Duality, criticality, anomaly, and topology in quantum spin-1 chains

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In quantum spin-1 chains, there is a nonlocal unitary transformation known as the Kennedy-Tasaki transformation U_{KT}, which defines a duality between the Haldane phase and the $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry-breaking phase. In this paper, we find that U_{KT} also defines a duality between a topological Ising critical phase and a trivial Ising critical phase, which provides a "hidden symmetry breaking" interpretation of the topological criticality. Moreover, since the duality relates different phases of matter, we argue that a model with self-duality (i.e., invariant under U_{KT}) is natural to be at a critical or multicritical point. We study concrete examples to demonstrate this argument. In particular, when H is the Hamiltonian of the spin-1 antiferromagnetic Heisenberg chain, we prove that the self-dual model $H + U_{\mathrm{KT}} H U_{\mathrm{KT}}$ is exactly equivalent to a gapless spin-1/2 XY chain, which also implies an emergent quantum anomaly. On the other hand, we show that the topological and trivial Ising critical phases that are dual to each other meet at a multicritical point which is indeed self-dual.

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I. INTRODUCTION

Symmetry-protected topological (SPT) phases are distinct from trivially gapped phases, provided that certain symmetry is imposed. A paradigm of SPT phases is the Haldane phase in the spin-1 antiferromagnetic (AFM) Heisenberg model in (1+1) dimension (D) [1–3]. Protected by the $\mathbb{Z}_2 \times \mathbb{Z}_2$ spin rotation symmetry, the Haldane phase is characterized by a unique gapped ground state (GS) in the bulk, nonlocal string order, and gapless edge states [1,2]. These properties generally hold for SPT phases of (1+1)D quantum systems protected by an on-site, unitary, and linear representation of an arbitrary symmetry group G, and the most general understanding of the (1+1)D SPT phases protected by G is based on the projective representations classified by the second cohomology group $H^2[G, U(1)]$ [4–7]. Nevertheless, a broad class of (1+1)D SPT phases including the Haldane phase can also be understood from a different perspective: *hidden symmetry breaking* [2,8–11]. For example, for any short-range interacting odd-integer-spin chains respecting the $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry, a nonlocal unitary transformation, known as the Kennedy-Tasaki (KT) transformation U_{KT}, defines a duality between the Haldane phase and the $\mathbb{Z}_2 \times \mathbb{Z}_2$ spontaneous symmetry-breaking (SSB) phase [2,9–11]. The SPT order of the Haldane phase is thus interpreted as *hidden* $\mathbb{Z}_2 \times \mathbb{Z}_2$ *symmetry breaking*.

While it has been a well-known fact that gapped phases can be further classified with additional symmetries imposed, it was recently realized that for critical systems, a universality class can also split into distinct subclasses when additional symmetries are imposed, yielding the concept of symmetry-protected (or symmetry-enriched) quantum criticality [12–18]. In particular, when two subclasses can be distinguished by symmetry properties of certain nonlocal operators, an SPT/trivial classification of quantum criticalities becomes possible [12]. In this work, we find that the KT transformation also defines a duality between an SPT Ising criticality and a trivial Ising criticality. We thus argue that the "topological" nature of the SPT Ising criticality can also be interpreted as *hidden symmetry breaking*.

When a duality becomes a symmetry (i.e., the system is *self-dual*), the self-duality must force the system to stay on the phase boundary between the two duality-related phases, often leading to criticality or multicriticality [19–23]. A prominent example is the quantum transverse field Ising chain $H_{\mathrm{Ising}} = -\sum_j (\sigma_j^z \sigma_{j+1}^z + h\sigma_j^x)$, in which the Kramers-Wannier duality [24,25] exchanges the symmetric phase and the $\mathbb{Z}_2$ SSB phase. At the self-dual point $h = 1$, H_{Ising} is at a critical point described by the Ising conformal field theory (CFT).

In this paper, we focus on the KT duality and study a Hamiltonian of the form $H(\lambda) = (1-\lambda)H_{\mathrm{Hal}} + (1+\lambda)U_{\mathrm{KT}} H_{\mathrm{Hal}} U_{\mathrm{KT}}$, where H_{Hal} is an SO(3) symmetric and short-range interacting spin-1 chain in the SPT Haldane phase (for example, the AFM Heisenberg model). In other words, $H(\lambda)$ with $-1 \leqslant \lambda \leqslant 1$ interpolates between the Haldane phase and its KT-dual phase ($\mathbb{Z}_2 \times \mathbb{Z}_2$ SSB phase). Note that $U_{\mathrm{KT}} H(\lambda) U_{\mathrm{KT}} = H(-\lambda)$. Surprisingly, we find that the self-dual model $H(0)$ is *exactly equivalent* to a $(1+1)$D spin-1/2 XXZ model doped by immobile holes, and the holes are completely absent from the low-energy theory. This means that the self-dual point is indeed a critical point described by a Gaussian CFT (with central charge $c = 1$). Furthermore, we find that the effective model for $H(|\lambda| \ll 1)$ is given by the famous $(1+1)$D spin-1/2 XYZ model, which implies that

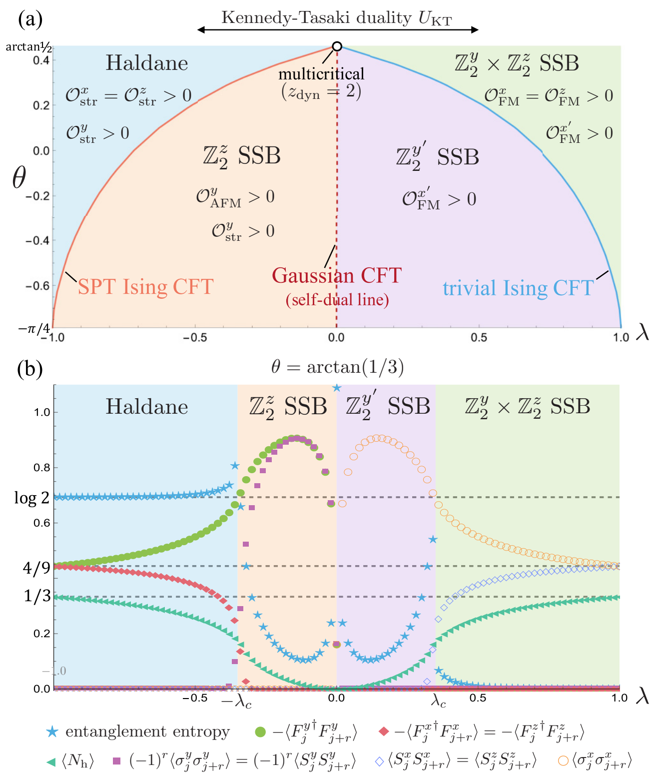

FIG. 1. (a) Schematic phase diagram of $H(\lambda, \theta)$ in the region $(\lambda, \theta) \in [-1, 1] \times (-\pi/4, \arctan \frac{1}{2}]$. There is an emergent Lieb-Schultz-Mattis (LSM) anomaly when $|\lambda| \ll 1$. Note that $\mathbb{Z}_2^y$ is a normal subgroup of $\mathbb{Z}_4^y$, while $\mathbb{Z}_2^{y'} = \mathbb{Z}_4^y/\mathbb{Z}_2^y$. Due to the global $\mathbb{Z}_4^y$ symmetry, $\mathcal{O}^x_{\rm str} = \mathcal{O}^z_{\rm str}$ and $\mathcal{O}^x_{\rm FM} = \mathcal{O}^z_{\rm FM}$. The $\mathbb{Z}_2^{y'}$ SSB phase can also be viewed as a $\mathbb{Z}_2^z$ SSB phase, while the $\mathbb{Z}_2^y \times \mathbb{Z}_2^z$ SSB phase is also a fully $\mathbb{Z}_4^y \rtimes \mathbb{Z}_2^z$ breaking phase. (b) DMRG results at $\theta = \arctan(1/3)$. The total hole number $\langle N_{\rm h} \rangle$ is obtained by infinite DMRG, while the correlation functions are calculated on an open chain with $L = 64$ and $r = 40$. Due to Eq. (36), $\mathcal{O}^y_{\rm str} \approx \mathcal{O}^y_{\rm AFM}$ when $-\lambda_c \ll \lambda < 0$. It is also clear that $\mathcal{O}^y_{\rm str} > 0$ at the SPT Ising critical point $-\lambda_c$. The half-chain entanglement entropy ($L = 64$) shows a sudden change at the critical points.

there is an *emergent quantum anomaly* around the self-dual point $\lambda = 0$. To our knowledge, the idea of emergent anomaly can be found in Refs. [18,26–29].

In fact, $H(\lambda)$ hosts more phases other than the Haldane and the $\mathbb{Z}_2 \times \mathbb{Z}_2$ SSB phases: there are two $\mathbb{Z}_2$ SSB phases in the region $-\lambda_c < \lambda < 0$ and $0 < \lambda < \lambda_c$ (with $\lambda_c < 1$), where $\pm\lambda_c$ are two Ising critical points; the former one is a $\mathbb{Z}_2 \times \mathbb{Z}_2$ trivial Ising criticality, while the latter is a $\mathbb{Z}_2 \times \mathbb{Z}_2$ SPT Ising criticality. This means that the KT transformation also defines a duality between the SPT and trivial Ising criticalities. If we introduce an additional parameter θ to the model, then the two critical lines $\pm\lambda_c(\theta)$ meet at $(\lambda, \theta) = (0, \arctan \frac{1}{2})$, on which the model $H(\lambda, \theta)$ is exactly equivalent to a spin-1/2 ferromagnetic (FM) Heisenberg chain doped by immobile holes. This means that the two KT duality-related Ising critical lines meet at a self-dual point which is indeed multicritical. See Fig. 1(a) for the phase diagram of $H(\lambda, \theta)$.

