfloor Q(1-\frac{t_0}{T})+\frac{1}{2}\rfloor} \bmod Q \\ +1 & \alpha \equiv \tau_{\lfloor Q(\frac{1}{2}-\frac{t_0}{T})+\frac{1}{2}\rfloor} \bmod Q \\ 0 & \text{otherwise} \end{cases}. \tag{149}$$

It implies a series of topological transitions associated with the shift of the dimerization t_0.

The Chern numbers of the generic path $\ell_{V_\alpha G V_\beta}$, ($1 \leqslant \alpha < \beta \leqslant Q$), $C^{\ell_{V_\alpha G V_\beta}}$ is simply given by

$$C^{\ell_{V_\alpha G V_\beta}} = \sum_{\gamma=\alpha}^{\beta-1} C^\gamma. \tag{150}$$

VI. NUMERICAL EVALUATION OF TOPOLOGICAL NUMBERS

In this section, extensive data for the numerical evaluation of the low-energy spectra, the CoMs, and the topological numbers (the sum of the jumps and the Chern numbers) are shown. They are given by the DMRG and the exact diagonalization.

A. Low-energy spectra of the finite-size systems

Low-energy spectra of the $Q = 3$, $L = 12$ system and the $Q = 5$, $L = 10$ system are shown in Figs. 11 and 12. The results for the $Q = 3$ case are consistently compared with the DMRG calculation shown in Sec. VI B.

B. Discontinuities of CoM by DMRG

In this section, the CoM's as discussed are directly calculated by using the DMRG method. The $Q = 3$ fermion system can be simulated by mapping to the following biquadratic spin model with open boundary condition [see Appendix A, Eq. (A1)],

$$H_S = H_S^{(2)} + H_B^S, \tag{151}$$

where

$$H_S^{(2)} = \sum_{\ell=1}^{L/2} [J_o(t)(\vec{S}_{2\ell-1} \cdot \vec{S}_{2\ell})^2 + J_e(t)(\vec{S}_{2\ell} \cdot \vec{S}_{2\ell+1})^2],$$

$$H_B^S(t) = -\sum_{j,\alpha} \Delta_\alpha \left(S_j^\alpha\right)^2. \tag{152}$$

Here $J_o = J_{i\in\text{odd}}$, $J_e = J_{i\in\text{even}}$ and Δ_α is given by Eq. (138).

We calculate the staggered quadratic CoM, given by

$$P_1^s(t) = \sum_{j=1}^{L} (-1)^j x_j \langle S_j^{x2}\rangle = \delta_L + P_1,$$

$$P_2^s(t) = \sum_{j=1}^{L} (-1)^j x_j \langle S_j^{y2}\rangle = \delta_L + P_2,$$

$$P_3^s(t) = \sum_{j=1}^{L} (-1)^j x_j \langle S_j^{z2}\rangle = \delta_L + P_3, \tag{153}$$

where $\delta_L = 0\,(L: \text{odd})$, $1/2\,(L: \text{even})$, $j_0 = (L+1)/2$ and $\langle\cdot\rangle$ means taking the expectation value for the ground state

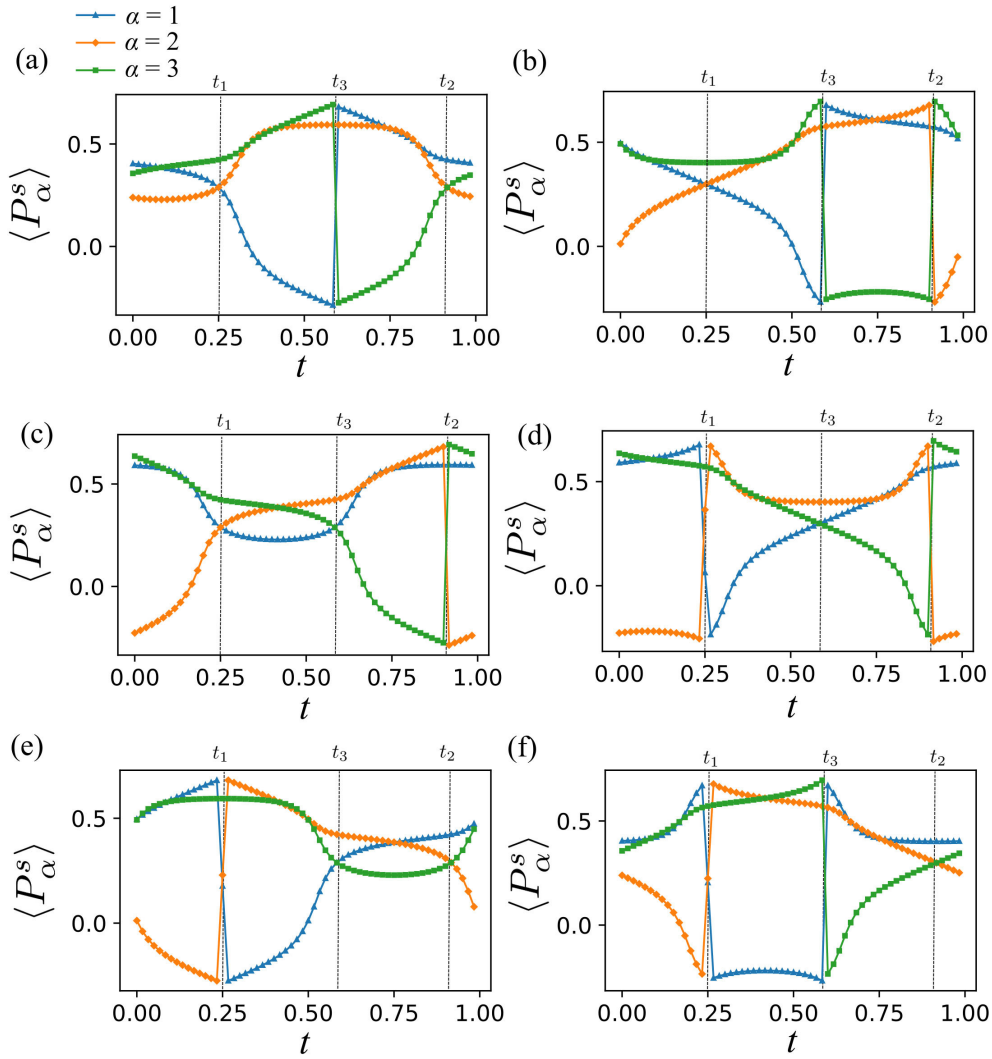


FIG. 13. The behavior of the sq-CoM's P^s_α for various t_0 and components. The system size is $L = 64$ with open boundary condition. The data shows $t_0/T = 1/12$ (a), $3/12$ (b), $5/12$ (c), $7/12$ (d), $9/12$ (e), and $11/12$ (f).

at the time t. The CoM of the spins P^s_α, $(\alpha = 1, 2, 3)$ is directly related to that of fermions P_α. Note that the factor $(-1)^{j-1}$ in Eq. (76) and the summation over j. In the following, we fix the parameters as $J_0 = -1$, $\delta J = 0.5$, and $\Delta = 0.5$, same to the parameter set of the results in Figs. 8–10.

To calculate the behavior of the CoM, we employ DMRG algorithm in TeNPy package [135]. The numerical results for various t_0 are shown in Fig. 13. We observe the behavior of the CoM in all data divides into two parts, continuous part, and jump part. The time evolution of the CoM in the continuous part indicates the presence of the bulk current. We verify that

TABLE VII. Jumps of the CoM, the sum of them in Fig. 13 and values of the numerically obtained Chern numbers picked up from Figs. 15(a), 15(b), and 15(c).

t_0	$(\Delta P_1, \Delta P_2, \Delta P_3)\|_{t=t_1}$	$(\Delta P_1, \Delta P_2, \Delta P_3)\|_{t=t_2}$	$(\Delta P_1, \Delta P_2, \Delta P_3)\|_{t=t_3}$	(I^1, I^2, I^3)	$(C^1, C^2, C^3 = C^0)$
$\frac{1}{12}T$	$(0, 0, 0)$	$(0, 0, 0)$	$(+1, 0, -1)$	$(+1, 0, -1)$	$(+1, 0, -1)$
$\frac{3}{12}T$	$(0, 0, 0)$	$(0, -1, +1)$	$(+1, 0, -1)$	$(+1, -1, 0)$	$(+1, -1, 0)$
$\frac{5}{12}T$	$(0, 0, 0)$	$(0, -1, +1)$	(0,0,0)	$(0, -1, +1)$	$(0, -1, +1)$
$\frac{7}{12}T$	$(-1, +1, 0)$	$(0, -1, +1)$	(0,0,0)	$(-1, 0, +1)$	$(-1, 0, +1)$
$\frac{9}{12}T$	$(-1, +1, 0)$	$(0, 0, 0)$	(0,0,0)	$(-1, +1, 0)$	$(-1, +1, 0)$
$\frac{11}{12}T$	$(-1, +1, 0)$	$(0, 0, 0)$	$(+1, 0, -1)$	$(0, +1, -1)$	$(0, +1, -1)$

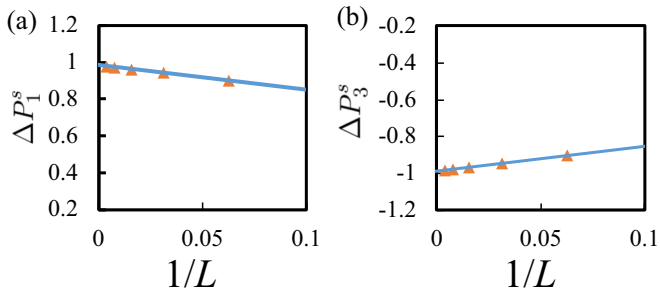


FIG. 14. System size dependence of the jumps of CoM's at $t = t_3$ and $t_0/T = 1/12$. Results for (a) ΔP_1^s and (b) ΔP_3^s.

each jump point is identical to the level crossing point (t_1, t_2, and t_3), expected from Figs. 9 and 11. The jumps of the CoM P_α and the sum them I^α are summarized in Table VII. This is consistent with the low-energy spectra shown in Fig. 11 (see the caption).

As for a finite-size effect mentioned in the Appendix G, we have shown a concrete example, in Fig. 14. The jump of ΔP_1^s and ΔP_3^s approach to 1 and –1 for $L \to \infty$. This agrees to the exponential localization of the edge states as discussed.

