Spin diffusion in spin glasses requires two magnetic variables, $\vec{M}$ and $\vec{m}$

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Experiment has established that spin glasses can support a steady-state spin current $\vec{j}_i$. However, the accepted theory of spin-glass dynamics permits oscillations but no steady-state spin current. Onsager's irreversible thermodynamics implies that the spin current is proportional to the gradient of a magnetization. We argue, however, that the magnon distribution function associated with the local equilibrium magnetization $\vec{M}$ cannot diffuse because it represents 10^{23} variables. We therefore invoke the nonequilibrium magnetization $\vec{m}$, which in spintronics is called the *spin accumulation*. Applying the theory of irreversible thermodynamics, we indeed find that it predicts spin diffusion, and we consider other experimental consequences of the theory, including a wavelength-dependent coupling between the reactive and the diffusive degrees of freedom.

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I. INTRODUCTION

Spintronics, i.e., control and manipulation of spin degrees of freedom [1,2], especially spin currents, has received great attention in recent years. Various magnetic materials including paramagnets, ferromagnets, and antiferromagnets has been used to realize spintronics. The present work considers spintronics in spin glasses, which dates to 2011. At that time Iguchi *et al.* studied spin current absorption in the spin glass AgMn using the spin pumping method, and proposed spin current injection as a method to investigate spin-glass properties [3].

More recently, Wessenberg *et al.* presented evidence for steady-state spin currents traversing an amorphous sample of the magnetic insulator yttrium iron garnet (YIG) [4]. In their work, x-ray diffraction on a 200-nm sputtered sample indicated no medium-range or long-range order. From this it was inferred that the system was polycrystalline and perhaps, for length scales larger than 100 nm, effectively a spin glass [5]. For these thin films they found spin currents much larger than for crystalline YIG. This experiment has motivated us to develop a theory for spin currents in spin glasses.

Unfortunately, further experiment is needed in this area, such as—all other variables fixed—a study of the dependence of the spin current on sample thickness. For dc spin current such a study has been performed on an antiferromagnet, which like a spin glass has no net magnetization. That study found two characteristic decay lengths [6]. (Reference [7] uses a kinetic theory to discuss spin diffusion for insulating antiferromagnets.) For a spin glass we expect that, due to its isotropic nature, there would be only one characteristic decay length.

At the microscopic level we expect that in a spin glass the equilibrium spin orientations vary rapidly from site to site, due to the frustrating effect of competing exchange interactions. This is illustrated explicitly in the simulations of Walker and Walstedt [8,9]. Previous theories of the macroscopic dynamics of spin glasses included as macroscopic variables the magnetization $\vec{M}$ and an orientation $\vec{\theta}$ [10,11]. The variable $\vec{\theta}$ represents a local rotation of the noncollinear magnetic order within the spin glass. For a spin-glass sample with no magnetization in zero external field H (i.e., no remanence), the local spins tip to give a local net magnetization $\vec{M}$. However, these theories give no spin current $\vec{j}_i$, where i is a spatial index. Note that a spin glass repeatedly prepared in the same field $\vec{H}$ and the same sample orientation (thus the same local anisotropies) will in each case go into distinct microscopic states. However, all of these microscopic states will have the same macroscopic properties. In equilibrium we take $\vec{\theta} = \vec{0}$.

The present work applies the spintronics idea [12,13] that spin currents are associated with the nonequilibrium magnetization $\vec{m}$ known as the *spin accumulation*. To our knowledge there is no previous theoretical work on spin currents (i.e., spin diffusion) in spin glasses, and the present work is directed at filling this gap to provide a framework to understand the experimental results in Ref. [4].

As background, Sec. II discusses the spin accumulation $\vec{m}$, Sec. III discusses the energy density when $\vec{m}$ is included, and Sec. IV discusses the corresponding thermodynamics. Section V uses irreversible thermodynamics to obtain the equations of motion. As a check, Sec. VI obtains the pure spin waves, where $\vec{m}$ is neglected, and Sec. VII obtains the pure spin diffusion modes, where $\vec{M}$ and $\vec{\theta}$ are neglected. Section VIII discusses the fully coupled modes, and Sec. IX discusses the weakly coupled modes, where either a spin wave is accompanied by a small amplitude for spin diffusion, or where spin diffusion is accompanied by a small amplitude spin wave. Section X presents a summary and discussion that relates the present work, for insulating spin glasses, to spin glasses with a conductor like Cu that has been doped with Mn. An Appendix discusses the Ruderman-Kittel-Kasuya-

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Yosida (RKKY) interaction in both nuclear and electron-spin systems.

II. SPIN ACCUMULATION

For magnetic insulators with magnon modes labeled by α, in local equilibrium we write the thermal occupation number $n_\alpha^{l.e.}$. A deviation from equilibrium δn_α of the thermal occupation number n_α can give a local value for the nonequilibrium magnetization, which we identify with $\vec{m}$ rather than $\vec{M}$. $\vec{M}$ may correspond to some equilibrium state of the system, although perhaps not for the actual $\vec{H}$, as can happen if $\vec{H}$ is rapidly changed. This approach permits a near-equilibrium spin glass to support spin diffusion in terms of a diffusive nonreactive magnetic variable, $\vec{m}$, while permitting $\vec{M}$ to remain a nondiffusive reactive variable.

