Chiral current induced by torsional Weyl anomaly

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Torsion can be realized as dislocation in the crystal lattice of material. It is particularly interesting if the material has fermions in the spectrum, such as graphene, topological insulators, and Dirac and Weyl semimetals, as it's transport properties can be affected by the torsion. In this paper, we find that, due to Weyl anomaly, torsion in Dirac and Weyl semimetals can induce novel chiral currents, either near a boundary or in a "conformally flat space." We briefly discuss how to measure this interesting effect in experiment. It is remarkable that these experiments can help to clarify the theoretical controversy of whether an imaginary Pontryagin density could appear in the Weyl anomaly.

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I. INTRODUCTION

The study of anomaly induced transport is an interesting subject (see [1] for a recent review). Although anomaly was originally discovered in particle physics, due to its universal nature anomaly has nontrivial implications to a large number of physical phenomena ranging over vastly different scales. For example, the chiral anomaly of nonabelian gauge theory imposes nontrivial constraints on the fundamental interaction of chiral fermions in the standard model [2]. Chiral anomaly also affects the transport dynamics of systems with chiral fermions [3–12] due to the well-known chiral magnetic and chiral vortical effects (see [13–15] for review). Interestingly, this kind of anomalous transport occurs only in a material system since nonvanishing chemical potentials are required. As anomaly itself is intrinsic to the quantum vacuum, it is an interesting question to ask if anomaly induced transport can occur independent of the chemical potentials.

Recently, chiral response associated with chiral anomaly in a torsional background has been a subject of intensive study. In general, a curved spacetime is equipped with a metric which fixes the causal structure and metric relations, and a connection which defines the parallel transport of tensors on the manifold. While the path dependence of parallel transport is measured by the curvature, the nonclosure of parallelism of parallel transport is measured by the torsion. For the parallel transport,

$$\nabla_\mu V^\rho = \partial_\mu V^\rho + \Gamma^\rho{}_{\mu\nu} V^\nu, \tag{1}$$

the torsion

$$T^\rho{}_{\mu\nu} := \Gamma^\rho{}_{\mu\nu} - \Gamma^\rho{}_{\nu\mu} \tag{2}$$

is given by the antisymmetric part of the connection. In Einstein general relativity (GR), the geometry of spacetime is taken to be torsion free since there is seemingly no observational evidence for torsion in the spacetime of our universe [16]. However, perhaps unexpectedly, torsion finds a legitimate position in condensed matter physics since torsion appears to be naturally suited for the geometrical description of dislocation defects in crystals [17–20]. Torsion has been realized and studied in diverse material systems such as graphene [21,22], topological insulators [23–25], and Dirac and Weyl semimetals [26–31]. As a result of the specific manner, fermion is coupled to torsion, and chiral anomaly could emerge and give rise to a novel chiral response in torsional material systems [32–40].

Just as a system may possess a chiral anomaly which characterizes the quantum chiral dependence of the system, generally a system may also possess a Weyl anomaly which characterizes the quantum scale dependence of the system. In general, the Weyl anomaly is defined as a difference between the trace of a renormalized stress tensor and the renormalized trace of a stress tensor [41,42]:

$$\mathcal{A} = \int_M \sqrt{-g}\, [g^{\mu\nu}\langle T_{\mu\nu}\rangle - \langle g^{\mu\nu} T_{\mu\nu}\rangle]. \tag{3}$$

In the presence of a background gauge field, the Weyl anomaly receives a contribution

$$\mathcal{A} = \int_M \sqrt{-g}\, b_1 F_{\mu\nu} F^{\mu\nu} \tag{4}$$

whose form is universal and is entirely determined by the coefficient b_1, a bulk central charge of the theory. For the normalization of the gauge field kinetic term $S = -1/4 \int F^2$, b_1 is given by the beta function of the theory as $b_1 = -\beta/2$.

