Topological order in interacting semimetals

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It has recently been demonstrated that it is possible to open a gap in a magnetic Weyl semimetal, while preserving the chiral anomaly along with the charge conservation and translational symmetries, which all protect the gapless nodes in a weakly interacting semimetal. The resulting state was shown to be a nontrivial generalization of a non-Abelian fractional quantum Hall liquid to three dimensions. Here we point out that a second fractional quantum Hall state exists in this case. This state has exactly the same electrical and thermal Hall responses as the first, but a distinct (fracton) topological order. Moreover, the existence of this second fractional quantum Hall state necessarily implies a gapless phase, which has identical topological response to a noninteracting Weyl semimetal, but is distinct from it. This may be viewed as a generalization (in a weaker form) of the known duality between a noninteracting two-dimensional Dirac fermion and QED$_3$ to $3 + 1$ dimensions. In addition we discuss a $(3 + 1)$-dimensional topologically ordered state, obtained by gapping a nodal line semimetal without breaking symmetries.

I. INTRODUCTION

Topological order, a concept that originated in the study of the fractional quantum Hall effect (FQHE) in two-dimensional (2D) electron gas systems [1], continues to be a subject of intense interest. From the fundamental physics perspective, topologically ordered states provide perfect examples of emergent macroscopic quantum phenomena, with fractionally quantized electromagnetic and thermal responses, that are impossible to explain based on textbook models of weakly interacting electrons. Instead, such fractionally quantized observable responses necessarily imply excitations with fractional charges, fractional and non-Abelian statistics, which cannot be constructed out of any finite number of elementary constituents [2]. In addition, such exotic excitations may have future potential practical uses in quantum computing and quantum simulation, as their nonlocal topological nature makes them highly resistant to decoherence and noise [3].

Topologically ordered states in 2D are by now well understood. Various theoretical models [4–7], as well as complete formal classifications of 2D topological orders exist [8]. Although significant progress has been made in recent years [9–19], less is known about topologically ordered states in three dimensions (3D). 3D topologically ordered states are significantly different from the 2D ones. On the one hand, fractional statistics is impossible in 3D, and quasiparticle excitations may only be bosons or fermions. This could make one doubt that, for example, fractional quantum Hall (FQH) states may even in principle be generalized to 3D, as the existence of anyons, i.e., quasiparticles with fractional statistics, is an essential feature of the 2D FQHE. On the other hand, in addition to point quasiparticles, one-dimensional loop excitations exist in 3D, which both adds complexity and opens up new interesting possibilities.

We recently demonstrated that a promising way to achieve 3D topologically ordered states is through gapping topological semimetals without breaking the protecting symmetries [20–22] (see Refs. [23–29] for related work). Topological semimetals [30–35] are intermediate phases between insulators of different electronic structure topology. They may be characterized by unquantized anomalies [36,37], i.e., topological terms with noninteger and continuously tunable coefficients, similar to the electron filling parameter, characterizing ordinary Fermi liquids. Much like fractional filling in a Fermi liquid mandates the existence of a Fermi surface of gapless particle-hole excitations [38], these unquantized anomalies necessarily imply gapless modes and corresponding long-range entanglement. The only way gaplessness may be circumvented in the absence of broken symmetries is through the formation of a topologically ordered state, which preserves the anomaly and the long-range entanglement of the gapless semimetal.

Specifically, in Ref. [20] we presented an explicit construction of a 3D topologically ordered state in a gapped magnetic Weyl semimetal, which exhibits a nontrivial generalization of the FQHE to 3D. This state is obtained starting from a magnetic Weyl semimetal with a single pair of nodes, separated by half a reciprocal lattice vector. These nodes may be gapped by breaking the $U(1)$ charge conservation symmetry while forming a superconducting state with intranodal pairing. In general, such states break translational symmetry since the Weyl nodes exist at nontrivial momenta in the first Brillouin zone (BZ). However, when the nodes are separated by exactly half a reciprocal lattice vector, such a pairing leads to density

modulation at the reciprocal lattice vector, which does not break the crystal translational symmetry. Restoring the charge conservation symmetry by proliferating flux $2hc/e = 4\pi$ (we will be using $\hbar = c = e = 1$ units throughout this paper) vortices in the superconducting order parameter leads to a featureless fractionalized insulator with $\mathbb{Z}_4$ topological order, that has the same electrical and thermal Hall conductivities as the original noninteracting Weyl semimetal, i.e., exhibits FQHE in 3D. Unlike in 2D FQH liquids, quasiparticle excitations in this state are bosons and fermions. What plays the role of the anyons in the 2D FQHE are intersections of the vortex-loop excitations with atomic planes. These behave as fractionally charged particles with semionic statistics, which may be sharply defined by considering three-loop braiding processes [15], involving a line defect of translational symmetry, i.e., an edge dislocation.

In this paper we show that, in addition to the 3D FQH state of Ref. [20], another state exists, which has identical topological response, but distinct topological order, which turns out to be of a fracton type. The existence of these two distinct states turns out to be closely related to a very similar property of gapped symmetric 2D Dirac surface states of 3D time-reversal (TR) invariant topological insulators (TIs) [39–43]. In this case, two distinct topologically ordered states exist. One, called Pfaffian-antisemion [40,42], is closely related to the 3D FQH states of Ref. [20] (more precisely, the relation is with the TR broken version of this state). The second one, T-Pfaffian [39,41], is related to the other 3D state we will construct in the present paper (again, more precisely, the relation is with the TR breaking version of this state, which is usually called PH, which stands for particle-hole-symmetric, Pfaffian [44,45]).

