Random magnetic field and the Dirac Fermi surface

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We study a single two-dimensional Dirac fermion at finite density, subject to a quenched random magnetic field. At low energies and sufficiently weak disorder, the theory maps onto an infinite collection of 1D chiral fermions (associated to each point on the Fermi surface) coupled by a random vector potential. This low-energy theory exhibits an exactly solvable random fixed line, along which we directly compute various disorder-averaged observables without the need for the usual replica, supersymmetry, or Keldysh techniques. We find the longitudinal dc conductivity in the collisionless $\hbar\omega/k_B T \to \infty$ limit to be nonuniversal and to vary continuously along the fixed line.

I. INTRODUCTION

In this paper, we introduce and analyze an effective theory for studying the influence of a quenched random magnetic field on a single Dirac fermion, moving in two spatial dimensions (2D) and placed at finite density. Applications of this theory include the following: graphene when restricted to a single valley [1] (where the vector potential disorder describes the effects of ripples in the graphene sheet [2,3]) and the gapless surface states of a time-reversal invariant 3D topological insulator [4]; an integer quantum Hall plateau transition for spinless electrons in a periodic potential [5,6]; and Dirac composite fermion mean-field descriptions of the half-filled Landau level and topological insulator surface state [7–10], other even-denominator metallic states [11–13], and the superconductor-insulator transition [14]. In this last guise, the random vector potential arises from a random scalar potential perturbation to the dual electron system.

In contrast to a 2D nonrelativistic fermion [15,16], a single Dirac cone enjoys a sort of topological protection against localization [17–19]: A free Dirac fermion cannot be gapped without breaking, on average, either time-reversal symmetry[1] or charge conservation; the random vector potential is unique (among random perturbations quadratic in the fermions) in that it respects a particle-hole symmetry [6]. The Dirac theory therefore gives rise to a delocalized, critical state for all values

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[1]Here we are explicitly referring to the symmetry of the Lagrangian for a single Dirac fermion. This may be realized in a time-reversal invariant way as the surface state of a time-reversal invariant topological insulator. In a purely 2D theory, the simplest model breaks time-reversal microscopically [5,20,21]; the resulting theory preserves a nonlocal particle-hole symmetry [7,10], which we are viewing here as an effective time-reversal symmetry for the Dirac fermion sector of the theory.

of the Fermi energy. The problem here is to find an effective theory for this metal.

The conventional analytical approach to this problem, due to Pruisken and others [22–27] (see [28,29] for reviews and [30–35] for specific studies of a disordered Dirac fermion), coarse grains away the elementary fermionic excitations in favor of a nonlinear sigma model with topological term, in which renormalization group fixed points are parameterized by the (disorder-averaged) Dirac fermion conductivity tensor. Because the longitudinal resistivity $\sim 1/\sigma_{xx}$ plays the role of a coupling constant in this model, it is a challenge to quantitatively extend this description to the regime, $\sigma_{xx} \sim e^2/h$, relevant to experiment (e.g., [36,37]). Tsvelik [38] has conjectured a $PSL(2|2)_8$ Wess-Zumino-Novikov-Witten theory for the $\sigma_{xx} \sim e^2/h$ regime, based on its consistency with various numerical studies of the Chalker-Coddington model [39,40] for the integer quantum Hall transition. Unfortunately, there is as yet no consensus among the most recent numerical works (e.g., [41–45]). For instance, there are statistically significant deviations among the various predictions for the localization length exponent (see Fig. 1 of [45]). These deviations are understood to be the result of finite-size corrections to scaling [46,47]; effects that are exacerbated by a leading irrelevant perturbation that is close to marginality. Motivated in part by the lack of numerical consensus, Zirnbauer [48,49] has recently proposed a fixed point description in terms of a conformal field theory (CFT) with only marginal perturbations. Currently, this scenario appears to lack direct numerical support [45,50]. An alternative study of the Dirac theory using (non-)Abelian bosonization by Ludwig *et al.* [6] makes use of an SU(2) symmetry that is present only at zero density. The zero-density theory exhibits a fixed line (as a function of the disorder variance), along which the dynamical critical exponent varies continuously while the longitudinal conductivity is constant and of the order of e^2/h. Perturbation by a chemical potential within this treatment leads to difficulty: The

associated chemical potential deformation breaks the SU(2) symmetry and causes a flow to strong coupling.[2]

Here we present a different approach to studying the finite density theory. We first take the limit of the theory that focuses on the low-energy fluctuations about the Dirac Fermi surface before incorporating disorder. Taking inspiration from [6], the idea is to try to study the effects of disorder on the scale-invariant effective theory of (a system with) a Fermi surface [51,52], rather than the "microscopic" Dirac theory that includes both particle and antiparticle excitations. We show how this approach allows for an exact solution of the effective theory—a random fixed line—provided the quenched vector potential is suitably long ranged and weak, and for direct calculations of the effects of disorder on various physical observables without the need for replicas, supersymmetry, or Keldysh formalisms. In contrast to the zero-density theory [6], we find the longitudinal dc conductivity—calculated in the collisionless $\hbar\omega/k_B T \to \infty$ limit [53]—to vary continuously along this finite-density fixed line.

The remainder of the paper is organized as follows. In Sec. II, we derive the effective theory of a Dirac Fermi surface in a random magnetic field. This theory takes the form of an infinite collection of 1D chiral fermions—one fermion for each point on the Fermi surface—coupled by the quenched vector potential disorder. The large emergent symmetry U(N), with $N \to \infty$ of the clean theory, allows for an exact solution of the disordered theory, for each disorder realization, in which the randomness is removed entirely from the effective action by a gauge transformation. Operators not invariant under the U(N) symmetry, however, depend on the disorder. In Sec. III, we use the exact solution to directly compute various disorder-averaged physical observables. We warm up by showing that the average fermion Green's function is short ranged and that the system has a finite density of states. We then turn to a calculation of the longitudinal conductivity, finding the conductivity to vary continuously with the strength of the disorder. In Sec. IV, we summarize and discuss possible directions of future work. Appendices A and B contain relevant details of the calculations summarized in the main text.

II. LOW-ENERGY EFFECTIVE THEORY

In this section, we derive the low-energy effective theory of a 2D Dirac fermion in a random magnetic field in terms of an infinite collection of 1D chiral fermions, coupled via a random vector potential.

A. Low-energy limit

Our starting point is the theory of a two-component Dirac fermion $\Psi(t, \mathbf{x})$ at finite density, coupled to a quenched random U(1) vector potential $A_j(\mathbf{x})$,

$$S = S_0 + S_1, \tag{1}$$

where S_0 is the action of a free Dirac fermion in 2D,[3]

$$S_0 = \int dt\, d^2\mathbf{x}\, \Psi^\dagger(t, \mathbf{x})(i\sigma_a \partial_a + \mu)\Psi(t, \mathbf{x}), \tag{2}$$

and S_1 is the coupling of the Dirac fermion to the vector potential disorder,

$$S_1 = \int dt\, d^2\mathbf{x}\, \Psi^\dagger(t, \mathbf{x})\sigma_j A_j(\mathbf{x})\Psi(t, \mathbf{x}). \tag{3}$$

Above, $\mathbf{x} = (x_1, x_2)$, $a \in \{0, 1, 2\}$, σ_0 is the 2×2 identity matrix, and σ_j for $j = 1, 2$ are standard Pauli matrices; repeated indices are summed over unless otherwise specified. The chemical potential μ is finite and nonzero, and we have set the velocity of the Dirac fermion to unity. The vector potential $A_j(\mathbf{x})$ is chosen to be a zero-mean Gaussian random variable. We specify its disorder ensemble in Sec. II C; for the present discussion, we require that $A_j(\mathbf{x})$ be of sufficiently long wavelength that its Fourier transform $A_j(\mathbf{q})$[4] is only nonzero for wave vectors $|\mathbf{q}| \ll 2|\mu|$. This condition on the vector potential restricts to small-angle impurity scattering between fermionic excitations about nearby Fermi points.

Following [54], we derive the low-energy limit of this theory, in which we first focus on the low-energy excitations near the Fermi surface, defined by Eq. (2), and then incorporate scattering between Fermi points mediated by A_j in Eq. (3). This order of limits assumes that the fluctuations of A_j are sufficiently weak compared with $|\mu|$.