C. Direct evaluation of Chern numbers

Using the bulk-edge correspondence of the topological pump, the Chern number that is a total pumped charge of the bulk should be the same to the discontinuity of the CoMs, Eq. (149). We have confirmed it by a direct evaluation of the Chern number Eq. (96) obtained by the integral of the field strength $B_{\rm tw}$. This is done by using the Fukui-Hatsugai-Suzuki formula [136] for $Q = 3$, $\Phi = 1/3, 2/3$, $Q = 4$,

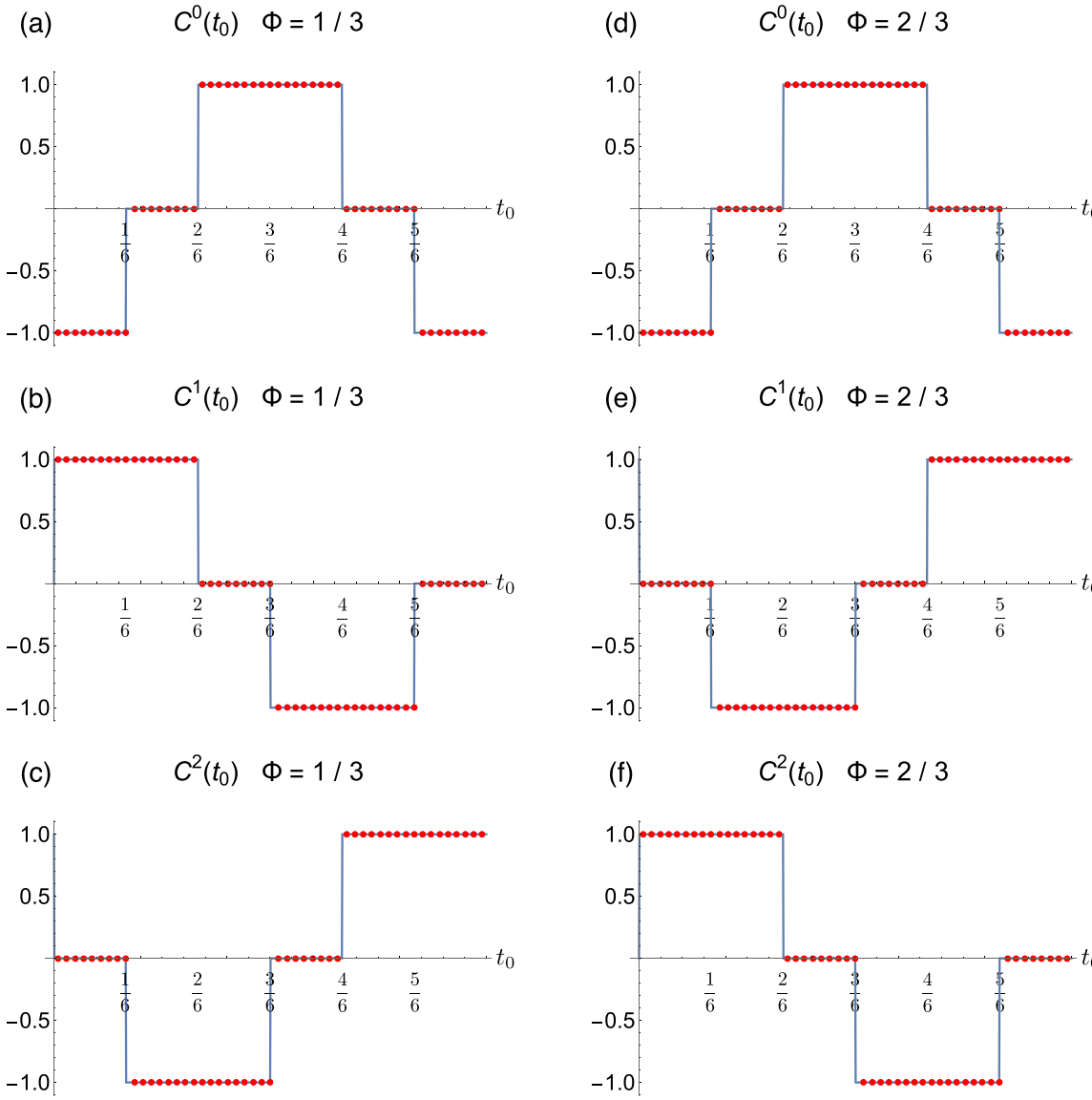


FIG. 15. t_0 dependence of the Chern numbers. $Q = 3$, $\Phi = \frac{1}{3}$ and $\frac{2}{3}$. $J_0 = -1.0$, $\delta = 0.5$ and $\Delta = 0.5$. The dimension of the Hilbert space is 4653 ($L = 10$). The data points are for $t_0 = (i - 0.5)/51$, $i = 1, \ldots, 51$. Red points are numerical evaluation of Eq. (96) by Fukui-Hatsugai-Suzuki formula [136] and the lines are analytical results, Eq. (149).

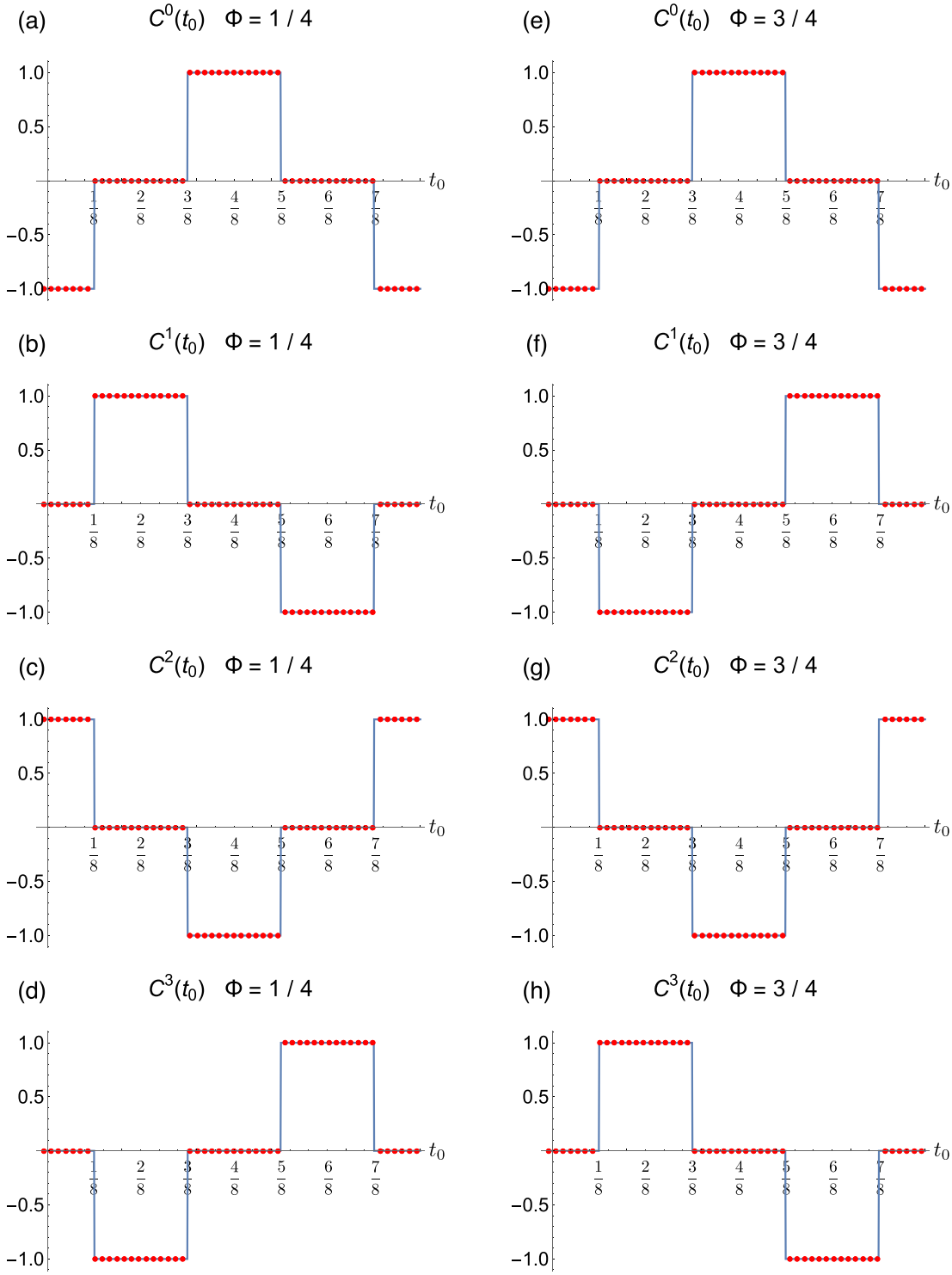


FIG. 16. t_0 dependence of the Chern numbers. $Q = 4$, $\Phi = \frac{1}{4}$ and $\frac{3}{4}$. $J_0 = -1.0$, $\delta = 0.5$, and $\Delta = 0.5$. The dimension of the Hilbert space is 2716 ($L = 8$). The data points are for $t_0 = (i - 0.5)/51$, $i = 1, \ldots, 51$. Red points are numerical evaluation of Eq. (96) by Fukui-Hatsugai-Suzuki formula [136] and the lines are analytical results, Eq. (149).

$\Phi = 1/4, 3/4$, and $Q = 5$, $\Phi = 1/5, 2/5, 3/5, 4/5$. We have plotted the Chern numbers as a function of t_0. The results are shown in Figs. 15–18. The analytic formula Eq. (149) is plotted by the solid lines and the numerical values obtained by discretized integration formula by the Fukui-Hatsugai-Suzuki formula are shown in the red circles. They agree with each other almost completely except a few points for $Q = 5$, $\Phi = 2/5$, and $\Phi = 3/5$ [137]. They are near the topological phase transitions where any numerical calculation can be unstable.

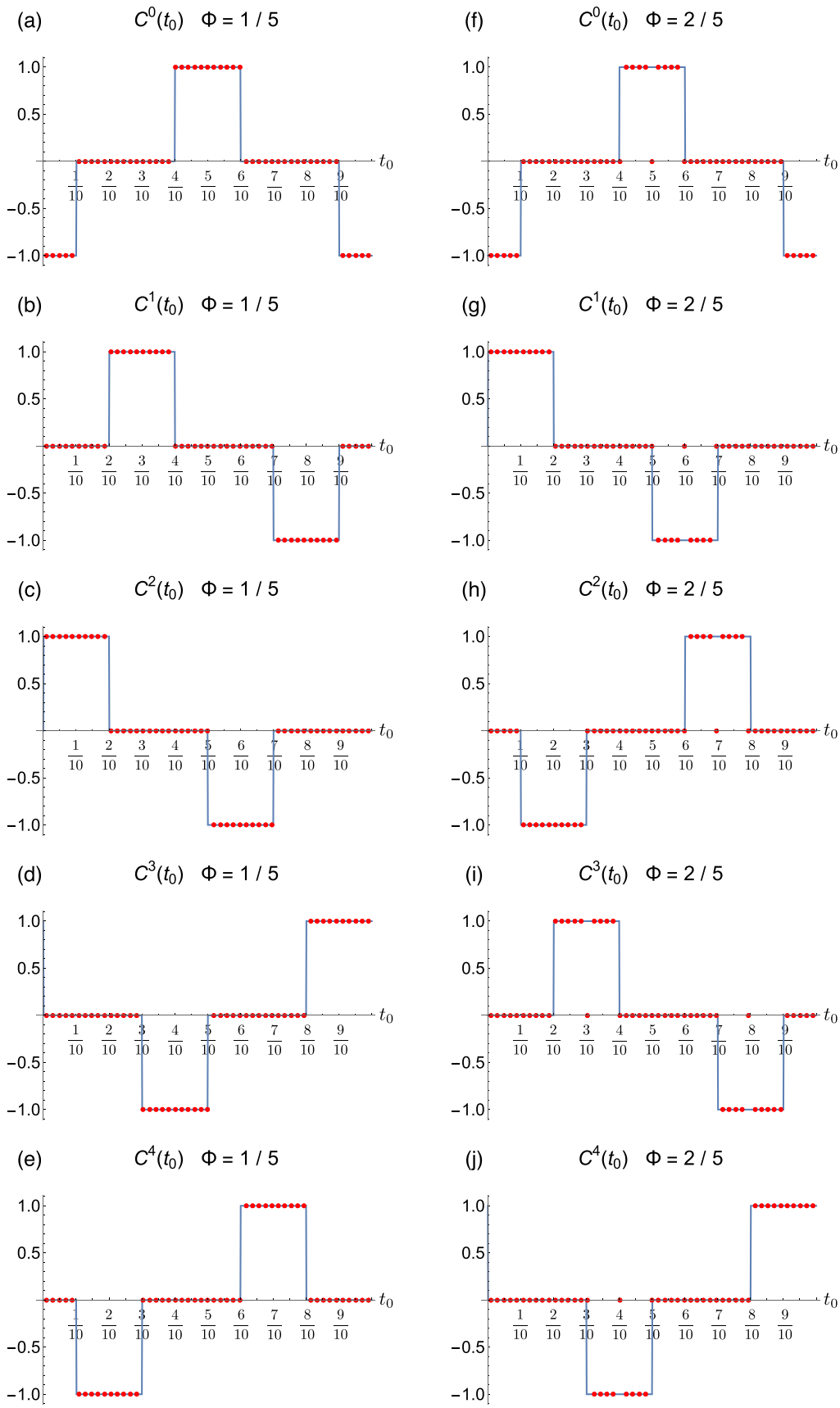


FIG. 17. t_0 dependence of the Chern numbers. $Q = 5$, $\Phi = \frac{1}{5}$ and $\frac{2}{5}$. $J_0 = -1.0$, $\delta = 0.5$ and $\Delta = 0.5$. The dimension of the Hilbert space is 545 ($L = 6$). The data points are for $t_0 = (i - 0.5)/51$, $i = 1, \ldots, 51$. Red points are numerical evaluation of Eq. (96) by Fukui-Hatsugai-Suzuki formula [136] and the lines are analytical results, Eq. (149).