Another interesting consequence emerges from these analogies to the 2D TR invariant TI surface state topological orders. It is well known that the PH-Pfaffian is closely related to the recently discovered duality relation between a massless noninteracting 2D Dirac fermion and QED₃ [44,46–51]. Namely, the PH-Pfaffian state is obtained when the dual Dirac fermion of QED₃ is gapped by pairing, which does not break the charge conservation symmetry since the dual fermion is neutral. The existence of the analog of the PH-Pfaffian state in our 3D system then also implies the existence of a gapless state, which is related to the noninteracting Weyl semimetal via a duality relation, somewhat similar to the 2D Dirac duality. We demonstrate that this is indeed the case. However, we find that the duality only applies to topological response in this case and not to the dynamics and is weaker than the 2D duality in this sense.

The path to topologically ordered insulators through gapping topological semimetals is quite general and is not limited to the magnetic Weyl semimetal case. To emphasize this point, here we also discuss a topologically ordered state, which is obtained by gapping a nodal line semimetal without breaking symmetries. This state has a topological order, distinct from a gapped Weyl semimetal, and is characterized by a fractional electric polarization, impossible in an ordinary weakly interacting insulator.

The rest of the paper is organized as follows. In Sec. II, after a preliminary discussion of the topological field theory description of the electromagnetic response of Weyl

semimetals, we recap the construction of the 3D analog of the Pfaffian-antisemion state of Refs. [20,21]. In Sec. III, we demonstrate the existence of a duality relation (which applies to topological response only) between a noninteracting Weyl semimetal and a QED₄, which describes a time-reversed Weyl semimetal, coupled to a dynamical gauge field. In Sec. IV we discuss a topologically ordered state, obtained by gapping a nodal line semimetal without breaking symmetries. This state is characterized by a fractional electric polarization, impossible in an ordinary insulator. We conclude in Sec. V with a brief discussion of our results.

II. GAPPED SYMMETRY-PRESERVING STATES IN WEYL SEMIMETALS

A. Preliminaries

To keep the paper self-contained, we will start by recapping the construction of the 3D FQH state of Refs. [20,21], which, as will be explained below, may be viewed as a TR breaking 3D analog of the Pfaffian-antisemion state on a strongly interacting 3D TI surface. We will also put the theory of Ref. [21] on a more rigorous footing by introducing the language of translation gauge fields [36,52–55], which allows one to use proper coordinate-free notation for the corresponding topological field theories.

We start from the simplest cubic lattice model of a magnetic Weyl semimetal with a pair of nodes [34]

$$H = \sum_{\mathbf{k}} \psi_{\mathbf{k}}^{\dagger}[\sigma_x \sin(k_x d) + \sigma_y \sin(k_y d) + \sigma_z m(\mathbf{k})]\psi_{\mathbf{k}}. \quad (1)$$

Here σ_i are Pauli matrices, describing the pair of touching bands and

$$m(\mathbf{k}) = \cos(k_z d) - \cos(Qd) - \tilde{m}[2 - \cos(k_x d) - \cos(k_y d)], \quad (2)$$

where d is the lattice constant, $\tilde{m} > 1$, and $m(\mathbf{k})$ vanishes at two points on the z axis with $k_z = \pm Q$, which correspond to the locations of the Weyl nodes.

Such a Weyl semimetal is characterized by the anomalous Hall conductivity

$$\sigma_{xy} = \frac{e^2}{h}\frac{2Q}{2\pi} = \frac{1}{2\pi}\frac{2Q}{2\pi}. \quad (3)$$

This may be expressed as a topological term in the effective action for probe electromagnetic gauge fields when the fermions are integrated out [56]:

$$\mathcal{L} = i\frac{2Q}{8\pi}\epsilon_{z\nu\alpha\beta}A_\nu \partial_\alpha A_\beta. \quad (4)$$

In its primitive form above, Eq. (4) does not actually look like a topological term, since it explicitly contains a preferred direction in space (z) and depends on a nonuniversal microscopic lattice constant d through the Weyl node separation $2Q$.

To fix these issues, it proves useful to introduce the concept of a translation gauge field [36,52–55]. Recall that Bravais lattice points $\mathbf{R}$ of a perfect crystal may be described as intersections of families of crystal planes, perpendicular to primitive reciprocal lattice vectors $\mathbf{b}^i$, where $i = 1, 2, 3$ (or x, y, z for a cubic crystal). Mathematically, this is expressed

by the equation

$$\theta^i(\mathbf{r}, t) = \mathbf{b}^i \cdot \mathbf{r} = 2\pi n^i, \qquad (5)$$

where n^i are sets of integers, labeling the crystal planes in a family i and the Bravais lattice vectors $\mathbf{r} = \mathbf{R}$ are the solutions of this equation. Equation (5) implies that the reciprocal lattice vectors in a perfect crystal may be expressed as gradients of the phases $b^i_j = \partial_j \theta^i$. This may be generalized to a distorted crystal, including time-dependent distortions, by introducing translation "gauge fields"

$$e^i_\mu = \frac{1}{2\pi} \partial_\mu \theta^i. \qquad (6)$$

The fields e^i_μ may in fact be viewed as true (strictly speaking, integer valued) gauge fields, if one explicitly takes account of the fact that the phases θ^i on crystal planes may be relabeled in arbitrary $2\pi \times \mathbb{Z}$ increments [54,55]. This will not make a significant difference in our case and either viewpoint is acceptable.

In a convenient differential form language, we may view e^i as a one-form

$$e^i = e^i_\mu dx^\mu. \qquad (7)$$

In a crystal without dislocations,

$$de^i = \tfrac{1}{2}(\partial_\mu e^i_\nu - \partial_\nu e^i_\mu) dx^\mu \wedge dx^\nu = 0, \qquad (8)$$

as clearly follows from the definition Eq. (6). On the other hand, if a dislocation with a Burgers vector along $\mathbf{b}^i$ is present, the integral of e^i around a loop, enclosing the dislocation line, is $\oint e^i = 1$.

