Theorem on extensive spectral degeneracy for systems with rigid higher symmetries in general dimensions

Zohar Nussinov [1,2,3,*] and Gerardo Ortiz [4]

[1] *Rudolf Peierls Centre for Theoretical Physics, University of Oxford, Oxford OX1 3PU, United Kingdom*
[2] *LPTMC, CNRS-UMR 7600, Sorbonne Universite, 4 Place Jussieu, 75252 Paris cedex 05, France*
[3] *Department of Physics, Washington University, St. Louis, Missouri 63160, USA*
[4] *Department of Physics, Indiana University, Bloomington, Indiana 47405,USA*

(Received 5 October 2022; revised 7 December 2022; accepted 22 December 2022; published 9 January 2023)

We establish, in the spirit of the Lieb-Schultz-Mattis theorem, lower bounds on the spectral degeneracy of quantum systems with higher (gaugelike) symmetries with rather generic physical boundary conditions in an arbitrary number of spatial dimensions. Contrary to applying twists or equivalent adiabatic operations, we exploit the effects of modified boundary conditions. When a general choice of boundary geometry is immaterial in determining spectral degeneracies while approaching the thermodynamic limit, systems that exhibit rigid noncommuting gaugelike symmetries, such as the orbital compass model, must have an exponential (in the size of the boundary) degeneracy *of each of their spectral levels*.

DOI: 10.1103/PhysRevB.107.045109

I. INTRODUCTION

In the current work, we extend earlier results concerning the degeneracy of quantum systems (principally, those relying on the Lieb-Schultz-Mattis theorem [1] and its higher dimensional generalizations [2–4]) to models with rigid "higher symmetries" that exhibit an exponential [in the linear ($d = 1$) system size, area ($d = 2$), or higher ($d > 2$) dimensional volume] spectral degeneracy [4–14]. Known arguments for establishing degeneracies (including, principally, the Lieb-Schultz-Mattis theorem) employ geometries with generalized Bloch type boundary conditions [15]. While, as we will explain, the proof of exponential degeneracy for systems exhibiting rigid higher symmetries is rather trivial in classical (large spin or similar) limits, this is not the case for quantum systems.

In our efforts to establish the exponential spectral degeneracy that these quantum systems feature, we will follow an approach different from that of the known proofs of the Lieb-Schultz-Mattis theorem and its extensions. Our proof will, instead, rely on altering the boundary geometry of the system from that of a pristine hypercubic (or other) lattice with (Born-von Karman or Bloch type [16]) periodic boundary conditions to the very same lattice having an exterior boundary that is not cleaved so perfectly as to be symmetric along all Cartesian directions. For such general "real life" boundaries, we will rigorously establish a degeneracy that is exponential in the size of the boundary. As we will spell out in some detail, this modified boundary geometry may be changed by the inclusion of terms forming a "zipper Hamiltonian." When added to the existing interactions (i.e., the system with generic non-Bloch type boundary conditions for which we may rigorously prove exponential degeneracy in a finite-size system), the zipper Hamiltonian effectively restores conventional periodic boundary conditions. We will illustrate that, under rather mild assumptions, in the thermodynamic limit of various systems, the zipper Hamiltonian may not lift the exponential degeneracy.

Recently, there has been a renewed surge of interest in physical theories whose extensive degeneracy is intimately related to associated higher symmetries (also heavily studied in myriad contexts as "d-dimensional gaugelike symmetries" [11–14,17–20] "unusual" [21], "stratified" [5,6], low-dimensional [7,22], "sliding" [8,23], "infinite but nonextensive set of conservation laws" [24], "generalized," "generalized global," or "higher form symmetries" [25–31] (and their "higher group" mixtures [32,33]), and "subdimensional" or "subsystem symmetries" [34–41]). In numerous theories with d-dimensional gaugelike symmetries, the spatial support of the symmetries is malleable and thus of a topological character while in many other theories, this spatial support cannot be deformed arbitrarily and is rigid. Some authors like to reserve the designation of "subsystem symmetries" to automatically imply these d-dimensional gaugelike symmetries associated with specific rigidly defined d-dimensional spatial regions that cannot be arbitrarily deformed. We will explicitly refer to these symmetries as "rigid higher symmetries." In this latter case, which will form the focus of our work, there may generally be an extensive number of independent symmetries. Numerous finer classifications of higher symmetries and their extensions exist (e.g., whether they are invertible or not) with nontrivial consequences, e.g., Refs. [42–45]. References [17,18] initiated a study explaining how the gauge like character of these symmetries mandates, in various circumstances, topological order. Subsequent, very penetrating, results linking these symmetries to topological phases appeared in Ref. [26] and other illuminating works. As befits their name(s), these higher or d-dimensional gaugelike symmetries act nontrivially on a d (or $d+1$)-dimensional