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As Weyl anomaly is independent of the chiral anomaly, it is interesting to ask if and how it give rises to any transport phenomena in a system. The answer is positive. Recently, a new kind of induced transport was discovered for boundary vacuum systems as a result of the Weyl anomaly. It was found that [43,44] for any renormalizable quantum field theory with a current coupled to an external electromagnetic (EM) field

$$S_A = \int_M \sqrt{-g}\, J^\mu A_\mu, \tag{5}$$

the Weyl anomaly give rises to an induced magnetization current in the vicinity of the boundary of the vacuum system

$$\langle J_\mu \rangle = \frac{-2\beta F_{\mu\nu} n^\nu}{x} + \cdots, \quad x \sim 0. \tag{6}$$

Here x is the proper distance to the boundary, n_μ is the inner normal vector, ... denote higher order terms in $O(x)$, and β is the beta function. Hereafter we will drop the symbol $\langle\ \rangle$ for the expectation value. It is instructive to review the derivation of this result to appreciate how it could be derived from the Weyl anomaly. In general, for a boundary quantum field theory, the renormalized current is generally singular near the boundary and the expectation value takes the asymptotic form near $x \sim 0$:

$$J_\mu = \frac{1}{x^3} J^{(3)}_\mu + \frac{1}{x^2} J^{(2)}_\mu + \frac{1}{x} J^{(1)}_\mu + J^{(0)} \log x + \cdots, \tag{7}$$

where $\cdots$ denotes terms regular at $x = 0$, and $J^{(n)}_\mu$ depend only on the background geometry, the background vector field strength, and the type of fields under consideration. For current that is conserved

$$D_\mu J^\mu = 0 \tag{8}$$

up to possibly an anomaly term, one can easily obtain the gauge invariant solution

$$\begin{aligned} &J^{(3)}_\mu = 0, \quad J^{(2)}_\mu = 0,\\ &J^{(1)}_\mu = \alpha_1 F_{\mu\nu} n^\nu + \alpha_2 \mathcal{D}_\mu k + \alpha_3 \mathcal{D}_\nu k^\nu_\mu + \alpha_4 \star F_{\mu\nu}\, n^\nu \end{aligned} \tag{9}$$

where $F_{\mu\nu}$, $\star F_{\mu\nu}$, n_μ, $\mathcal{D}_m$, $k_{\mu\nu}$, and $h_{\mu\nu}$ are respectively the background field strength, Hodge dual of field strength, the normal vector, induced covariant derivative, extrinsic curvature, and induced metric of the boundary. Now the Weyl anomaly $\mathcal{A}$ is a function of the background gauge field. Since it is related to the Logarithmic UV divergent term of effective action, one can establish the following "integrability" relation [43,54,60]:

$$(\delta \mathcal{A})_{\partial M_\epsilon} = \left(\int_{M_\epsilon} dx^4 \sqrt{g} J^\mu \delta A_\mu \right)_{\log \frac{1}{\epsilon}}, \tag{10}$$

where here a regulator $x \geqslant \epsilon$ to the boundary is introduced for the integral on the right hand side (RHS) of (10) and J^μ is the renormalized current. The relation (10) identifies the boundary contribution of the variation of the integrated anomaly $\mathcal{A}$ under an arbitrary variation of the gauge field δA_μ with the UV logarithmic divergent part of the integral involving the expectation value J^μ of the renormalized $U(1)$ current. The power of the relation (10) lies in the fact that the left hand side of (10) is a total variation and imposes constraints on the RHS of (10) that are powerful enough to to completely fix the asymptotic behavior of the current in terms of the Weyl anomaly of the theory. Using (10), one obtains the result (6) for the renormalized current immediately. We note that the result (6) is universal in two remarkable ways. First, it works for any quantum field theory, and not just conformal field theory. Moreover, it is independent of the choices of boundary conditions since only the bulk central charge, instead of boundary central charge, appears. It should be mentioned that the induced magnetization current for free theories in the vicinity of the the boundary was first obtained by Osborn and McAvity in [45], the universal result (6) as well as its intimate relation with the Weyl anomaly were originally established in [43]. The Weyl anomaly also has an interesting effect in cosmological spacetime. It was found that in a conformally flat spacetime $ds^2 = e^{2\sigma} \eta_{\mu\nu} dx^\mu dx^\nu$ without boundaries, the anomalous current is given by [46,47]

$$\langle J^\mu \rangle = -2\beta F^{\mu\nu} \partial_\nu \sigma + O(\sigma^2) \tag{11}$$

to the leading order of small σ. Generalization of the result (6) to higher dimensions and the result (11) for arbitrary finite σ can be found in [48,49] and [50], respectively. In addition, there is also an effect of vacuum spin transport induced by electromagnetic field [51].