The benefit of introducing translation gauge fields becomes apparent if we now replace a reciprocal lattice vector along the z direction in Eq. (4) with the corresponding translation gauge field

$$\frac{2\pi}{d} \delta^z_\mu \rightarrow 2\pi e^z_\mu. \qquad (9)$$

Then Eq. (4) becomes

$$\mathcal{L} = i\frac{\lambda}{4\pi} \epsilon_{\mu\nu\alpha\beta} e^z_\mu A_\nu \partial_\alpha A_\beta = i\frac{\lambda}{4\pi} e^z \wedge A \wedge dA, \qquad (10)$$

where $\lambda = 2Q/(2\pi/d)$ is a dimensionless separation between the Weyl nodes in units of the reciprocal lattice vector. Now Eq. (10) looks like a proper topological term, which only contains gauge fields and a universal coefficient. The nonuniversal and variable lattice constant d has been absorbed into the definition of the translation gauge field and we will henceforth set $d = 1$ for simplicity. Since we are using the imaginary-time formulation, upper and lower indices do not need to be distinguished and we will use lower indices for space-time components of the gauge fields throughout. Varying Eq. (10) with respect to e^z_z produces a response per atomic xy plane, which is determined by a universal numerical coefficient λ. A noninteger value of the coefficient λ requires gapless modes in the form of a pair of Weyl nodes to be present [36,37], since a fractional value (in units of e^2/h) of the Hall conductance per atomic plane is impossible in a noninteracting gapped insulator.

B. 3D analog of the Pfaffian-antisemion state

To derive the field theory of the gapped 3D FQH state of Ref. [20] we first move to a dual description of the noninteracting Weyl semimetal of Eq. (1), in which the electric charge is separated from the fermions and is represented in terms of a two-form gauge potential, which couples to the vortex loop excitations [21,57,58]. This approach is similar to what is known as "functional bosonization" [59–62], apart from unimportant technical details. We start by representing the fermion operators in Eq. (1) (after Fourier transforming them to real space) as

$$\psi_r = e^{i\theta_r} f_r, \qquad (11)$$

where r labels the sites of a cubic lattice, $e^{i\theta_r}$ represents a spinless charged boson (chargon), and f_r is a neutral fermion (spinon). After straightforward and standard manipulations [21,63,64], one obtains the following exact representation of the Weyl semimetal Lagrangian $\mathcal{L}$:

$$\mathcal{L} = \mathcal{L}_f + \mathcal{L}_b, \qquad (12)$$

where $\mathcal{L}_f$ is the Lagrangian of the spinons f_r, which has a form identical to the lattice Lagrangian of the original electrons ψ_r, except that f_r are coupled to a compact statistical gauge field a_μ rather than the probe electromagnetic field A_μ. The statistical field expresses $U(1)$ gauge invariance of the parton decomposition Eq. (11) and serves the purpose of gluing together the spinons and the chargons. The chargon Lagrangian has the form

$$\mathcal{L}_b = \frac{i}{4\pi}(A_\mu - a_\mu)\epsilon_{\mu\nu\alpha\beta}\Delta_\nu b_{\alpha\beta} + \frac{1}{8\pi^2\chi}(\epsilon_{\mu\nu\alpha\beta}\Delta_\nu b_{\alpha\beta})^2. \qquad (13)$$

Here $b_{\mu\nu} = -b_{\nu\mu}$ is a two-form $2\pi \times \mathbb{Z}$ valued lattice gauge field, which represents integer chargon currents J_μ as

$$J_\mu = \frac{1}{4\pi}\epsilon_{\mu\nu\alpha\beta}\Delta_\nu b_{\alpha\beta}. \qquad (14)$$

Δ_μ is a lattice derivative and χ is a positive constant. Lattice site indices r have been suppressed everywhere for brevity.

To avoid dealing with discrete variables, we may implement the $2\pi\mathbb{Z}$ constraint on $b_{\mu\nu}$ by adding a term $\frac{i}{2}\tilde{J}_{\mu\nu}b_{\mu\nu}$ to $\mathcal{L}_b$ and summing over integer-valued variables $\tilde{J}_{\mu\nu}$, which have the meaning of vortex-loop currents. Gauge invariance of Eq. (14) with respect to a transformation $b_{\mu\nu} \rightarrow b_{\mu\nu} + \Delta_\mu g_\nu - \Delta_\nu g_\mu$ implies a conservation law

$$\Delta_\mu \tilde{J}_{\mu\nu} = 0, \qquad (15)$$

which may be solved as

$$\tilde{J}_{\mu\nu} = \frac{1}{2\pi}\epsilon_{\mu\nu\alpha\beta}\Delta_\alpha c_\beta, \qquad (16)$$

where c_μ are $2\pi\mathbb{Z}$ valued one-form gauge fields. The constraint on c_μ may, in turn, be softened by adding a term $-t\cos(\Delta_\mu\phi + c_\mu)$, where the presence of a new compact angular variable ϕ takes account of the gauge invariance of Eq. (16) with respect to $c_\mu \rightarrow c_\mu + \Delta_\mu\phi$. In essence, the particle created by $e^{i\phi}$ is the original chargon.

Then, after taking the continuum limit, the chargon Lagrangian takes the dual form

$$\mathcal{L}_b = \frac{i}{4\pi}(A_\mu - a_\mu + c_\mu)\epsilon_{\mu\nu\alpha\beta}\partial_\nu b_{\alpha\beta} + \cdots, \quad (17)$$

where $(\cdots)$ contains both the higher-derivative terms for $b_{\mu\nu}$ and the additional terms for c_μ whose form depends on the value of the parameter t. In particular, when t is large, the $e^{i\phi}$ boson is condensed, leading to a mass term for c_μ (i.e., gap for vortices), which may then be ignored. Integration over $b_{\mu\nu}$ then simply sets as $A_\mu = a_\mu$, i.e., the electric charge is reattached to the spinons and we recover the original Weyl semimetal. In contrast, when t is small, the $e^{i\phi}$ particle is gapped, which leads to a Maxwell term, $(\epsilon\partial c)^2$, for the gauge field c_μ. In this case, integration over c_μ produces a mass term b^2 for the two-form gauge field, which corresponds to a charge gap. This state is a Mott insulator, which has gapless spinons that retain the Weyl semimetal band structure.

