Out-of-equilibrium finite-size scaling in generalized Kibble-Zurek protocols crossing quantum phase transitions in the presence of symmetry-breaking perturbations

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We study the effects of symmetry-breaking perturbations in the out-of-equilibrium quantum dynamics of many-body systems slowly driven by a time-dependent symmetry-preserving parameter, across the quantum critical regime associated with a continuous quantum transition (CQT). For this purpose, we analyze the out-of-equilibrium dynamics arising from generalized Kibble-Zurek (KZ) protocols, within a dynamic renormalization-group framework allowing for finite-size systems. We show that the time dependence of generic observables develops an out-of-equilibrium finite-size scaling (FSS) behavior, arising from the interplay between the timescale t_s of the parameter variations in the KZ protocol, the size L of the system, and the strength h of the symmetry-breaking perturbation, in the limit of large t_s and L. Moreover, scaling arguments based on the first-order adiabatic approximation of slow variations in quantum systems allow us to characterize the approach to the adiabatic regimes for some limits of the model parameters (for example, when we take $t_s \to \infty$ before $L \to \infty$), predicting asymptotic power-law suppressions of the nonadiabatic behaviors in the adiabatic limits. This out-of-equilibrium FSS is supported by numerical analyses for the paradigmatic quantum Ising chain along generalized KZ protocols, with a time-dependent transverse field crossing its CQT, in the presence of a static longitudinal field breaking the $\mathbb{Z}_2$ symmetry.

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I. INTRODUCTION

Out-of-equilibrium phenomena are generally observed in many-body systems when they are driven across phase transitions, where large-scale modes are not able to equilibrate even in the limit of very slow changes of the system parameters. We mention hysteresis, aging, out-of-equilibrium defect production, etc., which have been addressed both theoretically and experimentally, at classical and quantum phase transitions (see, e.g., Refs. [1–19] and references therein). Many-body systems develop out-of-equilibrium scaling behaviors when slowly crossing a phase transition, in the limit of a large timescale t_s of the parameter variations driving the dynamics. They generally depend on the nature of the transition, whether it is driven by thermal or quantum fluctuations, whether it is first-order or continuous, and in the latter case on some global properties determining the universality class of the critical behavior, see, e.g., Refs. [1,4,7,8,10,20–39].

In this paper, we analyze the out-of-equilibrium scaling behaviors emerging when one time-dependent symmetry-preserving parameter drives a quantum many-body system across the quantum critical regime associated with a continuous quantum transition (CQT) in the presence of a time-independent symmetry-breaking perturbation. For this purpose, we consider generalized Kibble-Zurek (KZ) protocols [30], analyzing the dynamics arising from variations of one symmetry-preserving model parameter $w(t)$ (defined so $w_c = 0$ is the critical point in the absence of the symmetry-breaking perturbation) in the presence of a nonzero symmetry-breaking perturbation, with a linear time dependence $w(t) = t/t_s$ and a large timescale t_s, starting from the ground state associated with the initial value $w(t_i)$. Analogous protocols have been generally considered to discuss the so-called KZ problem related to the defect production when crossing continuous transitions from disorder to order phases, see, e.g., Refs. [1,4,7,8,11,20–22,26,30–33,38,40–43].

As already noted in Ref. [30], the dynamic scenario arising from KZ protocols slowly crossing critical regimes is substantially affected by the presence of a nonzero symmetry-breaking external field h corresponding to a relevant renormalization group (RG) perturbation at the CQT. Typically, when $h \neq 0$, the correlation length ξ remains finite also at the critical value w_c of the symmetry-preserving Hamiltonian parameter w driving the CQT. However, at the critical point $w = w_c$, the correlation length ξ becomes large for small values of h, diverging as $\xi \sim h^{-1/y_h}$ for $h \to 0$, where $y_h > 0$ is an appropriate critical exponent, see later. Therefore, for sufficiently small values of h the quantum critical regime persists, and we can define an out-of-equilibrium finite-size scaling (FSS) limit, which allows us to get information on the universal effects of a small symmetry-breaking perturbation within the quantum critical region. Within this scaling regime, the symmetry-breaking perturbation gives rise to a universal distortion of the KZ scaling behavior observed in the absence of symmetry-breaking terms.

We investigate this issue within RG scaling frameworks at CQTs [10,44–46], which allow us to develop an out-of-equilibrium FSS theory [10,36,39] describing the intricate interplay among the Hamiltonian parameters, the timescale t_s of the KZ protocol and the lattice size L, in the limits of large t_s and large L, allowing for the effects of a further (sufficiently small) symmetry-breaking perturbation. The effects of symmetry-breaking perturbations within KZ protocols have been also addressed in Ref. [30], where the scaling behaviors in the infinite-volume (thermodynamic) limit were analyzed. Here we extend the characterization of the out-of-equilibrium scaling behavior to finite systems in an appropriate out-of-equilibrium FSS limit.

As a paradigmatic model where to test the out-of-equilibrium FSS framework, we consider the quantum Ising chain. We analyze the out-of-equilibrium FSS associated with generalized KZ protocols and provide numerical results that support it. In particular, we analyze the approach to the adiabatic regimes that can be realized for some limits of the parameters, involving the timescale t_s, the size of the system, and the strength of the symmetry-breaking perturbation. Within the same theoretical framework, we also discuss more general KZ protocols in which both symmetry-preserving and symmetry-breaking terms are time dependent and driven across the transition point.

We remark that equilibrium and out-of-equilibrium FSS frameworks generally simplify the study of the universal features of critical behaviors. This is essentially related to the fact that the general requirement of a large length scale ξ of the critical correlations is not subject to further conditions on the system size L. Indeed $\xi \sim L$ for FSS, while critical behaviors in the thermodynamic limit requires $\xi \ll L$. Therefore, much larger systems are necessary to probe analogous length scales ξ in the thermodynamic limit. The FSS scenarios are often observed for systems of moderately large size, see, e.g., Refs. [10,28,36,39]. Therefore, FSS behaviors may be more easily accessed by experiments where the coherent quantum dynamics of only a limited number of particles or spins can be effectively realized, such as experiments with quantum simulators in laboratories, e.g., by means of trapped ions [47,48], ultracold atoms [49,50], or superconducting qubits [51,52].

The paper is organized as follows. In Sec. II, we describe generalized KZ protocols in the presence of a further static symmetry-breaking perturbation. We also present paradigmatic d-dimensional quantum Ising models and, in particular, quantum Ising chains, which provide a theoretical laboratory to address the out-of-equilibrium dynamics along generalized KZ protocols. In Sec. III, we derive the out-of-equilibrium FSS laws emerging along the generalized KZ protocols in the presence of a static symmetry-breaking perturbation, which are supposed to apply to generic CQTs. This scaling framework is supported by numerical results for the quantum Ising chain, based on exact diagonalization up to size $L \approx 20$. In Sec. IV, we discuss the approach to adiabatic regimes, which are always possible in finite-size systems, and/or in the presence of a further external relevant perturbation, for a sufficiently large timescale t_s of the parameter variations. Section V presents a brief discussion of the more general case in which the KZ protocol is further extended to the case in which both the symmetry-preserving and the symmetry-breaking parameters are time dependent. Finally, in Sec. VI we summarize and draw our conclusions.

II. THE MODELS AND DYNAMIC PROTOCOLS

A. Kibble-Zurek protocols

We consider quantum many-body systems whose Hamiltonian can be written as

$$\hat{H}(w, h) = \hat{H}_c + w\,\hat{H}_w + h\hat{H}_h, \tag{1}$$

where $\hat{H}_c$ is a critical Hamiltonian (i.e., with its parameters tuned to their critical values), w is associated with a relevant RG perturbation $\hat{H}_w$ preserving the symmetry, and the parameter h is another parameter associated with a symmetry-breaking term $\hat{H}_h$, which gives rise to a further relevant RG perturbation at the CQT. Both Hamiltonian parameters w and h vanish at the critical point, i.e., $w_c = h_c = 0$. We assume that for $h = 0$ the critical point $w_c = 0$ separates disordered ($w < 0$) and ordered ($w > 0$) phases. On the other hand, nonzero values of the parameter h gives always rise to a gapped phase.

For sufficiently small values of h (we will make this condition more precise below), quasiadiabatic passages through the quantum critical regime associated with a CQT are obtained by slowly varying w across $w_c = 0$, following, e.g., the KZ protocol:

(i) The quantum evolution of finite systems of size L starts at the time t_i from the ground state $|\Psi_0(w_i, h)\rangle$ associated with the initial value $w_i < 0$ corresponding to the disordered phase.

(ii) Then the system evolves unitarily according to the Schrödinger equation (we set the Planck's constant $\hbar = 1$)

$$\begin{aligned} \frac{d\,|\Psi(t)\rangle}{dt} &= -i\,\hat{H}[w(t), h]\,|\Psi(t)\rangle, \\ |\Psi(t=0)\rangle &= |\Psi_0(w_i, h)\rangle, \end{aligned} \tag{2}$$

with a linear time dependence of $w(t)$,

$$w(t) = t/t_s, \tag{3}$$

up to a final value $w_f > 0$, corresponding to parameter values within the ordered phase. Thus we have $t_i = t_s\, w_i < 0$ and $t_f = t_s\, w_f > 0$. The parameter t_s of the KZ protocol represents the timescale of the time dependence of the Hamiltonian parameter w.

Across a phase transition, in particular, in the absence of further relevant perturbations, i.e., $h = 0$, the growth of an out-of-equilibrium dynamics is inevitable in the thermodynamic limit (i.e., $L \to \infty$ before taking the critical limit) even for very slow changes of the parameter w, because large-scale modes are unable to equilibrate the long-distance critical correlations emerging at the transition point. As a consequence, when starting from equilibrium states at the initial value w_i, the system cannot pass through equilibrium states associated with the values of $w(t)$ across the transition point, thus departing from an adiabatic dynamics. Such a departure from equilibrium develops peculiar out-of-equilibrium scaling phenomena in the limit of large timescale t_s of the time variation of $w(t)$.

This out-of-equilibrium scenario substantially changes in the presence of a finite nonzero symmetry-breaking perturbation, i.e., Hamiltonian Eq. (1) with $h \neq 0$. This is essentially due to the fact that the gap Δ does not generally close when $h \neq 0$. Indeed, at the critical point $w = w_c$, Δ remains finite in the large-L limit, behaving as $\Delta \sim |h|^{\varepsilon}$ ($\varepsilon > 0$) for small h (standard RG arguments show that $\varepsilon = z/y_h$, where z and y_h are appropriate critical exponents associated with the universality class of the CQT [45]; see below). Therefore, the adiabatic evolution through the ground states associated with the instantaneous values of $w(t)$ can be always realized for sufficiently large timescale t_s. In the presence of the symmetry-breaking perturbation h, a nontrivial out-of-equilibrium FSS limit can be still defined by appropriately rescaling h in the large-L limit, providing information for sufficiently small values of h. This issue will be addressed in the next sections within a general out-of-equilibrium FSS framework for quantum many-body systems driven across CQTs, see, e.g., Ref. [10].