In Fourier space, the equation of motion following from Eq. (2) is

$$[(\omega + \mu)\sigma_0 - p_j\sigma_j]\Psi(\omega, \mathbf{p}) = 0, \tag{4}$$

where $\Psi(\omega, \mathbf{p})$ is the Fourier transform of $\Psi(t, \mathbf{x})$ and $\mathbf{p} = (p_1, p_2)$. The equation of motion (4) implies that the particle/antiparticle excitations have the dispersion relation $\omega + \mu = \pm p$, with $p = |\mathbf{p}| \geqslant 0$. As such, Eq. (4) can be rewritten as

$$P^{(\mp)}(\mathbf{p})\Psi(\omega, \mathbf{p}) = 0, \tag{5}$$

where the projection matrices

$$P^{(\mp)}(\mathbf{p}) \equiv \frac{1}{2}\left(\sigma_0 \mp \frac{p_j\sigma_j}{p}\right). \tag{6}$$

Using the projection matrices, we may write Eq. (2) in Fourier space as

$$S_0 = \int \frac{d\omega\, d^2\mathbf{p}}{(2\pi)^3}\, \Psi^\dagger(\omega, \mathbf{p})[(\omega + \mu - p)P^{(+)}(\mathbf{p})$$
$$+ (\omega + \mu + p)P^{(-)}(\mathbf{p})]\Psi(\omega, \mathbf{p}). \tag{7}$$

For a spherical Fermi surface, we parametrize $\mathbf{p} = p\,\hat{r}_\theta \equiv p\,(\cos\theta, \sin\theta)$ with $\theta \in [0, 2\pi)$ labeling the points on the

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Fermi surface. The projection matrices become

$$P^{(\mp)}(\mathbf{p}) = \frac{1}{2}\begin{pmatrix} 1 & \mp e^{-i\theta} \\ \mp e^{+i\theta} & 1 \end{pmatrix}, \quad (8)$$

and we may expand Ψ in terms of its particle (R) and antiparticle (L) excitations as

$$\Psi(t, \mathbf{x}) = \int \frac{d\omega d^2\mathbf{p}}{(2\pi)^3} e^{-i\omega t + i\mathbf{p}\cdot\mathbf{x}} \left[\frac{1}{\sqrt{2}} \begin{pmatrix} e^{-i\theta} \\ 1 \end{pmatrix} R_\theta(\omega, p) \right.$$
$$\left. + \frac{1}{\sqrt{2}} \begin{pmatrix} -e^{-i\theta} \\ 1 \end{pmatrix} L_\theta(\omega, p) \right], \quad (9)$$

where

$$P^{(+)}(\mathbf{p})\Psi(\omega, \mathbf{p}) = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-i\theta} \\ 1 \end{pmatrix} R_\theta(\omega, p), \quad (10)$$

$$P^{(-)}(\mathbf{p})\Psi(\omega, \mathbf{p}) = \frac{1}{\sqrt{2}} \begin{pmatrix} -e^{-i\theta} \\ 1 \end{pmatrix} L_\theta(\omega, p). \quad (11)$$

Inserting this expansion (9) into Eq. (7), the free Dirac fermion action in Fourier space simplifies to

$$S_0 = \int \frac{d\omega d^2\mathbf{p}}{(2\pi)^3} [R_\theta^*(\omega, p)(\omega + \mu - p)R_\theta(\omega, p)$$
$$+ L_\theta^*(\omega, p)(\omega + \mu + p)L_\theta(\omega, p)]. \quad (12)$$

For $\mu > 0$, particles (R) with momentum $p \sim k_F \equiv |\mu|$ are light and antiparticles (L) have energy $\omega \geqslant k_F$; for $\mu < 0$, the antiparticles are light and the particles are heavy. Therefore, depending on the sign of μ, the low-energy effective theory for excitations with $p \sim k_F$ only retains the particles or antiparticles. Expanding about the Fermi momentum k_F,

$$p = k_F + p_\perp, \quad |p_\perp| \ll k_F, \quad (13)$$

we have

$$d^2\mathbf{p} = pdpd\theta \approx k_F dp_\perp d\theta, \quad (14)$$

and, for $\mu > 0$, S_0 becomes

$$S_0 = \int \frac{d\omega dp_\perp d\theta}{(2\pi)^3} R_\theta^*(\omega, p_\perp)(\omega - p_\perp)R_\theta(\omega, p_\perp), \quad (15)$$

where we replaced $R_\theta(\omega, p) \rightarrow \frac{1}{\sqrt{k_F}}R_\theta(\omega, p_\perp)$. For $\mu < 0$, we substitute $R \rightarrow L$ and $\omega - p_\perp \rightarrow \omega + p_\perp$. Note that the deviation $p_\perp$ of the momentum about k_F can be positive or negative. Depending on the sign of μ, we may interpret S_0 [Eq. (15)] as an infinite collection of 1D chiral fermions $R_\theta(\omega, p_\perp)$. We will only consider a single species of fermions with $\mu > 0$ so the low-energy theory will only involve R fermions.

The vector field $A_j(\mathbf{x})$ couples the chiral fermions $R_\theta(\omega, p_\perp)$ to one another according to S_1 [Eq. (3)]. To derive this coupling, we first decompose the Fourier transform of the vector potential $A_j(\mathbf{q})$ in terms of its longitudinal $A_L(\mathbf{q})$ and transverse $A_T(\mathbf{q})$ components,

$$A_j(\mathbf{q}) = i\frac{q_j}{q}A_L(\mathbf{q}) + i\epsilon_{jk}\frac{q_k}{q}A_T(\mathbf{q}). \quad (16)$$

The momentum transfer

$$\mathbf{q} = \mathbf{p} - \mathbf{p}', \quad (17)$$

where $\mathbf{p} = (k_F + p_\perp)\hat{r}_\theta$ and $\mathbf{p}' = (k_F + p'_\perp)\hat{r}_{\theta'}$. Since $p \sim p' \approx k_F$, we take

$$\frac{q_j}{q} \approx \frac{\hat{r}_\theta - \hat{r}_{\theta'}}{|\hat{r}_\theta - \hat{r}_{\theta'}|}. \quad (18)$$

Plugging the resulting decomposition (16) and the expansion (9) into Eq. (3), we obtain the low-energy vector potential coupling for $\mu > 0$,

$$S_1 = i \int \frac{d\omega d^2\mathbf{p}d^2\mathbf{p}'}{(2\pi)^5} R_\theta^*(\omega, p)R_{\theta'}(\omega, p')$$
$$\times \text{sign}(\theta - \theta')e^{\frac{i}{2}(\theta - \theta')}A_T(\mathbf{p} - \mathbf{p}'), \quad (19)$$

where $\text{sign}(X) = 1$ for $X \geqslant 0$ and $\text{sign}(X) = -1$ for $X < 0$. We have dropped a particle-antiparticle coupling between R and L fermions that is mediated by the longitudinal component A_L of the vector potential since such a term has an energy cost $\sim 2k_F$. The longitudinal component is therefore absent from this low-energy vector potential coupling. For $\mu < 0$, Eq. (19) acquires an overall minus sign and we substitute $R \rightarrow L$.

In general, $A_T(\mathbf{q})$ couples $R_\theta(\omega, p)$ and $R_{\theta'}(\omega, p')$ for arbitrary θ and θ'. By assuming that $A_T(\mathbf{q})$ is only nonzero for $|\mathbf{q}| \ll 2k_F$ (see Sec. II C for further discussion), we may further simplify the effective coupling (19). For $|\mathbf{q}| \ll 2k_F$, the momentum transfer $\mathbf{q} = \mathbf{p} - \mathbf{p}'$ can be approximated as

$$\mathbf{p} - \mathbf{p}' = (p_\perp - p'_\perp)\hat{r}_{\theta'} + k_F(\theta - \theta')\frac{d}{d\theta'}\hat{r}_{\theta'}, \quad (20)$$

where $|p_\perp - p'_\perp| \ll 2k_F$ and $|\theta - \theta'| \ll 1$. Here, $\hat{r}_{\theta'} = (\cos\theta', \sin\theta')$ and $\frac{d}{d\theta'}\hat{r}_{\theta'} = (-\sin\theta', \cos\theta')$ are orthogonal unit vectors about the Fermi surface point θ'. Defining

$$V_{\theta\theta'}(p_\perp - p'_\perp) \equiv i k_F \text{sign}(\theta - \theta') e^{\frac{i}{2}(\theta - \theta')} A_T(\mathbf{p} - \mathbf{p}'), \quad (21)$$

and given Eq. (20), the vector potential coupling (19) becomes

$$S_1 = \int \frac{d\omega dp_\perp d\theta dp'_\perp d\theta'}{(2\pi)^5} R_\theta^*(\omega, p_\perp)R_{\theta'}(\omega, p'_\perp)$$
$$\times V_{\theta\theta'}(p_\perp - p'_\perp). \quad (22)$$

[Recall the rescaling $R_\theta(\omega, p) \rightarrow \frac{1}{\sqrt{k_F}}R_\theta(\omega, p_\perp)$.] Note that the same $A_T(\mathbf{q})$ can enter different components of $V_{\theta\theta'}(p_\perp - p'_\perp)$ since a given $\mathbf{q}$ can correspond to the momentum transfer between distinct pairs of fermions.

We next perform the inverse Fourier transform[5] $(\omega, p_\perp) \rightarrow (t, z)$ on the fields appearing in the low-energy forms of S_0 [Eq. (15)] and S_1 [Eq. (22)]. For $\mu > 0$, $S = S_0 + S_1$ becomes

$$S = \int \frac{dtdzd\theta}{2\pi} R_\theta^*(t, z)i(\partial_t + \partial_z)R_\theta(t, z)$$
$$+ \int \frac{dtdzd\theta d\theta'}{(2\pi)^2} R_\theta^*(t, z)V_{\theta\theta'}(z)R_{\theta'}(t, z). \quad (23)$$

The first term describes an infinite collection of 1D chiral fermions, labeled by the Fermi point θ. The second term

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[5]We have $R_\theta(t, z) = \int \frac{d\omega dp_\perp}{(2\pi)^2} e^{-i\omega t + ip_\perp z}R_\theta(\omega, p_\perp)$ and $V_{\theta\theta'}(z) = \int \frac{dp_\perp}{2\pi} e^{ip_\perp z}V_{\theta\theta'}(p_\perp)$.

is a (quenched) random coupling $V_{\theta\theta'}(z)$ between fermions associated to the Fermi points θ and θ'.