To obtain a fully gapped state, which preserves topological response of the Weyl semimetal, Eq. (10), and does not break any symmetries, we place the spinons into a paired state. For weak pairing, only the intranodal pairing state opens a gap [65–68]. Such a pairing generally breaks translational symmetry, except when $2Q = \pi$ or $\lambda = 1/2$ [20], to which we now specialize. With such an intranodal pairing term, the spinon Hamiltonian may be brought to the form

$$H = \frac{1}{2}\sum_{\mathbf{k}} f_{\mathbf{k}}^\dagger \{\sigma_x \sin(k_x) + \sigma_y \sin(k_y)$$
$$+ [\sqrt{\Delta^2 + \cos^2(k_z)} - \tilde{m}[2 - \cos(k_x) - \cos(k_y)]]\sigma_z\} f_{\mathbf{k}}, \quad (18)$$

where Δ is the pairing amplitude. This Hamiltonian describes a 3D topological p-wave superconductor, which has a chiral Majorana mode, spanning the full extent of the BZ. This may also be viewed as a stack of 2D topological superconductors, since the pairing gap does not close at any value of k_z.

The spinon pairing produces a term $\propto -\cos(2a_\mu)$ for the statistical gauge field, which leaves only $a_\mu = 0, \pi \mod 2\pi$ possible values at low energies and makes it a $\mathbb{Z}_2$ gauge field. While nontrivial π-flux configurations of a_μ (visons [69]) are still possible, these may be easily shown to bind a gapless one-dimensional Majorana mode in their cores, which is a direct consequence of the fact that the spinon superconductor is topologically nontrivial. This means that in any fully gapped symmetry-preserving state such vison loop excitations must be gapped, which means that we may set $a_\mu = 0 \mod 2\pi$ at low energies. This detaches the boson and fermion sectors of the theory. The fermion sector thus contributes the same thermal Hall response as the noninteracting Weyl semimetal at $\lambda = 1/2$, which arises from the chiral Majorana mode, spanning the full BZ. The electrical response must entirely come from the boson sector.

In order to reproduce the electrical response of the noninteracting Weyl semimetal, it is necessary to condense double (i.e., flux 4π) vortices of the boson field $e^{i\theta}$. This is accomplished by the following modification of the field theory,

Eq. (17):

$$\mathcal{L}_b = \frac{i}{4\pi}(A_\mu + 2c_\mu)\epsilon_{\mu\nu\alpha\beta}\partial_\nu b_{\alpha\beta} + \frac{2i}{4\pi}\epsilon_{\mu\nu\alpha\beta}e_\mu^z c_\nu \partial_\alpha c_\beta$$
$$+ \frac{1}{2g}(\epsilon_{\mu\nu\alpha\beta}\partial_\alpha c_\beta)^2 + \frac{i}{2}b_{\mu\nu}\tilde{j}_{\mu\nu} + ic_\mu j_\mu. \quad (19)$$

The extra factor of 2 in front of c_μ, compared to Eq. (17), expresses the fact that double (flux 4π) vortices are being condensed. This also means that the quasiparticle, which is minimally coupled to the gauge field c_μ, carries a charge $1/2$. The second term is a topological term, which will give rise to the correct electrical Hall conductivity, as will be shown below. This term may be viewed as describing a layered integer quantum Hall state of the charge-$1/2$ bosonic quasiparticles. The third term is the Maxwell term. It is subdominant to the topological term at long wavelengths, but has been included explicitly since the topological term only contains components of c_μ, transverse to the translation gauge field e^z. In particular, if $e_\mu^z = \delta_\mu^z$, c_z does not enter into the topological term and its dynamics is governed by the Maxwell term. Finally, j_μ and $\tilde{j}_{\mu\nu}$ represent bosonic quasiparticle and vortex source currents, which are minimally coupled to the gauge fields c_μ and $b_{\mu\nu}$ correspondingly.

Let us now demonstrate that Eq. (19) indeed reproduces topological response of a noninteracting Weyl semimetal. Let us set $\tilde{j}_{\mu\nu} = 0$ and integrate out $b_{\mu\nu}$. This gives $c_\mu = -A_\mu/2$. Substituting this back into Eq. (19), we obtain

$$\mathcal{L}_b = \frac{i}{8\pi}\epsilon_{\mu\nu\alpha\beta}e_\mu^z A_\nu \partial_\alpha A_\beta - \frac{i}{2}A_\mu j_\mu. \quad (20)$$

The first term in Eq. (20) correctly reproduces the electrical Hall conductivity of a noninteracting Weyl semimetal with $\lambda = 1/2$, which is half conductance quantum $\sigma_{xy} = 1/4\pi$ per atomic plane. The second term tells us that quasiparticle excitations in the gapped state, described by Eq. (19), are bosons with electric charge $1/2$. To establish the gapped nature of this state it is important to note the following. If we reinsert the statistical gauge field a_μ into Eq. (19), it is clear that fluctuations of $b_{\mu\nu}$ effectively constrain $c_\mu = (a_\mu - A_\mu)/2$. This implies that, since a_μ is made a $\mathbb{Z}_2$ gauge field by spinon pairing, c_μ becomes a discrete $\mathbb{Z}_4$ gauge field. This is important, since, unlike in $2 + 1$ dimensions, a $(3 + 1)$-dimensional Maxwell-Chern-Simons theory with $U(1)$ gauge fields is gapless [70,71].