Central to these results is the fact that the anomalous currents [(6) and (11)] emerge as a direct response of the Weyl anomaly arising from background. Motivated by this observation, it is natural to expect that similar induced phenomena may occur if the fermions are allowed to couple to other external backgrounds. Now apart from EM background or spacetime curvature, spacetime torsion is another interesting background to consider. In this paper, we will study the quantum transport phenomena induced by the Weyl anomaly in a torsional material. In the following, we first derive in Sec. II the Weyl anomaly for Dirac fermions coupled to torsion. Using this result, we derive the Weyl anomaly induced chiral and vector currents in Sec. III. The discussion is extended to Weyl fermions in Sec. IV. We propose that measurements of the induced currents in Weyl semimetals could help to clarify the theoretical controversy of whether the Pontryagin density appears in the Weyl anomaly.

II. TORSION AND WEYL ANOMALY

Generally a spacetime is equipped with a metric and a connection. In Einstein general relativity (GR), the metricity condition and a symmetric connection are adopted so that connection is not independent but given by the metric. In general, departure from GR is characterized [16] by the nonmetricity tensor $Q_{\mu\nu\rho} := \nabla_\mu g_{\nu\rho}$ and the torsion tensor (2) defined as the antisymmetric part of the connection. For simplicity, we will focus in this paper on the particular interesting generalization of GR called the Einstein-Cartan theory, where $Q = 0$ and the connection is independently characterized by the torsion tensor. In this case, the gravitational coupling in four dimensions takes the general form

$$S = \int_M d^4x \sqrt{-g} \bar{\psi} i\gamma^\mu (\nabla_\mu - iV_\mu - i\gamma_5 S_\mu)\psi, \tag{12}$$

where ∇_μ is the covariant derivative defined with the standard Levi-Civita metric connection, and the components of torsion

$$V_\mu := T^\rho{}_{\rho\mu}, \quad S_\mu := \epsilon_{\mu\nu\rho\sigma} T^{\nu\rho\sigma} \tag{13}$$

behave effectively as vectors and axial vectors [21]. For three dimensions, there is no S_μ. Instead

$$\hat{S} := \epsilon_{\nu\rho\sigma} T^{\nu\rho\sigma} \tag{14}$$

behaves as pseudoscalar and the action in three dimensions becomes [21,22]

$$S = \int_M d^3x \sqrt{-g} \bar{\psi} i\gamma^\mu (\nabla_\mu - iV_\mu - i\gamma_5 \hat{S})\psi. \tag{15}$$

We use the mostly negative convention for the signature of the metric, and the torsion will be taken as a background.

The action (12) is classically Weyl invariant under the local scaling transformation: $\psi \to e^{-3\sigma/2}\psi$, $g_{\mu\nu} \to e^{2\sigma} g_{\mu\nu}$, $V_\mu \to V_\mu$, and $S_\mu \to S_\mu$. However, quantum mechanically there is an anomaly. For a manifold with a boundary, boundary conditions should be imposed on half of the spinor components. It can be shown that [52] Hermicity of the Dirac operator selects the following specific ones out of the general chiral bag boundary conditions:

$$(1 \pm i\gamma^n \gamma^5)\psi|_{\partial M} = 0, \tag{16}$$

where n denotes the normal direction. The one-loop Weyl anomaly can be obtained by applying the heat kernel expansion [52]. Let us focus on four dimensions; the Weyl anomaly reads

$$\mathcal{A} = \frac{1}{24\pi^2} \int_M d^4x \sqrt{-g} [F_{\mu\nu}F^{\mu\nu} + H_{\mu\nu}H^{\mu\nu}] + \frac{1}{12\pi^2} \int_{\partial M} d^3x \sqrt{-h} [B_1(S) - B_2(S) - B_3(S)], \tag{17}$$