*zohar.nussinov@physics.ox.ac.uk

spatial (or space-time) subvolume of a theory defined in D (or $D+1$) spatial (space-time) dimensions. The $d=0$ and $d=D$ (or $d=D+1$) dimensional spatial (or space-time) limits correspond to the standard local (gauge) and global symmetries, respectively.

The more nontrivial d-dimensional gaugelike symmetries lie between the diametrically opposite limits of local and global symmetries ($0<d<D$) (and thus act on a lower dimensional subvolume of the physical system). These symmetries may be associated with various groups. A general study of the consequences of these symmetries and, in particular, of the dimensional reductions that they imply in systems with topological order first appeared in Refs. [11–13,17–20]. The upshot is that, as a rule, when the dimension of these symmetries is smaller than that of the system ($d<D$), a generalization [11,12] of Elitzur's theorem [46] (physically capturing the existence of d-dimensional topological defects in D spatial dimensions) leads to an effective dimensional reduction from $D\to d$ dimensions for various observables and general spatio-temporal correlation functions [11–14,17–20]. This holds true regardless of whether the submanifolds on which the d-dimensional gaugelike symmetries operate are spatially rigid or not. For low d, the proliferation of d-dimensional topological defects at finite temperatures may, unfortunately, lead to a loss of memory in topological quantum memory schemes [13] in the absence of active error correction. This particular corollary concerning memory loss may be proven by application of the generalized dimensional reduction inequalities [12,13,17–20] to the autocorrelation functions of various systems subjected to thermal noise. For the special cases of stabilizer models (including the celebrated Kitaev toric code [47], the Chamon model [48], the Haah code [49], the X-cube [50], and other models, e.g., Refs. [13,51,52]) the lower dimensional symmetry inequalities are further augmented by exact dualities [12,13,17–20,51,52] that map the partition functions and the equations of motion of these higher ($D>1$) dimensional systems to those of dual one-dimensional systems. A trivial consequence of those dimension reducing dualities is that the equations of motion for general observables will exactly map to those of the lower dimensional dual systems [12,19,20,53]. This implies that their autocorrelation functions are identical to those of lower dimensional systems, suggesting that in the absence of active error correction some of these systems might not be immune to thermal fluctuations [54].

Intrinsically, the action of d-dimensional symmetries on a spatially smooth low-energy configuration [i.e., one with infrared (IR) Fourier components] may generate other degenerate low-energy states that are not, at all, necessarily slowly varying in space, and have significant weight associated with their ultraviolet (UV) components. This seemingly rather odd facet has reignited various questions and investigations as to how continuum field theories may describe such highly degenerate systems with unconventional IR-UV mixing [30,34,55,56]. Of particular current note are studies of "fracton" theories [28–30,34,35,38,48–52,56–68] and their hybrids [69,70] (including, using the above noted dualities, the exact finite temperature solutions [12,51,52] of all of the first fracton models [48,49,57,59]). Some qualitatively similar behaviors also appear in theories exhibiting a "fragmentation" of their Hilbert space into individual ergodic subspaces that do not readily enable transitions from one subspace to another [71]. Predecessors of current fracton-like models featuring an extensive number of d-dimensional gaugelike symmetries have been the "compass models" and their duals that we will study in Sec. III [[5–8,10–14,17–20,24],[72–77]], which include celebrated Kugel-Khomskii type models [14,21,72]. These decades-old models [72] and their extensions describe orbital (and spin) degrees of freedom in transition metal compounds [72–74]. Compass models are some of the simplest models capturing the quintessential physics associated with the physical connections between degeneracies and d-dimensional gaugelike symmetries that we wish to explore.