The most straightforward way to see that this theory also correctly captures the thermal Hall conductivity $\kappa_{xy} = 0$ is to consider the boundary theory, which corresponds to Eq. (19). To derive the boundary theory we follow the standard method [2]. We choose a gauge, in which on the boundary, taken to be in the xz plane, we set the temporal components of all the gauge fields to zero, i.e., $c_0 = 0$, $b_{0\mu} = 0$. Then, integrating out c_0, we obtain

$$\epsilon_{0\nu\lambda\rho}\partial_\nu b_{\lambda\rho} = \epsilon_{0\nu\lambda\rho}\partial_\nu(e_\lambda^z c_\rho - e_\rho^z c_\lambda), \quad (21)$$

while integrating $b_{0\nu}$ gives

$$\epsilon_{0\nu\lambda\rho}\partial_\lambda c_\rho = 0. \quad (22)$$

Equations (21) and (22) along with $de^z = 0$ imply that

$$\epsilon_{0\nu\lambda\rho}\partial_\nu b_{\lambda\rho} = 0. \quad (23)$$

Equation (23) may then be solved as

$$b_{ij} = \partial_i g_j - \partial_j g_i, \qquad (24)$$

where $i, j = x, z$ refer to spatial coordinates on the boundary, while Eq. (22) is solved as

$$c_i = \partial_i \varphi. \qquad (25)$$

Plugging this back into what remains of Eq. (19) after integrating out c_0 and $b_{0\mu}$, we obtain

$$\mathcal{L}_b = \frac{i}{2\pi} \epsilon_{0\nu\lambda\rho} e_\nu^z \partial_\lambda \varphi \partial_\tau \partial_\rho \varphi - \frac{i}{\pi} \epsilon_{0\nu\lambda\rho} \partial_\nu \varphi \partial_\tau \partial_\lambda g_\rho, \qquad (26)$$

where $\partial_\tau \equiv \partial_0$. Integrating this in the presence of a boundary, perpendicular to the y direction, gives

$$\mathcal{L}_{surf} = \frac{i}{2\pi} \epsilon_{ij} e_i^z \partial_\tau \partial_j \varphi - \frac{i}{\pi} \epsilon_{ij} \partial_\tau \varphi \partial_i g_j, \qquad (27)$$

where $i, j = x, z$. Adding symmetry-allowed nontopological terms and the electromagnetic field, we finally obtain the following surface state Lagrangian:

$$\mathcal{L}_{surf} = \frac{i}{2\pi} \epsilon_{ij} e_i^z \partial_\tau \varphi \partial_j \varphi - \frac{i}{\pi} \epsilon_{ij} \partial_\tau \varphi \partial_i g_j + \frac{v_\varphi}{2\pi}(\partial_i \varphi)^2$$
$$+ \frac{v_g}{2\pi}(\partial_i g_j - \partial_j g_i)^2 + \frac{i}{2\pi} \epsilon_{\mu\nu\lambda} A_\mu \partial_\nu g_\lambda. \qquad (28)$$

Setting $e_\mu^z = \delta_\mu^z$ and Fourier transforming, we obtain the following expression for the excitation spectrum of the surface modes:

$$\epsilon(\mathbf{k}) = -\frac{v_g k_x}{2} + \sqrt{\left(\frac{v_g^2}{4} + v_g v_\varphi\right)k_x^2 + v_g v_\varphi k_z^2}. \qquad (29)$$

This looks like an ordinary anisotropic 2D superfluid dispersion, except for a "tilt" in the x direction due to the first term. However, the dispersion is still nonchiral, since there is always a pair of left- and right-handed modes for every value of k_z. Consequently, a straightforward calculation gives a vanishing thermal Hall conductivity in this state:

$$\kappa_{xy} \sim \int dk_x dk_z v_x(\mathbf{k}) \epsilon(\mathbf{k}) \frac{\partial n_B[\epsilon(\mathbf{k})]}{\partial T} = 0, \qquad (30)$$

where $v_x(\mathbf{k}) = \frac{\partial \epsilon(\mathbf{k})}{\partial k_x}$ and $n_B(\epsilon)$ is the Bose-Einstein distribution. The integral over k_x in Eq. (30) vanishes since the left-handed ($k_x < 0$) and right-handed ($k_x > 0$) modes give a contribution that is equal in magnitude but opposite in sign.

By construction, this state is a fully gapped symmetric state, which has an identical topological response to a noninteracting Weyl semimetal at $\lambda = 1/2$. Note again that, while there does exist a close connection between this state and the 2D Pfaffian-antisemion state, it may not be viewed as a simple stack of such 2D states. In particular, there are no semion quasiparticles, but isolated intersections of 2π vortex-loop excitations with atomic xy planes do behave as semions.

C. 3D analog of the PH-Pfaffian state

Now we note that a second distinct gapped symmetric state, reproducing topological response of a noninteracting Weyl semimetal, actually exists. This state is, in a way, simpler than the 3D analog of the Pfaffian-antisemion above and, as we will

demonstrate, may be viewed as a 3D analog of the PH-Pfaffian [39,41,44,45].

To construct this state, we take a time-reversed copy of our Weyl semimetal with $\lambda = 1/2$. Writing its Lagrangian in terms of spinon and chargon variables, we have

$$\mathcal{L} = \bar{f}\gamma_\mu(\partial_\mu + ia_\mu)f - \frac{i}{8\pi}e^z \wedge a \wedge da + \frac{i}{4\pi}(A - a) \wedge db, \qquad (31)$$

where the first term is the contribution of the gapless Weyl fermions while the second term is the topological contribution from all the filled negative-energy states. We will switch to the index-free notation henceforth. We now place the chargons into a stack of independent $\nu = 1/2$ quantum Hall states in each xy-atomic plane. Technically, this means that we take the two-form gauge field b to be "foliated" [72–75]

$$b = e^z \wedge \tilde{b}, \qquad (32)$$

where $\tilde{b} = \tilde{b}_0 d\tau + \tilde{b}_x dx + \tilde{b}_y dy$ is a one-form field that lacks the z component, and add a term $-\frac{2i}{4\pi}e^z \wedge \tilde{b} \wedge d\tilde{b}$ to the Lagrangian equation (31). Furthermore, we place the spinons into the intranodal pairing state of Eq. (18), which leads to a 3D $p + ip$ topological superconductor with a chiral Majorana mode, spanning the surface BZ, whose chirality is, however, opposite to the chirality of the Fermi-arc state of the original noninteracting Weyl semimetal. This gaps out the gauge field a_μ and decouples the boson and fermion sectors.