It is worth mentioning that a related issue is the so-called KZ problem, i.e., the scaling behavior of the amount of final defects after slow passages through continuous transitions, from the disorder phase to the order phase [1,4,7,8,10,11,20–22,26,31–33,38–43]. The general features of the KZ scaling and, in particular, the KZ predictions for the abundance of residual defects, have been confirmed by several analytical and numerical studies, see, e.g., Refs. [7,8,10,26,33,43] and citing references, and by experiments for various physically interesting systems, see, e.g., Refs. [15–18,53–65].

B. The quantum Ising models

As paradigmatic quantum many-body systems, we consider the d-dimensional quantum Ising models in the presence of transverse and longitudinal fields, described by the Hamiltonian

$$\hat{H}(g, h) = -J \sum_{\langle \boldsymbol{xy} \rangle} \hat{\sigma}_{\boldsymbol{x}}^{(1)} \hat{\sigma}_{\boldsymbol{y}}^{(1)} - g \sum_{\boldsymbol{x}} \hat{\sigma}_{\boldsymbol{x}}^{(3)} - h \sum_{\boldsymbol{x}} \hat{\sigma}_{\boldsymbol{x}}^{(1)} \tag{4}$$

defined on a cubiclike lattice, where $\hat{\sigma}_{\boldsymbol{x}}^{(k)}$ are the Pauli matrices on the site $\boldsymbol{x}$ ($k = 1, 2, 3$ labels the three spatial directions), and the first sum runs on the bonds $\langle \boldsymbol{xy} \rangle$ of the lattice. In the following, we consider quantum Ising systems of size L with periodic boundary conditions (PBCs). We assume ferromagnetic nearest-neighbor interactions with $J = 1$.

Transforming the spin operators as $\hat{\sigma}_{\boldsymbol{x}}^{(1)} \to -\hat{\sigma}_{\boldsymbol{x}}^{(1)}$ and $\hat{\sigma}_{\boldsymbol{x}}^{(3)} \to \hat{\sigma}_{\boldsymbol{x}}^{(3)}$, the Hamiltonian $\hat{H}(g, h)$ maps into $\hat{H}(g, -h)$, thus $\hat{H}$ is $\mathbb{Z}_2$ symmetric for $h = 0$. The quantum Ising models are always gapped in the presence of a nonzero symmetry-breaking term, i.e., for $h \neq 0$. They undergo a CQT at $g = g_c$ and $h = 0$, whose quantum critical behavior belongs to the $(d + 1)$-dimensional Ising universality class, due to the quantum-to-classical mapping, see, e.g., Refs. [10,45]. The parameters $w \equiv g_c - g$ and h are relevant at the CQT, being, respectively, associated with the leading even (preserving the global $\mathbb{Z}_2$ symmetry) and odd (breaking the $\mathbb{Z}_2$ symmetry) RG perturbations at the $(d + 1)$-dimensional Ising fixed point.

Several exact results are known for one-dimensional models, such as the location of the critical point, at $g_c = 1$, and the RG dimensions of the Hamiltonian parameters w and h, which are $y_w = 1/\nu = 1$ and $y_h = 15/8$, respectively. Accurate estimates are available for two-dimensional quantum Ising systems, see, e.g., Refs. [66–72]; in particular, Ref. [70] reports $y_w = 1/\nu = 1.58737(1)$ and $y_h = 2.481852(1)$. For $d = 3$, the critical exponents take their mean-field values, $y_w = 2$ and $y_h = 3$, however, the critical singular behavior presents additional multiplicative logarithmic factors [10,45,66]. The length scale ξ of the critical modes behaves as $\xi \sim |w|^{-\nu}$ for $h = 0$, and $\xi \sim |h|^{-1/y_h}$ at $w = w_c = 0$. The dynamic exponent z, controlling the vanishing $\Delta \sim \xi^{-z}$ of the gap at the transition point, is given by $z = 1$ in any dimension. We also recall that the RG dimension of the order-parameter field, associated with the longitudinal operators $\hat{\sigma}_{\boldsymbol{x}}^{(1)}$, is given by

$$y_l = d + z - y_h, \tag{5}$$

while that associated with the transverse operators $\hat{\sigma}_{\boldsymbol{x}}^{(3)}$ is given by $y_t = d + z - y_w$.

The KZ protocols outlined in Sec. II A can be implemented within quantum Ising chains, by identifying the terms of the generic Hamiltonian Eq. (1) with

$$\begin{aligned} \hat{H}_c &= -\sum_{\langle \boldsymbol{xy} \rangle} \hat{\sigma}_{\boldsymbol{x}}^{(1)} \hat{\sigma}_{\boldsymbol{y}}^{(1)} - g_c \sum_{\boldsymbol{x}} \hat{\sigma}_{\boldsymbol{x}}^{(3)}, \\ w(t) &= g_c - g(t), \quad \hat{H}_w = \sum_{\boldsymbol{x}} \hat{\sigma}_{\boldsymbol{x}}^{(3)}, \\ \hat{H}_h &= -\sum_{\boldsymbol{x}} \hat{\sigma}_{\boldsymbol{x}}^{(1)}. \end{aligned} \tag{6}$$

C. Observables monitoring the quantum evolution

The out-of-equilibrium evolution of quantum many-body systems resulting from the KZ protocol can be monitored looking at the behavior of some observables and correlations at fixed time.

To characterize the departure from adiabaticity along the slow dynamic across the CQT, we monitor the adiabaticity function

$$A(t) = |\langle\, \Psi_0[w(t), h] \,|\, \Psi(t)\,\rangle|, \tag{7}$$

where $|\,\Psi_0[w(t), h]\,\rangle$ is the ground state of the Hamiltonian $\hat{H}[w(t), h]$, i.e., at the instantaneous value $w(t)$, while $|\,\Psi(t)\,\rangle$ is the actual time-dependent state evolving according to the Schrödinger Eqs. (2). The adiabaticity function measures the overlap of the time-dependent state at a time t with the ground state of the Hamiltonian at the corresponding $w(t)$. Since the KZ protocol starts from the ground state associated with $w_i = w(t_i)$, we have $A(t_i) = 1$ initially. Of course, the adiabaticity function for an adiabatic evolution takes the value $A(t) = 1$ at any time $t > t_i$.

In general, for protocols crossing transition points, $A(t)$ is expected to depart from the initial value $A(t_i) = 1$, due to the impossibility of the system to adiabatically follow the changes of the function $w(t)$ across its critical value $w = 0$. Note, however, that this is strictly true in the infinite-volume limit. In systems of finite size L, there is always a sufficiently large timescale t_s, so the system can evolve adiabatically, essentially because finite-size systems are strictly gapped, although the gap Δ at the CQT gets suppressed as $\Delta \sim L^{-z}$. The interplay between size L and timescale t_s gives rise to nontrivial out-of-equilibrium scaling behaviors, which can be studied within out-of-equilibrium FSS frameworks [10,36,39].

Another global observable monitoring the departure from adiabaticity is provided by the surplus energy of the system with respect to its instantaneous ground state at $w(t)$, often called *excitation* energy in earlier works, i.e.,

$$E_s(t) = \langle\Psi(t)|\,\hat{H}\,|\Psi(t)\rangle - \langle\Psi_0[w(t)]|\,\hat{H}\,|\Psi_0[w(t)]\rangle. \quad (8)$$

Since the KZ protocols that we consider start from a ground state at t_i, the excitation energy $E_s(t)$ vanishes along adiabatic evolutions, while nonzero values $E_s(t) > 0$ are related to the degree of out-of-equilibrium of the dynamics across the transition. One may also consider the corresponding density of excitation energy:

$$D_e(t) = L^{-d} E_s(t). \quad (9)$$

To monitor the out-of-equilibrium dynamics of the spin expectation values and correlations, we consider the evolution of the local and global average magnetization

$$m_{\boldsymbol{x}}(t) \equiv \langle\Psi(t)|\,\hat{\sigma}^{(1)}_{\boldsymbol{x}}\,|\Psi(t)\rangle, \quad M(t) \equiv \frac{1}{L^d}\sum_{\boldsymbol{x}} m_{\boldsymbol{x}}(t), \quad (10)$$

and the fixed-time connected correlation function

$$G(t,\boldsymbol{x},\boldsymbol{y}) \equiv \langle\Psi(t)|\,\hat{\sigma}^{(1)}_{\boldsymbol{x}}\,\hat{\sigma}^{(1)}_{\boldsymbol{y}}\,|\Psi(t)\rangle_c. \quad (11)$$

In the absence of boundaries, such as the case of PBCs, translation invariance implies $m_{\boldsymbol{x}}(t) = M(t)$ and $G(t,\boldsymbol{x},\boldsymbol{y}) \equiv G(t,\boldsymbol{x}-\boldsymbol{y})$.

In the following, we outline the out-of-equilibrium FSS scenario applying to the generalized KZ protocol described in Sec. II A. We contextualize it within quantum Ising models. However, most scaling arguments can be straightforwardly extended to generic CQTs. To support the emerging out-of-equilibrium FSS behaviors, we also report numerical analyses for the one-dimensional Ising chain with a time-dependent transverse field $g(t)$ and a static longitudinal field h. They are based on exact-diagonalization methods. The corresponding Schrödinger equation is solved using afourth-order Runge-Kutta method. This approach allows us to compute the out-of-equilibrium dynamics for lattice sizes up to $L \approx 20$, with high accuracy (practically exact). As we shall see, such moderately large (or relatively small) systems turn out to be already sufficient to provide a robust evidence of the out-of-equilibrium FSS outlined in the paper.

III. OUT-OF-EQUILIBRIUM SCALING

We discuss here the out-of-equilibrium scaling behaviors emerging along the KZ protocol outlined in Sec. II A, within a dynamic RG framework [10]. Out-of-equilibrium FSS laws are expected to develop in the limit of a large timescale t_s of the driving parameter $w(t)$, large size L of the system, and for sufficiently small values of the symmetry-breaking parameter h. They describe the interplay of the various scales of the problem, such as the time t and time scale t_s of the KZ protocol, the size L of the system, the energy scale $\Delta \sim L^{-z}$ of the system at the critical point, and the external longitudinal field h.

A. Homogenous scaling laws

Let us consider observables constructed from a local operator $\hat{O}(\boldsymbol{x})$. The general working hypothesis underlying out-of-equilibrium FSS frameworks is that the expectation value of $\hat{O}(\boldsymbol{x})$ and its correlation functions along KZ protocols obey asymptotic homogeneous scaling laws, [10] such as

$$O(t,t_s,w_i,h,L) \equiv \langle\Psi(t)|\hat{O}(\boldsymbol{x})|\Psi(t)\rangle$$
$$\approx b^{-y_o}\mathcal{O}(b^{-z}t, b^{y_w}w_i, b^{y_w}w(t), b^{y_h}h, L/b), \quad (12)$$

$$G_O(\boldsymbol{x},t,t_s,w_i,h,L) \equiv \langle\Psi(t)|\hat{O}(\boldsymbol{x}_1)\,\hat{O}(\boldsymbol{x}_2)|\Psi(t)\rangle_c$$
$$\approx b^{-2y_o}\mathcal{G}(\boldsymbol{x}/b, b^{-z}t, b^{y_w}w_i, b^{y_w}w(t), b^{y_h}h, L/b), \quad (13)$$

where b is an arbitrary (large) length scale, the RG dimension y_o of the local operator $\hat{O}$, and the RG exponents y_w, y_h, and z, are determined by the universality class of the CQT. We assumed translation invariance, i.e., systems without boundaries such as those with PBCs, so the expectation value O does not depend on $\boldsymbol{x}$, and the two-point function depends on the difference $\boldsymbol{x} \equiv \boldsymbol{x}_1 - \boldsymbol{x}_2$ only.