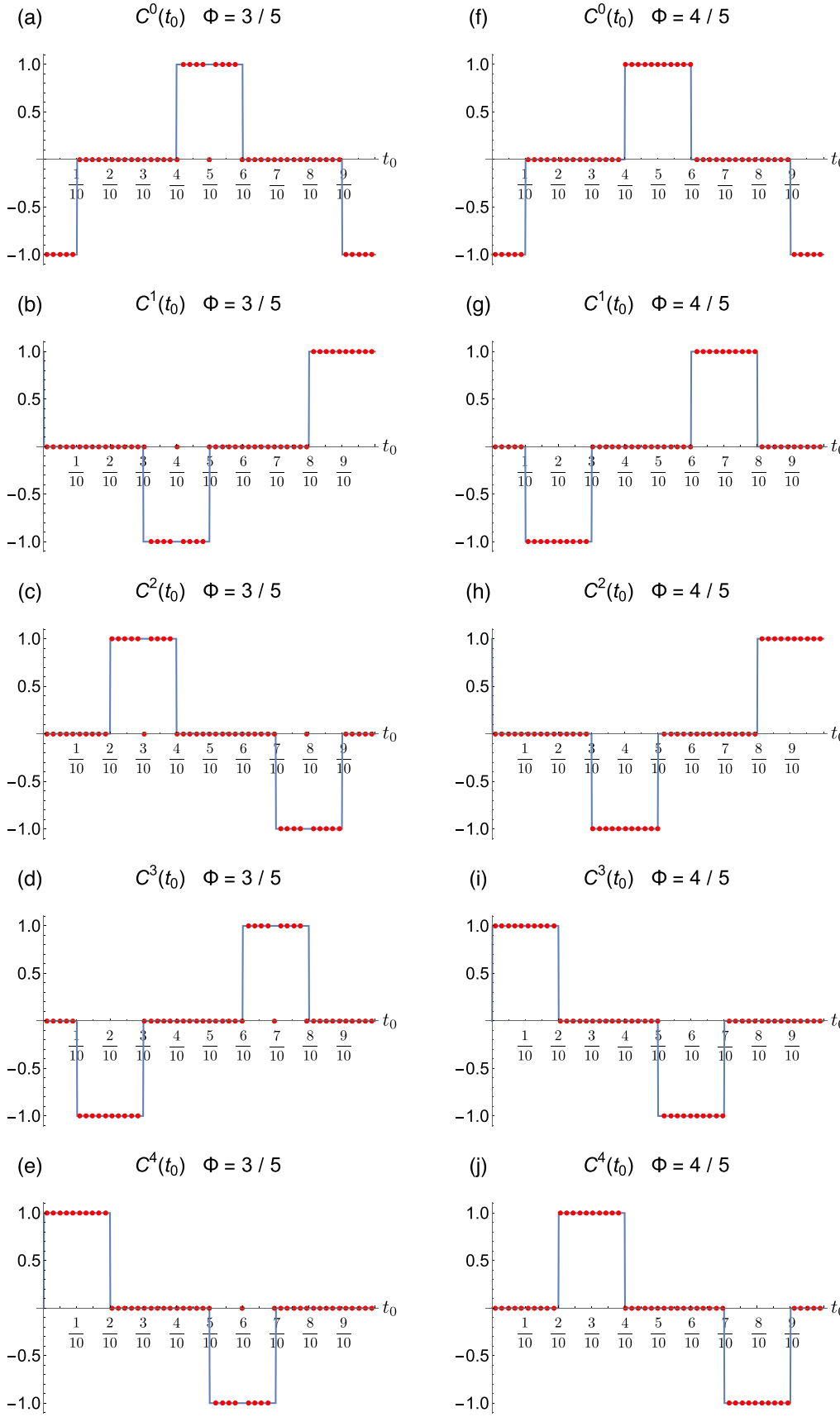


FIG. 18. t_0 dependence of the Chern numbers. $Q = 5$, $\Phi = \frac{3}{5}$ and $\frac{4}{5}$. $J_0 = -1.0$, $\delta = 0.5$ and $\Delta = 0.5$. The dimension of the Hilbert space is 545 ($L = 6$). The data points are for $t_0 = (i - 0.5)/51$, $i = 1, \ldots, 51$. Red points are numerical evaluation of Eq. (96) by Fukui-Hatsugai-Suzuki formula [136] and the lines are analytical results, Eq. (149).

VII. SUMMARY

A general scheme to construct a topological pump for a correlated system is proposed assuming the system is gapped and possesses a local gauge symmetry. Following the general idea, a topological pump of the $SU(Q)$ invariant quantum chain is discussed where the gauge invariance of the colored fermions plays a central role. Introducing a symmetry breaking perturbation, Q Chern numbers, which are given by the integral over the torus defined by the symmetric path in the synthetic Brillouin zone and the time cycle, characterize the bulk pump topologically. As for an open boundary condition, the sums of the discontinuities in Q different center-of-masses, which generate the large gauge transformation, give topological numbers of the pump as well. These discontinuities are topological numbers due to the edge states. Relations among the open/periodic/twisted boundary conditions are discussed in detail that justifies the bulk-edge correspondence. Using this bulk-edge correspondence, an explicit analytic formula for the Q Chern numbers that is associated with the Diophantine equation due to TKNN, is given. The low-energy spectra and the topological quantities are numerically evaluated. It confirms the consistency of the whole discussion.

The validity of the general scheme as demonstrated in detail for the $SU(Q)$ quantum chain opens a new paradigm of the topological pump for correlated systems.

ACKNOWLEDGMENTS

The work is supported in part by KAKENHI from JSPS Grants No. 17H06138, No. 23H01091 (Y.H.); Grants No. 21K13849, No. 23K13026 (Y.K.); and CREST JPMJCR19T1 (Y.H.).

APPENDIX A: COLOR DESCRIPTION OF $S = 1$ SPINS [74,75]

Let us first summarize a vector representation of $S = 1$ angular momentum in relation to the rotations in 3D as

$$R^x(\alpha) = \begin{bmatrix} 1 & & \\ & \cos\alpha & -\sin\alpha \\ & \sin\alpha & \cos\alpha \end{bmatrix} = 1 \oplus (\cos\alpha - i\sigma_y \sin\alpha)$$
$$= 1 \oplus e^{-i\alpha\sigma_y} \equiv e^{-i\alpha T^1},$$
$$T^1 = -i\begin{bmatrix} 0 & & \\ & 0 & 1 \\ & -1 & 0 \end{bmatrix}, \quad T^1_{ij} = -i\epsilon_{1ij}.$$

Similarly,

$$R_y(\beta) = e^{-i\beta T^2},$$
$$R_z(\gamma) = e^{-i\gamma T^3},$$

where

$$T^a_{ij} = -i\epsilon_{aij}.$$

To summarize, they are explicitly defined by

$$T^1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix}, \quad T^2 = \begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{bmatrix},$$
$$T^3 = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

They are spins as

$$([T^a, T^b])_{ij} = -\epsilon_{aik}\epsilon_{bkj} + \epsilon_{bik}\epsilon_{akj} = \epsilon_{aik}\epsilon_{bjk} - \epsilon_{bik}\epsilon_{ajk}$$
$$= \delta_{ab}\delta_{ij} - \delta_{aj}\delta_{ib} - \delta_{ba}\delta_{ij} + \delta_{bj}\delta_{ai}$$
$$= \delta_{ai}\delta_{bj} - \delta_{aj}\delta_{ib},$$
$$i\epsilon_{abc}T^c_{ij} = \epsilon_{abc}\epsilon_{cij} = \epsilon_{abc}\epsilon_{ijc}$$
$$= \delta_{ai}\delta_{bj} - \delta_{aj}\delta_{bi}.$$

Thus

$$[T^a, T^b] = i\epsilon_{abc}T^c.$$

Further

$$(T^aT^a)_{ij} = T^a_{ik}T^a_{kj} = -\epsilon_{aik}\epsilon_{akj} = \epsilon_{aki}\epsilon_{akj} = 2\delta_{ij},$$
$$T^aT^a = 2E_3.$$

This implies $S = 1$.

Let us consider a bilinear-biquadratic Hamiltonian of $S = 1$ quantum spin chain as

$$H = \sum_n [\cos\omega_S(\boldsymbol{S}_n \cdot \boldsymbol{S}_{n+1}) + \sin\omega_S(\boldsymbol{S}_n \cdot \boldsymbol{S}_{n+1})^2].$$

The spin-1 operators are written by color fermions, $c^\dagger = (c^\dagger_{+1}, c^\dagger_0, c^\dagger_{-1})$, due to Affleck as

$$S^a = c^\dagger T^a c = c^\dagger_\alpha T^a_{\alpha\beta} c_\beta, \quad \alpha, \beta = 1, 2, 3$$

with a constraint

$$\sum_\alpha n_\alpha = 1, \quad n_\alpha = c^\dagger_\alpha c_\alpha.$$

Let us check here.

Using a useful relations [138]

$$T^a_{\alpha\beta}T^a_{\gamma\delta} = \delta_{\alpha\delta}\delta_{\beta\gamma} - \delta_{\alpha\gamma}\delta_{\beta\delta},$$
$$(T^aT^b)_{\alpha\beta}(T^aT^b)_{\gamma\delta} = \delta_{\alpha\beta}\delta_{\gamma\delta} + \delta_{\alpha\gamma}\delta_{\beta\delta},$$

and noting the constraint, we have

$$[S^a, S^b] = c^\dagger_\alpha c_\beta c^\dagger_\gamma c_\delta \left(T^a_{\alpha\beta}T^b_{\gamma\delta} - T^b_{\alpha\beta}T^a_{\gamma\delta}\right)$$
$$= c^\dagger_\alpha(\delta_{\beta\gamma} - c^\dagger_\gamma c_\beta)c_\delta\left(T^a_{\alpha\beta}T^b_{\gamma\delta} - T^b_{\alpha\beta}T^a_{\gamma\delta}\right)$$
$$= c^\dagger_\alpha c_\delta\left(T^a_{\alpha\beta}T^b_{\beta\delta} - T^b_{\alpha\beta}T^a_{\beta\delta}\right)$$
$$= c^\dagger_\alpha c_\delta([T^a, T^b])_{\alpha\delta} = i\epsilon_{abc}S^c,$$
$$\boldsymbol{S}^2 = S^aS^a = c^\dagger_\alpha c_\beta c^\dagger_\gamma c_\delta T^a_{\alpha\beta}T^a_{\gamma\delta}$$
$$= c^\dagger_\alpha(\delta_{\beta\gamma} - c^\dagger_\gamma c_\beta)c_\delta T^a_{\alpha\beta}T^a_{\gamma\delta}$$
$$= c^\dagger_\alpha c_\delta T^a_{\alpha\beta}T^a_{\beta\delta} = c^\dagger_\alpha c_\beta(T^{a2})_{\alpha\beta} = 2\sum_\alpha n_\alpha = 2.$$