where $F = dV$, $H = dS$, $h_{\mu\nu}$ is the induced metric on the boundary ∂M; $k_{\mu\nu} = h^\rho_\mu h^\sigma_\nu \nabla_\rho n_\sigma$ is the extrinsic curvature; $\bar{k}_{\mu\nu}$ and k denote the traceless part and the trace of extrinsic curvatures, respectively; and $B_1(S) := \frac{2}{3} k(h^{\mu\nu} + n^\mu n^\nu) S_\mu S_\nu + S_n \nabla_\mu S^\mu + 2S_\mu h^{\mu\nu} \nabla_n S_\nu$, $B_2(S) := \frac{1}{3} k S_\mu S^\mu + n^\mu S^\nu \nabla_\nu S_\mu$, and $B_3(S) := \frac{1}{5} \bar{k}_{\mu\nu} S^\mu S^\nu$. Here we choose the normal vector so that $n^\mu = -n_\mu = (0, -1, 0, 0)$ in a flat half space. The Weyl anomaly (17) is Weyl invariant and satisfies the Wess-Zumino consistency condition [53]. Note that the bulk contribution to the torsional Weyl anomaly is discussed in [55–57].

We are interested in the expectation value of the chiral current and vector current in the theory. In four dimensions, the renormalized vacuum expectation value of the chiral current is derived by the variation of effective action with respect to the background "axial vector"

$$J^\mu_S = \langle \bar{\psi} \gamma^\mu \gamma_5 \psi \rangle = \frac{1}{\sqrt{-g}} \frac{\delta I_{\text{eff}}}{\delta S_\mu}. \tag{18}$$

In the following, we show that the knowledge of the Weyl anomaly (17) allows one immediately to determine (18) in closed analytic form.

III. CHIRAL CURRENT

A. Boundary theory

Let us first study the chiral current in four dimensional spacetime with a boundary, say, at $x = 0$ of the coordinate system. We follow the methods of [43,54], where we have studied the expectation value of current and stress tensor in boundary quantum field theories [58]. To start with, we note that since the mass dimension of chiral current is three, it takes the asymptotic form [59]

$$J^\mu_S = \frac{J^\mu_0}{x^3} + \frac{J^\mu_1}{x^2} + \frac{J^\mu_2}{x} + O(\ln x) \tag{19}$$

near the boundary. Here x is the proper distance from the boundary, J^μ_{n} have mass dimension n and depend on only the background geometry and the background torsion. J^μ_{n} can be solved by imposing the conservation law $\nabla_\mu J^\mu_S = O(1)$ [43], where $O(1)$ denotes the finite part of the chiral anomaly which is irrelevant to the divergent part of renormalized current. We obtain

$$J^\mu_0 = 0, \quad J^\mu_1 = \lambda\, h^{\mu\nu} S_\nu, \tag{20}$$

where λ is some constant. The key point in the above derivations is that the leading term of chiral current cannot be proportional to the normal vector n^μ, otherwise it cannot satisfy the conservation law $\nabla_\mu(n^\mu/x^3) \sim 1/x^4 \neq O(1)$. We note that unlike the case of gauge field [43], the transformation $\delta S_\mu = \partial_\mu \alpha$ changes the torsion and is not required to be a symmetry of the theory, therefore a nonvanishing J^μ_1 term as in (20) is allowed.

Now let us use the "integrability" relation (10) for the renormalized chiral current J^μ_S. To proceed, let us employ the Gauss normal coordinates to express the metric $ds^2 = -dx^2 + (h_{ij} - 2xk_{ij} + \cdots)dy^i dy^j$ and expand $S_\mu(x) = S_{0\ \mu} + xS_{1\ \mu} + O(x^2)$, where $x \in [0, +\infty)$ and S_i give the ith derivatives of S at $x = 0$. Substituting (17), (19), and (20) into (10), after some calculations we obtain the chiral current near the boundary:

$$J^a_S = \frac{S^a_0}{6\pi^2 x^2} + \frac{2k^a{}_b S^b_0 + k S^a_0}{10\pi^2 x} + O(\ln x),$$
$$J^n_S = \frac{(D_a S^a)_0}{6\pi^2 x} + O(\ln x), \quad x \sim 0, \tag{21}$$

where n and a respectively denote the normal and tangential directions, D_a is the covariant derivative on the boundary, and we denote for any function $F(x)$ that $F_0 = F(x = 0)$. A couple of remarks are in order. We note that as in the discussion [43,54], the total chiral current is finite since there are boundary contributions to the chiral current which cancel the divergence from the bulk contribution (21). We note that the result (21) applies not only to conformal field theory (CFT) but also the general quantum field theory (QFT) since the Weyl anomaly is welldefined for general quantum field theories [41,42]. We remark that (21) can be verified by the Green's function method [61,62]. In a flat half space with $k_{ab} = 0$, the correction of Green function due to torsion is given by

$$G_c(x, x') = -\int_M d^4y\sqrt{|g|}G_0(x, y)\gamma^\mu\gamma_5 S_\mu(y)G_0(y, x') + O(S^2), \quad (22)$$

where G_0 is Green's function without torsion. From (22), we can obtain the chiral current by

$$J_S^\mu(x) = -i \lim_{x'\to x} \mathrm{Tr}_{\mathrm{reg}}[\gamma^\mu\gamma_5 G_c(x, x')], \quad (23)$$

which agrees with (21). Here $\mathrm{Tr}_{\mathrm{reg}}$ means we have subtracted the reference current without boundaries. Another interesting remark is about the universal nature of the boundary current (21). In [38] it was shown that torsion does not lead to new chiral transport effects in the bulk since the response to torsion can be viewed as a manifestation of the chiral vortical effect. To see this, it was noted that in the presence of torsion, the chiral anomaly receives in addition to the electromagnetic contribution a torsion contribution:

$$\partial_\mu J_S^\mu - T^\lambda_{\lambda\mu}J_S^\mu = c_F\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} + c_T\epsilon^{\mu\nu\rho\sigma}\eta_{ab}T^a_{\mu\nu}T^b_{\rho\sigma}, \quad (24)$$

where the last term denotes the famous Nieh-Yan term (here we have focused on the space without curvatures for simplicity) and the coefficients c_F and c_T have mass dimensions zero and two,, respectively. As a result, unlike c_F, c_T depends on regularization and can be removed by a suitable local counterterm in the background fields [1]. Thus, the Nieh-Yan anomaly is not independent, and can be considered a manifestation of the chiral anomaly [38]. In fact, the form of current nonconservation will depend on the precise definition of the current [38]. Moreover, a choice of current that is based on physical symmetries was suggested and it was shown that the Neh-Yan anomaly does not appear [38].

Remarkably, the boundary chiral current induced by the Weyl anomaly (17) is different as, unlike the effect of torsion on the bulk current in (24), the effect of torsion on the boundary current (17) is nonremovable. Note that all the coefficients of the Weyl anomaly (17) are dimensionless. As a result, the Weyl-anomaly induced chiral current is universal near the boundary, and cannot be removed by local counterterms. The universality of the current (21) near the boundary does not contradict the nonuniversality of torsion-induced chiral current in the bulk. It arises from a novel boundary effect that is independent of renormalization scheme.

Let us briefly comment on how the chiral current (21) may be measured in Dirac semimetals. As shown in Fig. 1, we perform the Screw dislocation of lattices so that the red parallelogram does not close, and the missing part is defined by the blue Burgers vector. The density of the Burgers vector $\vec{b}$ behaves as the axial vector in Dirac and Weyl semimetals [21]. From (21) together with $\vec{S} \sim \vec{b}$, we draw conclusions that the Screw dislocation induces an anomalous chiral current near the boundary in Dirac and Weyl semimetals:

$$\vec{J}_S \sim \frac{\vec{b}}{x^2}, \quad x \gtrsim a, \quad (25)$$

where a denotes the lattice length, and we mainly focus on spatial $\vec{b}$ in this paper. Note that our result (21) for the continuum is UV finite and is independent of regularization. However, as we go from the continuum to a lattice, the details of the lattice will enter in general, such as in (25). In principle, other parameters of the system such as mass, temperature, hydrodynamic velocity, etc., may also appear in dimensionless combination and correct the overall coefficient of the induced current. Note that, to obey the bag boundary condition (16), we should place an insulator on the boundary of the materials so that no current can flow out of the boundary $x = 0$.