The boson sector Lagrangian now reads

$$\mathcal{L}_b = -\frac{2i}{4\pi}e^z \wedge \tilde{b} \wedge d\tilde{b} + \frac{i}{2\pi}e^z \wedge A \wedge d\tilde{b}. \qquad (33)$$

Integrating over $\tilde{b}$ leaves the effective action

$$\mathcal{L}_b = \frac{i}{8\pi}e^z \wedge A \wedge dA, \qquad (34)$$

which describes topological electrical response, which is identical to that of the original [i.e., not the time-reversed one of Eq. (31)] noninteracting Weyl semimetal, Eq. (10). The thermal Hall effect, coming from $\mathcal{L}_b$, is twice that of the noninteracting Weyl semimetal; however, minus a half is contributed by the opposite-chirality Majorana surface state of the paired time-reversed spinons. Thus we fully reproduce both electrical and thermal topological responses of the original noninteracting gapless Weyl semimetal.

This state may be viewed as a 3D generalization of the 2D PH-Pfaffian state. Note that, unlike the 3D analog of the Pfaffian-antisemion state, described above, this state is not a 3D incompressible liquid, but exhibits a fracton-type order [72–75]. If we ignore fermions, Eq. (33) describes a stack of independent 2D PH-Pfaffian states. The charge-1/2 anyon excitations in these 2D states are only able to move within a given plane and cannot tunnel between the planes. Neutral fermions propagate in 3D and connect the individual layers together, but the anyons remain confined within 2D layers.

III. "DUAL" WEYL SEMIMETAL

The existence of a 3D analog of the PH-Pfaffian has an important implication, which we will now discuss. Let us first return back to the 3D Pfaffian-antisemion state. Let us note

that, in this case, the topological response of a noninteracting Weyl semimetal is only reproduced when the fermionic spinons are gapped by pairing and vison vortex-loop excitations are gapped. If the pairing gap is taken to zero, the statistical field a is no longer massive and its coupling to the gapless Weyl spinons produces a topological term $\frac{i}{8\pi}e^z \wedge a \wedge da$, so that the Lagrangian may be written as

$$\mathcal{L} = \bar{f}\gamma_\mu(\partial_\mu + ia_\mu)f + \frac{i}{8\pi}e^z \wedge a \wedge da$$
$$+ \frac{i}{4\pi}(A - a + 2c) \wedge db + \frac{i}{2\pi}e^z \wedge c \wedge dc. \quad (35)$$

Integrating out b and c gives

$$\mathcal{L} = \bar{f}\gamma_\mu(\partial_\mu + ia_\mu)f + \frac{i}{4\pi}e^z \wedge a \wedge da$$
$$- \frac{i}{4\pi}e^z \wedge A \wedge da + \frac{i}{8\pi}e^z \wedge A \wedge dA. \quad (36)$$

To obtain the electromagnetic response, we now integrate out a. This may be done perturbatively, treating the response of the gapless low-energy modes, i.e., the first term in Eq. (36), as a perturbation, compared to the second term. This is possible because the response of the gapless modes, treated in the random-phase approximation (RPA), is given by

$$S_f = \frac{1}{2}\sum_q a_\mu(q)\Pi_{\mu\nu}(q)a_\nu(-q), \quad (37)$$

where

$$\Pi_{\mu\nu}(q) = (q^2\delta_{\mu\nu} - q_\mu q_\nu)f(q^2) \quad (38)$$

is the polarization operator of the massless 3D Dirac fermion and

$$f(q^2) = \frac{1}{12\pi^2}\ln\left(\frac{4\Lambda^2}{q^2}\right) + O(1). \quad (39)$$

Here $\Lambda \gg q$ is the cutoff momentum, and a convention $q_0 = -\Omega$ is used (Ω is the Matsubara frequency). Note that $\Pi_{\mu\nu}(q)$ is almost the same as the polarization operator of the massive 3D Dirac fermion, in which case $f(q^2)$ would have been a constant at small q. Even with the log nonanalyticity, $\Pi_{\mu\nu}(q)$ is still much smaller, in the long-wavelength limit, than the topological contributions, which are of first order in q.

At leading order we may then ignore the gapless fermions and vary the Lagrangian with respect to a, which gives at the saddle point $a = A/2$ and leaves the Lagrangian

$$\mathcal{L} = \bar{f}\gamma_\mu(\partial_\mu + iA_\mu/2)f + \frac{i}{16\pi}e^z \wedge A \wedge dA, \quad (40)$$

which clearly corresponds to half of the Hall conductivity of a noninteracting Weyl semimetal, i.e., the theory with gapless spinons does not reproduce topological response of the noninteracting Weyl semimetal.