Translated back into the language of site localized spins, we distinguish between the local value of the equilibrium part of the spin $\vec{S}_i$ and the nonequilibrium part of the spin $\vec{s}_i$. Summing the former over spins gives the local equilibrium $\vec{M}$ and summing over the latter gives the local nonequilibrium $\vec{m}$. Even before the term *spintronics* was coined, the idea of a local nonequilibrium $\vec{m}$ was employed by Dyakonov and Perel to predict, for semiconductors with a spin-orbit interaction, what later became known as the spin Hall effect and the inverse spin Hall effect [14,15]. They used the term *accumulation of spin*.

We recently applied the idea of spin accumulation $\vec{m}$ to the longitudinal magnetization M of a ferromagnet. There we argued that spin diffusion of M from position A to position B takes place indirectly, first at A via conversion of M to m, then by diffusion of m from A to B, and finally by conversion at B of m to M [16]. We also argued, implicitly, that any nonconserved order parameter (call it Y; it could be a superconducting order parameter), being encoded in a statistical distribution function of 10^{23} variables, is unable to diffuse, and therefore diffusion of Y from A to B occurs indirectly, first at A by conversion from Y to the "Y accumulation" y, then by diffusion of y from A to B, and finally by conversion at B from y to Y.

It is essential to recognize that spin glasses, with no magnetization and isotropic macroscopic properties, have a different symmetry than do ferromagnets. Ferromagnets have a single longitudinal magnetic variable and two transverse magnetic variables. On the other hand, spin glasses have three rotationally equivalent magnetic variables. In the absence of spin diffusion and anisotropy, ferromagnets have a decaying, nonpropagating longitudinal mode, and two degenerate transverse modes with quadratic dependence on the wave vector k. On the other hand, under the same conditions spin glasses have three degenerate modes with a linear dependence on k. This is much more like that for antiferromagnets, which, however, have a more complex symmetry than do spin glasses. In part for that reason the present work does not consider spin diffusion in antiferromagnets.

The simplest system supporting a spin current is a paramagnet, with spin current proportional to the gradient of the nonequilibrium magnetization $\vec{m}$ [12,17–19]. Both a spin diffusion constant D and a spin relaxation time τ are present, and give a characteristic and dissipative spin decay length $l_s \sim (D\tau)^{1/2}$. In the end, for a spin glass with $\vec{M}$, $\vec{\theta}$, and in addition $\vec{m}$, we find a similar decay length, but complicated by dimensionless factors related to ratios of additional decay times and ratios of the susceptibilities of $\vec{M}$ and $\vec{m}$.

III. ENERGY DENSITY

We consider that a spin glass is described by a set of local atomic magnet orientations that, when only exchange interactions are included, can be uniformly rotated by an infinitesimal angle $\vec{\theta}$ to produce an inequivalent microscopic state with no change in energy [8–10]. For a given microscopic configuration, when weak anisotropy interactions are included the spins will locally reorient slightly to give a local minimum that we will characterize by $\vec{\theta} = \vec{0}$.

If cooled in zero external field (zfc), the system will develop no magnetization, and when a nonzero $\vec{H}$ is applied, in equilibrium it will develop a *magnetization* $\vec{M}$ with

$$\vec{M} = \chi_M \vec{H} \quad \text{(zfc)}, \tag{1}$$

where χ_M is the susceptibility of $\vec{M}$.

If a spin glass is prepared in a finite external field $\vec{H}_{fc}$ (field cooling), then on removing $\vec{H}_{fc}$ there is a remanent magnetization M_0 along $\vec{H}_c$. When an additional field $\vec{H}$ is applied along $\vec{H}_{fc}$ we have $\vec{M}$ along $\vec{H}_{fc}$. Therefore $\vec{M}_0$ and $\vec{H}$ are collinear, so we may write the scalar equation

$$M = M_0 + \chi_M H. \tag{2}$$

A three-dimensional anisotropy (see below), due to single ion anisotropy or Dzyaloshinsky-Moriya anisotropy [20–22], tends to align $\vec{M}_0$ with the direction of $\vec{H}_{fc}$, and defines a three-dimensional (3D) baseline set of coordinates relative to which the spin can be rotated. For small rotations we employ the 3D angle $\vec{\theta}$. Compared to the saturation magnetization, M_0 typically is small, so that the system is only slightly affected by it.

A. Energy density with $\vec{M}$ and $\vec{\theta}$

In the absence of $\vec{m}$, following previous work we employ variables $\vec{M}$ and $\vec{\theta}$. We assume an energy density of the form

$$\frac{\varepsilon_0}{\mu_0} = \frac{(\vec{M} - \vec{M}_0)^2}{2\chi_M} - \vec{M} \cdot \vec{H} + \frac{K}{2}\vec{\theta}^{\,2} + \frac{\rho_s}{2}(\partial_i \vec{\theta})^2, \tag{3}$$

where μ_0 is the vacuum permeability. The first term permits a frozen-in remanent magnetization (common in spin glasses) and the second term is the Zeeman energy. The third term, with anisotropy constant K, represents the macroscopic anisotropy energy, which is minimized for $\vec{\theta} = \vec{0}$. The last term, with spin stiffness coefficient ρ_s, represents the increase in microscopic exchange energy for a *nonuniform* rotation $\vec{\theta}$.