II. KENNEDY-TASAKI (KT) TRANSFORMATION

For a quantum spin-S chain where S is a nonzero *integer*, let $\boldsymbol{S}_j = (S_j^x, S_j^y, S_j^z)$ be the spin-S operator on the lattice site $j \in \{1, 2, \ldots, L\}$. The on-site spin rotation operators can be written as

$$\begin{aligned} Y_\theta &= \prod_j \exp\left(-i\theta S_j^y\right), \\ Z_\theta &= \prod_j \exp\left(-i\theta S_j^z\right), \\ X_\pi &= Y_\pi Z_\pi = \prod_j \exp\left(-i\pi S_j^x\right). \end{aligned} \tag{1}$$

We define several rotation groups as [30]

$$\begin{aligned} \mathbb{Z}_4^y &= \{1, Y_{\pi/2}, Y_\pi, Y_{3\pi/2}\}, \\ \mathbb{Z}_2^y &= \{1, Y_\pi\}, \\ \mathbb{Z}_2^z &= \{1, Z_\pi\}, \\ \mathbb{Z}_2^y \times \mathbb{Z}_2^z &= \{1, X_\pi, Y_\pi, Z_\pi\}, \\ \mathbb{Z}_4^y \rtimes \mathbb{Z}_2^z &= \{1, X_\pi, Y_\pi, Z_\pi, Y_{\pi/2}, Y_{3\pi/2}, Z_\pi Y_{\pi/2}, Y_{\pi/2} Z_\pi\}. \end{aligned} \tag{2}$$

The "symmetry flux" of $\mathbb{Z}_2^y \times \mathbb{Z}_2^z$ is a nonlocal operator defined as

$$F_j^\alpha = \exp\left(-i\pi \sum_{k<j} S_k^\alpha\right) S_j^\alpha, \quad \alpha = x, y, z. \tag{3}$$

The correlation of two symmetry fluxes gives the nonlocal *string order parameter* [31],

$$\mathcal{O}^\alpha_{\rm str} = -\lim_{r\to\infty} \left\langle F_j^\alpha F_{j+r}^\alpha \right\rangle, \quad \alpha = x, y, z. \tag{4}$$

It is known that $\mathcal{O}^\alpha_{\rm str} > 0$ serves as an order parameter for the Haldane phase protected by $\mathbb{Z}_2^y \times \mathbb{Z}_2^z$, while $\mathcal{O}^\alpha_{\rm str} = 0$ for the trivial phase [9–11,32,33].

The KT transformation is defined on a spin-S chain with open boundary condition (OBC) as [9–11,34]

$$U_{\rm KT} = \prod_{1 \leqslant u < v \leqslant L} \exp\left(i\pi S_u^z S_v^x\right), \tag{5}$$

which satisfies $U_{\rm KT} = U_{\rm KT}^\dagger$ and $U_{\rm KT}^2 = 1$. The operator $U_{\rm KT}$ obviously has the on-site $\mathbb{Z}_2^y \times \mathbb{Z}_2^z$ symmetry, which guarantees a nice property of $U_{\rm KT}$: If a (1 + 1)D Hamiltonian H has the on-site $\mathbb{Z}_2^y \times \mathbb{Z}_2^z$ symmetry, then the dual Hamiltonian $\widetilde{H} = U_{\rm KT}\ H\ U_{\rm KT}$ must also have the same on-site $\mathbb{Z}_2^y \times \mathbb{Z}_2^z$ symmetry. Spin operators transform under $U_{\rm KT}$ as [10,35]

$$\begin{aligned} S_j^x &\xleftrightarrow{U_{\rm KT}} U_{\rm KT} S_j^x U_{\rm KT} = S_j^x\, e^{i\pi \sum_{k=j+1}^L S_k^x}, \\ S_j^y &\xleftrightarrow{U_{\rm KT}} U_{\rm KT} S_j^y U_{\rm KT} = e^{i\pi \sum_{k=1}^{j-1} S_k^z}\, S_j^y\, e^{i\pi \sum_{k=j+1}^L S_k^x}, \\ S_j^z &\xleftrightarrow{U_{\rm KT}} U_{\rm KT} S_j^z U_{\rm KT} = e^{i\pi \sum_{k=1}^{j-1} S_k^z}\, S_j^z. \end{aligned} \tag{6}$$

We can thus see that in the x and z directions, the following duality holds:

$$-F_j^\alpha F_{j+r}^\alpha = -S_j^\alpha\, e^{i\pi \sum_{k=j+1}^{j+r-1} S_k^\alpha} S_{j+r}^\alpha \underset{\alpha = x,z}{\xleftrightarrow{U_{\rm KT}}} S_j^\alpha S_{j+r}^\alpha. \tag{7}$$

[As for the y direction, see Eq. (35).] It is thus clear that the Haldane phase with $\mathcal{O}^\alpha_{\rm str} > 0$ is KT dual to a $\mathbb{Z}_2^y \times \mathbb{Z}_2^z$ SSB phase with an FM order $\mathcal{O}^\alpha_{\rm FM} = \lim_{r\to\infty} \langle S_j^\alpha S_{j+r}^\alpha \rangle > 0$ ($\alpha =$

$x, z)$ [9–11]. As an example, it can be easily seen that for the $S = 1$ case, an AFM Heisenberg interaction $\boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1}$ is KT dual to an FM interaction in both the x and z directions [36],

$$\boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1} \xleftrightarrow[S=1]{U_{\rm KT}} -S^x_j S^x_{j+1} + S^y_j e^{i\pi(S^z_j + S^x_{j+1})} S^y_{j+1} - S^z_j S^z_{j+1}. \quad (8)$$

On the contrary, Eq. (7) suggests that the KT dual of a trivial phase with $\mathcal{O}^\alpha_{\rm str} = 0$ is again a trivial phase with $\mathcal{O}^\alpha_{\rm FM} = 0$. This means that the trivial phase is distinct from the SPT phase in that the former has *no* hidden symmetry breaking. As a simple example, it can be seen that the trivial model

$$H_{\rm triv} = \sum_j \left(S^z_j\right)^2 \quad (9)$$

is invariant under the KT transformation.

III. MODEL

From now on, let us focus on the case with $S = 1$. For a spin-1 chain with only nearest-neighbor interaction and SO(3) spin rotation symmetry, the most general Hamiltonian is the bilinear-biquadratic (BLBQ) model [37–39],

$$H_{\rm BLBQ}(\theta) = \sum_{j=1}^{L-1} [\cos\theta(\boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1}) + \sin\theta(\boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1})^2]. \quad (10)$$

In particular, $\theta = 0$ and $\arctan(1/3)$ correspond to the Heisenberg model and the Affleck-Kennedy-Lieb-Tasaki (AKLT) model [40,41], respectively. In fact, the GS of $H_{\rm BLBQ}(\theta)$ is in the SPT Haldane phase protected by $\mathbb{Z}^y_2 \times \mathbb{Z}^z_2$ as long as $-\pi/4 < \theta < \pi/4$ [38], and in that case the dual Hamiltonian $\widetilde{H}_{\rm BLBQ}(\theta) = U_{\rm KT} H_{\rm BLBQ}(\theta) U_{\rm KT}$ is in the $\mathbb{Z}^y_2 \times \mathbb{Z}^z_2$ SSB phase. In the following, we study a one-parameter interpolation between the two duality-related models as

$$H(\lambda, \theta) = (1 - \lambda) H_{\rm BLBQ}(\theta) + (1 + \lambda) \widetilde{H}_{\rm BLBQ}(\theta), \quad (11)$$

where we have assumed the model is defined on a chain of length L with OBC and $-1 \leqslant \lambda \leqslant 1$. The Hamiltonian actually has the on-site $\mathbb{Z}^y_4 \rtimes \mathbb{Z}^z_2$ symmetry due to the fact that the right-hand side of Eq. (8) respects the on-site $\mathbb{Z}^y_4 \rtimes \mathbb{Z}^z_2$ symmetry [42]. In the thermodynamic limit $L \to \infty$, the model has translation symmetry (denote the group as $\mathbb{Z}^{\rm trn}$), and thus the whole symmetry group G of $H(\lambda, \theta)$ is

$$G = \mathbb{Z}^y_4 \rtimes \mathbb{Z}^z_2 \times \mathbb{Z}^{\rm trn}. \quad (12)$$

A phase diagram for $H(\lambda, \theta)$ is presented in Fig. 1(a). Note that the $\mathbb{Z}^y_2 \times \mathbb{Z}^z_2$ SSB phase can alternatively be regarded as a fully $\mathbb{Z}^y_4 \rtimes \mathbb{Z}^z_2$ breaking phase since both $\mathbb{Z}^y_2 \times \mathbb{Z}^z_2$ and $\mathbb{Z}^y_4 \rtimes \mathbb{Z}^z_2$ have four different 1D representations which give rise to four (quasi)degenerate GSs.

IV. SELF-DUALITY

Since $U_{\rm KT} H(\lambda, \theta) U_{\rm KT} = H(-\lambda, \theta)$, the model $H_{\rm SD}(\theta) = H(0, \theta)$ is self-dual at $\lambda = 0$. Let $\{|+\rangle_j, |0\rangle_j, |-\rangle_j\}$ be a basis of local Hilbert space satisfying $S^z_j|\pm\rangle_j = \pm|\pm\rangle_j$ and $S^z_j|0\rangle = 0$. We define a "p-wave basis" $\{|\uparrow\rangle_j, |\downarrow\rangle_j, |{\rm h}\rangle_j\}$ as [43,44]

$$\begin{aligned} |\uparrow\rangle_j &= \frac{1}{\sqrt{2}}(|+\rangle_j - |-\rangle_j), \\ |\downarrow\rangle_j &= |0\rangle_j, \\ |{\rm h}\rangle_j &= \frac{1}{\sqrt{2}}(|+\rangle_j + |-\rangle_j). \end{aligned} \quad (13)$$

In the following, we will often abbreviate $|\cdot\rangle_j$ to $|\cdot\rangle$ for simplicity. Let us treat $|\uparrow\rangle/|\downarrow\rangle$ as the up/down spin of a spin-1/2 particle (qubit) and $|{\rm h}\rangle$ as a hole. Define Pauli operators as

$$\begin{aligned} \sigma^x_j &= |\uparrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\uparrow|, \\ \sigma^y_j &= -i|\uparrow\rangle\langle\downarrow| + i|\downarrow\rangle\langle\uparrow| = S^y_j, \\ \sigma^z_j &= |\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|. \end{aligned} \quad (14)$$

Also define two number operators

$$\begin{aligned} n_j &= |\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|, \\ h_j &= |{\rm h}\rangle\langle{\rm h}| \end{aligned} \quad (15)$$

satisfying $n_j + h_j = 1$. The self-dual Hamiltonian can then be *exactly* written as

$$H_{\rm SD}(\theta) = H_{\rm XXZ} + \sin\theta \sum_{j=1}^{L-1} (2h_j h_{j+1} + n_j n_{j+1} + 2), \quad (16)$$

where

$$\begin{aligned} H_{\rm XXZ} = (\cos\theta - \sin\theta) &\sum_{j=1}^{L-1} \left(-\sigma^x_j \sigma^x_{j+1} + \sigma^y_j \sigma^y_{j+1} \right) \\ &+ \sin\theta \sum_{j=1}^{L-1} \sigma^z_j \sigma^z_{j+1} \end{aligned} \quad (17)$$

is the *spin*-1/2 *XXZ model*. The minus sign in front of $\sigma^x_j \sigma^x_{j+1}$ can be eliminated by a unitary transformation,

$$V = \prod_{k={\rm odd}} \sigma^y_k. \quad (18)$$

The Hilbert space of a spin-1 chain is given by

$$\mathscr{H}_1 = \bigotimes_{j=1}^{L} {\rm span}(|+\rangle_j, |0\rangle_j, |-\rangle_j), \quad (19)$$

while we define the Hilbert space of a spin-1/2 chain by

$$\mathscr{H}_{1/2} = \bigotimes_{j=1}^{L} {\rm span}(|\uparrow\rangle_j, |\downarrow\rangle_j). \quad (20)$$