B. Electrical current

The electrical current (density) is

$$J_j(t, \mathbf{x}) = \Psi^\dagger(t, \mathbf{x})\sigma_j\Psi(t, \mathbf{x}). \tag{24}$$

We are interested in the contribution to this current that arises from the low-energy excitations near the Fermi surface when $\mu > 0$. The simplest way to obtain this low-energy current is to replace in Eq. (3) the quenched vector potential $A_j(\mathbf{x})$ by a slowly varying background vector field $\mathcal{A}_j(t, \mathbf{x})$ and to take the low-energy limit described in the previous section. ("Slowly varying" means $|q_0|, |\mathbf{q}| \ll k_F$.) This produces the coupling [compare with Eq. (19)],

$$S_2 = \int \frac{d\omega d^2\mathbf{p} d\omega' d^2\mathbf{p}'}{(2\pi)^6} R_\theta^*(\omega, p) R_{\theta'}(\omega', p')$$
$$\times \left[\frac{e^{i\theta} + e^{-i\theta'}}{2}\mathcal{A}_x(q_0, \mathbf{q}) + \frac{e^{i\theta} - e^{-i\theta'}}{2i}\mathcal{A}_y(q_0, \mathbf{q})\right], \tag{25}$$

where $q_0 = \omega - \omega'$ and $\mathbf{q} = \mathbf{p} - \mathbf{p}'$. As before, $p = k_F + p_\perp$ and $p' = k_F + p'_\perp$. The variational derivative $J_j(-q_0, -\mathbf{q}) = (2\pi)^3 \frac{\delta S_2}{\delta \mathcal{A}_j(q_0, \mathbf{q})}$ gives the Fourier transform of the low-energy current,

$$J_x(-q_0, -\mathbf{q}) = \int \frac{d\omega d^2\mathbf{p} d^2\mathbf{p}'}{(2\pi)^3}\delta(\mathbf{q} - \mathbf{p} + \mathbf{p}')R_\theta^*(\omega, p)$$
$$\times \left(\frac{e^{i\theta} + e^{-i\theta'}}{2}\right)R_{\theta'}(\omega - q_0, p'), \tag{26}$$

$$J_y(-q_0, -\mathbf{q}) = \int \frac{d\omega d^2\mathbf{p} d^2\mathbf{p}'}{(2\pi)^3}\delta(\mathbf{q} - \mathbf{p} + \mathbf{p}')R_\theta^*(\omega, p)$$
$$\times \left(\frac{e^{i\theta} - e^{-i\theta'}}{2i}\right)R_{\theta'}(\omega - q_0, p'). \tag{27}$$

In computing the electrical conductivity, we will use a mixed Fourier space representation of the $\mathbf{q} = 0$ component of these currents. Fourier transforming $J_j(-q_0, \mathbf{q} = 0)$ with respect to q_0,[6] we have

$$J_x(t) \equiv J_x(t, \mathbf{q} = 0) = \int \frac{dz d\theta}{2\pi} R_\theta^*(t, z)\cos(\theta)R_\theta(t, z), \tag{28}$$

$$J_y(t) \equiv J_y(t, \mathbf{q} = 0) = \int \frac{dz d\theta}{2\pi} R_\theta^*(t, z)\sin(\theta)R_\theta(t, z). \tag{29}$$

$J_j(t)$ is the sum over the Fermi surface of the fermion density, weighted by $\cos(\theta)$ or $\sin(\theta)$, according to the current component j.

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[6]We use our previous convention for the Fourier transform of $R_\theta(t, z)$ and $J_j(t, \mathbf{q}) = \int \frac{dq_0}{2\pi}e^{-iq_0 t}J_j(q_0, \mathbf{q})$.

C. Disorder ensemble

We take the vector potential $A_j(\mathbf{q})$ to be a zero-mean Gaussian random variable with variance,

$$\overline{A_j(\mathbf{q})A_k(\mathbf{q}')} = g\delta_{ij}f(|\mathbf{q}|)\delta(\mathbf{q} + \mathbf{q}'). \tag{30}$$

Here, g is a dimensionless constant that controls the overall scale of the fluctuations of the vector potential; $gf(|\mathbf{q}|)$ is a unit-normalized function with support $|\mathbf{q}| \ll k_F$. For definiteness, we choose $f(|\mathbf{q}|) = \exp(-|\mathbf{q}|/M)$, where $M \ll k_F$ provides a smooth cutoff on the momentum transfer $\mathbf{q}$ in a scattering process.

We would like to determine what Eq. (30) implies for the random matrix $V_{\theta\theta'}(z)$. We will argue that $V_{\theta\theta'}(z)$ gives rise to a local interaction in the effective 1D theory. To this end, we consider the disorder average,

$$\overline{V_{\theta_1\theta_2}(z)V_{\theta_3\theta_4}(z')}$$
$$= -k_F^2\,\text{sign}(\theta_1 - \theta_2)\text{sign}(\theta_3 - \theta_4)e^{\frac{i}{2}(\theta_1 - \theta_2)}e^{\frac{i}{2}(\theta_3 - \theta_4)}$$
$$\times \int \frac{dp_\perp dp'_\perp}{(2\pi)^2}e^{ip_\perp z}e^{ip'_\perp z'}\overline{A_T(\mathbf{q})A_T(\mathbf{q}')}, \tag{31}$$

where [using our conventions from Eq. (20)]

$$\mathbf{q} = p_\perp\hat{r}_{\theta_2} + k_F(\theta_1 - \theta_2)\frac{d}{d\theta_2}\hat{r}_{\theta_2}, \tag{32}$$

$$\mathbf{q}' = p'_\perp\hat{r}_{\theta_4} + k_F(\theta_3 - \theta_4)\frac{d}{d\theta_4}\hat{r}_{\theta_4}. \tag{33}$$

We decompose $A_T(\mathbf{q})$ in terms of its Cartesian components by replacing

$$i\,\text{sign}(\theta_1 - \theta_2)e^{\frac{i}{2}(\theta_1 - \theta_2)}A_T(\mathbf{q})$$
$$= \frac{e^{i\theta_1} + e^{-i\theta_2}}{2}A_x(\mathbf{q}) + \frac{e^{i\theta_1} - e^{-i\theta_2}}{2i}A_y(\mathbf{q}), \tag{34}$$

$$i\,\text{sign}(\theta_3 - \theta_4)e^{\frac{i}{2}(\theta_3 - \theta_4)}A_T(\mathbf{q}')$$
$$= \frac{e^{i\theta_3} + e^{-i\theta_4}}{2}A_x(\mathbf{q}') + \frac{e^{i\theta_3} - e^{-i\theta_4}}{2i}A_y(\mathbf{q}'), \tag{35}$$

and then use Eq. (30) to find

$$\overline{V_{\theta_1\theta_2}(z)V_{\theta_3\theta_4}(z')} = \frac{k_F^2}{2}\int \frac{dp_\perp dp'_\perp}{(2\pi)^2}e^{ip_\perp z}e^{ip'_\perp z'}f(|\mathbf{q}|)\delta(\mathbf{q} + \mathbf{q}')$$
$$\times (e^{i(\theta_1 - \theta_4)} + e^{-i(\theta_2 - \theta_3)}). \tag{36}$$

We will approximate (36) by

$$\overline{V_{\theta_1\theta_2}(z)V_{\theta_3\theta_4}(z')} = gk_F\delta(\theta_1 - \theta_4)\delta(\theta_2 - \theta_3)f(z - z'), \tag{37}$$

where

$$f(z) = \int \frac{dp_\perp}{2\pi}e^{ip_\perp z}f(p_\perp), \tag{38}$$

and it is to be understood that $|\theta_1 - \theta_2| \ll 1$ and $|\theta_3 - \theta_4| \ll 1$. The disorder (37) is local in z since it depends on the relative coordinate $z - z'$. We expect Eq. (37) to be a good approximation to Eq. (36) at small momentum transfers and sufficiently low energies. More precisely, we require $M/k_F \ll 1$ and that the cutoff on $p_\perp \in (-\Lambda, \Lambda)$ satisfies $\Lambda/M \ll 1$.

A detailed explanation for the approximation (37) goes as follows. We begin by noting that for $M/k_F \ll 1$ and using $\delta(\mathbf{q} + \mathbf{q}')$, $f(\mathbf{q}) = f(\mathbf{q}')$ fixes $|\theta_1 - \theta_2| \ll 1$ and $|\theta_3 - \theta_4| \ll 1$.