In the absence of K, to produce a twist in $\vec{\theta}$ alone would require the spin analog of what, in numerical simulations for atoms, the distinguished computational physicist Aneesur Rahman called "ether pegs" [23]. With "spin ether pegs" to twist the spin texture, the new twisted equilibrium state will have characteristic twist length $l_n = (\rho_s/K)^{1/2}$. This nondissipative length does not relate to the dissipative spin decay

length, associated with both spin decay and spin diffusion, that we will find below.

B. Energy density with $\vec{m}$

When spin accumulation is included, we append the additional energy density

$$\frac{\varepsilon_1}{\mu_0} = \frac{m^2}{2\chi_m} - \vec{m}\cdot\vec{H} - \lambda_M \vec{m}\cdot(\vec{M}-\vec{M}_0), \tag{4}$$

where χ_m is the susceptibility of $\vec{m}$. Here the first term has its origin in exchange, and the second term is a Zeeman interaction. The last term, involving the effective exchange constant λ_M, is of a statistical nature. Enforcing the equilibrium conditions $\vec{M}_{eq} = \vec{M}_0 + \chi_M \vec{H}$ and $\vec{m}_{eq} = \vec{0}$ will constrain the exchange constant λ_M.

IV. THERMODYNAMICS

We now give the thermodynamic differential energy density $\varepsilon = \varepsilon_0 + \varepsilon_1$ more generally, in terms of the appropriate thermodynamic variables. We have, with entropy density s and temperature T,

$$d\varepsilon = T\,ds - \mu_0 \vec{H}^*\cdot d\vec{M} - \mu_0 \vec{h}^*\cdot d\vec{m} - \vec{\Gamma}\cdot d\vec{\theta} - \vec{\Gamma}_i \cdot d(\partial_i\vec{\theta}), \tag{5}$$

where $\vec{H}^*$, $\vec{h}^*$, and $\vec{\Gamma}_i$ are thermodynamic conjugates (effective fields) associated with $\vec{M}$, $\vec{m}$, and $\partial_i\vec{\theta}$. When (3) applies, $\vec{\Gamma} = -K\vec{\theta}$, and $\vec{\Gamma}_i = -\rho_s\partial_i\vec{\theta}$.

We will require that the conditions $\vec{H}^*_{eq} = \vec{0}$ and $\vec{h}^*_{eq} = \vec{0}$ recover the equilibrium conditions $\vec{M} = \vec{M}_0 + \chi_M\vec{H}$ and $\vec{m} = \vec{0}$.

The variational derivative of the energy with respect to $\vec{\theta}$ defines the net torque density

$$\vec{\Gamma}' \equiv \vec{\Gamma} - \partial_i\vec{\Gamma}_i, \tag{6}$$

and from (3), we have

$$\vec{\Gamma}' = -K\vec{\theta} + \rho_s\nabla^2\vec{\theta}. \tag{7}$$

This permits us to simplify (5) to, on neglecting a pure divergence term,

$$d\varepsilon = T\,ds - \mu_0\vec{H}^*\cdot d\vec{M} - \mu_0\vec{h}^*\cdot d\vec{m} - \vec{\Gamma}'\cdot d\vec{\theta}. \tag{8}$$

A. The effective field $\vec{H}^*$

We define the effective field $\vec{H}^*$ by

$$\vec{H}^* \equiv -\frac{\partial}{\partial\vec{M}}\frac{\varepsilon}{\mu_0} = \vec{H} - \frac{\vec{M}-\vec{M}_0}{\chi_M} + \lambda_M\vec{m}. \tag{9}$$

$\vec{H}^*_{eq} = \vec{0}$ is consistent with the equilibrium conditions.

We now introduce the magnetization deviation from local equilibrium, $\delta\vec{M}$, via

$$\delta\vec{M} \equiv -\chi_M\vec{H}^* = \vec{M} - \vec{M}_0 - \chi_M(\vec{H} + \lambda_M\vec{m}). \tag{10}$$

B. The effective field $\vec{h}^*$

We define the effective field $\vec{h}^*$ by

$$\vec{h}^* \equiv -\frac{\partial}{\partial\vec{m}}\frac{\varepsilon}{\mu_0} = \vec{H} - \frac{\vec{m}}{\chi_m} + \lambda_M(\vec{M}-\vec{M}_0). \tag{11}$$

We can make this consistent with the equilibrium conditions if we take

$$\lambda_M = -\frac{1}{\chi_M}, \tag{12}$$

so

$$\vec{h}^* \equiv -\frac{\partial}{\partial\vec{m}}\frac{\varepsilon}{\mu_0} = \vec{H} - \frac{\vec{m}}{\chi_m} - \frac{(\vec{M}-\vec{M}_0)}{\chi_M}. \tag{13}$$

We now introduce the spin accumulation deviation from local equilibrium, $\delta\vec{m}$, via

$$\delta\vec{m} \equiv -\chi_m\vec{h}^* = \vec{m} - \frac{\chi_m}{\chi_M}(\vec{M}-\vec{M}_0-\chi_M\vec{H}). \tag{14}$$