For $H_{\rm SD}$, the holes are completely decoupled from the qubits, making $\mathscr{H}_{1/2}$ an invariant subspace. We emphasize that a system is specified by a pair consisting of the Hamiltonian and its underlying Hilbert space. The pair $(H_{\rm SD}, \mathscr{H}_1)$ completely specifies the self-dual model. On the other hand, $(H_{\rm SD}, \mathscr{H}_{1/2})$ is equal to $(H_{\rm XXZ}, \mathscr{H}_{1/2})$ up to a constant, meaning that

$$PH_{\rm SD}P = PH_{\rm XXZ}P + 3(L-1)\sin\theta P, \quad (21)$$

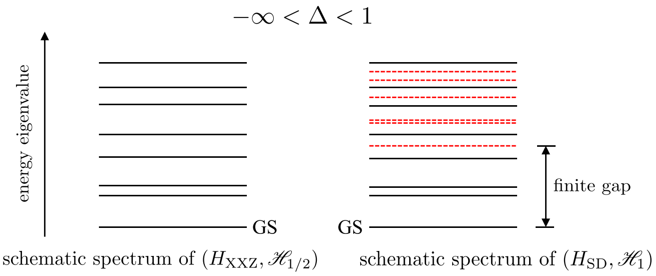

FIG. 2. An intuitive picture of the case $-\infty < \Delta < 1$ described in the Proposition. According to Eq. (21), the spectrum of $(H_{\rm XXZ}, \mathscr{H}_{1/2})$ is completely embedded in that of $(H_{\rm SD}, \mathscr{H}_1)$. When $-\infty < \Delta < 1$, in the thermodynamic limit, eigenstates of $(H_{\rm SD}, \mathscr{H}_1)$ with holes are separated from the GS by a finite-energy gap, making the low-energy physics of $(H_{\rm XXZ}, \mathscr{H}_{1/2})$ and $(H_{\rm SD}, \mathscr{H}_1)$ identical. In this figure, we have ignored the constant energy shift $3(L-1)\sin\theta$.

where

$$P = \bigotimes_{j=1}^{L} n_j = \bigotimes_{j=1}^{L} (|\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|) \tag{22}$$

is the projection operator onto $\mathscr{H}_{1/2}$. Luckily, $(H_{\rm XXZ}, \mathscr{H}_{1/2})$ is exactly solvable by the Bethe ansatz [45,46]: Let

$$\Delta(\theta) = \frac{\sin\theta}{|\cos\theta - \sin\theta|}. \tag{23}$$

The GS of $(H_{\rm XXZ}, \mathscr{H}_{1/2})$ is

(i) a Gaussian CFT ($c = 1$) when $-1 \leqslant \Delta < 1$;

(ii) gapped and degenerate when $|\Delta| > 1$;

(iii) gapless, degenerate, and has a dynamical critical exponent $z_{\rm dyn} = 2$ when $\Delta = 1$. (In this case, the model is unitarily equivalent to the FM Heisenberg chain.)

For the low-energy theory of $(H_{\rm SD}, \mathscr{H}_1)$ in the thermodynamic limit, we have the following:

Proposition. When $-\infty < \Delta < 1$, holes do not appear in the low-energy eigenstates of $(H_{\rm SD}, \mathscr{H}_1)$, meaning that states with holes are "gapped degrees of freedom (DOF)." On the other hand, when $\Delta \geqslant 1$ or $\Delta \to -\infty$, let W_1 and $W_{1/2}$ be the GS eigenspace of $(H_{\rm SD}, \mathscr{H}_1)$ and $(H_{\rm SD}, \mathscr{H}_{1/2})$, respectively. Then $W_1 \supsetneq W_{1/2}$, implying that holes appear in W_1.

An intuitive picture of the case $-\infty < \Delta < 1$ is shown in Fig. 2. A "general proof" of the Proposition is presented in Appendix A, but let us take a look at two special points as intuitive examples. At $\theta = 0$, $(VH_{\rm SD}V, \mathscr{H}_1)$ simply becomes a spin-1/2 XX model doped by immobile holes. A spin-1/2 XX chain in $\mathscr{H}_{1/2}$ can be exactly mapped to a free fermion chain. When $L \to \infty$, the GS energy density of the fermion chain is given by $e_\infty = -4/\pi$ [45,46]. In this case, if we cut the fermion chain somewhere, two edges will be created, which will raise the GS energy by $f = 2 - 4/\pi$ [47]. Now at certain site j, if we replace a qubit by a hole $|{\rm h}\rangle_j$, the total length of the spin-1/2 chain will be shortened by one, and in the meantime two edges will be created on both sides of j. This in total changes the GS energy by $f - e_\infty = 2 > 0$. Therefore, holes are energetically unfavorable when $\theta = 0$. On the other hand, at $\theta = \pi/4$, $(H_{\rm SD}, \mathscr{H}_1)$ is precisely the classical AFM three-state Potts model when representing in the p-wave basis [see Eq. (A12)], implying that $\{|{\rm h}\rangle_j\}$ are involved in the GS eigenspace.

A direct corollary of the Proposition is that following the same Δ dependence of the spin-1/2 XXZ model, the GS of $(H_{\rm SD}, \mathscr{H}_1)$ can only be in any of the three cases (i), (ii), and (iii) as $(H_{\rm XXZ}, \mathscr{H}_{1/2})$. Let us now focus on the region where $-\infty < \Delta(\theta) < 1$; in such a case, the low-energy theories of $(H_{\rm SD}, \mathscr{H}_1)$ and $(H_{\rm XXZ}, \mathscr{H}_{1/2})$ are identical. Let

$$Y'_\pi = Y_{\pi/2} = \prod_j \exp\left(-i\pi\sigma_j^y/2\right), \quad Z'_\pi = \prod_j \exp\left(-i\pi\sigma_j^z/2\right). \tag{24}$$

Since

$$\exp\left(-i\pi\sigma_j^y\right) = \exp\left(-i\pi\sigma_j^z\right) = -n_j + h_j, \tag{25}$$

we see that $Y_\pi = (Y'_\pi)^2$ and $(Z'_\pi)^2$ can only act nontrivially on the gapped DOF. We thus call $\mathbb{Z}_2^y$ a "gapped symmetry." In the low-energy theory (which lies in $\mathscr{H}_{1/2}$), $\mathbb{Z}_4^y$ reduces to the quotient group

$$\mathbb{Z}_2^{y'} = \mathbb{Z}_4^y/\mathbb{Z}_2^y. \tag{26}$$

Similarly, one can also define

$$\mathbb{Z}_2^{z'} = \{1, Z'_\pi, (Z'_\pi)^2, (Z'_\pi)^3\}/\{1, (Z'_\pi)^2\}. \tag{27}$$

The fact that the GS of $(H_{\rm XXZ}, \mathscr{H}_{1/2})$ always belongs to the cases (i), (ii), and (iii) is nowadays understood as a *Lieb-Schultz-Mattis (LSM) anomaly*. The anomaly essentially states that a spin-1/2 system with certain symmetries can never have a unique gapped GS [48–54]. The LSM anomaly of $H_{\rm XXZ}$ (and also $H_{\rm SD}$) is a result of the $\mathbb{Z}_2^{y'} \times \mathbb{Z}_2^{z'} \times \mathbb{Z}^{\rm trn}$ symmetry in $\mathscr{H}_{1/2}$ [51–54]. Since a self-duality in various cases implies a quantum phase transition, one may wonder if the anomaly of $H_{\rm SD}$ can also be regarded as a result of the KT self-duality $\mathbb{Z}_2^{\rm KT} = \{1, U_{\rm KT}\}$. Let us consider a trivial model $H_{\rm triv} = \sum_j (S_j^z)^2$ satisfying $[H_{\rm triv}, U_{\rm KT}] = 0$. Clearly, $(H_{\rm triv}, \mathscr{H}_{1/2})$ has a unique gapped GS because $PH_{\rm triv}P = \sum_j P(\sigma_j^z/2 + 1/2)P$. This tells us that neither $\mathbb{Z}_2^{\rm KT}$ nor $\mathbb{Z}_2^{\rm KT} \times \mathbb{Z}^{\rm trn}$ in $\mathscr{H}_{1/2}$ has an anomaly. However, a direct calculation shows that $PY'_\pi U_{\rm KT} Y'_\pi U_{\rm KT} P = (-i)^L P Z'_\pi P$ when L is even, which means that within $\mathscr{H}_{1/2}$, a system with both $\mathbb{Z}_2^{\rm KT}$ and $\mathbb{Z}_2^{y'}$ symmetries must also have $\mathbb{Z}_2^{y'} \times \mathbb{Z}_2^{z'}$ symmetry. Therefore, we claim that the anomaly of $H_{\rm SD}$ is protected by $\mathbb{Z}_2^{\rm KT}$, $\mathbb{Z}_2^{y'}$, and $\mathbb{Z}^{\rm trn}$ symmetries together in $\mathscr{H}_{1/2}$. This is actually an emergent anomaly; details will be discussed in Sec. VI.

The remainder of this paper will focus on the θ where $H(-1,\theta)$ is in the Haldane phase while $H_{\rm SD}(\theta)$ is critical, namely,

$$\theta \in \mathcal{R} = \left(-\pi/4, \arctan\frac{1}{2}\right), \tag{28}$$

which is included in the region $-\infty < \Delta(\theta) < 1$.

Within $\mathcal{R}$, it follows from $(VH_{\rm XXZ}V, \mathscr{H}_{1/2})$ that the low-energy theory of $(VH_{\rm SD}V, \mathscr{H}_1)$ can be exactly mapped to a spinless fermion chain with U(1) symmetry. See Appendix B for the details of $U_{\rm KT}$ in the spinless fermion language.