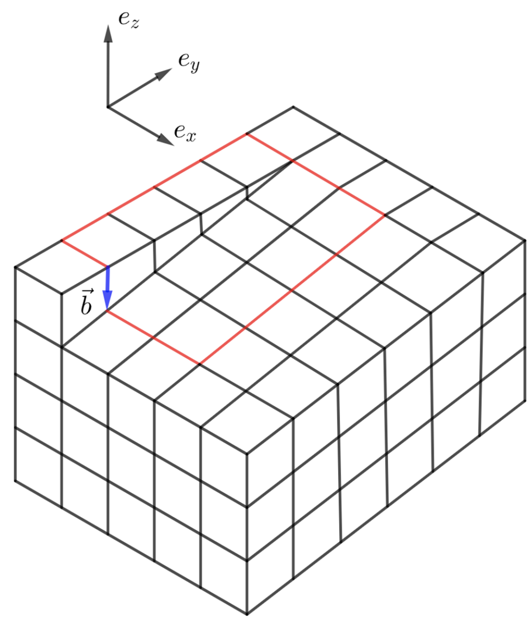

FIG. 1. Chiral current $\vec{J}_S \sim \vec{b}/x^2$ induced by Screw dislocation, where $\vec{b}$ is the Burgers vector.

Finally we make a remark for three dimensions. Following the same analysis as for the derivation of (19) and (20), the renormalized expectation value of the vector current $J_V^\mu = \langle\bar{\psi}\gamma^\mu\psi\rangle$ takes the form

$$J_V^\mu = F^{n\mu}(\alpha_1 + \alpha_2 \ln x), \quad (26)$$

where α_1, α_2 are constant parameters which are sensitive to the boundary condition. We note that in dimensions $d < 4$, the current is not related to the Weyl anomaly. Hence the parameters α_1, α_2 are not determined just by the central charges, but by further specific details of theory.

B. Conformally flat spacetime

There are also chiral currents in four dimensional conformally flat spacetime without boundaries. To demonstrate this, let us start by deriving the anomalous transformation rule for the chiral current. Consider the theory (12) with metric and chiral vector field given by $(g_{\mu\nu}, S_\mu)$. Due to the anomaly, the renormalized effective action I_{eff} is not invariant under the Weyl transformation. Generally, we have [63]

$$\frac{\delta}{\delta\sigma}I_{\mathrm{eff}}(e^{-2\sigma}g_{\mu\nu}) = \mathcal{A}(e^{-2\sigma}g_{\mu\nu}) \quad (27)$$

for arbitrary finite $\sigma(x)$. This can be integrated to give the effective action [53,64,65]. Using the fact that the anomaly (17) is a Weyl invariant, we immediately obtain the transformation rule for the effective action:

$$I_{\text{eff}}(e^{-2\sigma}g_{\mu\nu}) = I_{\text{eff}}(g_{\mu\nu}) + \frac{1}{24\pi^2}\int_M d^4x\sqrt{-g}H_{\mu\nu}H^{\mu\nu}\sigma \tag{28}$$

plus a boundary term $\frac{1}{12\pi^2}\int_{\partial M}\sqrt{-h}[\frac{-1}{5}\bar{k}_{\mu\nu}S^\mu S^\nu + B_1 - B_2]\sigma$, which we drop in spacetime without boundaries. One can check that the dilaton effective action satisfies the Wess-Zumino consistency $[\delta_{\sigma_1}, \delta_{\sigma_2}]I_{\text{eff}} = 0$. Using (28), we finally obtain the transformation rule for the chiral current (18) under Weyl transformation $g_{\mu\nu} \to g'_{\mu\nu} = e^{-2\sigma}g_{\mu\nu}$, $S_\mu \to S'_\mu = S_\mu$,