C. Deviations from local equilibrium

For completeness we rewrite $\vec{H}^*$ and $\delta\vec{M}$ as

$$\vec{H}^* = \vec{H} - \frac{\vec{M}-\vec{M}_0}{\chi_M} - \frac{\vec{m}}{\chi_M}, \tag{15}$$

$$\delta\vec{M} = \vec{M} - \vec{M}_0 - \chi_M\vec{H} + \vec{m}. \tag{16}$$

With (12) and the definition

$$\xi \equiv \frac{\chi_m}{\chi_M}, \tag{17}$$

we may write, in defining $\Delta\vec{m}$ and $\Delta\vec{M}$,

$$\delta\vec{m} \equiv -\chi_m\vec{h}^* = \vec{m} + \frac{\chi_m}{\chi_M}(\vec{M}-\vec{M}_0-\chi_M\vec{H}) \equiv \Delta\vec{m} + \xi\Delta\vec{M}, \tag{18}$$

$$\delta\vec{M} \equiv -\chi_M\vec{H}^* = (\vec{M}-\vec{M}_0-\chi_M\vec{H}) + \vec{m} \equiv \Delta\vec{M} + \Delta\vec{m}. \tag{19}$$

For uniform systems the equilibrium conditions $\vec{H}^* = \vec{0}$ and $\vec{h}^* = \vec{0}$ yield equations like those for two interpenetrating and interacting paramagnets, so for $\vec{H} \neq \vec{0}$ both $\vec{M} \neq \vec{0}$ and $\vec{m} \neq \vec{0}$.

V. IRREVERSIBLE THERMODYNAMICS AND EQUATIONS OF MOTION

We now employ Onsager's irreversible thermodynamics to study the dynamics of this system, including dissipation. For that we first write down the conservation laws and equations of motion for the thermodynamic variables. We then require that, if (5) holds initially, then it holds in the future, subject to the nondecreasing nature of s. We take $\vec{M}$ to be nondiffusive because it represents an equilibrium distribution. However, $\vec{m}$ can have a diffusive flux term.

This system has various unknown dissipative fluxes j and dissipative sources R. We assume that

(a) energy has only a flux j_i^ε;

(b) entropy has a flux j_i^s and a source $R_s \geqslant 0$;

(c) $\vec{M}$ has only a source $\vec{R}_M$ (it has no flux because we believe that a macroscopic variable should not be able to diffuse), and is driven by torque both from the effective field and from the anisotropy, or $-\gamma\vec{M}\times\vec{H}^* - \gamma\vec{\Gamma}'$;

(d) $\vec{m}$ has a flux $\vec{j}_i^m$ and a source $\vec{R}_m$, and is driven by $-\gamma\vec{m}\times\vec{h}^*$, with no lattice torque $\vec{\Gamma}'$ analogous to $\vec{\Gamma}$; and

(e) $\vec{\theta}$ has no flux but a source $\vec{R}_\theta$ and is driven by an unknown $\vec{\omega}$. We expect that $\vec{\omega} = \gamma\mu_0\vec{H}^*$, because the atomic moments in a spin glass precess in the effective field.

Thus, with gyromagnetic ratio due to electrons (properly as $-\gamma < 0$) we take

$$\partial_t \varepsilon + \partial_i j_i^\varepsilon = 0, \tag{20}$$

$$\partial_t s + \partial_i j_i^s = R_s \geqslant 0, \tag{21}$$

$$\partial_t \vec{M} = -\gamma\mu_0 \vec{M} \times \vec{H}^* - \gamma\vec{\Gamma}' + \vec{R}_M, \tag{22}$$

$$\partial_t \vec{m} + \partial_i \vec{j}_i^m = -\gamma\mu_0 \vec{m} \times \vec{h}^* + \vec{R}_m, \tag{23}$$

$$\partial_t \vec{\theta} = \vec{\omega} + \vec{R}_\theta. \tag{24}$$

Using (8) and the above equations gives

$$\begin{aligned} -\partial_i j_i^\varepsilon = {} & -T\partial_i j_i^s + TR_s - \mu_0\vec{H}^* \cdot \vec{R}_M - \mu_0\vec{h}^* \cdot \vec{R}_m \\ & + \mu_0\vec{h}^* \cdot \partial_i \vec{j}_i^m - \vec{\Gamma}' \cdot (\vec{\omega} - \gamma\mu_0\vec{H}^* + \vec{R}_\theta). \end{aligned} \tag{25}$$

Then

$$\begin{aligned} 0 \leqslant TR_s = {} & -\partial_i\left[j_i^\varepsilon - Tj_i^s + \mu_0\vec{h}^* \cdot \vec{j}_i^m\right] - j_i^s\partial_i T \\ & + \mu_0\vec{H}^* \cdot \vec{R}_M + \mu_0\vec{h}^* \cdot \vec{R}_m + \mu_0\vec{j}_i^m \cdot \partial_i\vec{h}^* \\ & + \vec{\Gamma}' \cdot (\vec{\omega} - \gamma\mu_0\vec{H}^* + \vec{R}_\theta). \end{aligned} \tag{26}$$