V. PERTURBATION THEORY

Our model can be written as

$$H(\lambda,\theta) = H_{\rm SD}(\theta) + \lambda H_{\rm pert}(\theta), \tag{29}$$

where, in the p-wave basis, $H_{\rm SD}$ is given by Eq. (16), and the second term reads

$$\begin{aligned} H_{\rm pert}(\theta) &= U_{\rm KT} H_{\rm BLBQ}(\theta) U_{\rm KT} - H_{\rm BLBQ}(\theta) \\ &= -\cos\theta \sum_j \left(\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y\right) - 2\cos\theta \sum_j (|\uparrow \mathrm{h}\rangle\langle \mathrm{h}\uparrow| + |\downarrow \mathrm{h}\rangle\langle \mathrm{h}\downarrow| + |\mathrm{h}\uparrow\rangle\langle\uparrow \mathrm{h}| + |\mathrm{h}\downarrow\rangle\langle\downarrow \mathrm{h}|) \\ &\quad - 2(\cos\theta - \sin\theta)\sum_j (|\uparrow\uparrow\rangle\langle \mathrm{hh}| + |\downarrow\downarrow\rangle\langle \mathrm{hh}| + |\mathrm{hh}\rangle\langle\uparrow\uparrow| + |\mathrm{hh}\rangle\langle\downarrow\downarrow|), \end{aligned} \tag{30}$$

where $|\cdot\rangle$ stands for a two-site state $|\cdot\rangle_{j,j+1}$. Around the self-dual point, we can treat $\lambda H_{\rm pert}$ as a perturbation to $H_{\rm SD}$. Thanks to the Proposition and Eq. (30), we know that holes are absent from the low-energy states of $(H(\lambda,\theta), \mathscr{H}_1)$ when $|\lambda| \ll 1$ and $\theta \in \mathcal{R}$. Let $N_{\rm h} = \sum_j h_j$. Holes being absent means that $\lim_{\lambda\to 0}\langle N_{\rm h}\rangle_{\rm GS} = 0$, which is also verified by our numerical calculations; see Fig. 1(b). Up to first order in λ, we find that the effective theory for $\left(H(\lambda,\theta), \mathscr{H}_1\right)$ is given by $(H_{\rm XYZ}, \mathscr{H}_{1/2})$, where

$$H_{\rm XYZ}(\lambda,\theta) = -[(1+\lambda)\cos\theta - \sin\theta]\sum_{j=1}^{L-1}\sigma_j^x\sigma_{j+1}^x + [(1-\lambda)\cos\theta - \sin\theta]\sum_{j=1}^{L-1}\sigma_j^y\sigma_{j+1}^y + \sin\theta\sum_{j=1}^{L-1}\left(\sigma_j^z\sigma_{j+1}^z + 3\right). \tag{31}$$

The *spin-1/2 XYZ model* $H_{\rm XYZ}$ obviously has $\mathbb{Z}_2^{y'} \times \mathbb{Z}_2^{z'} \times \mathbb{Z}^{\rm trn}$ symmetry in $\mathscr{H}_{1/2}$. Let us note that

$$PZ_\pi P = i^L P Z'_\pi P, \tag{32}$$

which means that we can identify Z_π with Z'_π in the low-energy theory. The exact solutions by Bethe ansatz [55–58] tell us that when $\lambda < 0$, $(H_{\rm XYZ}, \mathscr{H}_{1/2})$ is in a phase with

$$\mathcal{O}^y_{\rm AFM} = \lim_{r\to\infty}(-1)^r\left\langle\sigma_j^y\sigma_{j+r}^y\right\rangle = \lim_{r\to\infty}(-1)^r\left\langle S_j^y S_{j+r}^y\right\rangle > 0, \tag{33}$$

implying the breaking of $\mathbb{Z}_2^{z'}$ in $\mathscr{H}_{1/2}$ (or, equivalently, $\mathbb{Z}_2^z$ SSB in $\mathscr{H}_1$). On the other hand, when $\lambda > 0$, the XYZ model is in the $\mathbb{Z}_2^{y'}$ SSB phase with

$$\mathcal{O}^{x'}_{\rm FM} = \lim_{r\to\infty}\left\langle\sigma_j^x\sigma_{j+r}^x\right\rangle > 0. \tag{34}$$

In fact, from the following duality:

$$-F_j^y F_{j+r}^y \overset{U_{\rm KT}}{\longleftrightarrow} \sigma_j^x\sigma_{j+r}^x, \tag{35}$$

it is clear that the two $\mathbb{Z}_2$ SSB phases are dual to each other because

$$-PF_j^y F_{j+r}^y P = (-1)^r P\sigma_j^y\sigma_{j+r}^y P. \tag{36}$$

The whole phase diagram for $(\lambda,\theta) \in [-1,1]\times\mathcal{R}$ is determined by density matrix renormalization group (DMRG) calculations, and the results are presented in Fig. 1. The DMRG results suggest that a direct transition between the $\mathbb{Z}_2^y$ SSB phase and the $\mathbb{Z}_2^y \times \mathbb{Z}_2^z$ SSB phase happens at $\lambda_c(\theta) > 0$. Due to the KT duality, there is also a direct transition between the Haldane phase and the $\mathbb{Z}_2^z$ SSB phase at $-\lambda_c(\theta) < 0$.

On the other hand, when $\theta \in \mathcal{S} = [\arctan\frac{1}{2}, \pi/2]$, although the GS eigenspace of $H_{\rm SD}$ contains holes, Eq. (A16) tells us that two holes are never adjacent. Therefore, the effective Hamiltonian around the self-dual point is given by $(H_{tJ}, \mathscr{H}_1)$, where

$$\begin{aligned} H_{tJ}(\lambda,\theta\in\mathcal{S}) &= -[(1+\lambda)\cos\theta - \sin\theta]\sum_j\sigma_j^x\sigma_{j+1}^x + [(1-\lambda)\cos\theta - \sin\theta]\sum_j\sigma_j^y\sigma_{j+1}^y + \sin\theta\sum_j\left(\sigma_j^z\sigma_{j+1}^z + n_j n_{j+1} + 2\right) \\ &\quad + 2\lambda\cos\theta\sum_j(|\uparrow \mathrm{h}\rangle\langle \mathrm{h}\uparrow| + |\downarrow \mathrm{h}\rangle\langle \mathrm{h}\downarrow| + |\mathrm{h}\uparrow\rangle\langle\uparrow \mathrm{h}| + |\mathrm{h}\downarrow\rangle\langle\downarrow \mathrm{h}|). \end{aligned} \tag{37}$$

We can see that H_{tJ} is like a t-J model without double occupancy. A detailed study of H_{tJ} will be a future direction. In the following, we will keep focusing on the region $\mathcal{R} = (-\pi/4, \arctan\frac{1}{2})$.

VI. EMERGENT ANOMALY

The $G = \mathbb{Z}_4^y \rtimes \mathbb{Z}_2^z \times \mathbb{Z}^{\text{trn}}$ symmetry of the complete theory reduces to

$$G' = \mathbb{Z}_2^{y'} \times \mathbb{Z}_2^{z'} \times \mathbb{Z}^{\text{trn}} \tag{38}$$

in the low-energy theory. In other words, in the low-energy theory, $G = \mathbb{Z}_4^y \rtimes \mathbb{Z}_2^z \times \mathbb{Z}^{\text{trn}}$ leads to an LSM anomaly, which accounts for the absence of a unique gapped GS for $(H_{\text{XYZ}}, \mathscr{H}_{1/2})$ [51–54]. However, G in $\mathscr{H}_1$ has no anomaly, which can be seen from the toy model $H_{\text{toy}} = \sum_j (S_j^y)^2$ whose GS is trivially gapped. In other words, $H(|\lambda| \ll \lambda_c, \theta \in \mathcal{R})$ has an *emergent* anomaly. In order to recover the complete anomaly-free theory in $\mathscr{H}_1$, the emergent anomaly has to be canceled by some mechanism. Note that for the gapped symmetry, $\mathbb{Z}_2^y$, $\exp(-i\pi S_j^y)$ is identical to -1 in the low-energy theory. This indicates that the GS is "stacked" on a gapped (weak) SPT phase protected by $\mathbb{Z}_2^y \times \mathbb{Z}^{\text{trn}}$. It is this SPT phase that cancels the emergent anomaly because $Y'_\pi Z'_\pi = Y_\pi Z'_\pi Y'_\pi$. Below we will explain how this works in detail.

The model $H(|\lambda| \ll 1, \theta \in \mathcal{R})$ is effectively described by $(H_{\text{XYZ}}, \mathscr{H}_{1/2})$ and hence has an emergent LSM anomaly protected by G' in the low-energy eigenspace. Let M_d be a dD manifold in the real space and S^1 be a circle standing for the imaginary time with periodic boundary condition (PBC). Now let us put the model on a circle $M_1 = S^1$ [i.e., a chain with PBC. In the low-energy theory, the KT duality also holds for PBC; see Eq. (B3).], and consider the anomaly as the boundary of an SPT phase defined on $M_2 \times S^1$ with $\partial M_2 = M_1$. Due to the bulk-boundary correspondence [59,60], the SPT phase is also protected by $G' = \mathbb{Z}_2^{y'} \times \mathbb{Z}_2^{z'} \times \mathbb{Z}^{\text{trn}}$. The partition function on $M_2 \times S^1$ coupled to the G'-gauge field should be [61–63]

$$\begin{aligned} &Z[A^{y'}, A^{z'}, A^{\text{trn}}] \\ &= Z[0,0,0] \exp\left(i\pi \int_{M_2\times S^1} A^{y'} \wedge A^{z'} \wedge A^{\text{trn}}\right), \end{aligned} \tag{39}$$

where $Z[0, 0, 0]$ is the partition function in the absence of the G'-gauge field. $A^{y'}$, $A^{z'}$, and A^{trn} are gauge fields associated with $\mathbb{Z}_2^{y'}$, $\mathbb{Z}_2^{z'}$, and $\mathbb{Z}^{\text{trn}}$, respectively [64]. In general, a $\mathbb{Z}_2$-gauge field should satisfy the following restriction (taking $A^{y'}$ as an example) [65]:

$$\begin{aligned} \int A_\mu^{y'}\, dx^\mu &= 0, 1 \mod 2, \quad \forall \mu, \\ dA^{y'} &= 0 \quad \text{(almost everywhere)}, \\ dA^{y'} &\neq 0 \quad \text{(at monodromy defects)}, \\ \int_{N_2} dA^{y'} &= 0 \mod 2, \quad \forall N_2 \subset M_2 \times S^1, \end{aligned} \tag{40}$$

where N_2 is any 2D closed submanifold and the gauge field $A^{y'}$ is a 1-form [66],

$$A^{y'} = \sum_{\mu=1}^{3} A_\mu^{y'}(x^1, x^2, x^3)\, dx^\mu. \tag{41}$$

On the other hand, the $\mathbb{Z}$-gauge field A^{trn} satisfies the restriction

$$\begin{aligned} \int A_\mu^{\text{trn}}\, dx^\mu &= 0, 1, 2, 3, \ldots, \quad \forall \mu, \\ dA^{\text{trn}} &= 0. \end{aligned} \tag{42}$$

Equation (39) is not gauge invariant due to the (emergent) anomaly on ∂M_2. Be aware that introducing the G'-gauge field is sort of "illegal" because G' is not really the symmetry of the complete theory, which is anomaly free. The true symmetry of the complete theory is $G = \mathbb{Z}_4^y \rtimes \mathbb{Z}_2^z \times \mathbb{Z}^{\text{trn}}$ in $\mathscr{H}_1$. Nevertheless, G reduces to G' in the low-energy theory. Therefore, for consistency, the partition function coupled to the G-gauge field should take the form

$$Z[A^{\bar{y}}, A^z, A^{\text{trn}}] = Z[A^{y'}, A^{z'}, A^{\text{trn}}]\, Z_{\text{others}}, \tag{43}$$

where $A^{\bar{y}}$ is a $\mathbb{Z}_4^y$-gauge field, and Z_{others} is some other phase factor that should be able to cancel the emergent anomaly on $\partial M_2 \times S^1$ and thus guarantees the gauge invariance of $Z[A^{\bar{y}}, A^z, A^{\text{trn}}]$.