Note that in the above homogenous scaling laws, the dynamic features are essentially encoded in the time dependence of the scaling functions and, in particular, through the time-dependent Hamiltonian parameter $w(t)$. The other features are analogous to those arising from equilibrium FSS at CQTs [10,46], where the arguments of the scaling functions take into account the RG dimensions y_w and y_h of the relevant parameters w and h at the RG fixed point associated with the CQT. In this respect, the RG scaling framework for KZ protocols is obtained by replacing w with $w(t)$ in the equilibrium homogenous scaling laws. The scaling functions $\mathcal{O}$ and $\mathcal{G}_O$ are expected to be universal, i.e., largely independent of the microscopic details of the models and the KZ protocols (apart from a multiplicative factor and normalizations of the arguments).

We remark that the homogenous scaling laws Eqs. (12) and (13), which we use as the starting point of our theoretical framework, are expected to apply to generic CQTs. Analogous scaling frameworks have been constructed to describe other out-of-equilibrium phenomena at CQTs, in various situations such as after sudden quenches of the Hamiltonian parameters or in the presence of dissipative interactions [10].

By taking the ratio between the arguments $b^{-z}t$ and $b^{y_w}w(t)$ of the scaling functions reported in Eqs. (12) and (13), we obtain the scaling variable $b^{-(y_w+z)}t_s$, which tells us how the timescale t_s must be rescaled to observe the out-of-equilibrium scaling behavior. Then, by exploiting the arbitrariness of b, we may set

$$b^{-(y_w+z)}t_s = 1 \quad (14)$$

to derive the length scale

$$\lambda = t_s^{1/\zeta}, \quad \zeta = y_w + z, \quad (15)$$

corresponding to the length scale arising from the linear time dependence $w(t) = t/t_s$.

Out-of-equilibrium FSS can be straightforwardly derived by fixing $b = L$ in Eqs. (12) and (13). Then, the asymptotic out-of-equilibrium FSS limit is obtained by taking $t_s \to \infty$

and $L \to \infty$, while appropriate scaling variables are kept fixed, such as [10]

$$\Upsilon = t_s/L^\zeta = (\lambda/L)^\zeta, \quad K = w(t)L^{y_w},$$
$$\Phi = hL^{y_h}, \quad \Sigma = h\,\lambda^{y_h},$$
$$\Theta_i = w_i\,\lambda^{y_w}, \quad \Theta = w(t)\,\lambda^{y_w} = t/t_s^\kappa, \tag{16}$$

where

$$0 < \kappa = z/\zeta < 1. \tag{17}$$

We obtain $\kappa = 1/2$ for one-dimensional Ising chain, $\kappa \approx 0.386$ for $d = 2$, and $\kappa = 1/3$ for $d = 3$. Note that the above scaling variables are not all independent. Indeed, one can easily check that $K = \Upsilon^{\kappa-1}\Theta$, $\Sigma = \Phi\Upsilon^{y_h/\zeta}$, and $\Theta \geqslant \Theta_i$. Moreover, the timescaling variable $t\,\Delta$, where $\Delta \sim L^{-z}$ is the critical gap of the system, can be straightforwardly related to Θ and Υ by $t\,\Delta \sim \Theta\Upsilon^\kappa$. On the other hand, the out-of-equilibrium FSS keeping Υ fixed implies that the KZ timescale t_s is generally much larger that the inverse energy scale extracted from the energy differences of the lowest levels at the critical point. Indeed, we have that

$$t_s\Delta(L) \sim t_s L^{-z} = \Upsilon L^{y_w} \to \infty. \tag{18}$$

B. Out-of-equilibrium FSS

We can use the general homogenous scaling laws reported in Eqs. (12) and (13) to derive an out-of-equilibrium FSS limit and the behaviors of the observables of the quantum Ising models in this limit, such as the longitudinal magnetization M and the correlation function G of the quantum Ising systems, the adiabaticity function and the excitation energy, defined in Sec. II C.

By fixing $b = L$, we write their asymptotic behavior in the out-of-equilibrium FSS limit in terms of the scaling variables Υ, Θ, Φ, and Θ_i defined in Eqs. (16) as [10]

$$M(t, t_s, w_i, h, L) \approx L^{-y_l}\mathcal{M}_i(\Upsilon, \Theta, \Phi, \Theta_i), \tag{19}$$

$$G(\boldsymbol{x}, t, t_s, w_i, h, L) \approx L^{-2y_l}\,\mathcal{G}_i(\boldsymbol{x}/L, \Upsilon, \Theta, \Phi, \Theta_i), \tag{20}$$

where $\mathcal{M}_i$ and $\mathcal{G}_i$ are scaling functions, expected to be largely universal with respect to the details of the model and the KZ protocol.

Since the KZ protocol runs within the interval $t_i \leqslant t \leqslant t_f$, corresponding to the interval $w_i \leqslant w(t) \leqslant w_f$, the scaling variable Θ takes values within the interval

$$\Theta_i \leqslant \Theta \leqslant \Theta_f \equiv w_f t_s^{1-\kappa} > 0. \tag{21}$$

We omit the dependence on Θ_f, because the out-of-equilibrium FSS limit at fixed $\Theta < \Theta_f$ does not depend on Θ_f, but only on Υ and Θ_i. Of course, if we keep w_f fixed in the large-t_s limit, i.e., if we do not scale w_f to zero to keep Θ_f fixed, then $\Theta_f \to \infty$.

With increasing L, the out-of-equilibrium FSS develops within a smaller and smaller interval δw of values of $|w|$ around $w = 0$. In particular, for the *most critical* case when $h = 0$, the time interval of the out-of-equilibrium process described by the scaling laws scales as $t_{\rm KZ} \sim t_s^\kappa$, thus the relevant interval δw of values of $|w|$, where a nontrivial out-of-equilibrium scaling behavior is observed, must shrink as

$$\delta w \sim t_{\rm KZ}/t_s \sim L^{-y_w}, \tag{22}$$

when keeping Υ fixed. Therefore, assuming that the KZ protocol starts from a gapped phase, see Sec. II A, and that the initial $w_i < 0$ is kept fixed (corresponding to $\Theta_i \to -\infty$), the same out-of-equilbrium FSS is expected to hold, irrespective of the value of w_i. Therefore, the out-of-equilibrium FSS at fixed $w_i < 0$ simplifies to

$$M(t, t_s, w_i, h, L) \approx L^{-y_l}\mathcal{M}(\Upsilon, \Theta, \Phi), \tag{23}$$

$$G(\boldsymbol{x}, t, t_s, w_i, h, L) \approx L^{-2y_l}\,\mathcal{G}(\boldsymbol{x}/L, \Upsilon, \Theta, \Phi), \tag{24}$$

being independent of w_i. The scaling functions $\mathcal{M}$ and $\mathcal{G}$ are expected to match the $\Theta_i \to -\infty$ limit of the scaling functions depending on Θ_i (when w_i gets appropriately rescaled to keep Θ_i fixed), cf. Eqs. (19) and (20). Thus,

$$\mathcal{M}(\Upsilon, \Theta, \Phi) = \mathcal{M}_i(\Upsilon, \Theta, \Phi, \Theta_i \to -\infty), \tag{25}$$

and analogously for the correlation function G.

Since the magnetization vanishes for $h = 0$ for symmetry reasons, we must have that

$$\mathcal{M}(\Upsilon, \Theta, \Phi = 0) = 0. \tag{26}$$

Actually, in finite systems the magnetization, as well as any other correlation and observables, is expected to be an analytical function of h, thus $M \sim h$ at small h. Therefore, when keeping the other scaling argument fixed (in particular, for $\Upsilon > 0$, since $\Upsilon \to 0$ corresponds to the thermodynamic limit), we should have that

$$\mathcal{M}(\Upsilon, \Theta, \Phi) = c\,\Phi + O(\Phi^3), \tag{27}$$

where the coefficient c depends on the other scaling variables [an analogous behavior is expected at equilibrium, i.e., $\mathcal{M}_{\rm eq}(K, \Phi) = c(K)\,\Phi + O(\Phi^3)$]. Note also that $M(t, t_s, w_i, h, L) = -M(t, t_s, w_i, -h, L)$, thus

$$\mathcal{M}(\Upsilon, \Theta, \Phi) = -\mathcal{M}(\Upsilon, \Theta, -\Phi). \tag{28}$$

The out-of-equilibrium FSS relations can be also written in alternative ways, using other equivalent sets of scaling variables, cf. Eqs. (16). As we shall see, some interesting limits, such as the thermodynamic and adiabatic limits, are defined for particular choices of the scaling variables. For example, one may write them in terms of K instead of Θ, such as

$$M(t, t_s, w_i, h, L) \approx L^{-y_l}\widetilde{\mathcal{M}}(\Upsilon, K, \Phi), \tag{29}$$

by replacing $\Theta = \Upsilon^{1-\kappa}K$ in Eq. (23). As we shall see, the scaling variables Υ, K, and Φ are most appropriate to discuss the adiabatic limit.