As for the interaction between the sites i, j, it is written as

$$\boldsymbol{S}_i \cdot \boldsymbol{S}_j = c^\dagger_{i,\alpha}T^a_{\alpha\beta}c_{i,\beta}c^\dagger_{j,\gamma}T^a_{\gamma\delta}c_{j,\delta}$$
$$= c^\dagger_{i,\alpha}c_{i,\beta}c^\dagger_{j,\gamma}c_{j,\delta}T^a_{\alpha\beta}T^a_{\gamma\delta}$$
$$= c^\dagger_{i,\alpha}c_{i,\beta}c^\dagger_{j,\gamma}c_{j,\delta}(\delta_{\alpha\delta}\delta_{\beta\gamma} - \delta_{\alpha\gamma}\delta_{\beta\delta}),$$

$$
\begin{aligned}
&= c^\dagger_{i,\alpha}c_{i,\beta}c^\dagger_{j,\beta}c_{j,\alpha} - c^\dagger_{i,\alpha}c_{i,\beta}c^\dagger_{j,\alpha}c_{j,\beta},\\
(\boldsymbol{S}_i\cdot\boldsymbol{S}_j)^2 &= \big(c^\dagger_{i,\alpha}c_{i,\beta}c^\dagger_{j,\gamma}c_{j,\delta}T^a_{\alpha\beta}T^a_{\gamma\delta}\big)\big(c^\dagger_{i,\alpha'}c_{i,\beta'}c^\dagger_{j,\gamma'}c_{j,\delta'}T^b_{\alpha'\beta'}T^b_{\gamma'\delta'}\big)\\
&= (c^\dagger_{i,\alpha}c_{i,\beta}c^\dagger_{i,\alpha'}c_{i,\beta'})(c^\dagger_{j,\gamma}c_{j,\delta}c^\dagger_{j,\gamma'}c_{j,\delta'})T^a_{\alpha\beta}T^a_{\gamma\delta}T^b_{\alpha'\beta'}T^b_{\gamma'\delta'}\\
&= c^\dagger_{i,\alpha}(\delta_{\beta\alpha'} - c^\dagger_{i,\alpha'}c_{i,\beta})c_{i,\beta'}\cdot c^\dagger_{j,\gamma}(\delta_{\gamma'\delta} - c^\dagger_{j\gamma'}c_{j,\delta})c_{j,\delta'}\\
&\quad T^a_{\alpha\beta}T^a_{\gamma\delta}T^b_{\alpha'\beta'}T^b_{\gamma'\delta'}\\
&= c^\dagger_{i,\alpha}c_{i,\beta'}\cdot c^\dagger_{j,\gamma}c_{j,\delta'}\delta_{\beta\alpha'}\delta_{\gamma'\delta}T^a_{\alpha\beta}T^a_{\gamma\delta}T^b_{\alpha'\beta'}T^b_{\gamma'\delta'}\\
&= c^\dagger_{i,\alpha}c_{i,\beta'}c^\dagger_{j,\gamma}c_{j,\delta'}T^a_{\alpha\beta}T^a_{\gamma\delta}T^b_{\beta\beta'}T^b_{\delta\delta'}\\
&= c^\dagger_{i,\alpha}c_{i,\beta'}c^\dagger_{j,\gamma}c_{j,\delta'}(T^aT^b)_{\alpha\beta'}(T^aT^b)_{\gamma\delta'}\\
&= c^\dagger_{i,\alpha}c_{i,\beta'}c^\dagger_{j,\gamma}c_{j,\delta'}(\delta_{\alpha\beta'}\delta_{\gamma\delta'} + \delta_{\alpha\gamma}\delta_{\beta'\delta'})\\
&= c^\dagger_{i,\alpha}c_{i,\alpha}c^\dagger_{j,\gamma}c_{j,\gamma} + c^\dagger_{i,\alpha}c_{i,\beta}c^\dagger_{j,\alpha}c_{j,\beta}\\
&= c^\dagger_{i,\alpha}c_{i,\beta}c^\dagger_{j,\alpha}c_{j,\beta} + 1.
\end{aligned}
$$

Note that the fermion number operators are written by the spin-1 operators as

$$n_{j,\alpha} = 1 - \big(S^\alpha_j\big)^2. \tag{A1}$$

Then omitting the constant, the Hamiltonian is given as

$$
\begin{aligned}
H &= \sum_{i<j}\cos\omega_S\, c^\dagger_{i,\alpha}c_{i,\beta}c^\dagger_{j,\beta}c_{j,\alpha}\\
&\quad + (\sin\omega_S - \cos\omega_S)c^\dagger_{i,\alpha}c_{i,\beta}c^\dagger_{j,\alpha}c_{j,\beta}\\
&= \cos\omega_S H^{(1)}(\{1\}) + (\sin\omega_S - \cos\omega_S)H^{(2)}(\{1\})
\end{aligned}
$$

where (slightly extending the parameter space) with $J^{1,2}_{\alpha\beta} = 1$ as

$$
\begin{aligned}
H^{(1)}\big(\big\{J^{(1)}_{i,\alpha;j,\beta}\big\}\big) &= \sum_{i<j}J^{(1)}_{i,\alpha j;j,\beta}c^\dagger_{i\alpha}c_{i,\beta}c^\dagger_{j,\beta}c_{j,\alpha},\\
H^{(2)}\big(\big\{J^{(2)}_{i,\alpha j;j,\beta}\big\}\big) &= \sum_{i<j}J^{(2)}_{i,\alpha j;j,\beta}c^\dagger_{i,\alpha}c_{i,\beta}c^\dagger_{j,\alpha}c_{j,\beta}.
\end{aligned}
$$

Hermiticity implies

$$
\begin{aligned}
\big[H^{(1)}\big(J^{(1)}_{i,\alpha;j,\beta}\big)\big]^\dagger &= \sum_{i<j}\big(J^{(1)}_{i,\alpha;j\beta}\big)^* c^\dagger_{j,\alpha}c_{j,\beta}c^\dagger_{i,\beta}c_{i,\alpha}\\
&= \sum_{i<j}\big(J^{(1)}_{i,\beta;j,\alpha}\big)^* c^\dagger_{i,\alpha}c_{i,\beta}c^\dagger_{j,\beta}c_{j,\alpha},\\
J^{(1)}_{i,\alpha;j,\beta} &= J^{(1)}_{i,\beta j,\alpha},\\
\big[H^{(2)}\big(\big\{J^{(2)}_{i,\alpha;j,\beta}\big\}\big)\big]^\dagger &= \sum_{i<j}\big(J^{(2)}_{i,\alpha;j\beta}\big)^* c^\dagger_{j,\beta}c_{j,\alpha}c^\dagger_{i,\beta}c_{i,\alpha}\\
&= \sum_{i<j}\big(J^{(2)}_{i,\beta;j,\alpha}\big)^* c^\dagger_{i,\alpha}c_{i,\beta}c^\dagger_{j,\alpha}c_{j,\beta},\\
J^{(2)}_{i,\alpha;j,\beta} &= J^{(2)}_{i,\beta j,\alpha}.
\end{aligned}
$$

It is written as a matrix form $(\boldsymbol{J}^{(1,2)}_{ij})^\dagger = \boldsymbol{J}^{(1,2)}_{ij}$ where $(\boldsymbol{J}^{(1,2)}_{ij})_{\alpha,\beta} = J^{(1,2)}_{i,\alpha;j,\beta}$.

APPENDIX B: *SU*(2) CASE

For the $N = 2$ case, let us perform a particle-hole transformation for the odd site j as

$$\mathcal{U}_\Theta = \prod_{j:\text{odd}}^{\leftarrow}\xi_j = \cdots\xi_5\xi_3\xi_1,\quad \mathcal{U}_\Theta^{-1} = \prod_{j:\text{odd}}^{\rightarrow}\xi_j = \xi_1\xi_3\xi_5\ldots,$$

$$\mathcal{U}_\Theta c_{j,\alpha}\mathcal{U}_\Theta^{-1} = \begin{cases} c_{j,\alpha} & j:\text{even}\\ c^\dagger_{j,\alpha} & j:\text{odd}\end{cases},$$

where $\xi_j = c_j + c_j^\dagger$ and $\xi_j^2 = 1$. The Hamiltonian $H^{(2)}$ ($J^2_{j,\alpha\beta} = J^{(2)}_j$) is transformed as

$$
\begin{aligned}
\mathcal{U}_\Theta H^{(2)}\big(\big\{J^{(2)}_j\big\}\big)\mathcal{U}_\Theta &= \sum_{j:\text{odd},\alpha\beta}J^{(2)}_j c_{j,\alpha}c^\dagger_{j+1,\alpha}c_{j+1,\beta}c^\dagger_{j,\beta}\\
&\quad + \sum_{j:\text{even},\alpha\beta}J^{(2)}_j c^\dagger_{j,\alpha}c_{j+1,\alpha}c^\dagger_{j+1,\beta}c_{j,\beta}\\
&= \sum_{j:\text{odd},\alpha\beta}J^{(2)}_j(\delta_{\alpha\beta}c^\dagger_{j+1,\alpha}c_{j+1,\beta}\\
&\quad - c^\dagger_{j,\beta}c^\dagger_{j+1,\alpha}c_{j+1,\beta}c_{j,\alpha})\\
&\quad - \sum_{j:\text{even},\alpha\beta}J^{(2)}_j c^\dagger_{j,\alpha}c^\dagger_{j+1,\beta}c_{j+1,\alpha}c_{j,\beta}\\
&= -H^{(1)}\big(\big\{J^{(2)}_j\big\}\big) + \text{const.}
\end{aligned}
$$

Note that $H^{(1)}$ is the $SU(2)$ Heisenberg model as confirmed by writing $c_{j,\alpha} = a_\alpha$ and $c_{j+1,\alpha} = b_\alpha$

$$
\begin{aligned}
\sum_{\alpha\beta}a^\dagger_\alpha a_\beta b^\dagger_\beta b_\alpha &= a^\dagger_\uparrow a_\uparrow b^\dagger_\uparrow b_\uparrow + a^\dagger_\uparrow a_\downarrow b^\dagger_\downarrow b_\uparrow + a^\dagger_\downarrow a_\uparrow b^\dagger_\uparrow b_\downarrow\\
&\quad + a^\dagger_\downarrow a_\downarrow b^\dagger_\downarrow b_\downarrow\\
&= n^a_\uparrow n^b_\uparrow + n^a_\downarrow n^b_\downarrow + S^a_+S^b_- + S^a_-S^b_+\\
&= 2\boldsymbol{S}^a\cdot\boldsymbol{S}^b + \frac{1}{2},
\end{aligned}
$$

where $4S^a_zS^b_z = (n^a_\uparrow - n^a_\downarrow)(n^b_\uparrow - n^b_\downarrow) = n^a_\uparrow n^b_\uparrow + n^a_\downarrow n^b_\downarrow - n^a_\uparrow n^b_\downarrow - n^a_\downarrow n^b_\uparrow = 2(n^a_\uparrow n^b_\uparrow + n^a_\downarrow n^b_\downarrow) - (n^a_\uparrow + n^a_\downarrow)(n^a_\uparrow + n^b_\downarrow) = 2(n^a_\uparrow n^b_\uparrow + n^a_\downarrow n^b_\uparrow) - 1$ due to the constraint.