Next, we expect the dominant contributions to Eq. (36) to arise when the terms in the sum $(e^{i(\theta_1-\theta_4)} + e^{-i(\theta_2-\theta_3)})$ are in-phase. This sets $\theta_1 = \theta_4 + \theta_3 - \theta_2 + 2\pi m$, for an arbitrary integer m. Using the earlier two conditions, the in-phase requirement implies $|\theta_1 - \theta_4 - \pi m| \ll 1$: This is modeled by $\delta(\theta_1 - \theta_4 - \pi|m|)$. Substituting $\theta_1 = \theta_4 + \pi m$ into $f(\mathbf{q})$ and $\theta_4 = \theta_1 - \pi m$ into $f(\mathbf{q}')$ fixes $|\theta_4 - \theta_2 + \pi m| \ll 1$ and $|\theta_3 - \theta_1 + \pi m| \ll 1$. Since $\theta_i \in [0, 2\pi)$, these two conditions allow either $m = 0$ or $|m| = 1, 2$. First consider $m = 0$. Since $\theta_2 \approx \theta_1$ and $\theta_4 \approx \theta_1$, $\mathbf{q} \approx p_\perp \hat{r}_{\theta_1} + k_F(\theta_1 - \theta_2)\frac{d}{d\theta_1}\hat{r}_{\theta_1}$ and $\mathbf{q}' \approx p'_\perp \hat{r}_{\theta_1} + k_F(\theta_3 - \theta_4)\frac{d}{d\theta_1}\hat{r}_{\theta_1}$. Because $\hat{r}_{\theta_1}$ and $\frac{d}{d\theta_1}\hat{r}_{\theta_1}$ are orthonormal, $\delta(\mathbf{q} + \mathbf{q}') = \delta(p_\perp + p'_\perp)\delta[k_F(\theta_1 - \theta_2 + \theta_3 - \theta_4)] = \frac{1}{k_F}\delta(p_\perp + p'_\perp)\delta(\theta_3 - \theta_2)$, using $\delta(\theta_1 - \theta_4 - \pi|m|)$. For Gaussian $f(|\mathbf{q}|)$, the dependence on $p_\perp$ and $\theta_1 - \theta_2$ factorizes. Absorbing into g the variation of this Gaussian on $|\theta_1 - \theta_2| \approx 0$, we perform the integral over $p'_\perp$ using $\delta(p_\perp + p'_\perp)$ to arrive at Eq. (37). Next consider $m = 1$; the $m = -1$ and $m = \pm 2$ cases work similarly and will not be discussed. Following the $m = 0$ logic, the replacement $\theta_4 = \theta_1 - \pi$ introduces a relative phase in the angular δ function, $\delta(\mathbf{q} + \mathbf{q}') = \frac{1}{k_F}\delta(p_\perp - p'_\perp)\delta[\theta_1 - \theta_2 - (\theta_3 - \theta_4)] = \frac{1}{k_F}\delta(p_\perp - p'_\perp)\delta(2\theta_1 - \theta_2 - \theta_3 - \pi) \approx \frac{1}{k_F}\delta(p_\perp - p'_\perp)\delta(\theta_1 - \theta_2)$, using $\theta_3 \approx \theta_1 - \pi$. The δ function has support when $\theta_1 = \theta_2$ [and similarly requires using the in-phase δ function $\delta(\theta_1 - \theta_4 - \pi)$ and $f(\mathbf{q}')$ that $\theta_3 = \theta_4$]. For $\Lambda/M \ll 1$, scattering along the Fermi surface dominates and the relative contributions of the $m \neq 0$ terms should be suppressed. We therefore ignore the $|m| = 1, 2$ terms in the remainder.

Before further studying the effect of the vector potential disorder, we would like to make some remarks about other types of disorders: quenched scalar potential A_0 and mass m disorders. Applying the same logic that lead to the low-energy theory (23), these couplings take the form

$$
S^{A_0+m} = \int d^2x dt \, \Psi^\dagger(t, \mathbf{x})[A_0(\mathbf{x})\sigma_0 + m(\mathbf{x})\sigma_3]\Psi(t, \mathbf{x})
$$
$$
\approx \int \frac{dt dz d\theta d\theta'}{(2\pi)^2} R^*_\theta(t, z)[V^{A_0}_{\theta\theta'}(z) + V^m_{\theta\theta'}(z)]R_{\theta'}(t, z),
\tag{39}
$$

where $V^{A_0}_{\theta\theta'}(z)$ and $V^m_{\theta\theta'}(z)$ are the Fourier transforms ($p_\perp \to z$) of

$$
V^{A_0}_{\theta\theta'}(p_\perp - p'_\perp) \equiv \tfrac{1}{2}(e^{i(\theta-\theta')} + 1)A_0(\mathbf{p} - \mathbf{p}'), \tag{40}
$$

$$
V^m_{\theta\theta'}(p_\perp - p'_\perp) \equiv \tfrac{1}{2}(e^{i(\theta-\theta')} - 1)m(\mathbf{p} - \mathbf{p}'), \tag{41}
$$

with $A_0(\mathbf{q})$ and $m(\mathbf{q})$ the Fourier transforms of $A_0(\mathbf{x})$ and $m(\mathbf{x})$. Taking $A_0(\mathbf{q})$ and $m(\mathbf{q})$ to be zero-mean Gaussian random variables, we find the disorder averages,

$$
\overline{V^{A_0}_{\theta_1\theta_2}(z)\,V^{A_0}_{\theta_3\theta_4}(z')} \propto (e^{i(\theta_1-\theta_2+\theta_3-\theta_4)} + e^{i(\theta_1-\theta_2)} + e^{i(\theta_3-\theta_4)} + 1),
\tag{42}
$$

$$
\overline{V^m_{\theta_1\theta_2}(z)\,V^m_{\theta_3\theta_4}(z')} \propto (e^{i(\theta_1-\theta_2+\theta_3-\theta_4)} - e^{i(\theta_1-\theta_2)} - e^{i(\theta_3-\theta_4)} + 1).
\tag{43}
$$

Denote $\theta_1 - \theta_2 = \alpha$ and $\theta_3 - \theta_4 = \beta$. Under the same assumptions we used before, $|\alpha|, |\beta| \ll 1$, and excluding

large-angle scattering, the most dominant in-phase (coherent) contribution to the random scalar potential average occurs when $\alpha = \beta = 0$. This turns out to make no contribution to disorder-averaged quantities in the large-N limit considered in the next section, being suppressed by a factor of $1/N$, where N is the number of points on the Fermi surface. The random mass average vanishes when $\alpha = \beta = 0$. A subdominant contribution to the random mass average occurs when $\alpha = -\beta$. The right-hand side of Eq. (43) then becomes $2 - 2\cos\alpha \approx \frac{\alpha^2}{2}$. This is an order of magnitude smaller than the contribution of the random magnetic field that we focus on in the remainder of this paper.

D. Random fixed point and its discrete approximation

Under the renormalization group transformation[7] [51,52] that leaves the S_0 part of Eq. (23) invariant, the leading flow equation [55] for the disorder variance g is

$$
\frac{dg}{d\ell} = (3 - 2\Delta)g, \tag{44}
$$

where $\Delta = 1$ is the scaling dimension of $R^*_\theta(t, z)R_{\theta'}(t, z)$ and ℓ is the renormalization group length scale that increases as the energy is reduced. We have substituted $f(z - z') = \delta(z - z')$ in deriving Eq. (44). Randomness is therefore a relevant perturbation that drives the clean Dirac theory towards strong disorder.

Because the randomness is $\mathcal{O}(1)$ relevant, perturbation theory about the clean fixed point cannot access the strong-disorder regime. Luckily, the low-energy action (23) admits an exact solution for arbitrary g (such that the derivation in Sec. II A holds), in which $V_{\theta\theta'}(z)$ is eliminated via the field redefinition [56–58],

$$
\tilde{R}_\theta(t, z) = \int \frac{d\theta'}{2\pi} U_{\theta\theta'}(z)R_{\theta'}(t, z), \tag{45}
$$

where the unitary matrix

$$
U_{\theta\theta'}(z) \equiv \left(\mathcal{T}_z e^{-i\int_{z_0}^z dz' V(z')}\right)_{\theta\theta'}. \tag{46}
$$

Here, $\mathcal{T}_z$ denotes path ordering along z and z_0 is an arbitrary base point. The resulting action simplifies to

$$
S = \int \frac{dt dz d\theta}{2\pi} \tilde{R}^*_\theta(t, z)i(\partial_t + \partial_z)\tilde{R}_\theta(t, z). \tag{47}
$$

The action (47) exactly describes the strong-disorder regime of a Dirac fermion at finite density, subject to a random vector potential of sufficiently long wavelength. The random vector potential has been eliminated from the effective action using the infinite-dimensional symmetry of the Fermi surface [59–61].

The formal manipulations above are made concrete by discretizing the θ coordinate (say, by putting the system in a finite-size box). To this end, we take the Fermi surface to consist of N discrete points: $\theta \to \theta_I = 2\pi I/N$ with $I = 1, \ldots, N$.

---

[7]The scale transformation is the following: $z \to \lambda z, t \to \lambda t, \theta \to \theta, R_\theta(t, z) \to \lambda^{-1/2}R_\theta(t, z)$.

The fields and disorder are therefore replaced as

$$R_\theta(t, z) \rightarrow \sqrt{N} R_{\theta_I}(t, z) \equiv \sqrt{N} R_I(t, z), \quad (48)$$

$$V_{\theta\theta'}(z) \rightarrow V_{\theta_I \theta_J}(z) \equiv V_{IJ}(z). \quad (49)$$

(The scaling of $R_{\theta_I}(t, z)$ by $\sqrt{N}$ is for notational simplicity.) We are specifically interested in the limit $N \rightarrow \infty$. Substituting in the discrete form of the angular integration $\int \frac{d\theta}{2\pi} \rightarrow \frac{1}{N} \sum_I$, we have the (equivalent) discrete forms for the low-energy action,

$$S = \int dt\, dz \sum_{I=1}^{N} R_I^*(t, z) i(\partial_t + \partial_z) R_I(t, z)$$

$$+ \frac{1}{N} \int dt\, dz \sum_{I,J=1}^{N} R_I^*(t, z) V_{IJ}(z) R_J(t, z) \quad (50)$$

$$= \int dt\, dz \sum_{I=1}^{N} \tilde{R}_I^*(t, z) i(\partial_t + \partial_z) \tilde{R}_I(t, z), \quad (51)$$

where the rotated fermions, in discrete form, are

$$\tilde{R}_I(t, z) = \sum_{J=1}^{N} U_{IJ}(z) R_J(t, z), \quad U_{IJ}(z) = \left(\mathcal{T}_z e^{-\frac{i}{N} \int_{z_0}^{z} dz' V(z')} \right)_{IJ}. \quad (52)$$

The discrete action has an emergent $U(N)$ symmetry. In discrete form, the low-energy currents (28) and (29) become

$$J_x(t) = \int dz \sum_I R_I^*(t, z) \cos(\theta_I) R_I(t, z), \quad (53)$$

$$J_y(t) = \int dz \sum_I R_I^*(t, z) \sin(\theta_I) R_I(t, z). \quad (54)$$

For discrete θ, the disorder variance (37) becomes

$$\overline{V_{IJ}(z) V_{KL}(z')} = g k_F N \delta_{IL} \delta_{JK} f(z - z'). \quad (55)$$

The overall factor of N arises from the discrete form of $\delta(\mathbf{q} + \mathbf{q}')$ in Eq. (36) with $\theta_I = 2\pi I/N$. It is the discrete form of the action (51) that we will use in the next section.