An out-of-equilibrium FSS behavior analogous to that in Eq. (23) is put forward for the adiabaticity function, cf. Eq. (7),

$$A(t, t_s, w_i, h, L) \approx \mathcal{A}(\Upsilon, \Theta, \Phi) = \widetilde{\mathcal{A}}(\Upsilon, K, \Phi), \tag{30}$$

when keeping $w_i < 0$ fixed. Due to the initial condition of the KZ protocol, cf. Eqs. (2), i.e., the ground state at w_i, we must have $A(t_i, t_s, w_i, h, L) = 1$, and therefore $\mathcal{A}(\Upsilon, \Theta \to -\infty, \Phi) = 1$. It is therefore natural to assume that its scaling

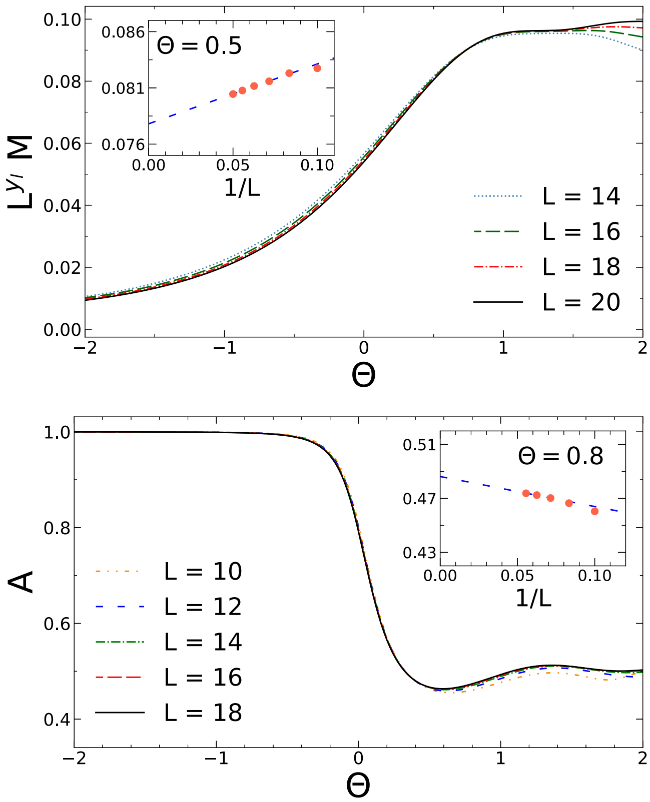

FIG. 1. Some results for the time evolution of quantum Ising chains along the generalized KZ protocol outlined in Sec. II A, keeping $w_i = -3$, $\Upsilon \equiv t_s/L^{\zeta} = 0.01$ (we recall that $\zeta = 2$), and $\Phi \equiv hL^{y_h} = 1$ ($y_h = 15/8$) fixed, versus $\Theta = t/t_s^{\kappa}$ with $\kappa = 1/2$. We show results for the magnetization (top), for which $y_l = 1/8$, and the adiabaticity function (bottom), up to $L = 20$ and $L = 18$, respectively. Their behaviors nicely agree with the asymptotic out-of-equilibrium FSS reported in Eqs. (23) and (30). The insets of both figures report data at fixed Θ versus $1/L$. They appear substantially consistent with an $1/L$ convergence to the asymptotic FSS.

behavior does not require a power of the size as prefactor, unlike the magnetization, cf. Eq. (23). Using standard RG arguments, we may also derive an ansatz for the out-of-equilibrium FSS behavior of the excitation energy defined in Eq. (8), which turns out to be

$$E_s(t, t_s, w_i, h, L) \approx L^{-z}\mathcal{E}(\Upsilon, \Theta, \Phi) = L^{-z}\widetilde{\mathcal{E}}(\Upsilon, K, \Phi), \quad (31)$$

where $z = 1$ is the RG exponent associated with the energy differences of the lowest states of the spectrum. Note that the leading analytic background contributions [10,46], generally arising at the critical point, get canceled by the difference of the two terms in the definition of E_s, cf. Eq. (8), thus justifying the scaling ansatz Eq. (31), where the excitation energy is assumed to scale as the energy gap at the transition point, i.e., $\Delta \sim L^{-z}$.

The scaling behaviors predicted by the above out-of-equilibrium FSS theory are strongly supported by numerical analyses of the quantum Ising chain along the generalized KZ protocols outlined in Sec. II A. Some results are reported in in Fig. 1 for the magnetization and the adiabaticity function as a function of the scaling time $\Theta = t/t_s^{\kappa}$ (where $\kappa = 1/2$), keeping the scaling variables $\Upsilon \equiv t_s/L^{\zeta}$ and $\Phi \equiv hL^{y_h}$ (where $\zeta = 2$ and $y_h = 15/8$) and the initial value w_i fixed. The evident collapse of the data with increasing L nicely confirms the predicted out-of-equilibrium FSS behaviors, i.e., Eq. (23) for the magnetization and Eq. (30) for the adiabaticity function. We have also checked that the asymptotic FSS functions do not depend on the initial $w_i < 0$ (keeping it fixed with increasing L). Analogous results are obtained for other values of the scaling variables Υ and Φ, and other monitoring observables, such as the excitation energy Eq. (8).

C. Scaling in the thermodynamic limit

The scaling behavior in the infinite-size *thermodynamic* limit can be straightforwardly obtained by taking the $L \to \infty$ limit of the FSS equations, therefore in the limit $\Upsilon \to 0$ keeping Θ and Σ fixed, cf. Eqs. (16). Equivalently, one may set $b = \lambda$ in the homogeneous scaling laws Eqs. (12) and (13), and then send $L/\lambda \to \infty$. Thus, taking the large-t_s limit keeping the initial value w_i fixed, we expect the asymptotic out-of-equilibrium scaling behavior

$$M(t, t_s, w_i, h, L \to \infty) \approx \lambda^{-y_l}\mathcal{M}_\infty(\Theta, \Sigma), \quad (32)$$

$$G(\boldsymbol{x}, t, t_s, w_i, h, L \to \infty) \approx \lambda^{-2y_l}\mathcal{G}_\infty(\boldsymbol{x}/\lambda, \Theta, \Sigma), \quad (33)$$

where $\Sigma = h\lambda^{y_h}$ was defined in Eqs. (16). One can easily derive the relation

$$\mathcal{M}_\infty(\Theta, \Sigma) = \lim_{\Upsilon\to 0} \Upsilon^{y_l/\zeta}\mathcal{M}(\Upsilon, \Theta, \Upsilon^{-y_h/\zeta}\Sigma), \quad (34)$$

where $\mathcal{M}$ is the scaling function entering Eq. (23). An analogous relation can be also written for the two-point function G.

Concerning the excitation-energy density D_e, cf. Eq. (9), we expect that in the infinite-volume limit

$$D_e(t, t_s, w_i, h, L \to \infty) \approx \lambda^{-(z+d)}\mathcal{E}_\infty(\Theta, \Sigma). \quad (35)$$

This is analogous to the scaling behavior of the excitation-energy density reported in Ref. [30].

D. Scaling corrections

The out-of-equilibrium FSS limit is expected to be approached with power-law suppressed corrections. Scaling corrections to the asymptotic scaling behaviors arises for finite time scales t_s and finite size L, in particular, when they are moderately large.

The sources of scaling corrections when approaching the out-of-equilibrium FSS are expected to include those that are already present at equilibrium. The irrelevant RG perturbations at the fixed point associated with the $(d+1)$-dimensional Ising fixed point are sources of scaling corrections. The contributions of the leading irrelevant RG perturbation are generally suppressed by a power law, as $\xi^{-\omega}$ (where ξ is the diverging correlation length, or the KZ length scale λ) [10,66]. These corrections are expected to be the leading ones in two-dimensional quantum Ising systems, for which $\omega \approx 0.83$, see, e.g., Ref. [10] and references therein. For one-dimensional quantum Ising systems where $\omega = 2$ [46,73–77], other contributions may become more relevant, such as those arising from analytical backgrounds to the critical behavior [10,66]. Earlier studies of out-of-equilibrium behaviors of the Ising chains along standard KZ protocols [10] found that the leading corrections are typically $O(1/L)$ or, equivalently, $O(1/\lambda)$, for the observables considered in this paper. We therefore expect that the asymptotic out-of-equilibrium FSS of quantum Ising chains along the generalized KZ protocol is approached with $O(1/L)$ corrections.

The numerical results for the Ising chains are in substantial agreement with the above analysis of the expected scaling corrections. The approach to the large-t_s (or, equivalently, large-L) asymptotic behavior turns out to be substantially consistent with an $O(1/L)$ suppression of the corrections, see, e.g., the results reported in the insets of Fig. 1.

IV. APPROACH TO THE ADIABATIC REGIME

In this section, we focus on the adiabatic limits that can be obtained within the out-of-equilibrium FSS scenario outlined in the previous section. Within generalized KZ protocols, there are essentially two roads to adiabaticity: one is related to the finite size L and the other one to the symmetry-breaking perturbation.

In the limit $\Upsilon = t_s/L^\zeta \to \infty$, the evolution as a function of $w(t) = t/t_s$ becomes adiabatic, i.e., it passes through the ground states associated with the instantaneous values $w(t)$ (when starting from the ground state for the initial value w_i). Indeed, since the finite size L guarantees the presence of a gap between the lowest states, even at the critical point, the critical point can be adiabatically crossed if $\Upsilon \to \infty$, passing through the ground states of the finite-size system for $w(t)$. The adiabatic evolution across the transition point is prevented only when $L \to \infty$ (before the limit $t_s \to \infty$) and $h = 0$.

Another adiabatic limit is related to the presence of the longitudinal field h, due to the fact that, in the presence of a nonvanishing longitudinal field h, the gap never closes; indeed, at the critical point $w = w_c$ it remains finite in the large-L limit, behaving as $\Delta \sim |h|^{z/y_h}$ for small $|h|$.

A. The adiabatic limit

Within the FSS framework, the adiabatic limit is achieved by taking the $\Upsilon \to \infty$ limit keeping K fixed, cf. Eqs. (16). Therefore, in the adiabatic limit $\Upsilon \to \infty$ the scaling functions must tend to those of the equilibrium FSS. If we consider a generic observable whose equilibrium FSS behavior is given by [10,46]

$$O(w, h, L) \approx L^{-y_o}\mathcal{O}_{\rm eq}(wL^{y_w}, hL^{y_h}), \qquad (36)$$

then its behavior along the KZ protocol in the limit $\Upsilon \to \infty$ must be

$$O(t, t_s, w_i, h, L) \approx L^{-y_o}\mathcal{O}_{\rm eq}(K, \Phi), \qquad (37)$$

with the scaling variables K and Φ given in Eqs. (16).

The adiabaticity function must behave trivially in the adiabatic limit, so the large-Υ limit of the scaling function $\widetilde{A}$ entering Eq. (30) must be

$$\lim_{\Upsilon\to\infty} \widetilde{\mathcal{A}}(\Upsilon, K, \Phi) = 1. \qquad (38)$$

This limit is also supported by the numerical analyses within the quantum Ising chain, see, e.g., Fig. 2, where the scaling curves appear to approach the adiabatic limit Eq. (38) with increasing Υ. Analogously, the excitation energy must vanish in the adiabatic limit, by construction, i.e.,

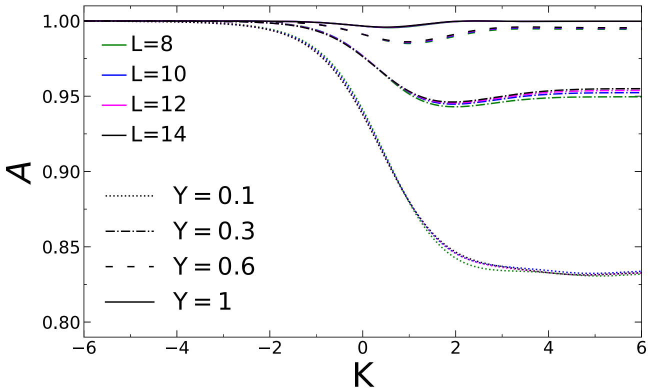

FIG. 2. The adiabaticity function A of the quantum Ising chain along generalized KZ protocols, versus $K = w(t)L^{y_w}$ at fixed $\Phi = hL^{y_h} = 1$ and $w_i = -1$, for various values of Υ and L. The curves for different values of Υ appear to follow different asymptotic curves, which show the expected approach to the adiabatic value $A = 1$ with increasing Υ, in agreement with Eq. (38). Note also the nonmonotonic dependence on K for sufficiently large Υ, likely due to a nontrivial interplay between effects related to the finite size and the presence of an external symmetry-breaking longitudinal field.

$$\lim_{\Upsilon\to\infty} \widetilde{\mathcal{E}}(\Upsilon, K, \Phi) = 0. \qquad (39)$$

B. Power-law approach to the adiabatic regime

The power-law approach to the adiabatic limit of the out-of-equilibrium FSS behaviors, i.e., for large values of Υ, can be inferred by exploiting the adiabatic perturbation theory, see, e.g., Refs. [8,78–80].