APPENDIX C: ADIABATIC APPROXIMATION AND THE CURRENT

Now let us here summarize a derivation of the current $j = \langle G|\hat{J}|G\rangle$ in the adiabatic approximation [40],

$$\hat{J} = \hbar^{-1}\partial_\theta H(\theta).$$

The many-body state $|G\rangle$ is adiabatically evolved from the snapshot ground state of a time-dependent $H(t)$ as

$$\mathrm{i}\hbar|\dot{G}(t)\rangle = H(t)|G(t)\rangle,\quad |G(0)\rangle = |g\rangle,$$

where $|\alpha\rangle = |\alpha(t)\rangle$ is a orthonormalized eigenstates of the snapshot Hamiltonian and $|g\rangle$ is its ground state as

$$H(t)|\alpha(t)\rangle = E_\alpha(t)|\alpha(t)\rangle,\quad \langle\alpha|\beta\rangle = \delta_{\alpha\beta}.$$

Writing as $|G\rangle = e^{-(\mathrm{i}/\hbar)\int_0^t dt' E_g(t')}\sum_\alpha |\alpha\rangle c_\alpha$, $c_g(0) = 1$, $c_\alpha(0) = 0$, $(\alpha \neq g)$, the Schrodinger equation is written as

$$E_g \sum_\alpha |\alpha\rangle c_\alpha + \mathrm{i}\hbar \sum_\alpha (|\alpha\rangle \dot{c}_\alpha + |\partial_t \alpha\rangle c_\alpha) = \sum_\alpha E_\alpha |\alpha\rangle c_\alpha.$$

Then multiplying $\langle g|$, and noting that $|c_\alpha| \ll |c_g|$, $\alpha \neq g$, it reduces to $\dot{c}_g + c_g \langle g|\partial_t g\rangle \approx 0$, that implies

$$c_g = e^{\mathrm{i}\gamma(t)}, \quad \gamma(t) = \mathrm{i}\int_0^t dt' \langle g|\partial_t g\rangle.$$

Also multiplying $\langle\alpha|$, $(\alpha \neq g)$, one obtains $E_g c_\alpha + \mathrm{i}\hbar\langle\alpha|\partial_t g\rangle c_g \approx E_\alpha c_\alpha$ that implies $c_\alpha = \mathrm{i}\hbar\frac{\langle\alpha|\partial_t g\rangle c_g}{E_\alpha - E_g}$. Now the time-dependent ground state is given as

$$|G\rangle = e^{-(\mathrm{i}/\hbar)\int_0^t dt' E_g(t')} e^{\mathrm{i}\gamma(t)} \left[|g\rangle + \mathrm{i}\hbar \sum_{\alpha\neq g} \frac{|\alpha\rangle\langle\alpha|\partial_t g\rangle}{E_\alpha - E_g} \right].$$

Then the expectation value of $\partial_\theta H$ is written as

$$\langle G|\partial_\theta H|G\rangle = \langle g|\partial_\theta H|g\rangle + \mathrm{i}\hbar \sum_{\alpha\neq g} \frac{\langle g|\partial_\theta H|\alpha\rangle\langle\alpha|\partial_t g\rangle - \langle\partial_t g|\alpha\rangle\langle\alpha|\partial_\theta H|g\rangle}{E_\alpha - E_g}.$$

Here let us remind a general relation $\langle g|\partial_\theta H|\alpha\rangle = (E_\alpha - E_g)\langle g|\partial_\theta\alpha\rangle$, that obeys from taking a derivative of the eigenequation $H|\alpha\rangle = E_\alpha|\alpha\rangle$ as $\langle\beta|\partial H|\alpha\rangle + E_\beta\langle\beta|\partial\alpha\rangle = \partial E_\alpha\langle\beta|\alpha\rangle + E_\alpha\langle\beta|\partial\alpha\rangle$,

$$\begin{aligned}
\delta j_x &= \langle G|J|G\rangle - \langle g|J|g\rangle, \\
&= \mathrm{i}\sum_{\alpha\neq g}(\langle g|\partial_\theta\alpha\rangle\langle\alpha|\partial_t g\rangle + \langle\partial_t g|\alpha\rangle\langle\alpha|\partial_\theta g\rangle) \\
&= -\mathrm{i}\sum_{\alpha\neq g}(\langle\partial_\theta g|\alpha\rangle\langle\alpha|\partial_t g\rangle - \langle\partial_t g|\alpha\rangle\langle\alpha|\partial_\theta g\rangle) \\
&= -\mathrm{i}\sum_{\alpha}(\langle\partial_\theta g|\alpha\rangle\langle\alpha|\partial_t g\rangle - \langle\partial_t g|\alpha\rangle\langle\alpha|\partial_\theta g\rangle) \\
&= -\mathrm{i}(\langle\partial_\theta g|\partial_t g\rangle - \langle\partial_t g|\partial_\theta g\rangle) = -\mathrm{i}B,
\end{aligned}$$

where the field strength B and the Berry connection A_μ are defined as

$$B = \partial_\theta A_t - \partial_t A_\theta, \quad A_\mu = \langle g|\partial_\mu g\rangle, \quad \mu = \theta, t.$$

To summarize, in the adiabatic approximation, we have

$$|G\rangle = C\left[|g\rangle + \mathrm{i}\hbar\sum_{\alpha\neq g}\frac{|\alpha\rangle\langle\alpha|\partial_t g\rangle}{E_\alpha - E_g}\right],$$

$$\langle G|\hat{J}|G\rangle = \langle g|\hat{J}|g\rangle - iB,$$

where $C = e^{-(\mathrm{i}/\hbar)\int_0^t dt' E_g(t')} e^{\mathrm{i}\gamma(t)}$.

APPENDIX D: BERRY PHASE AND GAUGE FIXING [63,106,108,109]

Let us start a D-dimensional Euclidean space $x = (x_1, x_2, \ldots, x_D) \in \mathbb{R}^D$ as a parameter space of the Hamiltonian $H(x)$. As for the Berry phase associated with a loop ℓ, we further assume that its ground state $|g(x)\rangle$,

$$H|g\rangle = |g\rangle E_g, \ \langle g|g\rangle = 1,$$

is gapped along the path ℓ,

$$H(x)|n(x)\rangle = |n(x)\rangle E_n(x), \ E_n(x) > E_g(x), \quad n \neq g, \ {}^\forall x \in \ell.$$

Note that the phase of the snapshot eigenstate is arbitrary

$$|g\rangle = |g'\rangle e^{i\Theta}, \quad \Theta \in \mathbb{R}$$

where $H|g'\rangle = |g'\rangle E_g$ and $\langle g'|g'\rangle = 1$.

As is well known the Berry connection $A_\mu = \langle g|\partial_\mu g\rangle$, $\mu = 1, \ldots, D$ depends of the phase of the ground state as

$$A_\mu = \langle g|\partial_\mu g\rangle = A'_\mu + i\partial_\mu\Theta,$$

where $A'_\mu = \langle g'|\partial_\mu g'\rangle$ and we assume that the parameter dependence of $|g\rangle$ and $|g'\rangle$ is smooth and differentiable. The Berry phase is define as

$$i\gamma_\ell = \int_\ell dx_\mu A_\mu,$$

which is gauge dependent (summation over μ is assumed). To be specific, let us assume the closed path ℓ is parameterized by $\theta \in [\theta_i, \theta_f]$ and take a different gauge when $\theta \in [\theta_1, \theta_2]$. Then we have

$$\begin{aligned}
\gamma_\ell &= -i\int_{t_i}^{t_f} d\theta\, \dot{x}_\mu A_\mu, \quad \left(\dot{x}_\mu = \frac{dx_\mu}{d\theta}\right) \\
&= -i\left[\int_{t_i}^{t_1} d\theta\, \dot{x}_\mu A_\mu + \int_{t_1}^{t_2} d\theta\, \dot{x}_\mu A_\mu + \int_{t_2}^{t_f} d\theta\, \dot{x}_\mu A_\mu\right] \\
&= \gamma'_\ell + \Delta\Theta,
\end{aligned}$$

where $\Delta\Theta = \Theta(x(\theta))|_{\theta_1}^{\theta_2}$ and

$$\gamma'_\ell = -i\left[\int_{t_i}^{t_1} d\theta\, \dot{x}_\mu A_\mu + \int_{t_1}^{t_2} d\theta\, \dot{x}_\mu A'_\mu + \int_{t_2}^{t_f} d\theta\, \dot{x}_\mu A_\mu\right].$$

Since $\Delta\Theta$ is arbitrary, γ_ℓ does not have a definite meaning unless one fixes the gauge globally.

The gauge is explicitly fixed by the scheme in Ref. [108]. Let us start by taking an arbitrary state $|\phi\rangle$ as a reference state. Taking a constant $|\phi\rangle$ is simple but it may not be necessarily constant but need to be single valued along the path ℓ. Then taking a gauge-independent projection $P = |g\rangle\langle g|$, the gauge fixing state by ϕ is given by

$$|g_\phi\rangle = P|\phi\rangle/\sqrt{N_\phi},$$

where $N_\phi = \langle\phi|P|\phi\rangle = \eta^*_\phi\eta_\phi$ and $\eta_\phi = \langle\phi|g\rangle$. This gauge fixing is only allowed if $N_\phi = |\eta_\phi|^2 \neq 0$. Since $\eta_\phi \in \mathbb{C}$, this condition is always satisfied all over the (one-dimensional) loop ℓ by a suitable choice of $|\phi\rangle$ (if $\eta_\phi = 0$ for ${}^\exists x \in \ell$, one may modify ϕ slightly).

Then by taking a different $|\phi_i\rangle$, $i = 1, 2$, they are related with each other as

$$|g_{\phi_1}\rangle = |g_{\phi_2}\rangle e^{i\Theta_{12}},$$

$$e^{i\Theta_{12}} = \sqrt{\frac{N_{\phi_2}}{N_{\phi_1}}}\frac{\langle g|\phi_1\rangle}{\langle g|\phi_2\rangle} = e^{i(\theta_1 - \theta_2)},$$

where $\langle g|\phi_i\rangle = |\langle g|\phi_i\rangle|e^{i\theta_i}$, $i = 1, 2$.

Assuming that $|g_{\phi_1}\rangle$ and $|g_{\phi_2}\rangle$ are single valued on the loop $x \in \ell$, Berry phases γ_1 and γ_1 are related as

$$\gamma_1 = \gamma_2 + \int_\ell d\Theta_{12} \equiv \gamma_2, \quad \mod 2\pi$$

since $e^{i\Theta}$ is single valued over the loop and $\int_\ell d\Theta_{12} = \int_{\theta_i}^{\theta_f} d\theta\, \dot{x}_\mu \partial_\mu \Theta(x) = \int_{\theta_i}^{\theta_f} d\theta\, \frac{d}{d\theta}\Theta(x(\theta)) = \Theta(x(\theta))|_{\theta_i}^{\theta_f} = 2\pi n$, $n \in \mathbb{Z}$.

This ambiguity also clears from the discretized expression of the Berry phase ($\lim_{L\to\infty}\gamma_L = \gamma$)

$$\gamma_L \equiv \mathrm{Arg}\,\langle g_0|g_1\rangle \cdots\cdots \langle g_n|g_{n+1}\rangle \cdots \langle g_{L-1}|g_L\rangle,$$

where $\theta_n = \theta_i + \frac{n}{L}(\theta_f - \theta_i)$, $n = 1, \cdots, L$ and $|g_0\rangle \equiv |g_L\rangle$. The expression is gauge invariant but the Arg is well defined only in modulo 2π.

APPENDIX E: DIMER LIMIT

Assuming L: even, let us consider a dimer limit (1)$J_o < 0$, $J_e = 0$ or (2)$J_o = 0$, $J_e < 0$ (o: odd and e: even) for the twisted Hamiltonian Eq. (40). The Berry phase defined on the canonical path $\ell = \ell_{V_\alpha G V_{\alpha+1}}$, $\alpha = 0, \ldots, Q$, $\gamma_Q = -i\int_\ell d\theta\, \langle g_{\rm tw}|\partial_\theta g_{\rm tw}\rangle$ is quantized due to the Z_Q symmetry as discussed.

In the limit (1), it is trivially $\gamma_Q = 0$ since the twist does not affect the Hamiltonian. Also inclusion of J_e unless the finite gap closes, $\gamma_Q = 0$ even for the finite coupling case. Only after the gap closing, the Berry phase may change. This is a topological symmetry protection.