Recall that $\mathbb{Z}_2^y$ is a symmetry associated with gapped DOF. An important observation is that $\exp(-i\pi S_j^y)$ is identical to -1 in the low-energy theory, which means that each lattice site is in a (0+1)D "gapped" SPT phase protected by $\mathbb{Z}_2^y$. Together with the translation symmetry, the GS of our model can be regarded as "stacking" on a (1+1)D gapped SPT phase protected by the $\mathbb{Z}_2^y \times \mathbb{Z}^{\text{trn}}$ symmetry [5,6]. [This is a weak SPT phase because it is essentially equivalent to a translational copy of (0+1)D SPT phases.] Under the G-gauge field, this (1+1)D weak SPT phase manifests itself via the following contribution to Z_{others}:

$$\exp\left(i\pi \int_{\partial M_2\times S^1} A^y \wedge A^{\text{trn}}\right), \tag{44}$$

where A^y is a $\mathbb{Z}_2^y$-gauge field. We now show that Eq. (44) cancels the emergent anomaly in Eq. (39). Note that there is an identity

$$Y'_\pi Z'_\pi = Y_\pi Z'_\pi Y'_\pi, \tag{45}$$

which implies that [67]

$$\int_{N_2} dA^y - \int_{N_2} A^{y'} \wedge A^{z'} = 0 \mod 2, \tag{46}$$

where N_2 refers to any 2D closed submanifold of $M_2 \times S^1$. Using Eq. (46) and the Stokes theorem, Eq. (39) becomes

$$Z[A^{y'}, A^{z'}, A^{\text{trn}}] = Z[0,0,0]\ \exp\left(i\pi \int_{\partial M_2\times S^1} A^y \wedge A^{\text{trn}}\right). \tag{47}$$

Since the two phases in Eq. (44) and Eq. (47) combine into

$$\begin{aligned} &\exp\left(2\pi i \int_{\partial M_2\times S^1} A^y \wedge A^{\text{trn}}\right) \\ &= \exp\left(2\pi i \int_{\partial M_2} A_1^y dx^1 \int_{S^1} A_3^{\text{trn}} dx^3 \right. \\ &\quad \left. - 2\pi i \int_{S^1} A_3^y dx^3 \int_{\partial M_2} A_1^{\text{trn}} dx^1\right) = 1, \end{aligned} \tag{48}$$

it is now clear that Eq. (43) is indeed gauge invariant and anomaly free.

In fact, Eq. (45) can also be written as $Y'_\pi Z_\pi = Y_\pi Z_\pi Y'_\pi$, which implies the following *short exact sequence*:

$$1 \to \mathbb{Z}_2^y \to \mathbb{Z}_4^y \rtimes \mathbb{Z}_2^z \to \mathbb{Z}_2^{y'} \times \mathbb{Z}_2^{z'} \to 1. \tag{49}$$

We say that $\mathbb{Z}_4^y \rtimes \mathbb{Z}_2^z$ is the *extension* of $\mathbb{Z}_2^{y'} \times \mathbb{Z}_2^{z'}$. We also notice that the idea of the anomaly cancellation by symmetry extension can be found in Refs. [18,26–29].

At the end of this section, we note that a *higher-dimensional emergent anomaly* might be realized by simply defining our model $H(\lambda,\theta)$ on higher-dimensional lattices, where $H_{\rm BLBQ}(\theta)$ is defined by the right-hand side of Eq. (8). If one can show that holes are absent from the low-energy theory (though that might not be an easy task), then the effective spin-1/2 Hamiltonian $H_{\rm XYZ}$ in Eq. (31) holds regardless of dimensions. In that case, we have emergent LSM anomaly protected by $\mathbb{Z}_2^{y'} \times \mathbb{Z}_2^{z'} \times$ (crystalline symmetry) in higher dimensions.

VII. DUALITY OF SPT/TRIVIAL ISING CRITICALITY

From Eq. (34), one can see that the $\mathbb{Z}_2^{y'}$ SSB phase also breaks the $\mathbb{Z}_2^{z'}$ symmetry. Furthermore, since the two symmetries $\mathbb{Z}_2^{z'}$ and $\mathbb{Z}_2^z$ are the same in the low-energy theory [see Eq. (32)], the $\mathbb{Z}_2^{y'}$ SSB phase is also a $\mathbb{Z}_2^z$ SSB phase. Thus the $\mathbb{Z}_2^y$ symmetry is broken or restored every time we cross the critical line $\lambda_c(\theta) > 0$, indicating the Ising universality class. Similarly, $-\lambda_c(\theta) < 0$ is also an Ising critical line. See Appendix C 2 for numerical evidence.

The transition at $\lambda_c > 0$ is a *trivial Ising criticality* (protected by $\mathbb{Z}_2^y \times \mathbb{Z}_2^z$) in the sense that the phases on both sides have SSB. At λ_c, $\mathbb{Z}_2^y$ is the "critical symmetry," and thus [68]

$$\langle S_j^x S_{j+r}^x \rangle = \langle S_j^z S_{j+r}^z \rangle \sim r^{-1/4}. \tag{50}$$

Due to the duality in Eq. (35), we know that $\mathcal{O}_{\rm FM}^{x'} > 0$ for the $\mathbb{Z}_2^y \times \mathbb{Z}_2^z$ SSB phase. Since $\mathcal{O}_{\rm FM}^{x'}$ is also nonzero for the $\mathbb{Z}_2^{y'}$ SSB phase, it is easy to believe that at the Ising critical point between the two SSB phases ($\lambda_c > 0$),

$$\mathcal{O}_{\rm FM}^{x'} = \lim_{r\to\infty} \langle \sigma_j^x \sigma_{j+r}^x \rangle > 0. \tag{51}$$

It then directly follows from Eq. (7) and Eq. (35) that at $-\lambda_c < 0$,

$$\langle F_j^x F_{j+r}^x \rangle = \langle F_j^z F_{j+r}^z \rangle \sim r^{-1/4}, \tag{52a}$$

$$\mathcal{O}_{\rm str}^y = -\lim_{r\to\infty} \langle F_j^y F_{j+r}^y \rangle > 0. \tag{52b}$$

Since the nonlocal symmetry fluxes F_j^x and F_j^z carry nontrivial charges under $\mathbb{Z}_2^y \times \mathbb{Z}_2^z$ (for example, $Z_\pi F_j^x Z_\pi = -F_j^x$) [69], we claim that the transition at $-\lambda_c$ represents a $\mathbb{Z}_2^y \times \mathbb{Z}_2^z$ *SPT Ising criticality* [12]. Since $\mathbb{Z}_2^z$ is broken as long as $0 < \lambda \leqslant 1$, it is obvious that the GS has twofold degeneracy at λ_c; so is that at $-\lambda_c$ due to the KT duality (remember that we always assume OBC). The twofold degeneracy at $-\lambda_c$ is actually associated with topological edge states [12]; this can be seen by noting that $0 \neq \langle F_1^y F_L^y \rangle = \langle S_1^y e^{i\pi S_1^y} Y_\pi e^{i\pi S_L^y} S_L^y \rangle = \pm \langle S_1^y e^{i\pi S_1^y} e^{i\pi S_L^y} S_L^y \rangle$ implies edge magnetization $\langle S_1^y e^{i\pi S_1^y} \rangle = -\langle S_1^y \rangle \neq 0$ and $\langle S_L^y \rangle \neq 0$, where we have used the clustering property $\langle S_1^y e^{i\pi S_1^y} e^{i\pi S_L^y} S_L^y \rangle \approx \langle S_1^y e^{i\pi S_1^y} \rangle \langle e^{i\pi S_L^y} S_L^y \rangle$ [18]. Moreover, $\mathcal{O}_{\rm str}^y > 0$ at $-\lambda_c$ indicates that the topological criticality is partially protected by the gapped symmetry $\mathbb{Z}_2^y$, which further implies that the twofold (quasi)degenerate GS has an energy splitting proportional to $e^{-L/\xi}$ [12]. From the above discussions, we can see that the KT duality also provides a *hidden* $\mathbb{Z}_2^y \times \mathbb{Z}_2^z$ *symmetry breaking* picture for the SPT criticality: The algebraic decay or the long-range order of $\langle F_j^\alpha F_{j+r}^\alpha \rangle$ at $-\lambda_c$ can be easily understood from the classical Landau transition at λ_c. For more details about the interpretation of hidden symmetry breaking, see Appendix D.