III. OBSERVABLES ALONG THE FIXED LINE

In general, in the presence of quenched disorder V with unit-normalized distribution $P[V]$, the disorder average of the correlation function of a physical observable $\mathcal{O}$ is defined as

$$\overline{\langle \mathcal{O} \rangle} \equiv \int DV\, P[V] \langle \mathcal{O} \rangle_V, \quad (56)$$

where the correlation function $\langle \mathcal{O} \rangle_V$ in the disorder realization V is

$$\langle \mathcal{O} \rangle_V \equiv \frac{\int D\Phi\, \mathcal{O}\, e^{iS[\Phi, V]}}{\int D\Phi\, e^{iS[\Phi, V]}}. \quad (57)$$

Here, we are momentarily denoting the dynamical fields of the theory by Φ and the action $S[\Phi, V]$ indicates a dependence upon both Φ and the disorder V. In most theories, the presence of V in the denominator of $\langle \mathcal{O} \rangle_V$ renders the direct analytic integration over all possible disorders in $\overline{\langle \mathcal{O} \rangle}$ difficult, if not impossible. As such, various ingenious tricks—such as replica, supersymmetric, and Keldysh formalisms—have been employed with various levels of success. In this paper,

we instead make use of the exact solution of the low-energy effective theory presented in Sec. II D to directly perform the disorder average.

To see how this works, consider the correlation function of a local observable $\mathcal{O}(R)$, which is a function of the unrotated $R_I(t, z)$ fermion,

$$\langle \mathcal{O}(R) \rangle_V = \frac{\int DR^\dagger DR\, \mathcal{O}(R)\, e^{iS[R, V]}}{\int DR^\dagger DR\, e^{iS[R, V]}}$$

$$= \frac{\int D\tilde{R}^\dagger D\tilde{R}\, \mathcal{O}(U^\dagger \tilde{R})\, e^{iS[\tilde{R}]}}{\int D\tilde{R}^\dagger D\tilde{R}\, e^{iS[\tilde{R}]}}. \quad (58)$$

In the first equality, $S[R, V]$ denotes the action (50); in the second equality, $S[\tilde{R}]$ denotes Eq. (51) with $\tilde{R}$ the rotated fermion (52) with rotation U a function of V. After the rotation, the denominator no longer depends on V and, in this case, yields the usual fermion determinant. The disorder V now only appears in the expression for the observable $\mathcal{O}(U^\dagger \tilde{R})$ and may, in principle, be averaged over.[8] A key point for us is that the local observables we consider separate into a sum of terms of the form

$$\mathcal{O}(U^\dagger \tilde{R}) = \sum_{a,b} c_{ab} \mathcal{A}_a(\tilde{R}) \mathcal{B}_b(V), \quad (59)$$

for some constants c_{ab}, such that the disorder-averaged correlation function $\overline{\langle \mathcal{O}(R) \rangle}$ factorizes,

$$\overline{\langle \mathcal{O}(R) \rangle} = \sum_{a,b} c_{ab} \frac{\int D\tilde{R}^\dagger D\tilde{R}\, \mathcal{A}_a(\tilde{R})\, e^{iS[\tilde{R}]}}{\int D\tilde{R}^\dagger D\tilde{R}\, e^{iS[\tilde{R}]}} \cdot \int dV\, P[V] \mathcal{B}_b(V)$$

$$\equiv \sum_{a,b} c_{ab} \langle \mathcal{A}_a(\tilde{R}) \rangle \cdot \overline{\mathcal{B}_b(V)}. \quad (60)$$

Terms such as $\langle \mathcal{A}_a(\tilde{R}) \rangle$ are calculated using the exact solution to the strong-disorder fixed point; terms such as $\overline{\mathcal{B}_b(V)}$ are calculated with respect to the given disorder ensemble. We will show how this factorization can be used to compute the fermion Green's function and the longitudinal conductivity at the random fixed point (51).

A. Diffusive Green's function and density of states

We begin by calculating the disorder-averaged Green's function and density of states. We will find that the average Green's function is short ranged and that the density of states is a positive constant. These calculations will introduce the technique we will later use to calculate the conductivity.

Using the low-energy action (51), the fermion two-point function averaged over the disorder is

$$\overline{\langle R_I(t, z) R_I^\dagger(0, 0) \rangle} = \sum_{A,B} \langle \tilde{R}_A(t, z) \tilde{R}_B^\dagger(0, 0) \rangle \cdot \overline{U_{IA}^\dagger(z) U_{BI}(0)}$$

$$= \sum_{A,B} \frac{i}{2\pi} \frac{\delta_{AB}}{(z - t) + i\alpha} \cdot \overline{U_{IA}^\dagger(z) U_{BI}(0)}, \quad (61)$$

---

[8] Note that any possible quantum anomalies [56] associated with the unitary rotations (46) of the chiral fermion path-integral measures, being a function of the disorder V only, mutually cancel between the numerator and denominator.

where $U(z)$ is defined in (52), α is a short-distance cutoff, and the index I is not summed over. Following [62], we compute the disorder average of U matrices by decomposing the interval $[0, z]$ into n steps $z_0 = 0, z_1 = \delta z, \dots, z_n = n\delta z$ of length $\delta z = |z|/n$ such that

$$U(z) = e^{-\frac{i}{N}\int_{z_{n-1}}^{z_n} dz'V(z')} \cdots e^{-\frac{i}{N}\int_{z_1}^{z_2} dz'V(z')} e^{-\frac{i}{N}\int_{z_0}^{z_1} dz'V(z')}. \quad (62)$$

We have chosen the arbitrary reference point $z_0 = 0$. For sufficiently large n with fixed $n\delta z = |z|$, the argument of each exponential can be approximated as

$$\frac{1}{N}\int_{z_{j-1}}^{z_j} dz'V(z') \approx \frac{1}{N}V(z_j)\delta z \equiv M_j. \quad (63)$$

Using Eq. (55), M_j is a zero-mean Gaussian random variable with variance,

$$\overline{(M_i)_{IJ}(M_j)_{KL}} = \frac{gk_F}{N}\delta_{IL}\delta_{JK}f(z_i - z_j)\delta z^2, \quad (64)$$

where $f(z_i - z_j)$ is given in Eq. (38). For discrete z, we take

$$f(z_i - z_j)\delta z = f_0\delta_{|i-j|,0} + f_1\delta_{|i-j|,1}. \quad (65)$$

The dimensionless coefficients f_0 and f_1 approximate a Gaussian $f(z_i - z_j)$ of finite width.

Using this, the disorder average of the product of U matrices in (61) becomes

$$\sum_A \overline{U_{IA}^\dagger(z)U_{AI}(0)} = \sum_A \overline{(e^{iM_1}\cdots e^{iM_n})_{IA} \times \mathbf{1}_{AI}}$$

$$= \overline{\left(\left[1 + iM_1 - \frac{M_1^2}{2} + \cdots\right]\cdots\left[1 + iM_n - \frac{M_n^2}{2} + \cdots\right]\right)_{II}}$$

$$= \delta_{II} - gk_F\left[\frac{f_0}{2}n + f_1(n-1)\right]\delta z\,\delta_{II}$$

$$= e^{-gk_F\left(\frac{f_0}{2}+f_1\right)|z|}\delta_{II}. \quad (66)$$

In the third equality, we have dropped higher-order terms in δz. In Appendix A, we check that these higher-order terms exponentiate to the form given in the fourth equality. Further details on such computations are given in Appendix B. Defining

$$g_{\text{eff}} \equiv g\left(\frac{f_0}{2} + f_1\right), \quad (67)$$

we obtain the disorder-averaged Green's function,

$$\overline{\langle R_I(t, z)R_I^\dagger(0, 0)\rangle} = \frac{i}{2\pi}\frac{e^{-g_{\text{eff}}k_F|z|}}{(z-t)+i\alpha}. \quad (68)$$

This Green's function has a spatial decay length $\lambda = 1/g_{\text{eff}}k_F$.