In contrast, let us return to Eq. (31), which describes a time-reversed Weyl semimetal, and add to it the foliated topological term of Eq. (33), without opening the spinon pairing gap:

$$\mathcal{L} = \bar{f}\gamma_\mu(\partial_\mu + ia_\mu)f - \frac{i}{8\pi}e^z \wedge a \wedge da$$
$$+ \frac{i}{2\pi}e^z \wedge (A - a) \wedge d\tilde{b} - \frac{2i}{4\pi}e^z \wedge \tilde{b} \wedge d\tilde{b}. \quad (41)$$

Integrating out $\tilde{b}$ now, we obtain

$$\mathcal{L} = \bar{f}\gamma_\mu(\partial_\mu + ia_\mu)f - \frac{i}{4\pi}e^z \wedge A \wedge da + \frac{i}{8\pi}e^z \wedge A \wedge dA. \quad (42)$$

This has identical electrical and thermal Hall responses to the original noninteracting Weyl semimetal. This means that Eq. (41) describes a distinct gapless state, which reproduces the topological response of a noninteracting Weyl semimetal. This statement is very closely analogous to the statement of duality between a noninteracting 2D Dirac fermion and QED$_3$ [44,46–51]. However, note that, in contrast to the 2D Dirac duality case, dynamically this system is quite different from a noninteracting Weyl semimetal. Indeed, integrating out f and then a in Eq. (42) using RPA produces a Meissner term for the components of A, transverse to z. The coefficient of the Meissner term, however, vanishes in the long-wavelength limit [it is equal to the inverse of the function $f(q^2)$, introduced in Eq. (39)]. The system thus behaves as a superconductor at finite length scales and in directions transverse to z, but with a phase stiffness that vanishes in the thermodynamic limit. In contrast, it behaves as an insulator along z.

IV. TOPOLOGICAL ORDER IN A GAPPED NODAL LINE SEMIMETAL

We will now extend the ideas, developed above, to the case of nodal line semimetals, which realize a distinct kind of (3 + 1)-dimensional topological order, when gapped without breaking the protecting symmetries. In the nodal line semimetals, only nodal lines which arise from touchings of pairs of nondegenerate bands, are topologically nontrivial. In this case, TR symmetry may be taken to be broken, while the nodal line is then protected by the mirror reflection symmetry in the plane, containing the line [76]. This may be described by the following two-band cubic-lattice Hamiltonian [77,78]:

$$\mathcal{H}(\mathbf{k}) = [6 - t_1 - 2(\cos k_x + \cos k_y + \cos k_z)]\sigma_x$$
$$+ 2t_2 \sin k_z\sigma_y. \quad (43)$$

The nodal line in this model appears in the xy plane of the momentum space and is protected by the mirror reflection symmetry within this plane, where the mirror reflection operator is σ_x. The band-touching line in the xy plane is given by the solution of the equation

$$4 - t_1 - 2(\cos k_x + \cos k_y) = 0. \quad (44)$$

In order construct a gapped symmetric state, it is useful to reinterpret Eq. (43) as a stacking of alternating electron- and holelike Fermi liquids with the band dispersions (see Fig. 1)

$$\epsilon_\pm(\mathbf{k}) = \pm[4 - t_1 - 2(\cos k_x + \cos k_y)], \quad (45)$$

where $\pm$ are the two eigenvalues of the mirror reflection operator σ_x [37]. The Luttinger volumes of the two Fermi liquids $\pm V_F$ are equal in magnitude to the area in momentum space, enclosed by the nodal line. For the two Fermi liquids, the topological response describes the filling of the charged particles

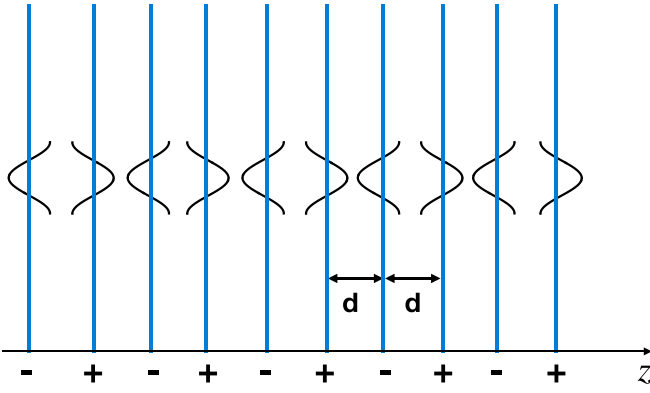$$\mathcal{L} = \pm\frac{iV_F}{4\pi^2}e^x \wedge e^y \wedge A. \quad (46)$$

FIG. 1. Construction of the nodal line semimetal as a stack (z is the stacking direction) of alternating coupled electron- and hole-like Fermi liquids, indicated schematically by their dispersions. The Luttinger volume of each 2D Fermi liquid is equal in magnitude to the area in momentum space, enclosed by the nodal line. The lattice constant d is set equal to unity in all the equations.

Consequently, the topological response of the nodal line takes the form of fractional electric polarization [37,79,80]

$$\mathcal{L} = \pm \frac{iV_F}{8\pi^2} e^x \wedge e^y \wedge dA, \tag{47}$$

impossible in an ordinary insulator without topological order.

The simplest way to obtain a gapped mirror-symmetric insulator with the same topological response, Eq. (47), is to stack gapped 2D Fermi-liquid states in a mirror-symmetric fashion. To gap the 2D Fermi liquids, we follow the same procedure as above. We represent an electron as a product of a neutral spinon f and a bosonic chargon $e^{i\theta}$ and place the spinons into a fully gapped paired state. The simplest fully gapped paired spinon state is p wave (since the Fermi liquids are spinless), described by the following Hamiltonian:

$$H_f = \sum_{\mathbf{k}} \left[\epsilon_{\pm}(\mathbf{k}) f_{\mathbf{k}}^{\dagger} f_{\mathbf{k}} + \frac{\Delta}{2} (\sin k_x + i \sin k_y) f_{\mathbf{k}}^{\dagger} f_{-\mathbf{k}}^{\dagger} + \text{H.c.} \right]. \tag{48}$$

Introducing Nambu spinor notation $\psi_{\mathbf{k}} = (f_{\mathbf{k}}, f_{-\mathbf{k}}^{\dagger})$, this may be represented as a massive 2D Dirac Hamiltonian

$$H = \frac{1}{2} \sum_{\mathbf{k}} \psi_{\mathbf{k}}^{\dagger} [\epsilon_{\pm}(\mathbf{k})\tau_z + \Delta(\tau_x \sin k_x - \tau_y \sin k_y)] \psi_{\mathbf{k}}, \tag{49}$$

where τ_a are Pauli matrices in the particle-hole space. This is the Hamiltonian of a Read-Green topological superconductor [81], which hosts chiral Majorana modes at the edges, with opposite chirality for electron- and holelike Fermi-liquid states. Consequently, an elementary flux $hc/2e = \pi$ vortex hosts a zero-energy localized Majorana bound state and cannot be condensed.