For R_s to be non-negative, we take

(a) the entropy flux to be

$$j_i^s = -\frac{\kappa}{T}\partial_i T, \tag{27}$$

where $\kappa \geqslant 0$ is the thermal conductivity, with units of entropy density diffusion constant;

(b) the $\vec{M}$ and $\vec{m}$ source terms to satisfy

$$\vec{R}_M = -\frac{\delta\vec{M}}{\tau_M} + \frac{\delta\vec{m}}{\tau_{mM}} = \frac{\chi_M}{\tau_M}\vec{H}^* - \frac{\chi_m}{\tau_{mM}}\vec{h}^*, \tag{28}$$

$$\vec{R}_m = -\frac{\delta\vec{m}}{\tau_m} + \frac{\delta\vec{M}}{\tau_{Mm}} = \frac{\chi_m}{\tau_m}\vec{h}^* - \frac{\chi_M}{\tau_{Mm}}\vec{H}^*, \tag{29}$$

with self-explanatory positive relaxation times;

(c) the magnetization flux to come only from diffusion of $\vec{m}$, as in

$$\vec{j}_i^m = \frac{D_m}{\chi_m}\partial_i\vec{h}^* = -D_m\partial_i\delta\vec{m}, \tag{30}$$

where $D_m \geqslant 0$ is a diffusion coefficient, with units of velocity times distance;

(d) the reactive part of the $\dot{\vec{\theta}}$ driving term to be

$$\vec{\omega} = \gamma\mu_0\vec{H}^*; \tag{31}$$

(e) the dissipative part of the $\dot{\vec{\theta}}$ driving term to be dissipative

$$\vec{R}_\theta = \alpha\gamma\vec{\Gamma}', \tag{32}$$

where $\alpha \geqslant 0$ is a dissipation coefficient, with units of inverse magnetization.

We also note the Onsager relation between cross-decay rates, which can be obtained by, in TR_s, equating the cross terms in (28) and (29):

$$\frac{\chi_m}{\tau_{mM}} = \frac{\chi_M}{\tau_{Mm}}. \tag{33}$$

VI. PURE SPIN WAVES: NEGLECT $\vec{m}$

In the absence of K, $\vec{H}$, dissipation, and $\vec{m}$, by (7) we have $\vec{\Gamma}' = \rho_s\nabla^2\vec{\theta}$. Further, by (19) we have $\vec{H}^* = -\Delta\vec{M}/\chi_M$, so

$$\partial_t\vec{M} = -\gamma\rho_s\nabla^2\vec{\theta}, \tag{34}$$

$$\partial_t\vec{\theta} = \gamma\mu_0\left(-\frac{\Delta\vec{M}}{\chi_M}\right). \tag{35}$$

At fixed $\vec{M}_0$ and fixed $\vec{H}$, $\Delta\vec{M} = \vec{M}$, so combining these gives

$$\partial_t^2\vec{M} = \frac{\gamma^2\mu_0}{\chi_M}\rho_s\nabla^2\vec{M}. \tag{36}$$

With $\vec{M} \sim e^{i(kx-\omega t)}$ we then have

$$\omega = vk, \quad v \equiv \left(\frac{\gamma^2\mu_0\rho_s}{\chi_M}\right)^{1/2}, \tag{37}$$

the spin-wave mode given in Ref. [10]. This might appear in an experiment by imposing a field gradient with wave vector k that oscillates at variable ω until there is a resonance.

If $K \neq 0$, then we may replace K by $K' = K + \rho_{ss}k^2$. In that case we have

$$\omega = \left(\omega_0^2 + v^2k^2\right)^{1/2}, \quad \omega_0 \equiv \gamma\left(\frac{\mu_0 K}{\chi_M}\right)^{1/2}. \tag{38}$$

This applies if ω_0 is much larger than the relaxation time for K [24–26]. If ω_0 is much smaller than the relaxation time for K, then (37) applies.

VII. PURE SPIN DIFFUSION: NEGLECT $\vec{M}$ AND $\vec{\theta}$

If a spin current $\vec{j}_i$ enters the system, and $\vec{M}$ and $\vec{\theta}$ can be neglected, then (23), (29), and (30) (on dropping the nonlinear term $\vec{m} \times \vec{h}^*$, and the term $\delta\vec{M}$), reduce to

$$\partial_t\vec{m} - D_m\nabla^2\vec{m} = -\frac{1}{\tau_m}\vec{m}. \tag{39}$$

For the dc case we expect that $\vec{m} \sim e^{-qx}$, where the solution of the above equation gives $q^{-1} = l_{SG} = (D_m\tau_m)^{1/2}$. It would be of interest to test this by studying spin flow through samples of different thicknesses.

VIII. FULLY COUPLED MODES

If a field gradient is added, then $\vec{M}$ should develop a gradient, which might twist $\vec{\theta}$. Let us consider how this might appear in an experiment.