$$J_S^\mu = \frac{1}{6\pi^2}\nabla_\nu(H^{\nu\mu}\sigma), \tag{29}$$

plus a trivial term $e^{-4\sigma}J'^\mu_S$. Here J_S^μ (J'^μ_S, respectively) denotes the vacuum expectation value of the chiral current of the theory (12) in the background spacetime $g_{\mu\nu}$ ($g'_{\mu\nu}$, respectively). Taking $g'_{\mu\nu}$ to be the flat spacetime metric and assuming that the chiral current vanishes in some region of the flat spacetime, we finally obtain (29) as the chiral current in conformally flat spacetime,

$$ds^2 = e^{2\sigma}\eta_{\mu\nu}dx^\mu dx^\nu. \tag{30}$$

Note that the conformal factor σ in (29) is arbitrary and needs not to be small. Therefore we can use (29) to calculate the current in general conformally flat spacetimes such as Anti-de-Sitter space, de-Sitter space, and Robertson-Walker universe. For Robertson-Walker universe $ds^2 = dt^2 - a(t)^2(dx^2 + dy^2 + dz^2)$, we have at time $t = t_*$,

$$J_S^\mu = \frac{1}{6\pi^2}H^{0\mu}H, \tag{31}$$

where $H = \dot{a}/a$ is the Hubble parameter. For simplicity we have chosen $a(t_*) = 1$. In materials, curvature and torsion can be mimicked by disclinations and dislocations, respectively. Thus, one may measure the effect (29) in Dirac semimetals with suitable disclinations and dislocations.

IV. WEYL FERMIONS

So far we have focused on Dirac fermions. The discussions can be generalized to Weyl fermions straightforwardly. The real part of the Weyl anomaly for Weyl fermions is half of that of Dirac fermions (17). As a result, the anomalous chiral current is also half of the Dirac fermions (21), (29). The imaginary part of the Weyl anomaly is parity odd and it is controversial whether such a term exists [66,67]. This imaginary part implies that the theory is nonunitary or there is absorption and dissipation in materials. For simplicity, let us take the vector parts of torsion V_μ as an example. The discussion for axial vector parts of torsion S_μ is similar. The Weyl anomaly of Weyl fermions related to V_μ is

$$\mathcal{A} = \frac{1}{48\pi^2}\int_M d^4x\sqrt{-g}\left[F_{\mu\nu}F^{\mu\nu} + i\frac{3}{2}\eta F_{\mu\nu}{}^*F^{\mu\nu}\right], \tag{32}$$

where ${}^*F^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F^{\alpha\beta}$, $\eta = 0$ or 1 denote the controversy. Following the above approach, we derive the currents

$$J_V^\mu = \frac{n_\nu F^{\nu\mu} + i\frac{3}{2}\eta\, n_\nu{}^*F^{\nu\mu}}{12\pi^2 x} + O(\ln x) \tag{33}$$

near a boundary and

$$J_V^\mu = e^{-4\sigma}J'^\mu_V + \frac{1}{12\pi^2}\nabla_\nu\left(F^{\nu\mu}\sigma + i\frac{3}{2}\eta\,{}^*F^{\nu\mu}\sigma\right) \tag{34}$$

in a conformally flat space without boundaries. Recall that the edge dislocations and screw dislocations can induce effective vectors and axial vectors coupled with fermions in materials, respectively. Thus, the vector V_μ can be realized by either an electromagnetic field or suitable edge dislocations in Weyl semimetals. It is interesting to measure the predicted current (33) and (34) in Weyl semimetals, which can help to clarify the theoretical controversy that an imaginary Pontryagin density could appear in the Weyl anomaly [66,67].

Summarizing, we have shown in this paper that, due to the Weyl anomaly, torsion can lead to novel currents and chiral currents for Dirac and Weyl fermions. We propose to measure these interesting effects in Dirac and Weyl semimetals with suitable dislocations. These experiments can help to clarify the theoretical controversy that an imaginary Pontryagin density could appear in the Weyl anomaly.

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