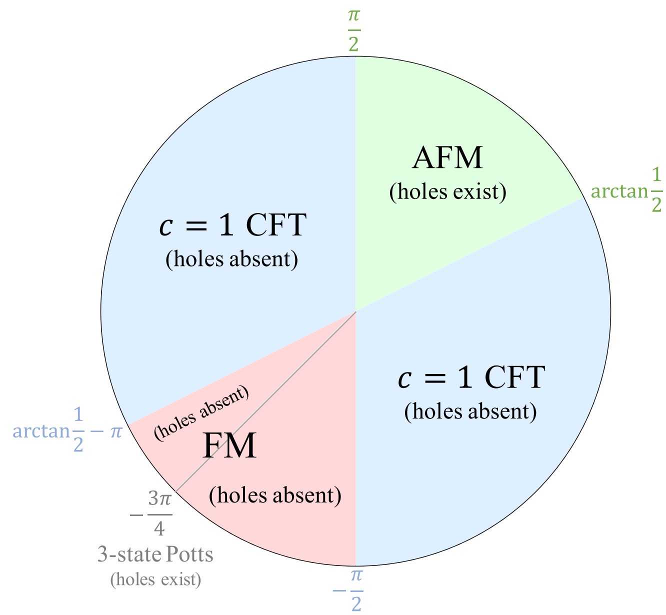

FIG. 3. Phase diagram for the GS of $H_{\rm SD}(\theta)$ in Eq. (16). Holes are absent from the low-energy eigenstates when $\theta \in (\pi/2, 2\pi + \arctan\frac{1}{2})\backslash\{5\pi/4\}$, which means that the low-energy physics of $H_{\rm SD}$ in this region is exactly the same as the spin-1/2 XXZ model. Note that an FM (AFM) GS of $H_{\rm SD}$ corresponds to an AFM (FM) GS of $H'_{\rm SD}$.

In Fig. 1(a), one can see that the trivial and topological Ising criticalities related by the KT duality meet at the self-dual point $(\lambda,\theta) = (0, \arctan\frac{1}{2})$, forcing the model $H_{\rm SD}(\arctan\frac{1}{2})$ to be at a multicritical point. Indeed, $\theta = \arctan\frac{1}{2}$ corresponds to $\Delta = 1$, in which case $(H_{\rm SD}, \mathscr{H}_1)$ is equivalent to a spin-1/2 FM Heisenberg model doped by immobile holes, which has $z_{\rm dyn} = 2$. On the other direction, the Ising critical lines terminate at $H(\pm 1, -\pi/4)$, whose low-energy physics is the CFT with $c = 3/2$ [70–72].

VIII. DISCUSSION

We have been focusing on the spin-1 chains, but in fact, the KT transformation directly applies to *any integer* spin quantum number S, as long as we take S_u^z and S_v^x in Eq. (5) to be the spin-S operators [11]. Let $H_S = \sum_j \boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1}$ be the spin-S AFM Heisenberg chain; it is believed that the GS of H_S is in the $\mathbb{Z}_2^y \times \mathbb{Z}_2^z$ SPT phase when S is odd, while the GS is trivial when S is even [11,35]. Note that the KT dual of the trivial phase is still trivial. Therefore, we propose the following:

Conjecture. Let $H_{\rm SD}^{(S)} = H_S + U_{\rm KT} H_S U_{\rm KT}$; the GS of $H_{\rm SD}^{(S)}$ is gapless when S is odd, while it is trivially gapped when S is even.

Moreover, the KT transformation can be generalized to (1+1)D systems with a broad class of symmetries beyond $\mathbb{Z}_2 \times \mathbb{Z}_2$, such as $\mathbb{Z}_n \times \mathbb{Z}_n$ and SO$(2n-1)$ [8,73–75], thus providing the hidden symmetry breaking picture for the SPT phases in such systems. Exploring the relationship between the KT duality, criticality, anomaly, and topology in such systems will also be interesting.

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APPENDIX A: EXISTENCE OR ABSENCE OF HOLES

A proof of the Proposition is presented here. Although our proof might not be entirely rigorous from a mathematical point of view, it nevertheless makes sense for physicists. Let us begin by first noting that by properly rotating the spins, $H_{\rm SD}$ in Eq. (16) is always unitarily equivalent to

$$H'_{\rm SD}(\theta) = 2\sqrt{2}\left|\sin\left(\frac{\pi}{4}-\theta\right)\right| \times H_{\rm XXZ}(\theta) + \sin\theta \sum_{j=1}^{L-1}(2h_j h_{j+1} + n_j n_{j+1} + 2), \tag{A1}$$

where

$$H_{\rm XXZ}(\theta) = \frac{1}{2}\sum_{j=1}^{L-1}\left[\sigma_j^x\sigma_{j+1}^x + \sigma_j^y\sigma_{j+1}^y - \Delta(\theta)\sigma_j^z\sigma_{j+1}^z\right]. \tag{A2}$$

Note that the definition of $H_{\rm XXZ}$ here is different from the main text, and within this Appendix we will adapt the definition in Eq. (A2). The parameter

$$\Delta(\theta) = \frac{\sin\theta}{|\cos\theta - \sin\theta|} \tag{A3}$$

determines the phase of $H_{\rm XXZ}(\theta)$ and hence $H_{\rm SD}(\theta)$. The results are summarized in Fig. 3.

1. Absence of holes

Let

$$\mathcal{A} = \left[-\frac{\pi}{2}, \arctan\frac{1}{2}\right) \cup \left(\frac{\pi}{2}, \arctan\frac{1}{2} + \pi\right], \tag{A4}$$

$$\mathcal{B} = \left(\arctan\frac{1}{2} - \pi, -\frac{\pi}{2}\right) \Big\backslash \left\{-\frac{3\pi}{4}\right\}. \tag{A5}$$

When $\theta \in \mathcal{A}$, we have $-1 \leqslant \Delta(\theta) < 1$, and thus $H_{\rm XXZ}$ is gapless and the low-energy physics is described by a $c = 1$ CFT. When $\theta \in \mathcal{B}$, $\Delta(\theta) < -1$ and $H_{\rm XXZ}$ has two degenerate and AFM ground states in the thermodynamic limit $L \to \infty$. Let $\gamma = \arccos[-\Delta(\theta)]$ when $\theta \in \mathcal{A}$ and $\xi = \operatorname{arccosh}[-\Delta(\theta)]$ when $\theta \in \mathcal{B}$. The ground-state (GS) energy density of $H_{\rm XXZ}$ in the thermodynamic limit, denoted as e_∞, was exactly obtained by Yang and Yang back in 1966 [45,46],

$$e_\infty(\theta) = \begin{cases} \frac{1}{2}\cos\gamma - (\sin\gamma)^2 \int_{-\infty}^{\infty} \frac{dx}{\cosh(\pi x)[\cosh(2\gamma x) - \cos\gamma]}, & -1 \leqslant -\cos\gamma = \Delta(\theta) < 1, \\ \frac{1}{2}\cosh\xi - \left[1 + 4\sum_{n=1}^{\infty}\frac{1}{1+e^{2\xi n}}\right]\sinh\xi, & -\cosh\xi = \Delta(\theta) < -1. \end{cases} \tag{A6}$$

When $1 \ll L < \infty$, we need to consider finite-size corrections. For OBC, the finite-size GS energy density $e_L^{\rm OBC}$ takes the form [47,76]

$$e_L^{\rm OBC} = e_\infty + \frac{f}{L} + o\left(\frac{1}{L}\right), \tag{A7}$$

where f is called the "surface energy" and is given by

$$f(\theta) = \begin{cases} \frac{\pi\sin\gamma}{2\gamma} - \frac{\cos\gamma}{2} - \frac{\sin\gamma}{4}\int_{-\infty}^{\infty} dx\left[1 - \tanh\left(\frac{\pi x}{4}\right)\tanh\left(\frac{\gamma x}{2}\right)\right], & -1 \leqslant -\cos\gamma = \Delta(\theta) < 1, \\ -\frac{1}{2}\cosh\xi + 4\left[\frac{1}{4} + \sum_{n=1}^{\infty}\frac{e^{2n\xi}-1}{1+e^{4n\xi}} + \sum_{n=1}^{\infty}\frac{(-1)^n}{1+e^{2n\xi}}\right]\sinh\xi, & -\cosh\xi = \Delta(\theta) < -1. \end{cases} \tag{A8}$$

Note that Eq. (A1) is defined on a chain with OBC. For a sufficiently long chain, in $\mathscr{H}_{1/2}$ (the subspace without holes), the GS energy of $H_{\rm SD}$ (up to the order of L^0) is given by

$$E_0 = 2\sqrt{2}\left|\sin\left(\frac{\pi}{4}-\theta\right)\right|(Le_\infty + f) + 3(L-1)\sin\theta. \tag{A9}$$

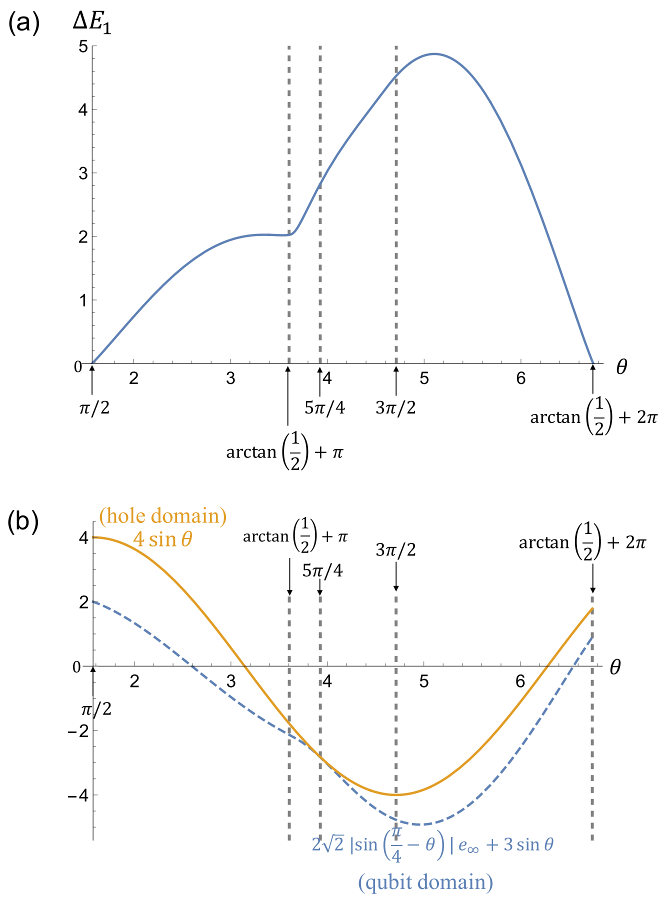

FIG. 4. (a) $\Delta E_1 > 0$ for all $\theta \in (\pi/2, \arctan\frac{1}{2} + 2\pi)$. (b) Energy density of the hole domain is higher than that of the qubit domain when $\theta \in A \cup B = (\pi/2, \arctan\frac{1}{2} + 2\pi)\backslash\{5\pi/4\}$, implying the absence of the phase separation.