For $\vec{H} = \vec{0} = \vec{M}_0$, by (18) and (19) we have

$$\delta\vec{M} = \vec{M} + \vec{m}, \quad \delta\vec{m} = \xi\vec{M} + \vec{m}. \tag{40}$$

Then, using (22), (28), and (15), and neglecting precession,

$$\partial_t\vec{M} = -\gamma\vec{\Gamma}' - \frac{\delta\vec{M}}{\tau_M} + \frac{\delta\vec{m}}{\tau_{mM}}; \tag{41}$$

using (24), (31), (32), and (16),

$$\partial_t \vec{\theta} = \gamma \mu_0 \left(-\frac{\delta \vec{M}}{\chi_M} \right) + \alpha \gamma \vec{\Gamma}'; \tag{42}$$

and using (23), (29), and (13),

$$\partial_t \vec{m} - D_m \nabla^2 \delta \vec{m} = -\frac{\delta \vec{m}}{\tau_m} + \frac{\delta \vec{M}}{\tau_{Mm}}. \tag{43}$$

Equation (7) should be used for $\vec{\Gamma}'$.

Consider the dc case, which is relevant for low-frequency studies using lock-in detectors. Then (41) and (42) can be used to eliminate $\vec{\Gamma}'$ and to obtain a linear relationship between $\delta \vec{M}$ and $\delta \vec{m}$, with a positive proportionality constant. Then (43) can be used to obtain an equation for $\vec{m}$, with a decay rate reduced from τ_m^{-1}. Despite the three equations, there is only a single doubly degenerate diffusion mode, with a modified spin decay length l_{SG}. (At finite frequency there also will be a nondiffusive mode that primarily involves $\delta \vec{M}$ and $\vec{\theta}$.)

Note that the anisotropy K can vary slowly with time [24]. In studying a dc spin current, we may take $K \to 0$. Then, setting the time derivatives to zero in (41) and (42) permits us to eliminate $\vec{\Gamma}'$ and to relate $\vec{M}$ and $\vec{m}$. Placing this in (43) then gives an effective wave-vector squared q^2 for the diffusion mode, with $\vec{M}$ and $\vec{m}$ related. $\vec{\Gamma}'$ then determines $\vec{\theta}$ because $\vec{\Gamma}'$ is related to $\vec{M}$ and $\vec{m}$, and includes the now-known wave vector q^2.

Using (40), we now rewrite (41), (42), and (43),

$$\partial_t \vec{M} = -\gamma \vec{\Gamma}' - \frac{\vec{M} + \vec{m}}{\tau_M} + \frac{\xi \vec{M} + \vec{m}}{\tau_{mM}}, \tag{44}$$

$$\partial_t \vec{\theta} = -\gamma \mu_0 \frac{\vec{M} + \vec{m}}{\chi_M} + \alpha \gamma \vec{\Gamma}', \tag{45}$$

$$\partial_t \vec{m} - D_m \nabla^2 (\xi \vec{M} + \vec{m}) = -\frac{\xi \vec{M} + \vec{m}}{\tau_m} + \frac{\vec{M} + \vec{m}}{\tau_{Mm}}, \tag{46}$$

where

$$\vec{\Gamma}' = -K \vec{\theta} + \rho_s \nabla^2 \vec{\theta}, \tag{47}$$

and $\vec{M}$ should be replaced by $\Delta \vec{M} = \vec{M} - \vec{M}_0 - \chi_M \vec{H}$ if $\vec{M}_0 - \chi_M \vec{H} \neq 0$.

We now assume that this physical system is subject to space and time variations of the form $e^{i(kx-\omega t)}$. This can be done either by driving the system at frequency ω, to which the system responds at some wave vector k that must be determined, or by turning on a disturbance at a fixed k, to which the system responds in time at some ω (which may be complex) that must be determined.

We will be particularly interested in fixed ω, and determining k. To that purpose it will be helpful to define

$$K' \equiv K + \rho_s k^2. \tag{48}$$

We also introduce

$$\frac{1}{\tau_M} \equiv \frac{1}{\tau_{ML}} + \frac{1}{\tau_{Mm}}, \quad \frac{1}{\tau_m} \equiv \frac{1}{\tau_{mL}} + \frac{1}{\tau_{mM}}, \tag{49}$$

$$r \equiv \frac{1}{\tau_{mM}} - \frac{1}{\tau_{Mm}}, \quad \xi \equiv \frac{\tau_{mM}}{\tau_{Mm}}. \tag{50}$$

With these terms defined, the equations of motion can be rewritten in matrix form as

$$-i\omega \begin{pmatrix} M \\ m \\ \theta \end{pmatrix} = \Gamma \begin{pmatrix} M \\ m \\ \theta \end{pmatrix}, \tag{51}$$

where the matrix Γ is

$$\Gamma = \begin{pmatrix} -\frac{1}{\tau_{ML}} & -\frac{1}{\tau_{ML}} + r & \gamma K' \\ -\xi \left(Dk^2 + \frac{1}{\tau_{mL}} \right) & -Dk^2 - \frac{1}{\tau_{mL}} - r & 0 \\ -\frac{\gamma \mu_0}{\chi_M} & -\frac{\gamma \mu_0}{\chi_M} & -\alpha \gamma K' \end{pmatrix}. \tag{52}$$

We will solve Eqs. (51), which define an eigenvalue problem, using perturbation theory around the propagating spin-wave mode and about the diffusive mode.