As for the case (2), the Hamiltonian $H_{\rm tw}^{J_o=0}$ is decoupled for each dimers and is written as [see Eq. (40)]

$$H_{\rm tw}^{J_o=0} = J_e \sum_{\alpha,\beta} e^{-i(\varphi_\alpha - \varphi_\beta)} c_{L,\alpha}^\dagger c_{1,\alpha}^\dagger c_{1,\beta} c_{L,\beta} + (\varphi_\alpha, \varphi_\beta \text{ independent terms}),$$

where the last terms do not include $c_{L,\alpha}$.

This is gauge out by the transformation at the site L [63,66,122],

$$\mathcal{U}_L = e^{-i\sum_\alpha \varphi_\alpha \hat{n}_{L,\alpha}},$$
$$\mathcal{U}_L c_{L,\alpha} \mathcal{U}_L^\dagger = e^{+i\varphi_\alpha} c_{L,\alpha},$$
$$\mathcal{U}_L H_0^{J_o=0} \mathcal{U}_L^\dagger = H_{\rm tw}^{J_o=0},$$

where $H_0^{J_o=0}$ is without twist.

Then the ground state of the twisted Hamiltonian $H_{\rm tw}$ is given by $|g\rangle = \mathcal{U}_L|g_0\rangle$ where $|g_0\rangle$ is φ_α independent as $H^0|g_0\rangle = |g_0\rangle E$. The Berry phase is given by

$$\gamma_Q = -i\int_\ell d\theta\, \langle g|\mathcal{U}_L^\dagger \partial_\theta \mathcal{U}_L|g\rangle = -\int_\ell d\theta\, \partial_\theta \varphi_\alpha \langle g_0|\hat{n}_{L,\alpha}|g_0\rangle = -\frac{1}{Q}\sum_\alpha \Delta\varphi_\alpha,$$

since $\langle g_0|\hat{n}_{L,\alpha}|g_0\rangle = 1/Q$ due to the $SU(Q)$ invariance of H^0. Noting the discussion in Sec. III C, $\sum_\alpha \Delta\varphi_\alpha = 2\pi(Q-1)$ for a path $\ell_{V_0 G V_1}$ and $\sum_\alpha \Delta\varphi_\alpha = -2\pi$ for a path, $\ell_{V_\beta G V_{\beta+1}}$ ($\beta = 2, \ldots, Q$),

$$\gamma_Q = +\frac{2\pi}{Q}, \quad \mod 2\pi.$$

APPENDIX F: SYMMETRY OF THE PATHS

Let us first discuss $N = 3$ case to be simple. The three paths are decomposed into $\ell_{V_j G}$ ($j = 0, 1, 2$) and explicitly parameterized by $\theta \in (0, \frac{2\pi}{3})$ as

$$\ell_{V_0 G} = \{(\theta_1, \theta_2) = (\theta, \theta)\},$$
$$\ell_{V_1 G} = \{(\theta_1, \theta_2) = (2\pi - 2\theta, \theta) \equiv (-2\theta, \theta)\},$$
$$\ell_{V_2 G} = \{(\theta_1, \theta_2) = (\theta, 2\pi - 2\theta) \equiv (\theta, -2\theta)\}.$$

The modification associated with the twist is given by the gauge transformation at the site L,

$$\boldsymbol{\theta} \in \ell_{V_0 G} \quad (\varphi_1, \varphi_2, \varphi_3) = (\theta, 2\theta, 0),$$
$$\boldsymbol{\theta} \in \ell_{V_1 G} \quad (\varphi_1, \varphi_2, \varphi_3) = (0, \theta, 2\theta),$$
$$\boldsymbol{\theta} \in \ell_{V_2 G} \quad (\varphi_1, \varphi_2, \varphi_3) = (2\theta, 0, \theta),$$

where ($\varphi_0 = \varphi_3$)

$$\theta_1 = \varphi_1 - \varphi_0,$$
$$\theta_2 = \varphi_2 - \varphi_1,$$
$$\theta_3 = \varphi_0 - \varphi_2.$$

Then Z_3 transformation of the Hamiltonian is as follows:

$$\mathcal{U}_{Z_3} H(V_0 G) \mathcal{U}_{Z_3}^\dagger = H(V_1 G),$$
$$\mathcal{U}_{Z_3} H(V_1 G) \mathcal{U}_{Z_3}^\dagger = H(V_2 G),$$
$$\mathcal{U}_{Z_3} H(V_2 G) \mathcal{U}_{Z_3}^\dagger = H(V_0 G).$$

In case of the generic $SU(Q)$, the Q paths are defined in a d-dimensional parameter space ($d = Q - 1$) and parameterized by $\theta \in (0, \frac{2\pi}{Q})$ as

$$\ell_{V_0 G} = \{(\theta_1, \ldots, \theta_d) = (\theta, \ldots, \theta)\},$$
$$\ell_{V_j G} = \{(\theta_1, \ldots, \theta_d) = (\theta, \ldots, \overbrace{-(Q-1)\theta}^{j}, \ldots, \theta)\},$$
$$j = 1, \ldots, Q-1 = d$$
$$\boldsymbol{\theta} \in \ell_{V_0 G} \quad (\varphi_1, \ldots, \varphi_N) = (\theta, \ldots, \theta, -(Q-1)\theta),$$
$$\boldsymbol{\theta} \in \ell_{V_j G} \quad (\varphi_1, \ldots, \varphi_Q) = (\theta, \ldots, \overbrace{-(Q-1)\theta}^{j}, \ldots, \theta)\}.$$

Again Z_Q invariance, $c_{j,\alpha} \to c_{j,\alpha+1}$, of the Hamiltonian is written as

$$H(\theta') = \mathcal{U}_{Z_Q}^j H(\theta) (\mathcal{U}_{Z_Q}^j)^\dagger, \quad j = 1, \ldots, Q = d+1, \quad \text{(F1)}$$

where $\theta \in \ell_{V_0 G}$ and $\theta' \in \ell_{V_j G}$. It implies a relation between the ground states

$$|g(\theta')\rangle = \mathcal{U}_{Z_Q}^j |g(\theta)\rangle.$$

Note that at the vertices V_j's and $G = (\frac{2\pi}{N}, \ldots)$, the Hamiltonian is invariant as

$$[H(V_j), \mathcal{U}_{Z_Q}] = 0, \tag{F2}$$

$$[H(G), \mathcal{U}_{Z_Q}] = 0. \tag{F3}$$

APPENDIX G: DISCONTINUITY OF THE CENTER OF MASS: $\Delta P = \pm\frac{1}{2}$

Noting that $N = \sum_j \hat{\rho}_j$, $\hat{\rho}_j = (-1)^j \hat{n}_j$ commutes with the Hamiltonian as

$$[H, N_\rho] = 0, \quad N_\rho = \sum_j \hat{\rho}_j,$$

let us assume that a (generic) level crossing of the ground state $|g(t)\rangle$ at $t = t_i$ between $|g_-\rangle$ and $|g_+\rangle$ as

$$\langle g_+|N|g_+\rangle - \langle g_-|N|g_-\rangle = \sum_j (\rho_j^+ - \rho_j^-) = \pm 1,$$
$$\rho_j^\pm = \langle g_\pm|\hat{\rho}_j|g_\pm\rangle.$$

We may further assume that this is due to the edge state localized near $j \sim L$ with a localization length ξ. This can be justified by the low-energy spectrum as discussed in Sec. VI A. Then it implies

$$\Delta\rho_j = \rho_j^+ - \rho_j^- \to Ce^{j/\xi}, \quad (L \to \infty).$$

The normalization constant is evaluated as $\pm C^{-1} = \sum_j e^{j\xi^{-1}} = e^{1/\xi}\frac{e^{L/\xi}-1}{e^{1/\xi}-1} = e^{L/\xi} + \mathcal{O}(1)$ and

$$\begin{aligned}\Delta P &= \langle g_+|P|g_+\rangle - \langle g_-|P|g_-\rangle \\ &= \sum_j \frac{1}{L}(j - j_0)\Delta\rho_j = CL^{-1}\sum_j (j - j_0)e^{j\xi^{-1}} \\ &\to CL^{-1}\left(\frac{d}{d\xi^{-1}}e^{L\xi^{-1}} - j_0 e^{L\xi^{-1}}\right) \\ &= \pm\frac{1}{2} + \mathcal{O}(L^{-1}), \ L \to \infty.\end{aligned}$$

[1] R. B. Laughlin, Phys. Rev. B **23**, 5632 (1981).
[2] D. J. Thouless, M. Kohmoto, M. P. Nightingale, and M. den Nijs, Phys. Rev. Lett. **49**, 405 (1982).
[3] J. E. Avron, R. Seiler, and B. Simon, Phys. Rev. Lett. **51**, 51 (1983).
[4] Q. Niu, D. J. Thouless, and Y. S. Wu, Phys. Rev. B **31**, 3372 (1985).
[5] M. Kohmoto, Ann. Phys. **160**, 343 (1985).
[6] F. D. M. Haldane, Phys. Rev. Lett. **61**, 2015 (1988).
[7] B. I. Halperin, Phys. Rev. B **25**, 2185 (1982).
[8] A. H. MacDonald, Phys. Rev. B **29**, 6563 (1984).
[9] Y. Hatsugai, Phys. Rev. B **48**, 11851 (1993).
[10] Y. Hatsugai, Phys. Rev. Lett. **71**, 3697 (1993).
[11] X.-L. Qi, Y.-S. Wu, and S.-C. Zhang, Phys. Rev. B **74**, 045125 (2006).
[12] H. Schulz-Baldes, J. Kellendonk, and T. Richter, J. Phys. A: Math. Gen. **33**, L27 (1999).
[13] G. M. Graf and M. Porta, Commun. Math. Phys. **324**, 851 (2013).
[14] E. Prodan and H. Schulz-Baldes, *Bulk and Boundary Invariants for Complex Topological Insulators*, Mathematical Physics Studies (Springer-Verlag, New York, 2016).
[15] V. Mathai and G. C. Thiang, Commun. Math. Phys. **345**, 675 (2016).
[16] C. Tauber, P. Delplace, and A. Venaille, Phys. Rev. Res. **2**, 013147 (2020).
[17] K. Yatsugi, T. Yoshida, T. Mizoguchi, Y. Kuno, H. Iizuka, Y. Tadokoro, and Y. Hatsugai, Commun. Phys. **5**, 180 (2022).
[18] X. G. Wen, Phys. Rev. Lett. **64**, 2206 (1990).
[19] F. D. M. Haldane and S. Raghu, Phys. Rev. Lett. **100**, 013904 (2008).
[20] Z. Wang, Y. D. Chong, J. D. Joannopoulos, and M. Soljačić, Phys. Rev. Lett. **100**, 013905 (2008).
[21] Z. Wang, Y. Chong, J. D. Joannopoulos, and M. Soljačić, Nature (London) **461**, 772 (2009).
[22] E. Prodan and C. Prodan, Phys. Rev. Lett. **103**, 248101 (2009).
[23] M. Hafezi, E. A. Demler, M. D. Lukin, and J. M. Taylor, Nat. Phys. **7**, 907 (2011).
[24] T. Ochiai, Phys. Rev. B **86**, 075152 (2012).
[25] L.-H. Wu and X. Hu, Phys. Rev. Lett. **114**, 223901 (2015).
[26] T. Ozawa, H. M. Price, A. Amo, N. Goldman, M. Hafezi, L. Lu, M. C. Rechtsman, D. Schuster, J. Simon, O. Zilberberg, and I. Carusotto, Rev. Mod. Phys. **91**, 015006 (2019).
[27] J. Ningyuan, C. Owens, A. Sommer, D. Schuster, and J. Simon, Phys. Rev. X **5**, 021031 (2015).
[28] V. V. Albert, L. I. Glazman, and L. Jiang, Phys. Rev. Lett. **114**, 173902 (2015).
[29] L. M. Nash, D. Kleckner, A. Read, V. Vitelli, A. M. Turner, and W. T. M. Irvine, Proc. Natl. Acad. Sci. USA **112**, 14495 (2015).
[30] R. Süsstrunk and S. D. Huber, Science **349**, 47 (2015).
[31] M. Mancini, G. Pagano, G. Cappellini, L. Livi, M. Rider, J. Catani, C. Sias, P. Zoller, M. Inguscio, M. Dalmonte, and L. Fallani, Science **349**, 1510 (2015).
[32] C. L. Kane and E. J. Mele, Phys. Rev. Lett. **95**, 146802 (2005).
[33] B. A. Bernevig, T. L. Hughes, and S.-C. Zhang, Science **314**, 1757 (2006).
[34] M. König, S. Wiedmann, C. Brüne, A. Roth, H. Buhmann, L. W. Molenkamp, X.-L. Qi, and S.-C. Zhang, Science **318**, 766 (2007).
[35] J. Moore, Nat. Phys. **5**, 378 (2009).
[36] M. Z. Hasan and C. L. Kane, Rev. Mod. Phys. **82**, 3045 (2010).
[37] P. Delplace, J. B. Marston, and A. Venaille, Science **358**, 1075 (2017).
[38] T. Yoshida, T. Mizoguchi, and Y. Hatsugai, Phys. Rev. E **104**, 025003 (2021).
[39] L. Yamauchi, T. Hayata, M. Uwamichi, T. Ozawa, and K. Kawaguchi, arXiv:2008.10852.
[40] D. J. Thouless, Phys. Rev. B **27**, 6083 (1983).
[41] Q. Niu and D. J. Thouless, J. Phys. A: Math. Gen. **17**, 2453 (1984).