By expanding the state $|\Psi(t)\rangle$ in terms of the instantaneous Hamiltonian eigenstates $|\Psi_n[w(t)]\rangle$ (assuming no degeneracy and that $n = 0$ is the lowest eigenstate),

$$|\Psi(t)\rangle = \sum_{n\geqslant 0} a_n(t)|\Psi_n[w(t)]\rangle, \qquad (40)$$

we can write the overlap with the ground state associated with the instantaneous value of $w(t)$ as

$$|\langle\, \Psi_0[w(t)]\,|\,\Psi(t)\,\rangle|^2 = 1 - \sum_{n>0}|a_n(t)|^2. \qquad (41)$$

We can use the first-order adiabatic approximation to estimate the amplitudes $a_n(t)$. This is essentially justified by the fact that the out-of-equilibrium FSS limit implies $t_s \gg L^{-z}$, cf. Eq. (18). Using the general results, see, e.g., Refs. [24,79,80], at the time $t = 0$ corresponding to the critical point when $w = w_c = 0$, we obtain

$$|\langle\, \Psi_0[w(0)]\,|\,\Psi(0)\,\rangle|^2 - 1 \approx t_s^{-2}\sum_{n>0}\frac{|\langle n|\hat{H}_w|0\rangle|^2}{(\Delta E_n)^4}, \qquad (42)$$

where $\Delta E_n \equiv E_n(w=0) - E_0(w=0)$ is the energy difference between the nth level and the ground state at $w = 0$.

Then, using the expected scaling behaviors at CQTs [10],

$$\Delta E_n \sim L^{-z}, \quad \langle n|\hat{H}_w|0\rangle \sim L^{d-y_t}, \tag{43}$$

where $y_t = d + z - y_w$, we obtain

$$A^2 - 1 \sim t_s^{-2} L^{2(z+y_w)} = \Upsilon^{-2}. \tag{44}$$

This result is expected to extend to generic values of K along the out-of-equilibrium evolution (we also checked it numerically, see below). Therefore, the out-of-equilibrium FSS keeping K and Φ fixed, cf. Eq. (30), should asymptotically behave as

$$\widetilde{\mathcal{A}}(\Upsilon \to \infty, K, \Phi) \approx 1 - a\,\Upsilon^{-2} \tag{45}$$

to match the asymptotic Eq. (42), where the coefficient a generally depends on K and Φ. Analogously for the excitation energy Eq. (8), using the first-order adiabatic approximation one arrives at the asymptotic power-law suppression

$$\widetilde{\mathcal{E}}(\Upsilon \to \infty, K, \Phi) \sim \Upsilon^{-2}. \tag{46}$$

We remark that the $O(t_s^{-2})$ behavior in the adiabatic limit is obtained assuming a discrete spectrum, which is appropriate in finite-size systems, and expected to extend to the out-of-equilibrium FSS limit, in particular, when the out-of-equilibrium FSS behavior approaches the equilibrium FSS, in the limit $\Upsilon \to \infty$, keeping K and Φ constant.

The above power-law approach to adiabaticity is confirmed by numerical analyses on the quantum Ising chain. Some results for the adiabaticity function and the excitation energy along the KZ protocol are shown in Fig. 3. Analogous results are obtained for other values of the scaling variables. Their behaviors clearly confirm the Υ^{-2} power-law suppression of nonadiabaticity in the adiabatic limit $\Upsilon \to \infty$, cf. Eqs. (45) and (46).

C. The adiabatic limit due to the longitudinal field

In the presence of an external longitudinal field, adiabaticity is generally recovered in the limit $t_s \to \infty$ when keeping the longitudinal field $h \neq 0$ fixed. Indeed, in the presence of a longitudinal field h, the gap does not close at the critical point $w = w_c$, i.e., $\Delta(w = w_c, h) \sim |h|^{z/y_h}$, and therefore the KZ protocol can follow an adiabatic evolution in the limit of large time scale $t_s \to \infty$, even in the thermodynamic limit. Therefore, the out-of-equilibrium FSS outlined in Sec. III must also have a corresponding adiabatic limit, which must be realized when $|\Phi| \equiv |h|L^{y_h} \to \infty$. Without loss of generality, we assume $h > 0$, thus $\Phi > 0$ and $\Sigma > 0$, in the following.

To study the $\Phi \to \infty$ limit, it is useful to introduce the length scale associated with the longitudinal field h, i.e.,

$$\lambda_h = h^{-1/y_h} = \Phi^{-1/y_h} L. \tag{47}$$

Since Φ can be written as $\Phi = (L/\lambda_h)^{y_h}$, the large-$\Phi$ limit corresponds to $L \gg \lambda_h$. Moreover, we introduce the scaling variable

$$\widetilde{\Upsilon} = t_s \lambda_h^{-\zeta} = \Sigma^{\zeta/y_h} = \Upsilon \Phi^{\zeta/y_h}. \tag{48}$$

Note that the adiabatic limit $\Upsilon \to \infty$ is equivalent to $\widetilde{\Upsilon} \to \infty$ when we keep $\Phi > 0$ constant. However, in the limit $\Phi \to \infty$, $\widetilde{\Upsilon} \to \infty$ even though Υ is kept finite.

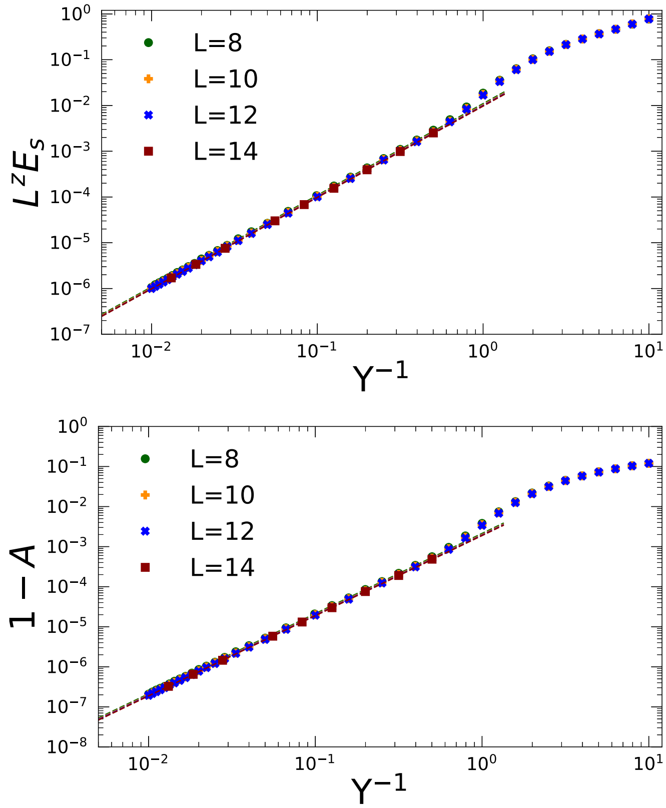

FIG. 3. Log-log plots of the adiabaticity function (bottom) and the excitation energy (top) for fixed scaling variables $K \equiv w(t)L^{y_w} = 1$ and $\Phi = hL^{y_h} = 1$, and fixed $w_i = -1$, versus $\Upsilon^{-1} = L^{\zeta}/t_s$. The data for different values of L are hardly distinguishable, demonstrating a good convergence to the corresponding asymptotic behavior. The approach to the adiabatic regime for large values of Υ turns out to be perfectly consistent with Eqs. (45) and (46), predicting an asymptotic $O(\Upsilon^{-2})$ suppression of the residual nonadiabatic effects (in both figures, the dashed line shows a linear fit to $b\Upsilon^{-2}$ of the data for the largest available values of Υ). Note that the asymptotic Υ^{-2} decay is observed for $\Upsilon \gtrsim 1$.

We expect that the scaling functions for the adiabaticity function and the excitation energy become trivial for $\Phi \to \infty$, i.e.,

$$\lim_{\Phi\to\infty} \widetilde{\mathcal{A}}(\Upsilon, K, \Phi) = 1, \quad \lim_{\Phi\to\infty} \widetilde{\mathcal{E}}(\Upsilon, K, \Phi) = 0. \tag{49}$$

The adiabatic limit of the magnetization is less trivial; indeed, we expect that in the limit $\Phi \to \infty$,

$$\widetilde{\mathcal{M}}(\Upsilon, K, \Phi \to \infty) = \mathcal{M}_{\rm eq}(K, \Phi \to \infty), \tag{50}$$

where $\mathcal{M}_{\rm eq}(K, \Phi)$ is the scaling function entering the equilibrium FSS [10,46], defined as in Eq. (36) with $y_o = y_l$. By matching its asymptotic behavior for $\Phi \to \infty$ with the power-law $M \sim h^{1/\delta}$ at the critical point $w = w_c$ and for finite h, one can easily derive the asymptotic behavior $\mathcal{M}_{\rm eq}(K, \Phi \to \infty) \sim \Phi^{1/\delta}$. Note that for the Ising chain $\delta = y_h/y_l = 15$, thus the $O(\Phi^{1/\delta})$ divergence turns out to be very slow.