Now consider the subspace of m holes. When the m holes are disjoint, sufficiently far away from each other and sufficiently far away from the boundary, the GS energy of $H_{\rm SD}$ in this subspace is (up to the order of L^0)

$$E_m = 2\sqrt{2}\left|\sin\left(\frac{\pi}{4}-\theta\right)\right|[(L-m)e_\infty + (m+1)f] + (L-1-2m)\sin\theta + 2(L-1)\sin\theta. \tag{A10}$$

The energy difference is given by

$$\begin{aligned}\Delta E_m &= E_m - E_0\\ &= m\left[2\sqrt{2}\left|\sin\left(\frac{\pi}{4}-\theta\right)\right|(f-e_\infty) - 2\sin\theta\right]\\ &= m\Delta E_1.\end{aligned} \tag{A11}$$

As long as $\Delta E_1 = 2\sqrt{2}|\sin(\pi/4-\theta)|(f-e_\infty) - 2\sin\theta > 0$, eigenstates with disjoint holes are gapped from the ground state. The value of ΔE_1 can be easily obtained numerically; see Fig. 4(a). We can see that ΔE_1 is indeed positive when $\theta \in \mathcal{A}\cup\mathcal{B}$.

We have not yet ruled out the possibility of "phase separation," meaning that holes form a domain, like $|\ldots\uparrow\uparrow \text{hhh}\ldots\text{hh}\downarrow\uparrow\ldots\rangle$. In fact, the energy density of the hole domain is given by $4\sin\theta$, while the energy density of the spin-1/2 domain is $2\sqrt{2}|\sin(\pi/4-\theta)|e_\infty + 3\sin\theta$ (up to the order of L^0). We can numerically show that the former energy density is always higher than the latter when $\theta \in \mathcal{A}\cup\mathcal{B}$ [see Fig. 4(b)], which means that the phase separation does not occur.

To sum up, holes $\{|\text{h}\rangle_j\}$ are gapped from the ground state of $H_{\rm SD}(\theta)$ when θ is in the region described by Eq. (A4) and Eq. (A5), which is in accordance with our DMRG results.

2. Existence of holes

At two special points $\theta = \pi/4$ and $-3\pi/4$, $H_{\rm SD}$ (but not $H'_{\rm SD}$) reduces to the classical *three-state Potts model*. This can be seen by noting that

$$\begin{aligned}&\sigma^z_j\sigma^z_{j+1} + 2h_jh_{j+1} + n_jn_{j+1}\\ &\quad= 2(|\uparrow\uparrow\rangle\langle\uparrow\uparrow| + |\downarrow\downarrow\rangle\langle\downarrow\downarrow| + |\text{hh}\rangle\langle\text{hh}|),\end{aligned} \tag{A12}$$

where $\theta = \pi/4$ is AFM and $\theta = -3\pi/4$ is FM. The ground states at these two points are thus degenerate, and holes $\{|\text{h}\rangle_j\}$ appear in the GS eigenspace.

In the region

$$\theta \in \left[\arctan\frac{1}{2}, \frac{\pi}{2}\right], \tag{A13}$$

$\Delta(\theta) \geqslant 1$, and the ground state of $H_{\rm XXZ}$ is FM. The GS energy of $H_{\rm XXZ}$ exactly equals $-(L-1)\Delta(\theta)/2$. In the subspace without holes, the GS energy of $H_{\rm SD}$ is then given by

$$\begin{aligned}E_0 &= 2\sqrt{2}\left|\sin\left(\frac{\pi}{4}-\theta\right)\right|\\ &\quad\times[-(L-1)\Delta(\theta)/2] + 3(L-1)\sin\theta\\ &= 2(L-1)\sin\theta.\end{aligned} \tag{A14}$$

In the subspace of m holes, when the holes are all disjoint, the GS energy becomes

$$\begin{aligned}E_m &= 2\sqrt{2}\left|\sin\left(\frac{\pi}{4}-\theta\right)\right| \times [-(L-1-2m)\Delta(\theta)/2]\\ &\quad+ (L-1-2m)\sin\theta + 2(L-1)\sin\theta\\ &= 2(L-1)\sin\theta = E_0,\end{aligned} \tag{A15}$$

which means that adding disjoint holes to the ground state does not cost energy. On the other hand, if the m holes form a domain,

$$\begin{aligned}E_m &= 2\sqrt{2}\left|\sin\left(\frac{\pi}{4}-\theta\right)\right| \times [-(L-m-2)\Delta(\theta)/2]\\ &\quad+ 2(m-1)\sin\theta + (L-m-2)\sin\theta + 2(L-1)\sin\theta\\ &= 2(L+m-2)\sin\theta.\end{aligned} \tag{A16}$$

We see that $E_m > E_0$ as long as $m \geqslant 2$. In other words, phase separation does not occur in the ground state.

APPENDIX B: KT DUALITY IN LOW-ENERGY THEORY

As shown in the previous sections, when

$$|\lambda| \ll 1, \quad \theta \in \mathcal{R} = \left(-\frac{\pi}{4}, \arctan\frac{1}{2}\right), \tag{B1}$$

the low-energy eigenspace of the model $\big(H(\lambda,\theta),\mathscr{H}_1\big)$ completely lies in $\mathscr{H}_{1/2}$. The projection onto $\mathscr{H}_{1/2}$ gives

$$P\,U_{\mathrm{KT}}P=\prod_{1\leqslant u<v\leqslant L}P\exp\left[\frac{i\pi}{4}\big(1+\sigma_u^z\big)\big(1-\sigma_v^z\big)\right]P. \quad \text{(B2)}$$

From Eq. (B2), it can be shown that within $\mathscr{H}_{1/2}$, the following duality holds:

$$\begin{aligned}-\sigma_j^y\sigma_{j+1}^y \;&\underset{1\leqslant j<L-1}{\overset{U_{\mathrm{KT}}}{\longleftrightarrow}}\; \sigma_j^x\sigma_{j+1}^x,\\ (-1)^{L-1}\sigma_L^y\sigma_1^y \;&\overset{U_{\mathrm{KT}}}{\longleftrightarrow}\; \sigma_L^x\sigma_1^x.\end{aligned} \quad \text{(B3)}$$

Interestingly, although we have been dealing with the case of OBC, Eq. (B3) shows that even if we impose PBC on Eq. (31), the effective theory around the self-dual point still respects the KT duality as long as L is even.

Within $\mathcal{R}$, it follows from $(H_{\mathrm{XXZ}},\mathscr{H}_{1/2})$ that the low-energy theory of $(\mathcal{H}_{\mathrm{SD}}=VH_{\mathrm{SD}}V,\mathscr{H}_1)$ can be exactly mapped to a U(1) symmetric spinless fermion chain by the Jordan-Wigner transformation,

$$c_j^\dagger=\sigma_j^+\prod_{k<j}\big(-\sigma_k^z\big), \quad \text{(B4)}$$

where V is defined in Eq. (18) and $c_j^\dagger$ is a fermion creation operator. Let $\mathcal{U}_{\mathrm{KT}}=VU_{\mathrm{KT}}V$ satisfying $[\mathcal{H}_{\mathrm{SD}},\mathcal{U}_{\mathrm{KT}}]=0$. It then follows from Eq. (B2) that within $\mathscr{H}_{1/2}$, the following duality holds for fermions:

$$c_j^\dagger \overset{\mathcal{U}_{\mathrm{KT}}}{\longleftrightarrow} (-1)^{L(L-1)/2}\,c_j^\dagger\,(-1)^F, \quad \text{(B5)}$$

where $F=\sum_j c_j^\dagger c_j$. Note that $(-1)^F$ cannot be simply regarded as a phase because it anticommutes with $c_j^\dagger$.

APPENDIX C: NUMERICAL RESULTS

1. Gaussian criticality

The fact that the self-dual point in the region $\theta\in\mathcal{R}$ stands for a Gaussian criticality ($c=1$ CFT) is also supported by our density matrix renormalization group (DMRG) calculations of the critical exponent η. According to $(H_{\mathrm{XXZ}},\mathscr{H}_{1/2})$ in Eq. (17) [see, also, the effective Hamiltonian in Eq. (31) with $\lambda=0$)], the spin correlation function behaves as

$$\big\langle\sigma_j^x\sigma_{j+r}^x\big\rangle\sim r^{-\eta(\theta)}, \quad \text{(C1)}$$

where [77]

$$\eta(\theta)=\frac{1}{2}-\frac{1}{\pi}\arcsin[\Delta(\theta)]. \quad \text{(C2)}$$

In particular, $\eta(0)=1/2$ and $\eta(\arctan\frac{1}{3})=1/3$, which are consistent with the numerical results presented in Fig. 5.

2. Ising criticality

Our DMRG results in Fig. 6 show that $\langle S_j^zS_{j+r}^z\rangle=\langle S_j^xS_{j+r}^x\rangle\sim r^{-1/4}$ at the critical point $\lambda_c>0$, which indeed suggests the Ising universality class [68]. The fact that $\mathcal{O}_{\mathrm{FM}}^{x'}>0$ at $-\lambda_c$ and $\mathcal{O}_{\mathrm{str}}^{y}>0$ at λ_c is also supported by DMRG calculations; see Fig. 1(b).

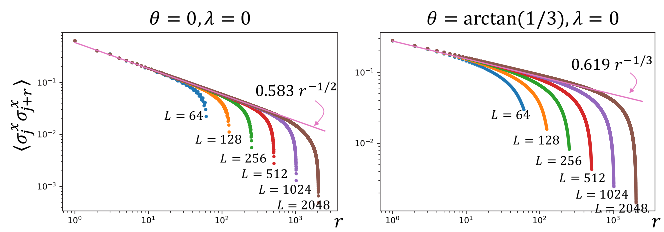

FIG. 5. The correlation function $\langle\sigma_j^x\sigma_{j+r}^x\rangle$ on an open chain with length L is calculated with various L at $\theta=0$ (left) and $\arctan(1/3)$ (right). From the log-log plots, it is clear that the data are well fitted by $r^{-\eta(\theta)}$ when $1\ll r\ll L$.

3. Direct transition between the SPT phase and the $\mathbb{Z}_2^y\times\mathbb{Z}_2^z$ SSB phase

To numerically show that there is indeed a direct transition between the Haldane phase and the $\mathbb{Z}_2^y\times\mathbb{Z}_2^z$ SSB phase at $(\lambda,\theta)=(0,\arctan\frac{1}{2})$, we estimate $\mathcal{O}_{\mathrm{str}}^{x,z}$ and $\mathcal{O}_{\mathrm{FM}}^{x,z}$ around that point; see Fig. 7. Compared to Fig. 1(b), the result in Fig. 7 suggests that the two $\mathbb{Z}_2$ SSB phases vanish at $\theta=\arctan(1/2)$.