[42] J. E. Avron, A. Raveh, and B. Zur, Rev. Mod. Phys. **60**, 873 (1988).
[43] P. W. Brouwer, Phys. Rev. B **58**, R10135 (1998).
[44] B. L. Altshuler and L. I. Glazman, Science **283**, 1864 (1999).
[45] R. Shindou, J. Phys. Soc. Jpn. **74**, 1214 (2005).
[46] E. Berg, M. Levin, and E. Altman, Phys. Rev. Lett. **106**, 110405 (2011).
[47] Y. E. Kraus, Y. Lahini, Z. Ringel, M. Verbin, and O. Zilberberg, Phys. Rev. Lett. **109**, 106402 (2012).
[48] S. Nakajima, T. Tomita, S. Taie, T. Ichinose, H. Ozawa, L. Wang, M. Troyer, and Y. Takahashi, Nat. Phys. **12**, 296 (2016).
[49] M. Lohse, C. Schweizer, O. Zilberberg, M. Aidelsburger, and I. Bloch, Nat. Phys. **12**, 350 (2016).
[50] Y. Hatsugai and T. Fukui, Phys. Rev. B **94**, 041102(R) (2016).
[51] L. Wang, M. Troyer, and X. Dai, Phys. Rev. Lett. **111**, 026802 (2013).
[52] M. Nakagawa, T. Yoshida, R. Peters, and N. Kawakami, Phys. Rev. B **98**, 115147 (2018).
[53] H. Watanabe and M. Oshikawa, Phys. Rev. X **8**, 021065 (2018).
[54] Y. Kuno and Y. Hatsugai, Phys. Rev. Res. **2**, 042024(R) (2020).
[55] C. Schweizer, M. Lohse, R. Citro, and I. Bloch, Phys. Rev. Lett. **117**, 170405 (2016).
[56] Y. Kuno and Y. Hatsugai, Phys. Rev. B **104**, 045113 (2021).
[57] T.-S. Zeng, W. Zhu, and D. N. Sheng, Phys. Rev. B **94**, 235139 (2016).
[58] D. González-Cuadra, A. Bermudez, P. R. Grzybowski, M. Lewenstein, and A. Dauphin, Nat. Commun. **10**, 2694 (2019).
[59] Y. Kuno and Y. Hatsugai, Phys. Rev. B **104**, 125146 (2021).
[60] K. Kudo, Y. Kuno, and Y. Hatsugai, Phys. Rev. B **104**, L241113 (2021).
[61] Z.-C. Gu and X.-G. Wen, Phys. Rev. B **80**, 155131 (2009).
[62] F. Pollmann, A. M. Turner, E. Berg, and M. Oshikawa, Phys. Rev. B **81**, 064439 (2010).
[63] Y. Hatsugai, J. Phys. Soc. Jpn. **75**, 123601 (2006).
[64] T. Hirano, H. Katsura, and Y. Hatsugai, Phys. Rev. B **77**, 094431 (2008).
[65] N. Chepiga, I. Affleck, and F. Mila, Phys. Rev. B **94**, 205112 (2016).
[66] T. Kariyado, T. Morimoto, and Y. Hatsugai, Phys. Rev. Lett. **120**, 247202 (2018).
[67] S. Fubasami, T. Mizoguchi, and Y. Hatsugai, Phys. Rev. B **100**, 014438 (2019).
[68] H. Araki, T. Mizoguchi, and Y. Hatsugai, Phys. Rev. Res. **2**, 012009(R) (2020).
[69] I. Affleck, T. Kennedy, E. H. Lieb, and H. Tasaki, Phys. Rev. Lett. **59**, 799 (1987).
[70] D. Maekawa and H. Tasaki, J. Math. Phys. **64**, 031901 (2023).
[71] A. Fabre, J.-B. Bouhiron, T. Satoor, R. Lopes, and S. Nascimbene, Phys. Rev. Lett. **128**, 173202 (2022).
[72] C.-H. Li, Y. Yan, S.-W. Feng, S. Choudhury, D. B. Blasing, Q. Zhou, and Y. P. Chen, PRX Quantum **3**, 010316 (2022).
[73] R. Citro and M. Aidelsburger, Nat. Rev. Phys. **5**, 87 (2023).
[74] I. Affleck, Nucl. Phys. B **265**, 409 (1986).
[75] I. Affleck, Nucl. Phys. B **305**, 582 (1988).
[76] I. Affleck, Phys. Rev. Lett. **54**, 966 (1985).
[77] C. Itoi and M.-H. Kato, Phys. Rev. B **55**, 8295 (1997).
[78] A. V. Chubukov, Phys. Rev. B **43**, 3337 (1991).
[79] G. Fáth and J. Sólyom, Phys. Rev. B **51**, 3620 (1995).
[80] N. Kawashima, Prog. Theor. Phys. Suppl. **145**, 138 (2002).
[81] A. Läuchli, G. Schmid, and S. Trebst, Phys. Rev. B **74**, 144426 (2006).
[82] H. Yang, L. Li, K. Okunishi, and H. Katsura, Phys. Rev. B **107**, 125158 (2023).
[83] This constraint is not necessary for general Q case *a priori*. Removing this constraint may provide a new (gapped) phase associated with additional interactions.
[84] E. Zohar, J. I. Cirac, and B. Reznik, Phys. Rev. A **88**, 023617 (2013).
[85] E. Zohar, J. I. Cirac, and B. Reznik, Rep. Prog. Phys. **79**, 014401 (2015).
[86] S. Taie, E. Ibarra-García-Padilla, N. Nishizawa, Y. Takasu, Y. Kuno, H.-T. Wei, R. T. Scalettar, K. R. A. Hazzard, and Y. Takahashi, Nat. Phys. **18**, 1356 (2022).
[87] Although we used a fermion representation with the constraint $\sum_\alpha n_\alpha = 1$, releasing this constraint and inclusion of the fermion hopping and interaction, one may consider a strongly correlated fermion systems. Since the Hamiltonian $H^{(2)}$ is a pair hopping or pairing interaction, its ground state can be superconducting and the pump, in principle, offers a new platform. Also when one replace the colored fermions by bosons, it also gives an interesting model, although it differs from the "$SU(Q)$ quantum spin" of the original model due to Affleck.
[88] We summarize the notation introduced by Affleck [74–77] for the "$SU(Q)$ spin", $(S_i)_{\alpha,\beta}$, $(\alpha,\beta = 1,\cdots Q)$ that satisfies the commutation $[(S_i)_{\alpha,\beta},(S_j)_{\gamma,\delta}] = \delta_{ij}(\delta_{\alpha,\delta}(S_i)_{\gamma,\beta} - \delta_{\gamma,\beta}(S_i)_{\alpha,\delta})$. The "$SU(Q)$-spin" is written by the canonical fermions $c_{i,\alpha}$ as $(S_i)_{\alpha,\beta} = c^\dagger_{i,\alpha}c_{i,\beta} - Q^{-1}\delta_{\alpha,\beta}$ supplemented by the constraint $\sum_\alpha n_\alpha = 1$. Then it implies $H^{(1)}(\{J\}) = J\sum_j (S_j)_{\alpha,\beta}(S_{j+1})_{\beta,\alpha} + \text{const}$. This is obtained by the strong coupling limit of the $SU(Q)$ Hubbard model [75] and realized experimentally as well [86]. Also it implies $H^{(2)}(\{J\}) = -J\sum_j (S_j)_{\alpha,\beta}(S_{j+1})_{\alpha,\beta} + \text{const.}$, where we assume the representation above for i: even and the other representation (by using the same symbol $(S_i)_{\alpha,\beta}$), $(S_i)_{\alpha,\beta} = -(c^\dagger_{i,\beta}c_{i,\alpha} - Q^{-1}\delta_{\alpha,\beta})$, for i: odd [74,76].
[89] When $Q = 3$, it is spanned by the Gell-Mann matrices

$$\lambda_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \lambda_2 = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = S^3,$$

$$\lambda_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \lambda_4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

$$\lambda_5 = \begin{bmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix} = -S^2, \quad \lambda_6 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

$$\lambda_7 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix} = S^1, \quad \lambda_8 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}/\sqrt{3}$$

.
[90] For an Hermite matrix $\boldsymbol{G}$, let us define $\mathcal{G} = \boldsymbol{c}^\dagger \boldsymbol{G}\boldsymbol{c}$ where $\boldsymbol{c}^\dagger = (c_1^\dagger, c_2^\dagger, \ldots)$. Assuming that $\boldsymbol{G}$ is diagonalized by a unitary matrix $\boldsymbol{U}$ as $\boldsymbol{G} = \boldsymbol{U}\boldsymbol{g}\boldsymbol{U}^\dagger$, $\boldsymbol{g} = \text{diag}\,(g_1, g_2, \ldots)$, $g_i \in \mathbb{R}$, we have $\mathcal{G} = \sum_i g_i d_i^\dagger d_i$ where $\boldsymbol{d} \equiv \boldsymbol{U}^\dagger \boldsymbol{c}$, $\boldsymbol{c} = \boldsymbol{U}\boldsymbol{d}$. Then for $\mathcal{U} = e^{-i\theta\mathcal{G}}$, $\mathcal{U}d_i\mathcal{U}^\dagger = e^{-i\theta g_i d_i^\dagger d_i} d_i e^{i\theta g_i d_i^\dagger d_i} = e^{i\theta g_i} d_i$ $(\theta \in \mathbb{R})$,

it reads $\mathcal{U}\boldsymbol{c}\mathcal{U}^\dagger = \boldsymbol{U}\mathcal{U}\boldsymbol{d}\,\mathcal{U}^\dagger = \boldsymbol{U}e^{i\theta \boldsymbol{g}}\boldsymbol{d} = \boldsymbol{U}e^{i\theta \boldsymbol{g}}\boldsymbol{U}^\dagger\boldsymbol{U}\boldsymbol{d} = e^{i\theta \boldsymbol{G}}\boldsymbol{c}$. Also it gives $\mathcal{U}\boldsymbol{c}^\dagger\mathcal{U}^\dagger = \boldsymbol{c}^\dagger e^{-i\theta \boldsymbol{G}}$.