Let us now discuss the approach to the adiabatic limit arising from the presence of a nonzero h, i.e., for $\Phi \to \infty$ or, equivalently, $\widetilde{\Upsilon} \to \infty$ when keeping Υ fixed. We recall that, in the presence of a nonzero h, the gap is always nonzero, its

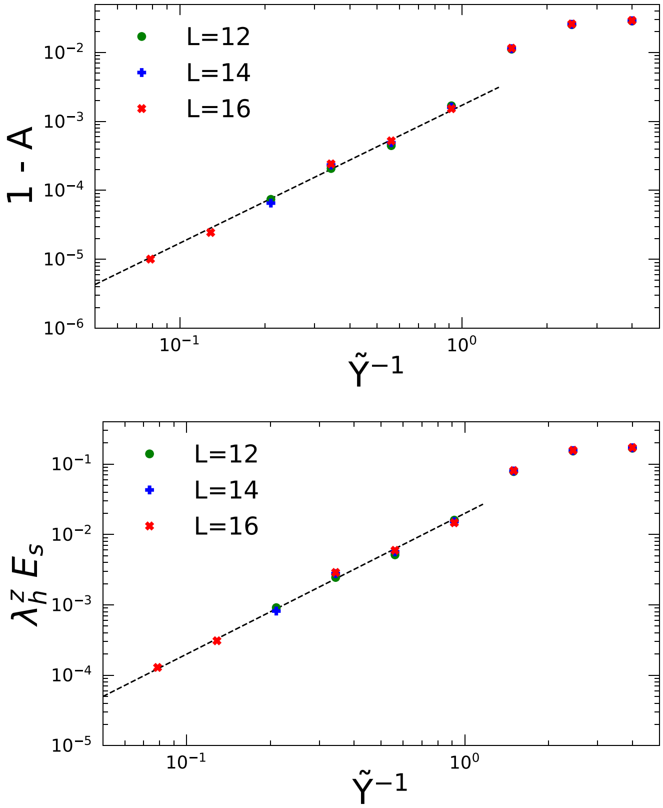

FIG. 4. Log-log plots of the adiabaticity function A (more precisely of $1 - A$, top figure) and the energy excitation E_s (actually the product $\lambda_h^z E_s$, bottom figure) for fixed scaling variables $K \equiv w(t)L^{y_w} = 0$, $\Upsilon = 0.25$, fixed $w_i = -1$, versus $\widetilde{\Upsilon}^{-1}$. Data for different values of L show a good convergence with increasing L. The resulting scaling behaviors in the large $\widetilde{\Upsilon}$ limit agree with Eqs. (51) and (52), predicting an asymptotic $O(\widetilde{\Upsilon}^{-2})$ suppression of the residual nonadiabatic contributions (the dashed lines show the $\widetilde{\Upsilon}^{-2}$ behavior in the log-log plots). Such a power-law approach is approximately observed for $\widetilde{\Upsilon} \gtrsim 1$, with some oscillations of relatively small amplitude (whose origin is not clear).

minimum at $w = w_c$ being $\Delta \sim h^{z/y_h}$. To discuss the adiabatic limit arising from a nonzero h, it is convenient to focus on the adiabaticity function A and the excitation energy E_s or the corresponding density D_e, in that they have a trivial adiabatic limit by construction. Using analogous arguments to those outlined for the adiabatic limit $\Upsilon \to \infty$ in Sec. IV B and, in particular, the fact that the adiabaticity violations are generally $O(t_s^{-2})$ within the first-order adiabatic approximation, the approach to the adiabatic limit is expected to be characterized by the power law

$$A(t, t_s, w_i, h, L) \approx \widetilde{\mathcal{A}}(\Upsilon, K, \Phi \to \infty) \approx 1 - a\widetilde{\Upsilon}^{-2}, \quad (51)$$

where $\widetilde{\Upsilon} = \Upsilon\Phi^{\zeta/y_h} \to \infty$, and the factor a generally depends on the other scaling variables K and Υ.

Numerical results for the quantum Ising chain confirm the above asymptotic behavior in the large-Φ limit, as, for example, shown by the data reported in the top Fig. 4, obtained keeping the scaling variables K and Υ fixed (in particular, for $K \equiv w(t)L^{y_w} = 0$ corresponding to $t = 0$, and $\Upsilon = 0.25$, and also fixed $w_i = -1$, analogous results have been obtained for other values of Υ and K). Again the data for different values of L show a good convergence to the corresponding asymptotic large-L scaling behavior, which agrees with the predicted $\widetilde{\Upsilon}^{-2}$ power-law approach to the adiabatic limit $A = 1$.

To derive the asymptotic behavior of the energy excitation, and consistently perform the large-Φ limit of the scaling Eq. (31), we must replace the L^{-z} prefactor on the right-hand side of Eq. (31) with a corresponding expression in terms of h and Φ, such as $L^{-z} = \lambda_h^{-z}\Phi^{-z/y_h}$, where λ_h is the length scale associated with the longitudinal field h, cf. Eq. (47). Then we expect

$$\lambda_h^z E_s(t, t_s, w_i, h, L) \approx \Phi^{-z/y_h}\widetilde{\mathcal{E}}(\Upsilon, K, \Phi) \approx c\widetilde{\Upsilon}^{-2}, \quad (52)$$

where c generally depends on the scaling variables K and Υ. Some numerical results for the quantum Ising chain are reported on the bottom Fig. 4, where the power-law approach Eq. (52) to the adiabatic limit of the excitation energy is clearly observed.

We finally note that the $\Phi \to \infty$ limit can be also seen as an infinite volume limit in that it corresponds to $L/\lambda_h \to \infty$. Let us consider the energy-excitation density Eq. (9), which is expected to have a well-defined infinite-volume limit. By simple manipulations of Eq. (31), one can write its scaling behavior as

$$D_e(t, t_s, w_i, h, L) \approx \lambda_h^{-(z+d)}\,\widehat{\mathcal{E}}(\widetilde{K}, \Sigma, L/\lambda_h), \quad (53)$$

where $\widetilde{K} = w(t)\lambda_h^{y_w}$, $\Sigma = h\lambda^{y_h} = (\lambda/\lambda_h)^{y_h} = \widetilde{\Upsilon}^{y_h/\zeta}$, and λ and λ_h are, respectively, the length scales defined in Eq. (15) and (47). Then we take the limit $L/\lambda_h \to \infty$, keeping $\widetilde{K}$ and Σ fixed. Thus, assuming that such a limit is regular:

$$D_e(t, t_s, w_i, h, L \to \infty) \approx \lambda_h^{-(z+d)}\,\widehat{\mathcal{E}}_\infty(\widetilde{K}, \Sigma). \quad (54)$$

Then, we note that taking the further limit Σ, $\widetilde{\Upsilon} \to \infty$, keeping $\widetilde{K}$ fixed, corresponds to an adiabatic limit, so

$$\lim_{\Sigma\to\infty} \widehat{\mathcal{E}}_\infty(\widetilde{K}, \Sigma) \to 0. \quad (55)$$

Assuming that the asymptotic behavior Eq. (52) persists in the infinite-volume limit corresponding to $\Upsilon \to 0$ (note that this is possible because Υ ad $\widetilde{\Upsilon}$ are not anymore related when $\Phi \to \infty$), we also expect

$$\widehat{\mathcal{E}}_\infty(\widetilde{K}, \Sigma \to \infty) \sim \widetilde{\Upsilon}^{-2}, \quad (56)$$

where $\widetilde{\Upsilon} = \Sigma^{\zeta/y_h} = t_s h^{\zeta/y_h}$. One can easily check that this asymptotic behavior agrees with the results reported in Ref. [30] for $t = 0$, corresponding to $\widetilde{K} = 0$. Indeed, we obtain

$$D_e \sim \lambda_h^{-(z+d)}\widetilde{\Upsilon}^{-2} = t_s^{-2}h^{(d-z-2y_w)/y_h}, \quad (57)$$

corresponding to $D_e \sim t_s^{-2}h^{-16/15}$ for the Ising chain.

V. EXTENDED KZ PROTOCOLS

We now consider a KZ protocol in which both Hamiltonian parameters w and h are assumed to be time dependent and cross their critical values with a linear time dependence. More precisely, we extend the time dependence of the Hamiltonian Eq. (1) to

$$\hat{H}[w(t), h(t)] = \hat{H}_c + w(t)\hat{H}_w + h(t)\hat{H}_h,$$
$$w(t) = t/t_s, \quad h(t) = t/t_{s,h}. \quad (58)$$

In this case, we allow for different timescales in the time dependence of w and h, but both of them take their critical value at $t = 0$. We consider a KZ protocol analogous to that outlined in Sec. II A, but allowing for the time changes of h too, starting at $t = t_i < 0$ from the ground state associated with $w(t_i)$ and $h(t_i)$, and then unitarily evolving using the Hamiltonian $\hat{H}[w(t), h(t)]$.

We first consider the simplest case in which $t_{s,h} = c\, t_s$ where c is a finite constant. To address the resulting out-of-equilibrium scaling behavior, one may again exploit homogenous scaling laws analogous to those reported in Eqs. (12) and (13), with one scaling variable for each time-dependent parameter $w(t)$ and $h(t)$, i.e., $K = w(t)L^{y_w}$ and $K_h = h(t)L^{y_h}$. However, since the RG dimensions of the even and odd model parameters w and h differ, i.e., $y_w \neq y_h$, and their time dependence is controlled by a unique timescale $t_s \sim t_{s,h}$, we cannot consistently rescale them to get nontrivial FSS behaviors keeping K and K_h fixed to any finite nonzero value. In particular, if we keep $K = (t/t_s)L^{y_w}$ fixed in the simultaneous large t_s and L limits, then $K_h \to \infty$ due to the fact that $y_h > y_w$ generally. Thus, no asymptotic FSS is expected to emerge when keeping K fixed. On the other hand, if we keep $K_h = (t/t_{s,h})L^{y_h}$ fixed in the FSS limit, then $K \to 0$. Therefore, the emerging scaling behavior is analogous to that obtained in KZ protocols keeping $w = w_c = 0$ fixed and varying only the symmetry-breaking parameter $h(t)$, see, e.g., Ref. [39].

In conclusion, we do not expect to observe further interesting out-of-equilibrium scaling phenomena when we allow for variations of both the even and odd Hamiltonian parameters w and h. If we insist on varying both parameters, to obtain a nontrivial scaling behavior we need to assume different timescales, tuned so $t_{s,h}/t_s \sim L^{y_h - y_w}$, to have $K \sim K_h$.

VI. CONCLUSIONS

We have studied the effects of static homogenous symmetry-breaking perturbations in the out-of-equilibrium quantum dynamics of many-body systems driven by a time-dependent symmetry-preserving parameter across the critical regime associated with a CQT.

The out-of-equilibrium scenario arising from KZ protocols, slowly crossing critical regimes, substantially changes in the presence of symmetry-breaking perturbations, such as those arising from the longitudinal field h in the quantum Ising models, cf. Eq. (4), giving rise to a relevant RG perturbation at their CQTs. The gap Δ does not generally close when $h \neq 0$, behaving as $\Delta \sim |h|^{z/y_h}$ at the critical point $w = w_c$ in the infinite-volume limit. Therefore, the adiabatic evolution across the critical point w_c, through ground states associated with the instantaneous values of $w(t)$, can be always realized in the infinite-volume limit, for sufficiently large timescale t_s of the KZ protocol. However, for sufficiently small values of h, the systems can be kept within the critical regime, giving rise to universal peculiar distortions of the KZ scaling behavior observed in the absence of symmetry-breaking terms.

We show that out-of-equilibrium FSS behaviors emerge along generalized KZ protocols even in the presence of the symmetry-breaking perturbation, described in Sec. II A. For this purpose, we exploit RG scaling frameworks implemented in terms of asymptotic homogenous scaling laws, cf. Eqs. (12) and (13), that are expected to hold along the generalized KZ protocol. This allows us to develop a generalized out-of-equilibrium FSS theory, allowing for the symmetry-breaking perturbation. The emerging out-of-equilibrium FSS scenario in the presence of a static symmetry-breaking perturbation requires an appropriate tuning of the corresponding Hamiltonian parameter, controlled by its RG dimension at the fixed point associated with the CQT.