Before proceeding further, we briefly comment on the relation between symmetries and degeneracies in classical (large spin or similar) limits with a particular focus on theories exhibiting rigid higher symmetries such as the classical variant of the 90° square lattice compass model [14] that we will specifically elaborate on later. In classical theories, the system is in a product type state. In these theories, no entanglement exists between the disjoint spatial regions where the different higher symmetries may operate. Thus, given a classical (spin, field, or other) configuration $\mathcal{C}$, one may turn, in a binary fashion, "on" or "off" the $\mathcal{M}_{\rm cl}$ independent classical higher symmetries that operate on different subregions of $\mathcal{C}$. The $2^{\mathcal{M}_{\rm cl}}$ different binary strings defined by the choice of applying/not applying these $\mathcal{M}_{\rm cl}$ different symmetries may, generally, be associated with $2^{\mathcal{M}_{\rm cl}}$ states that are degenerate with $\mathcal{C}$. Symmetries that have more than two group elements, allowing for more than just "on" or "off" applications, may yield a degeneracy that is larger yet. If the number of independent symmetries $\mathcal{M}_{\rm cl}$ scales as a lower dimensional subvolume of the system then the above implies sub-extensive exponential degeneracies. Apart from classical compass type systems and classical renditions of various quantum models with topological order [78], different subvolume exponential degeneracies were also found to appear in classical field theories with non-Abelian backgrounds [79], originally introduced as models of glasses [80], as well as a host of classical spin models with spiral and other ground states [78,81,82]. In both classical and quantum continuum gauge theory formulations that treat elasticity as that of a strongly anisotropic medium in space-time, defects are constrained to lower dimensional regions [83,84] matching more recent fracton inspired gauge theory formulations [66].

As is well known, in quantum theories the relation between symmetries and degeneracies is more intricate. Quantum fluctuations may lift classical degeneracies. These fluctuations can select and stabilize a particular lower energy ordered state in a space spanned by a plethora of many classically degenerate states—a mechanism often colloquially called "quantum order by disorder" [14,85–90]. Despite being notably different, this phenomenon is superficially reminiscent of anomalies in quantum mechanics. Anomalies arise in situations in which a symmetry that exists classically is no longer a symmetry of its regularized quantum theory. Although the classical d-dimensional symmetries at hand may remain symmetries of the quantum theory, unlike quantum anomalies, the degeneracies that the classical symmetries imply need not carry over to their quantum counterparts. Indeed, in the quantum arena, symmetries *do not* straightforwardly imply the existence of degeneracies (e.g., the eigenstates of the Hamiltonian may transform as singlets under the various symmetry operations). One of the simplest textbook examples illustrating this dictum is afforded by the absence of nontrivial transformations of the (even and odd) symmetric ($D = 1$) double well eigenstates under the parity symmetry operator. By sharp contrast, degeneracies always necessarily mandate the existence of symmetries (within any linear subspace spanned by $n \geqslant 2$ orthogonal degenerate eigenstates, there is an internal SU(n) "rotation" symmetry associated with general superpositions of these degenerate states) that, rather trivially, does not change the energy (see, e.g., Ref. [17]). Determining the spectral degeneracies is typically done on a case by case basis for numerous problems across diverse fields, both fundamental and applied [91–95]. Degeneracies that do not exist in finite size systems can, in some instances, rear their head so as to only emerge asymptotically in the thermodynamic limit. The latter situation arises in diverse situations including, e.g., those cases in which tunneling between edge states is suppressed in the thermodynamic limit [96]. In the context of the compass model (with periodic boundary conditions) that we will use as an example in this work, an exponential degeneracy was suggested to appear in the thermodynamic limit in Refs. [9,10].