We now use the Green's function (68) to check that the density of states is finite. For this, we need the retarded Green's function averaged over the disorder,

$$\overline{G_{II}^R(t, z; 0, 0)} = -i\Theta(t)\overline{\langle\{R_I(t, z), R_I(0, 0)\}\rangle}$$

$$= -\Theta(t)\frac{i}{\pi}\frac{\alpha e^{-g_{\text{eff}}k_F|z|}}{(z-t)^2 + \alpha^2}, \quad (69)$$

where $\Theta(t)$ is the step function. Fourier transforming $\overline{G^R(t, z; 0, 0)}$ for $\alpha \to 0$, we obtain

$$\overline{G_{II}^R(\omega, p_\perp)} = \frac{1}{\omega - p_\perp + ig_{\text{eff}}k_F}. \quad (70)$$

From Eq. (70), we obtain the density of states per unit volume,

$$\rho(\omega) = -\frac{1}{\pi}\text{Im}\int\frac{dp_\perp}{2\pi}\frac{1}{\omega - p_\perp + ig_{\text{eff}}k_F} = \frac{1}{2\pi}. \quad (71)$$

B. Longitudinal conductivity

We next turn to the disorder-averaged longitudinal conductivity $\sigma_{xx}(\omega)$. The Kubo formula reads

$$\sigma_{xx}(\omega) = \frac{1}{i\omega_n}\frac{1}{L}\frac{k_F}{N}\int_0^\beta d\tau\, e^{i\omega_n\tau}\overline{\langle\mathcal{T}_\tau J_x(\tau)J_x(0)\rangle}\bigg|_{i\omega_n\to\omega+i0^+}, \quad (72)$$

where $\tau = it$, $\omega_n = (2n+1)\pi\beta$ is a positive Matsubara frequency at temperature $1/\beta$, and $L\cdot N/k_F$ is a spatial volume factor equal to $\lim_{\mathbf{q}\to 0}\delta(\mathbf{q}) = \lim_{p_\perp\to 0}\delta(p_\perp)\cdot\frac{1}{2\pi}\delta(k_F\theta_I)$ for some I.

Plugging in the expression for the current $J_x(\tau)$ (53) and using the unitary (52), the current two-point function is

$$\langle\mathcal{T}_\tau J_x(\tau)J_x(0)\rangle$$

$$= \int dzdz' \sum_{I,J,A,B,C,D}\langle\tilde{R}_A^\dagger(\tau, z)\tilde{R}_B(\tau, z)\tilde{R}_C^\dagger(0, z')\tilde{R}_D(0, z')\rangle$$

$$\times \overline{U_{AI}(z)\cos(\theta_I)U_{IB}^\dagger(z)U_{CJ}(z')\cos(\theta_J)U_{JD}^\dagger(z')}$$

$$= \int dzdz'\left[\frac{i}{2\beta\sinh\left(\frac{\pi(z-z'+i(\tau-\tau'))}{\beta}\right)}\right]^2$$

$$\times \cdot\text{Tr}\,\overline{U(z)CU^\dagger(z)U(z')CU^\dagger(z')}. \quad (73)$$

In the first equality, we introduced the diagonal matrix $C_{II'} = \cos\left(\frac{2\pi I}{N}\right)\delta_{II'}$; in the second equality, we used Eq. (51) to compute the finite-temperature fermion four-point function and expressed the product of U and C matrices using standard matrix notation. The disorder average of the product of U and C matrices is computed as in the previous section. To this end, we partition the interval $[0, z]$ into n segments and the interval $[0, z']$ into m segments, each of size $\delta z = z/n = z'/m$, and

decompose each U matrix as in Eq. (62). We find

$$\mathrm{Tr}\overline{(e^{-iM_n}\cdots e^{-iM_1})C(e^{iM_1}\cdots e^{iM_n})(e^{-iM_m}\cdots e^{-iM_1})C(e^{iM_1}\cdots e^{iM_m})} = \sum_I \cos^2\left(\frac{2\pi I}{N}\right)e^{-2g_{\mathrm{eff}}k_F|z-z'|} = \frac{N}{2}e^{-2g_{\mathrm{eff}}k_F|z-z'|}, \quad (74)$$

where g_{eff} is defined in Eq. (67). The details for the evaluation of the disorder average are given in Appendix B. The sum over I in the second equality is performed for $N \to \infty$ using the continuum limit $\frac{1}{N}\sum_I \to \int \frac{d\theta}{2\pi}$.

Inserting Eq. (74) into the current two-point function (73), the conductivity (72) becomes

$$\sigma_{xx}(\omega) = \frac{1}{i\omega_n}\frac{k_F}{2L}\int dz\,dz'\,d\tau\,e^{i\omega_n\tau}\left[\frac{i}{2\beta\sinh^2\left(\frac{\pi(z-z'+i(\tau-\tau'))}{\beta}\right)}\right]^2$$
$$\times\,e^{-2g_{\mathrm{eff}}k_F|z-z'|}\Big|_{i\omega_n\to\omega+i0^+}. \quad (75)$$

Shifting $z \to z + z'$ and $\tau \to \tau + \tau'$, the integral over z' produces a factor of L; we next calculate the integral over τ in the zero-temperature limit $\beta \to \infty$. The relevant term is

$$\int_0^\infty d\tau\,e^{i\omega_n\tau}\left[\frac{i}{2\pi}\frac{1}{(z+i\tau)}\right]^2 = -\frac{1}{4\pi}|\omega_n|\,e^{-\omega_n z}\Theta(\omega_n z). \quad (76)$$

Performing the remaining integral over z, we obtain the conductivity

$$\sigma_{xx}(\omega) = \frac{1}{8}\cdot\frac{1}{g_{\mathrm{eff}}-i\frac{\omega}{2vk_F}} \quad \text{(units of } e^2/h\text{)}, \quad (77)$$

where we have restored the fermion velocity v, previously set to one. This is the main result of this paper. The dc longitudinal conductivity varies as $1/g_{\mathrm{eff}}$ along the random fixed line. An identical calculation with $\cos\left(\frac{2\pi I}{N}\right) \to \sin\left(\frac{2\pi I}{N}\right)$ produces $\sigma_{yy}(\omega) = \sigma_{xx}(\omega)$. The Hall conductivity $\sigma_{xy}(\omega)$ vanishes because there is no time-reversal symmetry breaking on average. This follows from an explicit computation similar to the above, in which $\sum_I \cos^2\left(\frac{2\pi I}{N}\right) \to \sum_I \cos\left(\frac{2\pi I}{N}\right)\sin\left(\frac{2\pi I}{N}\right) = 0$.

We may crudely estimate the regime of validity of Eq. (77) as follows. Recall from Eq. (30) that $g_{\mathrm{eff}} \sim g$ characterizes the scale of the fluctuations of the random vector potential $\mathbf{A}(\mathbf{q})$, which in turn determines the random coupling $|V_{\theta\theta'}(p_\perp - p'_\perp)| \sim k_F|\mathbf{A}(\mathbf{q})| \sim k_F\sqrt{g_{\mathrm{eff}}}$ in Eq. (22). General effective field theory considerations require $k_F\sqrt{g_{\mathrm{eff}}} \leqslant \Lambda$, where $\Lambda \ll k_F$ is the cutoff on momenta transverse to the Fermi surface. Inserting this inequality into Eq. (77) at $\omega = 0$, we find

$$\sigma_{xx}(\omega=0) \geqslant \frac{1}{8}\cdot\frac{k_F^2}{\Lambda^2}. \quad (78)$$

$\Lambda \ll k_F$ (rather than $\Lambda \leqslant k_F$) ensures that the scattering is primarily tangential to the Fermi surface, instead of perpendicular to it (see Sec. II C). A study of the effects of the various leading corrections to the effective theory (23) could potentially clarify the bound (78).

IV. DISCUSSION AND SUMMARY

We have studied the effects of a quenched random, transverse magnetic field on a 2D Dirac fermion placed at finite density. For weak disorder of sufficiently long wavelength, we showed how the effective theory reduces to an infinite collection of chiral fermions coupled by the vector potential. This simplification allows for an exact treatment of the effects of the disorder. We found a line of fixed points along which the longitudinal dc conductivity (77) varies continuously with the disorder variance.

The dc conductivity was calculated in the collisionless $\hbar\omega/k_B T \to \infty$ limit. It is important to extend our study to the opposite order of limits $\hbar\omega/k_B T \to 0$, the so-called incoherent regime [53], relevant to experiment. (See [63] for a study of distinct dc limits of the ac conductivity of a clean Dirac fermion at zero density.) This question is pertinent to the expected universality of the conductivity at a quantum phase transition [64]. Numerical studies [65–67] of the integer quantum Hall transition appear to be roughly consistent with experiment (e.g., [68,69]), giving a value for the dc longitudinal conductivity $\sigma_{xx} \sim (0.54 - 0.60)\,e^2/h$.

We focused exclusively on the point where the Dirac fermion is massless. In the clean limit at finite density, a metal intervenes between integer quantum Hall states with $\sigma_{xy} = \pm\frac{1}{2}\frac{e^2}{h}$ as the mass m is tuned between $\pm\mu$, where μ is the chemical potential. Based on numerics (e.g., [44]), the metallic region is absent in the presence of disorder and a direct integer quantum Hall transition should be obtained. It would be interesting to redo our analysis with a finite mass m to try to find the localization length exponent for this transition.

The disorder we studied was of sufficiently long wavelength that it mediated elastic scattering between nearby Fermi points only. The opposite regime, in which all Fermi points are coupled by the disorder, might be interesting to consider. The action of the theory at energy $\omega = 0$ takes the form

$$S_{\omega=0} = \int dz\sum_{I,J=1}^N R_I^*(z)(i\partial_z\delta_{IJ} + J_{IJ})R_J(z), \quad (79)$$

with random J_{IJ} coupling all Fermi points I, J, subject to a given ensemble. Interpreting z as "time," this action is reminiscent of the quadratic Sachdev-Ye-Kitaev model [70,71] with complex fermions.