As paradigmatic models, we consider the quantum Ising models, whose CQTs are characterized by the spontaneous breaking of their $\mathbb{Z}_2$ symmetry, and two relevant parameters: the symmetry-preserving transverse field g, or $w = g_c - g$, and the symmetry-breaking longitudinal field h, cf. Eq. (4). We consider generalized KZ protocols, where the out-of-equilibrium dynamics is induced by variations of the symmetry-preserving parameter $w(t)$ across its critical value w_c, with a linear time dependence $w(t) = t/t_s$, from the disordered to ordered phase, and a large timescale t_s. Unlike most earlier studies based on KZ protocols, we allow for a nonzero longitudinal field h. We extend the study of Ref. [30], where the out-of-equilibrium scaling behaviors of KZ protocols in the presence of a symmetry-breaking perturbation were discussed in the infinite-volume (thermodynamic) limit.

Within the out-of-equilibrium FSS framework of generalized KZ protocols in the presence of the symmetry-breaking perturbation, the adiabatic limit may arise for essentially two reasons, i.e., because of the finite size (thus the critical gap does not vanish, getting suppressed as $\Delta \sim L^{-z}$) and because of the longitudinal field h (which does not allow the gap to vanish even in the thermodynamic limit, behaving as $\Delta \sim |h|^{z/y_h}$ for sufficiently small values of h). We argue that the nonadiabaticity of the out-of-equilibrium dynamics gets suppressed by power laws, which can be inferred from the first-order adiabatic approximation.

The scaling laws obtained within the out-of-equilibrium FSS framework and, in particular, the predicted power-law approaches to the adiabatic regime, are confirmed by numerical analyses of generalized KZ protocols in the quantum Ising chain.

We have also briefly discussed the case in which the generalized KZ protocol is characterized by the time dependence of both even and odd parameters w and h. Within analogous scaling frameworks, we argue that no further interesting scaling behaviors emerge when both parameters are changed with the same timescales. Nontrivial FSS behaviors may only be obtained by appropriately rescaling the timescales of $w(t)$ and $h(t)$, proportionally to different powers of the size.

We remark that the emerging out-of-equilibrium scaling scenario, put forward for extended KZ protocols in the presence of symmetry-breaking perturbations, is expected to hold for generic CQTs in any spatial dimension, with the appropriate identifications of the symmetry-preserving and symmetry-breaking Hamiltonian parameters playing the role of w and h in Eq. (1). Moreover, analogous out-of-equilibrium scaling scenarios are also expected to emerge at classical finite-temperature transitions in the presence of perturbations breaking the global symmetry.

As already mentioned in the Introduction, the FSS approaches generally simplify the analyses of the universal features of critical behaviors, essentially because they do not require the further condition $\xi \ll L$ of the thermodynamic limit. In this paper, we have shown that the out-of-equilibrium FSS scenarios associated with generalized KZ protocols can already be observed for relatively small systems, with a few decades of spin operators. This fact may also provide a great help in studying higher-dimensional systems, where only relatively small systems can be numerically studied. See, for example, the study reported in Ref. [81] for two-dimensional quantum Ising systems, where some evidence of scaling behaviors along standard KZ protocols have already been observed for small systems. Scaling behaviors requiring a limited numbers of degrees of freedom may be more easily accessed by experiments where the coherent quantum dynamics of only a limited number of particles or spins can be effectively realized, such as experiments with quantum simulators in laboratories, e.g., by means of trapped ions [47,48], ultracold atoms [49,50], or superconducting qubits [51,52]. Moreover, our results may turn out to be particularly relevant for quantum simulations and quantum computing, where important experimental advances have been achieved recently, see, e.g., Refs. [82–87].

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[1] T. W. B. Kibble, Some implications of a cosmological phase transition, Phys. Rep. **67**, 183 (1980).
[2] K. Binder, Theory of first-order phase transitions, Rep. Prog. Phys. **50**, 783 (1987).
[3] A. J. Bray, Theory of phase-ordering kinetics, Adv. Phys. **43**, 357 (1994).
[4] W. H. Zurek, Cosmological experiments in condensed matter systems, Phys. Rep. **276**, 177 (1996).
[5] P. Calabrese and A. Gambassi, Ageing properties of critical systems, J. Phys. A: Math. Gen. **38**, R133 (2005).
[6] D. Boyanovsky, H. J. de Vega, and D. J. Schwarz, Phase transitions in the early and the present universe, Annu. Rev. Nucl. Part. Sci. **56**, 441 (2006).
[7] J. Dziarmaga, Dynamics of a quantum phase transition and relaxation to a steady state, Adv. Phys. **59**, 1063 (2010).
[8] A. Polkovnikov, K. Sengupta, A. Silva, and M. Vengalattore, Colloquium: Nonequilibrium dynamics of closed interacting quantum systems, Rev. Mod. Phys. **83**, 863 (2011).
[9] G. Biroli, Slow relaxations and nonequilibrium dynamics in classical and quantum systems, in *Strongly Interacting Quantum Systems Out of Equilibrium: Lecture Notes of the Les Houches Summer School* (Oxford University Press, Oxford, UK, 2016).
[10] D. Rossini and E. Vicari, Coherent and dissipative dynamics at quantum phase transitions, Phys. Rep. **936**, 1 (2021).
[11] T. W. B. Kibble, Topology of cosmic strings and domains, J. Phys. A: Math. Gen. **9**, 1387 (1976).
[12] I. Chuang, R. Durrer, N. Turok, and B. Yurke, Cosmology in the laboratory: Defect dynamics in liquid crystals, Science **251**, 1336 (1991).
[13] M. J. Bowick, L. Chandar, E. A. Schiff, and A. M. Srivastava, The cosmological Kibble mechanism in the laboratory: String formation in liquid crystals, Science **263**, 943 (1994).
[14] C. Bäuerle, Yu M. Bunkov, S. N. Fisher, H. Godfrin, and G. R. Pickett, Laboratory simulation of cosmic string formation in the early Universe using superfluid ^{3}He, Nature (London) **382**, 332 (1996).
[15] C. N. Weiler, T. W. Neely, D. R. Scherer, A. S. Bradley, M. J. Davis, and B. P. Anderson, Spontaneous vortices in the formation of Bose-Einstein condensates, Nature (London) **455**, 948 (2008).
[16] S. Ulm, S. J. Roßnagel, G. Jacob, C. Degünther, S. T. Dawkins, U. G. Poschinger, R. Nigmatullin, A. Retzker, M. B. Plenio, F. Schmidt-Kaler, and K. Singer, Observation of the Kibble-Zurek scaling law for defect formation in ion crystals, Nat. Commun. **4**, 2290 (2013).
[17] K. Pyka, J. Keller, H. L. Partner, R. Nigmatullin, T. Burgermeister, D. M. Meier, K. Kuhlmann, A. Retzker, M. B. Plenio, W. H. Zurek, A. del Campo, and T. E. Mehlstäubler, Topological defect formation and spontaneous symmetry breaking in ion Coulomb crystals, Nat. Commun. **4**, 2291 (2013).
[18] N. Navon, A. L. Gaunt, R. P. Smith, and Z. Hadzibabic, Critical dynamics of spontaneous symmetry breaking in a homogeneous Bose gas, Science **347**, 167 (2015).
[19] A. Trenkwalder, G. Spagnolli, G. Semeghini, S. Coop, M. Landini, P. Castilho, L. Pezzè, G. Modugno, M. Inguscio, A. Smerzi, and M. Fattori, Quantum phase transitions with parity-symmetry breaking and hysteresis, Nat. Phys. **12**, 826 (2016).
[20] A. Polkovnikov, Universal adiabatic dynamics in the vicinity of a quantum critical point, Phys. Rev. B **72**, 161201(R) (2005).
[21] W. H. Zurek, U. Dorner, and P. Zoller, Dynamics of a Quantum Phase Transition, Phys. Rev. Lett. **95**, 105701 (2005).
[22] J. Dziarmaga, Dynamics of a Quantum Phase Transition: Exact Solution of the Quantum Ising Model, Phys. Rev. Lett. **95**, 245701 (2005).
[23] A. Fubini, G. Falci, and A. Osterloh, Robustness of adiabatic passage through a quantum phase transition, New J. Phys. **9**, 134 (2007).
[24] C. De Grandi, V. Gritsev, and A. Polkovnikov, Quench dynamics near a quantum critical point, Phys. Rev. B **81**, 012303 (2010); Quench dynamics near a quantum critical point: Application to the sine-Gordon model, **81**, 224301 (2010).
[25] S. Gong, F. Zhong, X. Huang, and S. Fan, Finite-time scaling via linear driving, New J. Phys. **12**, 043036 (2010).
[26] A. Chandran, A. Erez, S. S. Gubser, and S. L. Sondhi, Kibble-Zurek problem: Universality and the scaling limit, Phys. Rev. B **86**, 064304 (2012).
[27] A. Pelissetto, D. Rossini, and E. Vicari, Out-of-equilibrium dynamics driven by localized time-dependent perturbations at quantum phase transitions, Phys. Rev. B **97**, 094414 (2018).
[28] A. Pelissetto, D. Rossini, and E. Vicari, Dynamic finite-size scaling after a quench at quantum transitions, Phys. Rev. E **97**, 052148 (2018).
[29] D. Rossini and E. Vicari, Scaling of decoherence and energy flow in interacting quantum spin systems, Phys. Rev. A **99**, 052113 (2019).

[30] M. M. Rams, J. Dziarmaga, and W. H. Zurek, Symmetry Breaking Bias and the Dynamics of a Quantum Phase Transition, Phys. Rev. Lett. **123**, 130603 (2019).

[31] W. H. Zurek, Cosmological experiments in superfluid helium? Nature (London) **317**, 505 (1985).

[32] A. Polkovnikov and V. Gritsev, Breakdown of the adiabatic limit in low-dimensional gapless systems, Nat. Phys. **4**, 477 (2008).

[33] A. Dutta, G. Aeppli, B. K. Chakrabarti, U. Divakaran, T. F. Rosenbaum, and D. Sen, *Quantum Phase Transitions in Transverse Field Spin Models: From Statistical Physics to Quantum Information* (Cambridge University Press, Cambridge, UK, 2015).

[34] A. Pelissetto and E. Vicari, Off-equilibrium scaling behaviors driven by time-dependent external fields in three-dimensional O(N) vector models, Phys. Rev. E **93**, 032141 (2016).

[35] A. Pelissetto and E. Vicari, Dynamic Off-Equilibrium Transition in Systems Slowly Driven Across Thermal First-order Transitions, Phys. Rev. Lett. **118**, 030602 (2017).

[36] D. Rossini and E. Vicari, Dynamic Kibble-Zurek scaling framework for open dissipative many-body systems crossing quantum transitions, Phys. Rev. Res. **2**, 023211 (2020).

[37] A. Pelissetto, D. Rossini, and E. Vicari, Scaling properties of the dynamics at first-order quantum transitions when boundary conditions favor one of the two phases, Phys. Rev. E **102**, 012143 (2020).