Our work aims to show how, in specific circumstances (including those of altered boundary conditions), higher symmetries may rigorously lead to degeneracies in quantum systems. We will discuss how *localized* changes (zipper interaction terms) in the system Hamiltonian may connect it to the Hamiltonian of a system with exactly provable degenerate states. Furthermore, if the choice of boundary conditions is immaterial (i.e., if this particular choice does not modify the spectral properties of the system in the thermodynamic limit) then exact spectral degeneracies may still emerge asymptotically in this limit for general boundary conditions.

Our principal findings can be succinctly summarized by two basic inter-related maxims for a system of linear size L featuring d-dimensional gaugelike symmetries.

(a) Rigid d-dimensional gaugelike symmetries mandate exponential degeneracies when the choice of boundary conditions is immaterial in the thermodynamic ($L \to \infty$) limit. That is, the logarithm of the degeneracy $g(E)$ of *each level* [97] of energy E scales asymptotically at least as fast as $L^{d'}$,

$$\lim_{L\to\infty} L^{-d'} \ln g(E) > 0, \tag{1}$$

with

$$d' = D - d \tag{2}$$

denoting the co-dimension of the spatial dimension (d) of the higher (gaugelike) symmetries.

(b) Given boundary conditions for which exponential degeneracy may be rigorously proven for finite system sizes, the same exponential degeneracy may persist in the thermodynamic limit in the presence of other boundary conditions or interactions with different environments.

In what follows, we provide simple formal proofs of these maxims for general systems before turning to model system applications. Readers preferring to have a concrete application in mind before reading these general results may instead first peruse Sec. III, and only then read the more general abstract discussion in Sec. II below. At the very end of this work, we will rationalize our formal exact results by general intuitive considerations.

II. ON EXPONENTIAL DEGENERACY AND BOUNDARY CONDITIONS

A. A fundamental theorem

In what follows, by symmetries we allude to unitary operators that commute with the system Hamiltonian. In order to establish maxim (a), we start with a simple Lemma.

Lemma 1. Consider a system governed by a Hamiltonian H_{open} on a D-dimensional spatial volume Ω_D, and free (open) boundaries, for which there are two "dual" sets of independent symmetries $\{U_a\}$ and $\{V_a\}$ (with $a = 1, 2, \ldots, \mathcal{M}$) satisfying the following two conditions:

(1) All operators in one of these two sets mutually commute with one another,

$$[U_a, U_{a'}] = 0. \tag{3}$$

(2) For any symmetry U_a in the above set there is only a single dual operator V_a that does not commute with it. Specifically,

$$[U_a, V_a] \equiv W_a \neq 0 \text{ and } [U_a, V_{a'}] = 0 \ \text{ for } a \neq a', \tag{4}$$

where the operator W_a does not have a null space.

When the above conditions are met, *each* eigenstate of H_{open} is, at least, $2^{\mathcal{M}}$-fold degenerate.

Proof. Given condition (1), we may label all of the eigenstates of H_{open} by the eigenvalues $\{\lambda_a\}$ of the $\mathcal{M}$ independent symmetries $\{U_a\}$ (along with any additional quantum numbers if additional degeneracies appear in a sector of fixed eigenvalues of all of the mutually commuting symmetry operators $\{U_a\}$ operators). That is, all of the energy eigenstates are of the form

$$|\psi\rangle = |\lambda_1 \ldots \lambda_a \ldots \lambda_{\mathcal{M}}, \{\nu\}\rangle, \tag{5}$$