IX. WEAKLY COUPLED MODES

In practice we expect the modes of this system to be neither noninteracting nor strongly coupled. Rather we expect one mode to be primarily propagating and one mode to be primarily diffusive. The goal of this section is to determine the extent to which the secondary degree of freedom of each mode is coupled into the primary degree of freedom, as in perturbation theory.

A. Spin-wave M-θ mode

The zeroth-order spin-wave mode is obtained by setting $m = 0$. Then

$$\partial_t \vec{M} = -\gamma \vec{\Gamma}' - \frac{\vec{M}}{\tau_{ML}}, \tag{53}$$

$$\partial_t \vec{\theta} = -\gamma \mu_0 \frac{\vec{M}}{\chi_M} + \alpha \gamma \vec{\Gamma}'. \tag{54}$$

Assuming the variation $e^{ikx-\omega t}$, we have

$$-i\omega \begin{pmatrix} M \\ \theta \end{pmatrix} = \begin{pmatrix} -\frac{1}{\tau_{ML}} & \gamma K' \\ -\frac{\gamma \mu_0}{\chi_M} & -\alpha \gamma K' \end{pmatrix} \begin{pmatrix} M \\ \theta \end{pmatrix}. \tag{55}$$

The equation determining the eigenvalues is

$$\det \begin{pmatrix} -\frac{1}{\tau_{ML}} + i\omega & \gamma K' \\ -\frac{\gamma \mu_0}{\chi_M} & -\alpha \gamma K' + i\omega \end{pmatrix} = -\omega^2 - i\omega \left(\frac{1}{\tau_{ML}} + \alpha \gamma K' \right) + \left(\frac{\alpha}{\tau_{ML}} + \frac{\gamma \mu_0}{\chi_M} \right) \gamma K' = 0, \tag{56}$$

whose solutions are

$$\omega = -\frac{i}{2} \left(\frac{1}{\tau_{ML}} + \alpha \gamma K' \right) \pm \sqrt{-\frac{1}{4} \left(\frac{1}{\tau_{ML}} - \alpha \gamma K' \right)^2 + \frac{\gamma^2 \mu_0 K'}{\chi_M}}. \tag{57}$$

We will consider the situation where ω is given. We then write K' in terms of ω:

$$K' = \frac{i\omega\left(-\frac{1}{\tau_{ML}} + i\omega\right)}{\gamma\left[\alpha\left(-\frac{1}{\tau_{ML}} + i\omega\right) - \frac{\gamma\mu_0}{\chi_M}\right]}. \tag{58}$$

Substituting this into the equation for M, we get

$$\left(-\frac{1}{\tau_{ML}} + i\omega\right)M + \gamma\frac{i\omega\left(-\frac{1}{\tau_{ML}} + i\omega\right)}{\gamma\left[\alpha\left(-\frac{1}{\tau_{ML}} + i\omega\right) - \frac{\gamma\mu_0}{\chi_M}\right]}\theta = 0, \tag{59}$$

so the unperturbed modes satisfy

$$\theta = -\left[\alpha\left(-\frac{1}{\tau_{ML}} + i\omega\right) - \frac{\gamma\mu_0}{\chi_M}\right]M. \tag{60}$$

We next use the unperturbed solution of M in the equation of m. We have

$$\left[-\xi\left(Dk^2 + \frac{1}{\tau_{mL}}\right)\right]M + \left[-\left(Dk^2 + \frac{1}{\tau_{mL}} + r\right) + i\omega\right]m = 0. \tag{61}$$

Using $k^2 = (K' - K)/\rho_s$, we can find m in terms M and ω:

$$m = \frac{\xi\left[\frac{D}{\rho_s}(K' - K) + \frac{1}{\tau_{mL}}\right]}{-\left[\frac{D}{\rho_s}(K' - K) + \frac{1}{\tau_{mL}} + r\right] + i\omega}M, \tag{62}$$

with K' given in (58) in terms of ω. For this perturbation calculation to be valid, the values of the material parameters and ω should make the proportionality coefficient in (62) much smaller than 1.

B. Diffusion mode from *m*

We next consider the situation where the amplitude m is much larger than those of M and θ. The unperturbed mode then corresponds to the solution of the equation on $\partial_t m$ and $\partial_t\theta$, with M set to zero:

$$\partial_t\vec{m} - D_m\nabla^2\vec{m} = -\frac{\vec{m}}{\tau_m} + \frac{\vec{m}}{\tau_{Mm}}. \tag{63}$$

Substituting in the wave form of the solutions, we have

$$\left(-Dk^2 - \frac{1}{\tau_{mL}} - r + i\omega\right)m = 0. \tag{64}$$

We will consider the situation where ω is given. We then write k^2 in terms of ω:

$$k^2 = \frac{1}{D}\left(-\frac{1}{\tau_{mL}} - r + i\omega\right). \tag{65}$$

We thus have

$$K' = \rho_0 + \frac{\rho_s}{D}\left(-\frac{1}{\tau_{mL}} - r + i\omega\right). \tag{66}$$