To condense higher-flux vortices, we need to consider the chargon sector of the theory. Suppose we attempt to condense flux-2π vortices. The chargon sector will be described by the following theory [82,83]:

$$\mathcal{L}_b = \frac{i}{2\pi}(A + c) \wedge db \pm \frac{iV_F}{(2\pi)^2} e^x \wedge e^y \wedge c. \tag{50}$$

Here b is a one-form gauge field, which determines the charge current

$$J_{\mu} = \frac{1}{2\pi} \epsilon_{\mu\nu\lambda} \partial_{\nu} b_{\lambda}, \tag{51}$$

while c is a one-form gauge field, which determines the vortex current

$$\tilde{J}_{\mu} = \frac{1}{2\pi} \epsilon_{\mu\nu\lambda} \partial_{\nu} c_{\lambda}. \tag{52}$$

The last term of the Lagrangian produces the correct electromagnetic response of a system with charge $\nu = \pm V_F/(2\pi)^2$ per unit cell when b is integrated out, setting $c = -A$. However, when the filling ν is not an integer, Eq. (50) cannot be the correct theory of a featureless insulator since the last term is not gauge invariant. With $\nu = \pm p/q$, a featureless insulator may be obtained only by condensing flux $2\pi q$ vortices, which is described by the theory

$$\mathcal{L}_b = \frac{i}{2\pi}(A + qc) \wedge db \pm ipe^x \wedge e^y \wedge c, \tag{53}$$

where all terms now have properly quantized integer coefficients and are gauge invariant. This is because the quasiparticle, minimally coupled to c_{μ}, carries charge $1/q$, as seen from the first term. Therefore, the filling of such quasiparticles is $q\nu = p$ (i.e., an integer), which is what the second term expresses.

Stacking such insulators with alternating sign of ν in the z direction in a mirror-symmetric fashion, we obtain

$$\mathcal{L}_b = \frac{i}{4\pi}(A + qc) \wedge db \pm \frac{ip}{2} e^x \wedge e^y \wedge dc, \tag{54}$$

where the factor of $1/2$ in front of the last term arises due to the fact that the unit cell of the stack contains a pair of electron- and holelike 2D Fermi liquids and the mirror symmetry requires that all neighboring 2D Fermi liquids in the stack are separated by an equal distance. The gauge field b in Eq. (54) has now been promoted to a two-form field, such that the $(3 + 1)$-dimensional charge current is given by

$$J_{\mu} = \frac{1}{4\pi} \epsilon_{\mu\nu\alpha\beta} \partial_{\nu} b_{\alpha\beta}, \tag{55}$$

while the two-form vortex current is

$$\tilde{J}_{\mu\nu} = \frac{1}{2\pi} \epsilon_{\mu\nu\alpha\beta} \partial_{\alpha} c_{\beta}. \tag{56}$$

Integrating out b in Eq. (54) gives $c = -A/q$ and the electromagnetic response described by

$$\mathcal{L} = \pm \frac{ip}{2q} e^x \wedge e^y \wedge dA = \pm \frac{iV_F}{8\pi^2} e^x \wedge e^y \wedge dA, \tag{57}$$

which coincides with Eq. (47). Thus we obtain a featureless insulator with topological order, which has an identical topological response to a weakly interacting nodal line semimetal. Note that the nodal line semimetal has no topological thermal response, which is also the case in the fractionalized insulator that we have constructed.

V. DISCUSSION AND CONCLUSIONS

In this paper we have presented a theory of $(3 + 1)$-dimensional topologically ordered states, obtained by gapping

3D topological semimetals without breaking protecting symmetries. We started by pointing out that a second gapped symmetric topologically ordered state, preserving the chiral anomaly of magnetic Weyl semimetals, exists, in addition to the state, originally proposed in Ref. [20]. We have shown that, while the state of Ref. [20] may be viewed as a 3D TR breaking analog of the Pfaffian-antisemion state in gapped 3D TI surface states, the new state is the 3D analog of the PH-Pfaffian. In contrast to the 3D Pfaffian-antisemion state, the 3D PH-Pfaffian does not exhibit a true 3D topological order, but a fractonlike order instead, with independent layers of 2D PH-Pfaffian liquid immersed in a 3D $p + ip$ topological superconductor of neutral composite fermions.

We then demonstrated that an interesting consequence of the existence of the 3D PH-Pfaffian, is a duality relation between a noninteracting Weyl semimetal and QED$_4$, in which a time-reversed electrically neutral Weyl semimetal is coupled to a dynamical gauge field, whose topological defects (intersections of flux lines with atomic planes) carry the electric charge. This duality relation may be viewed as a 3D generalization of the known Dirac fermion to QED$_3$ duality relation, but is weaker than in the 2D case, since the duality only applies to the topological response and not to the dynamics.

Finally, we have extended the theory to include topological orders in a gapped nodal line semimetal. Other extensions, in particular to TR invariant Weyl and Dirac semimetals, are also possible, but do not lead to any fundamentally new structure. One lesson we may highlight is that gapped symmetric topological semimetals provide a very simple and natural setting for $(3 + 1)$-dimensional topologically ordered states to appear. The simplicity stems, in part, from the fact that, due to the existence of a preferred direction, selected by either the separation between the Weyl points in momentum space, or the plane of the nodal line, there exists a natural connection to well-studied $(2 + 1)$-dimensional topological orders. The connection manifests either directly, in the form of layered fractonlike order, or less directly, when intersections of $(3 + 1)$-dimensional vortex-loop excitations with atomic planes behave as fractionally charged and sometimes anyonic $(2 + 1)$-dimensional quasiparticle excitations.

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