with $\{\nu\}$ additional labels (comprised of the energy eigenvalue and other quantum numbers if any) for all states within a given sector of $\{\lambda_a\}_{a=1}^{\mathcal{M}}$. To illustrate that for any a, there are, at least, two states with different eigenvalues of U_a having the same energy, we may next invoke condition (2). As the commutator W_a does not have a null space ($W_a|\psi\rangle \neq 0$), it follows that $|\psi\rangle$ is not an eigenstate of V_a. Thus the state $V_a|\psi\rangle$ is linearly independent of $|\psi\rangle$. Since V_a is a symmetry of the Hamiltonian, it further follows that the two states $|\psi\rangle$ and $V_a|\psi\rangle$ are linearly independent eigenstates of H_{open} sharing the same eigenvalue (energy). Lastly, the commutativity $[U_a, V_{a'}] = 0$ for $a \neq a'$ implies that $V_a|\psi\rangle$ is still an eigenstate of all other operators $U_{a'}$ with $a' \neq a$. Repeating the latter sequence of steps when the two states $|\psi\rangle$ and $V_a|\psi\rangle$ are acted by other symmetries $V_{a'}$ with $a' \neq a$, it follows that the states $V_a^{n_a} V_{a'}^{n_{a'}} |\psi\rangle$ (where $n_a = 0, 1$ and $n_{a'} = 0, 1$) constitute four linearly independent eigenstates of the Hamiltonian. Recursively iterating the above procedure for other symmetries of the V type, one sees that the $2^{\mathcal{M}}$ states $V_1^{n_1} V_2^{n_2} \ldots V_{\mathcal{M}}^{n_{\mathcal{M}}} |\psi\rangle$ (where for each $1 \leqslant a \leqslant \mathcal{M}$, we may set n_a to be either 0 or 1) are linearly independent degenerate eigenstates of the Hamiltonian. The existence of (at least) the above $2^{\mathcal{M}}$ independent degenerate eigenstates establishes the lemma. ■

If all quantum numbers $\{\nu\}$ in Eq. (5) are eigenvalues of operators that each commute with the symmetries $\{U_a\}$ and $\{V_a\}$ then the degeneracy of each state of fixed $\{\nu\}$ exhibits a degeneracy that is an integer multiple of $2^{\mathcal{M}}$.

A simple application of this lemma illustrates the well known fourfold degeneracy of the toric code model [47] on a simple 2-torus. In this model, there are only $\mathcal{M} = 2$ independent symmetries of the U type and $\mathcal{M} = 2$ of the V type that satisfy the conditions of lemma 1. Indeed, these are none other than the standard logical Z and X operators of the toric code model, realizing a $\mathbb{Z}_2 \times \mathbb{Z}_2$ ($d = 1$) gaugelike symmetry [47]. However, as we will detail, far more nontrivial degeneracies appear when there is a large number $\mathcal{M}$ of independent higher symmetries that satisfy the conditions of lemma 1.

In particular, this lemma leads us to our central theorem,

Theorem 2. General Hamiltonian systems (e.g., lattice theories on general graphs) that exhibit, at least,

$$\mathcal{M} \geqslant cL^{d'},\ c, d' > 0, \tag{6}$$

independent symmetries that satisfy the conditions of lemma 1, have for all energy eigenvalues E of the Hamiltonian a degeneracy

$$g(E) \geqslant 2^{cL^{d'}}. \tag{7}$$

Proof. This follows from an immediate application of lemma 1 given Eq. (6). ■

In Sec. III D, we will illustrate via an example of a compass model [5–8,10–14,17–21,72–76] with modified boundary conditions, so that the conditions of lemma 1 are satisfied, how higher symmetries imply a nontrivial exact exponential degeneracy. If the symmetries in lemma 1 correspond to spatially rigid independent higher (or gaugelike) symmetries of dimension d that, by virtue of being independent, cannot be continuously deformed into one another, then $d' = D - d$ as explained above. We note that since there may be additional multiplicities of $\{\nu\}$ associated with any set of the eigenvalues of the mutually commuting operators (including $H_{\rm open}$ itself), the degeneracy of each of the eigenvalues E of $H_{\rm open}$ may be an integer multiple of $2^{\mathcal{M}}$ associated with independent sign flips of each of the eigenvalues λ_a of the symmetries U_a. We emphasize that the theorem refers to *symmetries* that are *independent*. This is crucial since its number $\mathcal{M}$ is dependent upon the type of boundary condition, as we will explain below.

The above considerations may be sharpened and further generalized as follows.