[91] A. Kitazawa, J. Phys. A: Math. Gen. **30**, L285 (1997).

[92] R. D. King-Smith and D. Vanderbilt, Phys. Rev. B **47**, 1651 (1993).

[93] R. Resta, Rev. Mod. Phys. **66**, 899 (1994).

[94] R. Resta, Phys. Rev. Lett. **80**, 1800 (1998).

[95] This CoM is written by $SU(Q)$ quantum spins by substitution $\hat{n}_{j,\alpha} \to (S_j)_{\alpha,\alpha} = (\bar{S}_j)_{\alpha,\alpha}$.

[96] When L is even, $1-2+-\cdots+(L-1)-L = (1-2)+(3-4)+\cdots+((L-1)-L) = -1\frac{L}{2} = -L/2$ and $\sum_j (-1)^{j-1}x_j = L^{-1}\sum_j (-1)^{j-1}(j-j_0) = -\frac{1}{2}$. When L is odd, $1-2+-\cdots-(L-1)+L = (1-2)+(3-4)+\cdots+(L-2)-(L-1))+L = -\frac{L-1}{2}+L = \frac{L+1}{2}$ and $\sum_j (-1)^{j-1}x_j = L^{-1}(\sum_j (-1)^{j-1}j - j_0) = L^{-1}(\frac{L+1}{2} - \frac{L+1}{2}) = 0$.

[97] S. Greschner, S. Mondal, and T. Mishra, Phys. Rev. A **101**, 053630 (2020).

[98] L. Lin, Y. Ke, and C. Lee, Phys. Rev. A **101**, 023620 (2020).

[99] R. Slansky, Phys. Rep. **79**, 1 (1981).

[100] Y. Hatsugai and I. Maruyama, Europhys. Lett. **95**, 20003 (2011).

[101] $[\psi_{a,b}, \psi^\dagger_{a,b}] = 1 - Q^{-1}\sum_\alpha (n_{a,\alpha} + n_{b,\alpha})$.

[102] This symmetry breaking term is written by $SU(Q)$ spins as $H_B(t) = \sum_{j,\alpha}(-1)^j (S_j)_{\alpha,\alpha}\Delta_\alpha(t) + \text{const.}$

[103] $H = -\sum_{j,\alpha,\beta} J^{(2)}_{j,\alpha\beta}[(S_j)_{\alpha,\beta} + (-1)^j Q^{-1}\delta_{\alpha,\beta}][(S_{j+1})_{\alpha,\beta} - (-1)^j Q^{-1}\delta_{\alpha,\beta}] + \sum_{j,\alpha}(-1)^j (S_j)_{\alpha,\alpha}\Delta_\alpha(t) + \text{const.}$

[104] $P_\alpha(t) = -\sum_j x_j[\langle g_{0,\text{op}}|(S_j)_{\alpha,\alpha}|g_{0,\text{op}}\rangle + (-1)^j Q^{-1}]$. The boundary condition for the $SU(Q)$ spins is also open.

[105] The jump can be a multiple of $\pm\frac{1}{2}$ due to an additional symmetry. It is split into canonical jumps of the unit discontinuity $\pm\frac{1}{2}$ by inclusion of an infinitesimal perturbation.

[106] Y. Hatsugai, New J. Phys. **12**, 065004 (2010).

[107] This Z_2 quantization for the Berry phase for the noninteracting systems was also discussed before [139,140].

[108] Y. Hatsugai, J. Phys. Soc. Jpn. **73**, 2604 (2004).

[109] Y. Hatsugai, J. Phys.: Condens. Matter **19**, 145209 (2007).

[110] If $N_\phi = |\langle\phi|g_{0,\text{tw}}\rangle| = 0$, slightly modify $|\phi\rangle$, then $N_\phi \neq 0$. It is always possible if the dimension of the Hilbert space is larger than 1.

[111] F. Haldane, Phys. Lett. A **93**, 464 (1983).

[112] E. Lieb, T. Schultz, and D. Mattis, Ann. Phys. **16**, 407 (1961).

[113] I. Affleck and E. H. Lieb, Lett. Math. Phys. **12**, 57 (1986).

[114] J. des Cloizeaux and J. J. Pearson, Phys. Rev. **128**, 2131 (1962).

[115] J. B. Parkinson, J. Phys. C.: Solid State Phys. **20**, L1029 (1987).

[116] M. N. Barber and M. T. Batchelor, Phys. Rev. B **40**, 4621 (1989).

[117] A. Klümper, Europhys. Lett. **9**, 815 (1989).

[118] I. Affleck, J. Phys.: Condens. Matter **2**, 405 (1990).

[119] N. Chepiga and F. Mila, Phys. Rev. B **96**, 054425 (2017).

[120] T. Kariyado and Y. Hatsugai, Phys. Rev. B **91**, 214410 (2015).

[121] I. Maruyama and S. Miyahara, J. Phys. Soc. Jpn. **87**, 123703 (2018).

[122] T. Hirano, H. Katsura, and Y. Hatsugai, Phys. Rev. B **78**, 054431 (2008).

[123] We take a global single gauge for $|g^{L-1}\rangle$ (Appendix D). Then the phase of $|g^L\rangle \equiv \mathcal{U}_L|g^{L-1}\rangle$ is also globally unique. It implies Eq. (125) is without 2π ambiguity.

[124] It implies the gap closing at some twist by inclusion of the coupling between the dimers.

[125] T. Kennedy, J. Phys.: Condens. Matter **2**, 5737 (1990).

[126] M. Hagiwara, K. Katsumata, I. Affleck, B. I. Halperin, and J. P. Renard, Phys. Rev. Lett. **65**, 3181 (1990).

[127] M. Arikawa, S. Tanaya, I. Maruyama, and Y. Hatsugai, Phys. Rev. B **79**, 205107 (2009).

[128] N. Chepiga and F. Mila, Phys. Rev. B **97**, 174434 (2018).

[129] S. Ryu and Y. Hatsugai, Phys. Rev. Lett. **89**, 077002 (2002).

[130] T. Kennedy and H. Tasaki, Phys. Rev. B **45**, 304 (1992).

[131] M. Yamanaka, Y. Hatsugai, and M. Kohmoto, Phys. Rev. B **48**, 9555 (1993).

[132] J. E. Avron and L. G. Yaffe, Phys. Rev. Lett. **56**, 2084 (1986).

[133]
$$\sin 2\pi\left(\frac{j}{Q}+\frac{t_j}{T}\right) - \sin 2\pi\left(\frac{j+1}{Q}+\frac{t_j}{T}\right) = 2\sin 2\pi\frac{1}{2Q}\cos 2\pi\left(\frac{j+\frac{1}{2}}{Q}+\frac{t_j}{T}\right) = 0.$$

It implies $\frac{1}{4}$ or $\frac{3}{4} = \frac{j+\frac{1}{2}}{Q} + \frac{t_j}{T}$. Here we choose $\frac{3}{4}$ since $\frac{1}{4}$ gives the level crossing at the positive energies.

[134] The first condition is written as

$$\frac{t_0}{T}+\frac{1}{4} < \frac{3}{4} - \frac{j_{\text{ini}}+\frac{1}{2}}{Q} < \frac{t_0}{T}+\frac{1}{4}+\frac{1}{Q}$$
$$\frac{t_0}{T} < \frac{1}{2} - \frac{j_{\text{ini}}+\frac{1}{2}}{Q} < \frac{t_0}{T}+\frac{1}{Q}$$
$$Q\frac{t_0}{T} < \frac{Q}{2} - \left(j_{\text{ini}}+\frac{1}{2}\right) < Q\frac{t_0}{T}+1$$
$$Q\left(\frac{1}{2}-\frac{t_0}{T}\right) - \frac{3}{2} < j_{\text{ini}} < Q\left(\frac{1}{2}-\frac{t_0}{T}\right) - \frac{1}{2}.$$

Similarly the last one is written as follows:

$$\frac{t_0}{T}+\frac{3}{4}-\frac{1}{Q} < \frac{3}{4} - \frac{j_{\text{fin}}+\frac{1}{2}}{Q} < \frac{t_0}{T}+\frac{3}{4}$$
$$\frac{t_0}{T}-\frac{1}{Q} < -\frac{j_{\text{fin}}+\frac{1}{2}}{Q} < \frac{t_0}{T}$$
$$Q\frac{t_0}{T}-1 < -\left(j_{\text{fin}}+\frac{1}{2}\right) < Q\frac{t_0}{T}$$
$$-Q\frac{t_0}{T}-\frac{1}{2} < j_{\text{fin}} < -Q\frac{t_0}{T}+\frac{1}{2}.$$

Then it implies

$$j_{\text{ini}}+1 = \left\lfloor Q\left(\frac{1}{2}-\frac{t_0}{T}\right)+\frac{1}{2}\right\rfloor j_{\text{fin}} = \left\lfloor -Q\frac{t_0}{T}+\frac{1}{2}\right\rfloor.$$

[135] J. Hauschild and F. Pollmann, SciPost Phys. Lect. Notes 5 (2018).

[136] T. Fukui, Y. Hatsugai, and H. Suzuki, J. Phys. Soc. Jpn. **74**, 1674 (2005).

[137] Although changing P of the parameter $\Phi = P/Q$ induces a relabeling of the color index α for the symmetry breaking term, Eq. (43), the parameter space of the twists remains unchanged.

It implies that the agreement of the analytic formula of the Chern numbers, Eq. (149), with the numerical evaluation is highly nontrivial.

[138]

$$
\begin{aligned}
T^a_{\alpha\beta}T^a_{\gamma\delta} &= -\epsilon_{a\alpha\beta}\epsilon_{a\gamma\delta} = -\delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma}, (T^aT^b)_{\alpha\beta}(T^aT^b)_{\gamma\delta} \\
&= \epsilon_{a\alpha\kappa}\epsilon_{b\kappa\beta}\epsilon_{a\gamma\lambda}\epsilon_{\lambda\delta} = \epsilon_{a\alpha\kappa}\epsilon_{a\gamma\lambda}\epsilon_{b\kappa\beta}\epsilon_{b\lambda\delta} \\
&= (\delta_{\alpha\gamma}\delta_{\kappa\lambda} - \delta_{\alpha\lambda}\delta_{\gamma\kappa})(\delta_{\kappa\lambda}\delta_{\beta\delta} - \delta_{\kappa\delta}\delta_{\lambda\beta}) \\
&= (\delta_{\alpha\gamma}\delta_{\kappa\lambda} - \delta_{\alpha\lambda}\delta_{\gamma\kappa})\delta_{\kappa\lambda}\delta_{\beta\delta} - (\delta_{\alpha\gamma}\delta_{\kappa\lambda} - \delta_{\alpha\lambda}\delta_{\gamma\kappa})\delta_{\kappa\delta}\delta_{\lambda\beta} \\
&= (\delta_{\alpha\gamma}\delta_{\lambda\lambda} - \delta_{\alpha\lambda}\delta_{\gamma\lambda})\delta_{\beta\delta} - (\delta_{\alpha\gamma}\delta_{\delta\lambda} - \delta_{\alpha\lambda}\delta_{\gamma\delta})\delta_{\lambda\beta} \\
&= (3\delta_{\alpha\gamma} - \delta_{\alpha\gamma})\delta_{\beta\delta} - \delta_{\alpha\gamma}\delta_{\delta\beta} + \delta_{\alpha\beta}\delta_{\gamma\delta} \\
&= \delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\beta}\delta_{\gamma\delta}.
\end{aligned}
$$

[139] J. Zak, Phys. Rev. Lett. **62**, 2747 (1989).
[140] Y. Hatsugai, Solid State Commun. **149**, 1061 (2009).