The similarity of the effective theory (50) of the random Dirac Fermi surface to the theory of N chiral free fermions in 1D suggests a possible route towards a non-Fermi-liquid generalization, in which disorder may be studied simultaneously. For example, we may consider two independent Dirac fermions—with chemical potentials that are of equal magnitude and opposite sign—in the presence of quenched vector potential disorder. The effective action is $S_R + S_L$, where

$$S_R = \int dt\,dz\sum_{I,J=1}^N R_I^*(t,z)\left[i(\partial_t + \partial_z)\delta_{IJ} + \frac{1}{N}V_{IJ}(z)\right]R_J(t,z), \quad (80)$$

$$S_{\mathrm{L}} = \int dt dz \sum_{I,J=1}^{N} L_I^*(t,z) \left[i(\partial_t - \partial_z)\delta_{IJ} - \frac{1}{N} V_{IJ}(z) \right] L_J(t,z),$$

(81)

and R_I (L_J) is the low-energy excitation about the Fermi surface defined by positive (negative) chemical potential. Couplings between right ($R_I^* R_I$) and left ($L_J^* L_J$) densities lead to Luttinger-liquid-like behavior. So long as the couplings preserve a diagonal subgroup of the $\mathrm{U}(N) \times \mathrm{U}(N)$ symmetry of the Fermi surface, the interactions and disorder can be studied

simultaneously. The implications and possible microscopic origin of such a theory are unclear.

ACKNOWLEDGMENTS

APPENDIX A: DISORDER AVERAGE OF SINGLE-PARTICLE GREEN'S FUNCTION

In this Appendix, we check that the higher-order corrections to the third equality in Eq. (66) exponentiate. This ensures the disorder-averaged Green's function takes the short-range form (68). Beginning with Eq. (66), we write

$$\sum_A \overline{U_{IA}^\dagger(z) U_{AI}(0)} = \sum_A \overline{(e^{iM_1} \dots e^{iM_{n-1}} e^{iM_n})_{II}}$$

$$= \overline{\left(\left[1 + iM_1 - \frac{M_1^2}{2} + \cdots \right] \left[1 + iM_2 - \frac{M_2^2}{2} + \cdots \right] \cdots \left[1 + iM_n - \frac{M_n^2}{2} + \cdots \right] \right)_{II}}$$

$$\equiv \mathrm{Lin} + \mathrm{Quad} + \mathrm{Cubic} + \cdots,$$

(A1)

where Lin, Quad, and Cubic stand for terms with two, four, and six M contractions. To simplify the presentation here, we set $f_0 = 0$ in Eq. (65), which leads to

$$\overline{(M_i)_{IJ}(M_j)_{KL}} = \left(\frac{g k_F}{N} f_1 \right) \delta z \, \delta_{IL} \, \delta_{JK} \, \delta_{|i-j|,1}, \quad g_Y \equiv \frac{g k_F}{N} f_1.$$

(A2)

First, we compute the linear term,

$$\mathrm{Lin} = \overline{[(iM_1)(iM_2) + (iM_2)(iM_3) + \cdots (iM_{n-1})(iM_n)]_{II}} = \sum_{j=1}^{n-1} \overline{[(iM_j)(iM_{j+1})]_{II}} = i^2 g_Y N \delta_{II} (n-1) \delta z.$$

(A3)

Next we compute the quadratic term,

$$\mathrm{Quad} = \overline{\left[\sum_{j=1}^{n-1} \frac{1}{4} (iM_j)^2 (iM_{j+1})^2 + \sum_{j=1}^{n-2} \frac{1}{2} (iM_j)(iM_{j+1})^2 (iM_{j+2}) + \sum_{j<k, j+1\neq k} (iM_j)(iM_{j+1})(iM_k)(iM_{k+1}) \right]_{II}}$$

$$= \frac{1}{4} i^4 g_Y^2 N^2 (n-1)\delta z^2 + \frac{1}{2} i^4 g_Y^2 N^2 (n-2)\delta z^2 + i^4 g_Y^2 N^2 \frac{(n-2)(n-3)}{2} \delta z^2 \to i^4 g_Y^2 N^2 \frac{z^2}{2}.$$

(A4)

In the last line, we took the continuum limit $n\delta z \to z$, in which only the n^2 term survives; in this limit, all linear in n terms can be dropped. Finally, we compute the cubic term. Given the computation of the quadratic term, we only retain the n^3 terms here,

$$\mathrm{Cubic} = \sum_{j,k,\ell, j<k<\ell, j+1\neq k, k+1\neq \ell} \overline{[(iM_j)(iM_{j+1})(iM_k)(iM_{k+1})(iM_\ell)(iM_{\ell+1})]_{II}}$$

$$= i^6 g_Y^3 N^3 \frac{n^3 + (\text{subleading terms})}{6} \delta z^3 \to i^6 g_Y^3 N^3 \frac{z^3}{6}.$$

(A5)

Examining Lin, Quad, and Cubic, in particular, the combinational factors $\frac{1}{2!}$, $\frac{1}{3!}$, we have confidence that the average $\sum_A \overline{U_{IA}^\dagger(z) U_{AI}(0)}$ indeed takes the exponential form

$$\sum_A \overline{U_{IA}^\dagger(z) U_{AI}(0)} = 1 + i^2 g_Y N z + i^4 g_Y^2 N^2 \frac{z^2}{2} + i^6 g_Y^3 N^3 \frac{z^3}{6} + \cdots = e^{-g_Y N z} = e^{-g k_F f_1 z}.$$

(A6)

APPENDIX B: DISORDER AVERAGE OF PRODUCTS OF U AND C MATRICES

In this Appendix, we detail the evaluation of the disorder average of products of U and C matrices, focusing on the product that appears in Eq. (73),

$$\mathrm{Tr}\, \overline{U(z)CU^\dagger(z)U(z')CU^\dagger(z')} \equiv \overline{U_{AI}(z)C_{II'}U_{I'B}^\dagger(z)U_{BJ}(z')C_{JJ'}U_{J'A}^\dagger(z')},$$

(B1)

where $U_{IJ}(z)$ is defined in Eq. (52), $C_{II'} = \cos(\frac{2\pi I}{N})\delta_{II'}$, and the sums over $A, I, I', B, J, J' = 1, \ldots, N$ are understood. The computation of Eq. (66) is similar and will not be discussed.

To calculate Eq. (B1), we discretize the z direction into segments $[z_{k-1}, z_k]$ of length $\delta z > 0$, where $z_k = k\delta z$ for all integer k. We take $z = z_n$, $z' = z_m$, and consider the limit $n, m \to \infty$ with z_n and z_m fixed. $U(z)$ is decomposed as

$$U(z) = e^{-\frac{i}{N}\int_{z_{n-1}}^{z_n} dz' V(z')} \cdots e^{-\frac{i}{N}\int_{z_1}^{z_2} dz' V(z')} e^{-\frac{i}{N}\int_{z_0}^{z_1} dz' V(z')}, \tag{B2}$$

and similarly for $U(z')$. Since δz is infinitesimal, we approximate

$$\frac{1}{N}\int_{z_{j-1}}^{z_j} dz' V(z') \approx \frac{1}{N}V(z_j)\delta z \equiv M_j. \tag{B3}$$

The decomposition of U becomes

$$U(z) = e^{-iM_n}e^{-iM_{n-1}}\cdots e^{-iM_1}. \tag{B4}$$

From Eq. (55), M_j is a zero-mean Gaussian random variable with variance,

$$\overline{(M_i)_{IJ}(M_j)_{KL}} = \frac{gk_F}{N}\delta_{IL}\delta_{JK}f(z_i - z_j)\delta z^2, \tag{B5}$$

where $f(z_i - z_j)$ is defined by

$$f(z_i - z_j)\delta z = f_0\delta_{|i-j|,0} + f_1\delta_{|i-j|,1} + f_2\delta_{|i-j|,2} + \cdots + f_k\delta_{|i-j|,k}. \tag{B6}$$

We refer to f_0 as the on-site correlation coefficient, f_1 as the first-neighbor correlation coefficient, f_2 as the second-neighbor correlation coefficient, etc. Using Eq. (B6), we write the disorder average (B1) as

$$\mathrm{Tr}\,\overline{U(z)CU^\dagger(z)U(z')CU^\dagger(z')} = \mathrm{Tr}[C\,C] + W_0 + W_1 + \cdots + W_k, \tag{B7}$$

where W_j denotes the contribution from f_j. The first term $\mathrm{Tr}[C\,C]$ is the constant term without any Wick contraction due to the disorder averaging.