We next use the unperturbed solution of m in the equation of M and θ. We have

$$\left(-\frac{1}{\tau_{ML}} + i\omega\right)M + \gamma K'\theta = -\left(-\frac{1}{\tau_{ML}} + r\right)m, \tag{67}$$

$$\left(-\frac{\gamma\mu_0}{\chi_M}\right)M + (-\alpha\gamma K' + i\omega)\theta = \frac{\gamma\mu_0}{\chi_M}m, \tag{68}$$

whose solution is

$$M = -\frac{\left(-\frac{1}{\tau_{ML}} + r\right)(-\alpha\gamma K' + i\omega) + \frac{\gamma^2\mu_0}{\chi_M}K'}{\left(-\frac{1}{\tau_{ML}} + i\omega\right)(-\alpha\gamma K' + i\omega) + \frac{\gamma^2\mu_0}{\chi_M}K'}m, \tag{69}$$

$$\theta = \frac{(-r + i\omega)\frac{\gamma\mu_0}{\chi_M}}{\left(-\frac{1}{\tau_{ML}} + i\omega\right)(-\alpha\gamma K' + i\omega) + \frac{\gamma^2\mu_0}{\chi_M}K'}m, \tag{70}$$

with K' given in (66) in terms of ω. For this perturbation calculation to be valid, the values of the material parameters and ω should make the proportionality coefficients in these two equations much smaller than 1.

The total magnetization $M + m$ for this perturbation solution, which is what would be measured, is given by

$$M + m = \left(1 + \frac{M}{m}\right)m = \left[1 - \frac{\left(-\frac{1}{\tau_{ML}} + r\right)(-\alpha\gamma K' + i\omega) + \frac{\gamma^2\mu_0}{\chi_M}K'}{\left(-\frac{1}{\tau_{ML}} + i\omega\right)(-\alpha\gamma K' + i\omega) + \frac{\gamma^2\mu_0}{\chi_M}K'}\right]m, \tag{71}$$

with K' given in (66).

X. SUMMARY AND DISCUSSION

We have developed the theory of spin diffusion in spin glasses, finding it necessary to invoke the spin accumulation $\vec{m}$, due to a nonequilibrium distribution of excitations. Although prompted by experiments, at the moment there is a need for further experiments in order to compare with the general predictions of the theory, and to determine parameters appearing in the theory.

The present theory was developed with insulators in mind. A metallic spin glass, such as Cu doped with Mn, is more complex [5]. There the randomly located Mn has spin-glass order, with exchange between two localized Mn spins at irregular positions having irregular sign. This exchange is believed to be due to the RKKY interaction, which is mediated by the Cu host conduction electrons that communicate between the two Mn's (see the Appendix). In addition, however, each Mn spin polarizes the bath of Cu electrons in its vicinity, thus making the host Cu a (likely weak) itinerant spin glass that responds to the Mn spin order.

For a spin current to propagate through such a system, there must be a spin current in both the Cu and the Mn. Spin currents in Cu are carried by delocalized conduction electrons by means of a spatially varying nonequilibrium spin-distribution function. Spin currents in Mn are carried by localized Mn by means of a spatially varying nonequilibrium magnon-distribution function. This is a rather complex situation, likely requiring at the microscopic level a theory for the spin current that is spatially averaged over both the Cu and the Mn. That is well beyond the scope of the present work.

However, the present macroscopic theory serves the purpose of describing macroscopic spin currents even in CuMn. It does not consider the possibility of a charge current, as can occur in the conductor CuMn, as opposed to the insulator YIG.

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APPENDIX: ON THE RKKY INTERACTION

The RKKY interaction was initially developed for collinear nuclear spins interacting indirectly with conduction electrons. In order to obtain a mechanism for the line *broadening* observed for *nuclei* in nuclear magnetic resonance, Ruderman and Kittel considered indirect exchange mediated by the hyperfine interaction between conduction electrons and nuclei [27]. It took the form $A_{ij}\vec{I}_i \cdot \vec{I}_j$ between nuclear spins $\vec{I}_n$. With k_F the host Fermi wave vector and R_{ij} the separation between two nuclei, they found an oscillatory and inverse power-law dependence of the exchange constant A_{ij} on $2k_F R_{ij}$. They noted that although exchange *narrowing* will occur for pure isotopic samples, for naturally occurring isotopic mixtures such an interaction (being inhomogeneous) can lead to the observed line *broadening*.

Kasuya, noting Zener's phenomenology on the interaction of core electrons (d states) via the s-d interaction with s-state conduction electrons [28], then modified the Ruderman and Kittel approach to treat d-d electron interactions, now mediated by the s-d interaction [29]. Yosida applied these ideas to CuMn alloys [30]. noting that "the experimental results on the electronic g-value of the Mn ions and the Knight shift of the Cu-nuclei can be qualitatively accounted for." Later experiments on CuMn alloys at low temperatures indicated a complex magnetic structure. Van Vleck provided a clarifying discussion of theoretical details [31], and referred to an earlier paper by Fröhlich and Nabarro [32] that contains an exchange interaction between nuclear spins with no dependence on their relative position. Van Vleck also cites additional papers where Zener [33] expands on his earlier work [28].

It was later found that when the Mn spins are permitted to be noncollinear, the spatially oscillating interaction between randomly placed Mn impurities minimizes the Mn energy for a seemingly random noncollinear spin structure called a spin glass [34].

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