APPENDIX D: HIDDEN SYMMETRY BREAKING

The KT transformation U_{KT} provides a hidden symmetry-breaking interpretation of gapped SPT phases, not only because it defines an SPT-SSB duality, but also because the KT dual of a trivially gapped phase is again trivial, meaning that a trivially gapped phase has no hidden symmetry breaking [see Eq. (D1)],

$$\begin{aligned}\mathbb{Z}_2\times\mathbb{Z}_2\text{ gapped SPT}\;&\overset{U_{\mathrm{KT}}}{\longleftrightarrow}\;\mathbb{Z}_2\times\mathbb{Z}_2\text{ SSB},\\ \mathbb{Z}_2\times\mathbb{Z}_2\text{ gapped trivial}\;&\overset{U_{\mathrm{KT}}}{\longleftrightarrow}\;\mathbb{Z}_2\times\mathbb{Z}_2\text{ gapped trivial}.\end{aligned} \quad \text{(D1)}$$

Can U_{KT} also provide a hidden symmetry breaking interpretation of SPT Ising critical phases? To answer this question, let us first briefly review the classification of Ising criticalities with $\mathbb{Z}_2\times\mathbb{Z}_2$ symmetry. There are nine different $\mathbb{Z}_2\times\mathbb{Z}_2$ Ising criticalities, which can be divided into three subclasses A, B, and C [12]; see Fig. 8. Subclass A contains the criticalities between the trivially gapped phase and a $\mathbb{Z}_2$

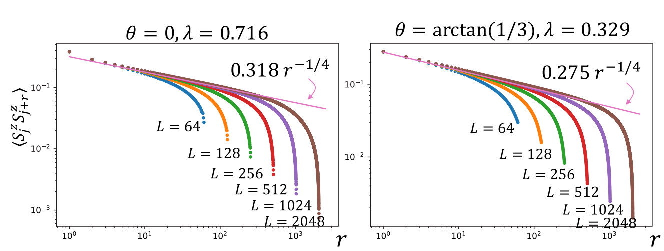

FIG. 6. The correlation function $\langle S_j^zS_{j+r}^z\rangle$ on an open chain with length L is calculated with various L at $\theta=0$ and $\arctan(1/3)$. From the log-log plots, it is clear that the data are well fitted by $r^{-1/4}$ when $1\ll r\ll L$.

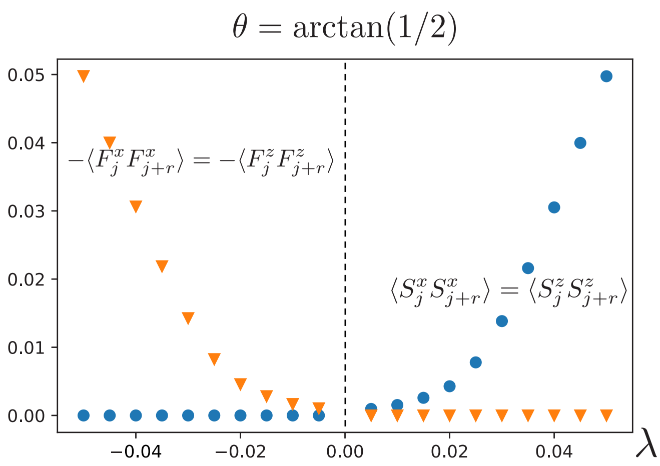

FIG. 7. DMRG calculations at $\theta = \arctan(1/2)$ and $-0.05 \leqslant \lambda \leqslant 0.05$. The order parameters $\mathcal{O}_{\text{str}}^{x,z}$ and $\mathcal{O}_{\text{FM}}^{x,z}$ are estimated by taking $L = 1024$ and $r = 512$.

SSB phase. There are three $\mathbb{Z}_2$ SSB phases represented by

$$H_x = \sum_j S_j^x S_{j+1}^x,\ H_y = \sum_j S_j^y S_{j+1}^y,\ H_z = \sum_j S_j^z S_{j+1}^z. \quad \text{(D2)}$$

Subclass B contains the criticalities between the gapped SPT phase and a $\mathbb{Z}_2$ SSB phase. Subclass C contains the criticalities between the fully $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry-breaking phase and a $\mathbb{Z}_2$ SSB phase. The critical line $-\lambda_c(\theta) < 0$ in Fig. 1(a) belongs to subclass B, while the critical line $\lambda_c(\theta) > 0$ belongs to subclass C.

An example of subclass A is given by [78,79]

$$H_A = \sum_j \left[\left(S_j^z\right)^2 + a\, S_j^x S_{j+1}^x \right], \quad \text{(D3)}$$

with $a = \pm 2$. According to Eq. (6), we have

$$H_A \overset{U_{\text{KT}}}{\longleftrightarrow} \widetilde{H}_A = \sum_j \left[\left(S_j^z\right)^2 - a\, S_j^x S_{j+1}^x \right], \quad \text{(D4)}$$

where $\widetilde{H}_A$ still belongs to subclass A. It is thus clear that as summarized in Eq. (D5), the $\mathbb{Z}_2 \times \mathbb{Z}_2$ SPT Ising criticalities (subclass B) have hidden symmetry breaking, while subclass A has no hidden symmetry breaking,

$$\begin{aligned} &\text{Subclass B} \overset{U_{\text{KT}}}{\longleftrightarrow} \text{Subclass C}, \\ &\text{Subclass A} \overset{U_{\text{KT}}}{\longleftrightarrow} \text{Subclass A}. \end{aligned} \quad \text{(D5)}$$

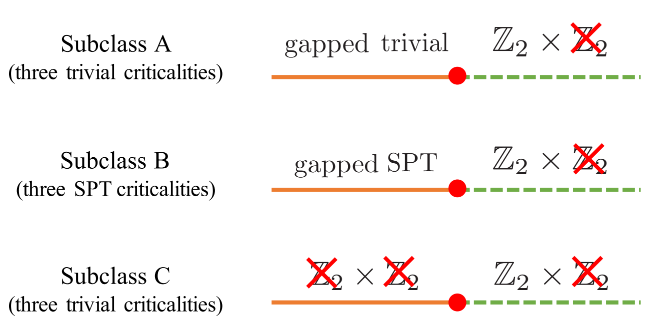

FIG. 8. With $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry, the Ising universality class splits into three subclasses, and each subclass contains three symmetry-enriched criticalities [12]. Subclasses A and C are trivial, while subclass B is SPT.

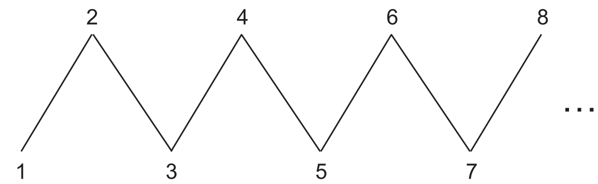

FIG. 9. Zigzag chain with a spin-1/2 on each vertex.

APPENDIX E: KT TRANSFORMATION, KRAMERS-WANNIER DUALITY, AND DOMAIN WALL DECORATION

In this section, we show that the KT transformation, the Kramers-Wannier (KW) duality, and the domain wall (DW) decoration are closely related to each other.

Let us consider spin-1/2 models defined on a zigzag chain; see Fig. 9. We require the models to have $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry, where

$$\begin{aligned} &\mathbb{Z}_2 \times \mathbb{Z}_2 = \{1, A, B, AB\}, \\ &A = \prod_{k=1}^{\infty} \sigma_{2k-1}^x,\ B = \prod_{k=1}^{\infty} \sigma_{2k}^x. \end{aligned} \quad \text{(E1)}$$

On the zigzag chain, the trivially gapped phase is represented by the Hamiltonian

$$H_{\text{triv}} = -\sum_j \sigma_j^x. \quad \text{(E2)}$$

The gapped SPT phase is represented by the cluster model [7,80,81],

$$H_{\text{SPT}} = -\sum_j \sigma_{j-1}^z \sigma_j^x \sigma_{j+1}^z. \quad \text{(E3)}$$

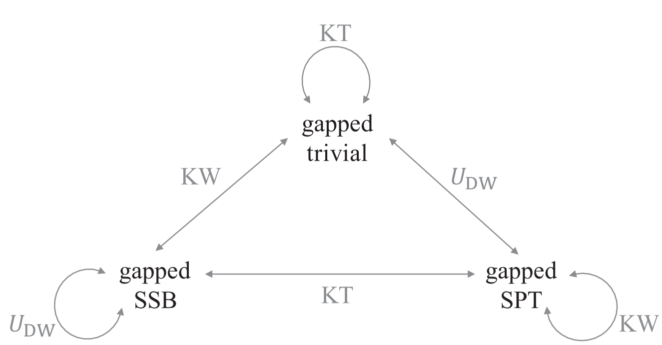

FIG. 10. Three $\mathbb{Z}_2 \times \mathbb{Z}_2$ gapped phases and the dualities between them. KT transformation can be defined as a combination of KW duality and U_{DW}. Note that the KW duality in the figure applies to both $\mathbb{Z}_2$ symmetries.

The fully $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry-breaking phase is represented by

$$H_{\rm SSB} = -\sum_k \left(\sigma^z_{2k-1}\sigma^z_{2k+1} + \sigma^z_{2k}\sigma^z_{2k+2}\right). \tag{E4}$$

It is well known that $H_{\rm triv}$ and $H_{\rm SSB}$ can be transformed into each other by applying the KW duality to both $\mathbb{Z}_2$ symmetries [24,25]. In fact, $H_{\rm triv}$ can also be transformed into $H_{\rm SPT}$ via the so-called DW decoration [21,82,83],

$$U^\dagger_{\rm DW} H_{\rm triv} U_{\rm DW} = H_{\rm SPT}, \tag{E5}$$

where

$$U_{\rm DW} = \exp\left[\frac{i\pi}{4}\sum_{j=2}^{\infty}(-1)^j \sigma^z_{j-1}\sigma^z_j\right]. \tag{E6}$$

KT transformation is a duality between the SPT and SSB phases. One may realize that by properly combining KW duality and DW decoration, the KT transformation can be defined. Indeed, if we define [84,85]

$$\mathrm{KT} = \mathrm{KW} \times U_{\rm DW} \times \mathrm{KW}, \tag{E7}$$

then KT in Eq. (E7) gives the correct SPT-SSB and trivial-trivial mappings. The relations between KT, KW, and DW are summarized in Fig. 10. For more details about Eq. (E7), see Refs. [84,85].

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[64] Strictly speaking, the gauge field of a discrete group is a cochain rather than a differential form. Therefore, the product of such gauge fields should be the cup product $\smile$ instead of the wedge product $\wedge$. See Appendix B of Ref. [86], Appendix J.4.e of Ref. [6], and Refs. [87,88] for details. Nevertheless, the calculus on a differential manifold and the calculus on a simplicial complex are almost parallel. In this article, we sacrifice the mathematical rigor and use differential forms and wedge products for simplicity.

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