[38] J. Rysti, J. T. Maäkinen, S. Autti, T. Kamppinen, G. E. Volovik, and V. B. Eltsov, Suppressing the Kibble-Zurek Mechanism by a Symmetry-Violating Bias, Phys. Rev. Lett. **127**, 115702 (2021).

[39] F. Tarantelli and E. Vicari, Out-of-equilibrium dynamics arising from slow round-trip variations of Hamiltonian parameters across quantum and classical critical points, Phys. Rev. B **105**, 235124 (2022).

[40] B. Damski, The Simplest Quantum Model Supporting the Kibble-Zurek Mechanism of Topological Defect Production: Landau-Zener Transitions from a New Perspective, Phys. Rev. Lett. **95**, 035701 (2005).

[41] M. Uhlmann, R. Schützhold, and U. R. Fischer, Vortex Quantum Creation and Winding Number Scaling in a Quenched Spinor Bose Gas, Phys. Rev. Lett. **99**, 120407 (2007).

[42] M. Uhlmann, R. Schützhold, and U. R. Fischer, System size scaling of topological defect creation in a second-order dynamical quantum phase transition, New J. Phys. **12**, 095020 (2010).

[43] T. Nag, A. Dutta, and A. Patra, Quench dynamics and quantum information, Int. J. Mod. Phys. B **27**, 1345036 (2013).

[44] S. L. Sondhi, S. M. Girvin, J. P. Carini, and D. Shahar, Continuous quantum phase transitions, Rev. Mod. Phys. **69**, 315 (1997).

[45] S. Sachdev, *Quantum Phase Transitions* (Cambridge University, Cambridge, England, 1999).

[46] M. Campostrini, A. Pelissetto, and E. Vicari, Finite-size scaling at quantum transitions, Phys. Rev. B **89**, 094516 (2014).

[47] R. Islam, E. E. Edwards, K. Kim, S. Korenblit, C. Noh, H. Carmichael, G.-D. Lin, L.-M. Duan, C.-C. Joseph Wang, J. K. Freericks, and C. Monroe, Onset of a quantum phase transition with a trapped ion quantum simulator, Nat. Commun. **2**, 377 (2011).

[48] S. Debnath, N. M. Linke, C. Figgatt, K. A. Landsman, K. Wright, and C. Monroe, Demonstration of a small programmable quantum computer with atomic qubits, Nature (London) **536**, 63 (2016).

[49] J. Simon, W. S. Bakr, R. Ma, M. E. Tai, P. M. Preiss, and M. Greiner, Quantum simulation of antiferromagnetic spin chains in an optical lattice, Nature (London) **472**, 307 (2011).

[50] H. Labuhn, D. Barredo, S. Ravets, S. de Leseleuc, T. Macri, T. Lahaye, and A. Browaeys, Tunable two-dimensional arrays of single Rydberg atoms for realizing quantum Ising models, Nature (London) **534**, 667 (2016).

[51] Y. Salathé, M. Mondal, M. Oppliger, J. Heinsoo, P. Kurpiers, A. Potocnik, A. Mezzacapo, U. Las Heras, L. Lamata, E. Solano, S. Filipp, and A. Wallraff, Digital Quantum Simulation of Spin Models with Circuit Quantum Electrodynamics, Phys. Rev. X **5**, 021027 (2015).

[52] A. Cervera-Lierta, Exact Ising model simulation on a quantum computer, Quantum **2**, 114 (2018).

[53] S. Ducci, P. L. Ramazza, W. Gonzáles-Viñas, and F. T. Arecchi, Order Parameter Fragmentation after a Symmetry-Breaking Transition, Phys. Rev. Lett. **83**, 5210 (1999).

[54] R. Monaco, J. Mygind, M. Aaroe, R. J. Rivers, and V. P. Koshelets, Zurek-Kibble Mechanism for the Spontaneous Vortex Formation in Nb-Al/Al_{ox}/Nb Josephson Tunnel Junctions: New Theory and Experiment, Phys. Rev. Lett. **96**, 180604 (2006).

[55] L. E. Sadler, J. M. Higbie, S. R. Leslie, M. Vengalattore, and D. M. Stamper-Kurn, Spontaneous symmetry breaking in a quenched ferromagnetic spinor Bose-Einstein condensate, Nature (London) **443**, 312 (2006).

[56] D. Chen, M. White, C. Borries, and B. DeMarco, Quantum Quench of an Atomic Mott Insulator, Phys. Rev. Lett. **106**, 235304 (2011).

[57] S. M. Griffin, M. Lilienblum, K. T. Delaney, Y. Kumagai, M. Fiebig, and N. A. Spaldin, Scaling Behavior and Beyond Equilibrium in the Hexagonal Manganites, Phys. Rev. X **2**, 041022 (2012).

[58] G. Lamporesi, S. Donadello, S. Serafini, F. Dalfovo, and G. Ferrari, Spontaneous creation of Kibble-Zurek solitons in a Bose-Einstein condensate, Nat. Phys. **9**, 656 (2013).

[59] S. Braun, M. Friesdorf, S. S. Hodgman, M. Schreiber, J. P. Ronzheimer, A. Riera, M. del Rey, I. Bloch, J. Eisert, and U. Schneider, Emergence of coherence and the dynamics of quantum phase transitions, Proc. Natl. Acad. Sci. USA **112**, 3641 (2015).

[60] L. Chomaz, L. Corman, T. Bienaimé, R. Desbuquois, C. Weitenberg, S. Nascimbéne, J. Beugnon, and J. Dalibard, Emergence of coherence via transverse condensation in a uniform quasi-two-dimensional Bose gas, Nat. Commun. **6**, 6162 (2015).

[61] J.-M. Cui, Y.-F. Huang, Z. Wang, D.-Y. Cao, J. Wang, W.-M. Lv, L. Luo, A. del Campo, Y.-J. Han, C.-F. Li, and G.-C. Guo, Experimental trapped-ion quantum simulation of the Kibble-Zurek dynamics in momentum space, Sci. Rep. **6**, 33381 (2016).

[62] M. Gong, X. Wen, G. Sun, D.-W. Zhang, D. Lan, Y. Zhou, Y. Fan, Y. Liu, X. Tan, H. Yu, Y. Yu, S.-L. Zhu, S. Han, and P. Wu, Simulating the Kibble-Zurek mechanism of the Ising model with a superconducting qubit system, Sci. Rep. **6**, 22667 (2016).

[63] M. Anquez, B. A. Robbins, H. M. Bharath, M. Boguslawski, T. M. Hoang, and M. S. Chapman, Quantum Kibble-Zurek Mechanism in a Spin-1 Bose-Einstein Condensate, Phys. Rev. Lett. **116**, 155301 (2016).
[64] L. W. Clark, L. Feng, and C. Chin, Universal space-time scaling symmetry in the dynamics of bosons across a quantum phase transition, Science **354**, 606 (2016).
[65] A. Keesling, A. Omran, H. Levine, H. Bernien, H. Pichler, S. Choi, R. Samajdar, S. Schwartz, P. Silvi, S. Sachdev, P. Zoller, M. Endres, M. Greiner, V. Vuletic, and M. D. Lukin, Quantum Kibble-Zurek mechanism and critical dynamics on a programmable Rydberg simulator, Nature (London) **568**, 207 (2019).
[66] A. Pelissetto and E. Vicari, Critical phenomena and renormalization group theory, Phys. Rep. **368**, 549 (2002).
[67] R. Guida and J. Zinn-Justin, Critical exponents of the N-vector model, J. Phys. A: Math. Gen. **31**, 8103 (1998).
[68] M. Campostrini, A. Pelissetto, P. Rossi, and E. Vicari, 25th-order high-temperature expansion results for three-dimensional Ising-like systems on the simple-cubic lattice, Phys. Rev. E **65**, 066127 (2002).
[69] M. Hasenbusch, A finite size scaling study of lattice models in the three-dimensional Ising universality class, Phys. Rev. B **82**, 174433 (2010).
[70] F. Kos, D. Poland, D. Simmons-Duffin, and A. Vichi, Precision islands in the Ising and O(N) models, J. High Energy Phys. 08 (2016) 036.
[71] M. V. Kompaniets and E. Panzer, Minimally subtracted six-loop renormalization of O(n)-symmetric φ^4 theory and critical exponents, Phys. Rev. D **96**, 036016 (2017).
[72] M. Hasenbusch, Restoring isotropy in a three-dimensional lattice model: The Ising universality class, Phys. Rev. B **104**, 014426 (2021).
[73] P. Pfeuty, The one-dimensional Ising model with a transverse field, Ann. Phys. **57**, 79 (1970).
[74] M. Caselle, M. Hasenbusch, A. Pelissetto, and E. Vicari, Irrelevant operators in the two-dimensional Ising model, J. Phys. A: Math. Gen. **35**, 4861 (2002).
[75] P. Calabrese, M. Caselle, A. Celi, A. Pelissetto, and E. Vicari, Nonanalyticity of the Callan-Symanzik β-function of two-dimensional O(N) models, J. Phys. A: Math. Gen. **33**, 8155 (2000).
[76] M. Caselle and M. Hasenbusch, Critical amplitudes and mass spectrum of the 2D Ising model in a magnetic field, Nucl. Phys. B **579**, 667 (2000).
[77] M. Campostrini, A. Pelissetto, P. Rossi, and E. Vicari, Two-point correlation function of three-dimensional O(N) models: The critical limit and anisotropy, Phys. Rev. E **57**, 184 (1998).
[78] L. I. Schiff, *Quantum Mechanics* (McGraw-Hill Kogakusha, Tokyo, 1977).
[79] G. Rigolin, G. Ortiz, and V. H. Ponce, Beyond the quantum adiabatic approximation: Adiabatic perturbation theory, Phys. Rev. A **78**, 052508 (2008).
[80] C. De Grandi and A. Polkovnikov, *Quantum Quenching, Annealing and Computation*, edited by A. Das, A. Chandra, and B. K. Chakrabarti, Lecture Notes in Physics Vol. 802 (Springer, Heidelberg, 2010).
[81] M. Schmitt, M. M. Rams, J. Dziarmaga, M. Heyl, and W. Zurek, Quantum phase transition dynamics in the two-dimensional transverse-field Ising model, Sci. Adv. **8**, eabl6850 (2022).
[82] J. I. Cirac and P. Zoller, Goals and opportunities in quantum simulation, Nat. Phys. **8**, 264 (2012).
[83] I. Bloch, J. Dalibard, and S. Nascimbeéne, Quantum simulations with ultracold quantum gases, Nat. Phys. **8**, 267 (2012).
[84] R. Blatt and C. F. Roos, Quantum simulations with trapped ions, Nat. Phys. **8**, 277 (2012).
[85] A. Aspuru-Guzik and P. Walther, Photonic quantum simulators, Nat. Phys. **8**, 285 (2012).
[86] A. A. Houck, H. E. Türeci, and J. Koch, On-chip quantum simulation with superconducting circuits, Nat. Phys. **8**, 292 (2012).
[87] I. M. Georgescu, S. Ashhab, and F. Nori, Quantum simulation, Rev. Mod. Phys. **86**, 153 (2014).