First-neighbor correlation: W_1

We begin with the first-neighbor contribution W_1. We substitute Eq. (B4) into Eq. (B7) and expand the exponentials to obtain

$$W_1 = \overline{(e^{-iM_n}\cdots e^{-iM_1})_{AI}C_{II'}(e^{iM_1}\cdots e^{iM_n})_{I'B}} \times \overline{(e^{-iM_m}\cdots e^{-iM_1})_{BJ}C_{JJ'}(e^{iM_1}\cdots e^{iM_m})_{J'A}}|_{f_1}$$

$$\approx \overline{[(1-iM_n)\cdots(1-iM_1)]_{AI}C_{II'}[(1+iM_1)\cdots(1+iM_n)]_{I'B}}$$

$$\times \overline{[(1-iM_m)\cdots(1-iM_1)]_{BJ}C_{JJ'}[(1+iM_1)\cdots(1+iM_m)]_{J'A}}|_{f_1}, \tag{B8}$$

where $|_{f_1}$ indicates that we only consider the contributions due to the first-neighbor correlation f_1 in Eq. (B6). Note that the disorder average in the first equality is being performed over the entire product, not separately over each term in the product. Assuming $n > m$, there are four possible contractions involving M_n that we isolate by writing W_1 as

$$\overline{[(-iM_n)(-iM_{n-1})]_{AI}}C_{II'}\delta_{I'B}\,\delta_{BJ}C_{JJ'}\delta_{J'A} + \overline{(-iM_n)_{AI}(iM_{n-1})_{I'B}}\,C_{II'}\delta_{BJ}C_{JJ'}\delta_{J'A}$$

$$+ \overline{(-iM_{n-1})_{AI}(iM_n)_{I'B}}C_{II'}\,\delta_{BJ}C_{JJ'}\delta_{J'A} + \delta_{AI}C_{II'}\overline{[(iM_{n-1})(iM_n)]_{I'B}}\,\delta_{BJ}C_{JJ'}\delta_{J'A}$$

$$+ \overline{(e^{-iM_{n-1}}\cdots e^{-iM_1})_{AI}C_{II'}(e^{iM_1}\cdots e^{iM_{n-1}})_{I'B}(e^{-iM_m}\cdots e^{-iM_1})_{BJ}C_{JJ'}(e^{iM_1}\cdots e^{iM_m})_{J'A}}|_{f_1}$$

$$= 2g\,f_1\,k_F\,\delta z\,(-\delta_{AI}\delta_{I'B} + \frac{1}{N}\delta_{II'}\delta_{AB})\delta_{BJ}\hat{C}_{II'}\hat{C}_{JJ'}\delta_{J'A}$$

$$+ \overline{(e^{-iM_{n-1}}\cdots e^{-iM_1})_{AI}C_{II'}(e^{iM_1}\cdots e^{iM_{n-1}})_{I'B}(e^{-iM_m}\cdots e^{-iM_1})_{BJ}C_{JJ'}(e^{iM_1}\cdots e^{iM_m})_{J'A}}|_{f_1}. \tag{B9}$$

The subleading term $\propto \frac{1}{N}$, which comes from "crossing-contractions," vanishes because $C_{II'}\delta_{II'} = \mathrm{Tr}[C] = 0$.

We continue as above sequentially contracting first-neighbor pairs involving $M_{n-1}, M_{n-2}, \ldots, M_{m+2}$ to arrive at

$$W_1 = -2g\,f_1\,k_F\mathrm{Tr}[CC][n - (m+2) + 1]\,\delta z + (\text{Contractions of } M_{m+1}, \ldots, M_1). \tag{B10}$$

There are eight contractions that involve M_{m+1} and M_m in the second term in Eq. (B10),

$$\{\overline{[(-iM_{m+1})(-iM_m)]_{AI}}C_{II'}\delta_{I'B}\delta_{BJ}C_{JJ'}\delta_{J'A} + \overline{(-iM_{m+1})_{AI}(iM_m)_{I'B}}C_{II'}\delta_{BJ}C_{JJ'}\delta_{J'A}$$

$$+ \overline{(-iM_{m+1})_{AI}(-iM_m)_{BJ}}C_{II'}\delta_{I'B}C_{JJ'}\delta_{J'A} + \overline{(-iM_{m+1})_{AI}(+iM_m)_{J'A}}C_{II'}\delta_{I'B}\delta_{BJ}C_{JJ'}\}$$

$$+ [\overline{(M_m)_{AI}(M_{m+1})_{I'B}}C_{II'}\delta_{BJ}C_{JJ'}\delta_{J'A} - \overline{(M_m M_{m+1})_{I'B}}\delta_{AI}C_{II'}\delta_{BJ}C_{JJ'}\delta_{J'A}$$

$$+ \delta_{AI}C_{II'}\overline{(M_{m+1})_{I'B}(M_m)_{BJ}}C_{JJ'}\delta_{J'A} - \delta_{AI}C_{II'}\overline{(M_{m+1})_{I'B}(M_m)_{J'A}}\delta_{BJ}C_{JJ'}]. \tag{B11}$$

In the first $[\cdot]$, the first term cancels with the fourth, and the second term cancels with the third; the same pairs mutually cancel in the second $[\cdot]$. Next we consider the contractions involving M_m and M_{m-1}, and we again obtain zero. These cancellations continue through to contractions involving M_1. This makes sense since we expect the result to only depend on the difference $(n-m)$. Summarizing, we have

$$W_1 = -\text{Tr}[CC]\{2g\,f_1\,k_F\,[n-(m+2)+1]\delta z\}. \tag{B12}$$

Second-neighbor correlation and beyond: $W_2, \ldots, W_k$

Following the logic used to compute the first-neighbor correlations, we find

$$W_2 = \overline{(e^{-iM_n}\cdots e^{-iM_1})_{AI}C_{II'}(e^{iM_1}\cdots e^{iM_n})_{I'B}(e^{-iM_m}\cdots e^{-iM_1})_{BJ}\,C_{JJ'}(e^{iM_1}\cdots e^{iM_m})_{J'A}}\big|_{f_1}$$

$$\approx \overline{[(1-iM_n)\cdots(1-iM_1)]_{AI}C_{II'}[(1+iM_1)\cdots(1+iM_n)]_{I'B}[(1-iM_m)\cdots(1-iM_1)]_{BJ}C_{JJ'}[(1+iM_1)\cdots(1+iM_m)]_{J'A}}\big|_{f_2}$$

$$= -\text{Tr}[CC]\{2g\,f_2\,k_F\,[n-(m+3)+1]\delta z\}. \tag{B13}$$

Generalizing to kth-neighbor correlations, we have

$$W_k = -\text{Tr}[CC]\{2g\,f_k\,k_F\,[n-(m+k+1)+1]\delta z\}. \tag{B14}$$

On-site correlation: W_0

Unlike the off-site correlations discussed above, we have to expand the exponentials to quadratic order to obtain the contribution from on-site correlations,

$$W_0 = \overline{(e^{-iM_n}\cdots e^{-iM_1})_{AI}C_{II'}(e^{iM_1}\cdots e^{iM_n})_{I'B}(e^{-iM_m}\cdots e^{-iM_1})_{BJ}\,C_{JJ'}(e^{iM_1}\cdots e^{iM_m})_{J'A}}\big|_{f_0}$$

$$\approx \overline{\left[\left(1-iM_n-\frac{M_n^2}{2}\right)\cdots\left(1-iM_1-\frac{M_1^2}{2}\right)\right]_{AI}\hat{C}_{II'}\left[\left(1+iM_1-\frac{M_1^2}{2}\right)\cdots\left(1+iM_n-\frac{M_n^2}{2}\right)\right]_{I'B}}$$

$$\times \overline{\left[\left(1+iM_m-\frac{M_m^2}{2}\right)\cdots\left(1+iM_1-\frac{M_1^2}{2}\right)\right]_{BJ}\hat{C}_{JJ'}\left[\left(1+iM_1-\frac{M_1^2}{2}\right)\cdots\left(1+iM_m-\frac{M_m^2}{2}\right)\right]_{J'A}}\big|_{f_0}$$

$$= g f_0 k_F \delta z\left(-\delta_{AI}\delta_{I'B}+\frac{2}{N}\delta_{II'}\delta_{AB}\right)\delta_{BJ}\hat{C}_{II'}\hat{C}_{JJ'}\delta_{J'A}+(\text{Contractions of }M_{n-1},\ldots,M_1). \tag{B15}$$

The crossing term $\propto \frac{1}{N}$ again vanishes. Continuing in this way, we arrive at

$$W_0 = -\text{Tr}[CC]\{g\,f_0\,k_F\,[n-(m+1)+1]\delta z\}+(\text{Contractions of }M_m,\ldots,M_1). \tag{B16}$$

Similar to the cancellations that were discussed in the context of the first-neighbor correlations, the remaining contractions of $M_m,\ldots,M_1$ equal zero. Thus, we have

$$W_0 = -\text{Tr}[CC]\{g\,f_0\,k_F\,[n-(m+1)+1]\delta z\}. \tag{B17}$$

Final result

Collecting all correlations $f_0, f_1, \ldots, f_k$ and replacing $\lim_{n\to\infty}n\delta z = z$ and $\lim_{m\to\infty}m\delta z = z'$, we have

$$\text{Tr}\,\overline{U(z)CU^\dagger(z)U(z')CU^\dagger(z')} = \text{Tr}[CC]\left\{1-2gk_F\delta z\left[\frac{f_0}{2}(n-m)+\sum_{j=1}^{k}f_j(n-m-k)\right]\right\} = \text{Tr}[CC]e^{-2gk_F|z-z'|\left(\frac{f_0}{2}+f_1+\cdots f_k\right)}. \tag{B18}$$

Note that each term proportional to k in the sum vanishes at large n and m. This concludes our calculation of the disorder average in Eq. (74). In the main text, f_0 and f_1 are nonzero, and $f_{k\geqslant 2}=0$. Since the calculation of this disorder average is essentially the same as that of the fermion two-point function, the demonstration in Appendix A carries over and ensures that the exponential form above holds.

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