Corollary 3. Consider the situation in which (i) the operators $\{U_a\}$ and $\{V_a\}$ are *not symmetries* of the full Hamiltonian $H_{\rm open}$ and commute with $H_{\rm open}$ only in a sector of given fixed energy E and (ii) the operators $\{U_a\}$ and $\{V_a\}$ satisfy Eqs. (4) and (5). In this case, the proofs of lemma 1 and theorem 2 apply for the projected Hamiltonian $P_E H_{\rm open} P_E$ with P_E being the projection operator onto the space of energy E to establish exponential degeneracy. Thus, even if $\{U_a\}$ and $\{V_a\}$ are not exact symmetries of the full Hamiltonian $H_{\rm open}$ and only commute with it within a given energy subspace (i.e., these operators are *emergent symmetries* [98]) then in that energy subspace the exponential degeneracy of theorem 2 is ensured. Similar results apply if $\{U_a\}$ and $\{V_a\}$ satisfy Eqs. (4) and (5) and only become symmetries in projected sectors of fixed quantum numbers other than those of constant energy.

In the systems that we will investigate in more detail, including compass model [5–8,10–14,17–21,72–76] examples in Sec. III, the many-body Hamiltonian $H_{\rm open}$ is a sum of few body interactions or "bonds" b_γ,

$$H_{\rm open} = \sum_\gamma b_\gamma. \tag{8}$$

We stress that the exact exponential degeneracy of theorem 2 applies to all eigenstates and is not limited to the ground-state sector of the system. Indeed, later on we will discuss finite temperature theories.

We next outline the reason why, for *typical spin and bosonic systems*, boundary conditions can be invoked such that the conditions of lemma 1 apply for $\mathcal{M} \geqslant cL^{d'}$ independent symmetries with the co-dimension d' of Eq. (2). In fact, *common open boundary realizations of real physical systems satisfy Eq. (6)*. In what follows, we will denote the generic local spin or Bose operators by $\{\phi_r^\mu\}$. Here, μ is an internal label and r the appropriate external position (site) index. These operators will explicitly become the Pauli operators $\{\sigma_r^\mu\}$ (with $\mu = x, y, z$) in spin $S = 1/2$ models. For generic finite size lattices (or graphs) different from those with periodic boundary conditions, such an extensive number of independent symmetries trivially appear as a result of geometric considerations. In Bose and spin systems, whenever the symmetry operators $\{U_a\}$ have their support on d-dimensional regions $\{\mathcal{R}_a\}$ that are spatially disjoint (i.e., when these regions share no common sites), then they will commute since Bose and spin operators on different spatial sites trivially commute and Eq. (3) is satisfied. This is generally not true when they do overlap and nontrivial commutators will appear. Indeed, if the regions $\mathcal{R}_a$ and $\tilde{\mathcal{R}}_a$ (respectively, the spatial supports of the symmetries U_a and V_a) share common sites while $\mathcal{R}_a$ and $\tilde{\mathcal{R}}_b$ (respectively, the spatial supports of the symmetries U_a and V_b with $a \neq b$) share no common sites then, generally, U_a and V_a need not commute with one another. Lastly, U_a and V_b with $a \neq b$ do not have overlapping spatial support and may trivially commute [giving rise to Eq. (4)]. In mathematical terms,

$$\begin{aligned} &\mathcal{R}_a \cap \mathcal{R}_b = \emptyset, \text{ for } a \neq b, \\ &\mathcal{R}_a \cap \tilde{\mathcal{R}}_b = \emptyset, \text{ for } a \neq b, \\ &\mathcal{R}_a \cap \tilde{\mathcal{R}}_a \neq \emptyset. \end{aligned} \tag{9}$$

In many systems, geometry alone [whether the spatial (or spatiotemporal) support of the symmetry operators overlaps or does not] mandates Eqs. (3) and (4). That is, geometry determines whether the symmetry operators commute. By definition, for a system in D spatial dimensions exhibiting d-dimensional gaugelike symmetries, the symmetries U_a or V_a have their support on a d-dimensional spatial region $\mathcal{R}_a$ and $\tilde{\mathcal{R}}_a$. For a generic open volume Ω_D in D dimensions (i.e., not one with square or cubic boundaries), there are foliations of Ω_D into $\mathcal{O}(L^{d'})$ regions $\{\mathcal{R}_a\}$ and into $\mathcal{O}(L^{d'})$ regions $\{\tilde{\mathcal{R}}_b\}$ on which the symmetries $\{U_a\}$ and $\{V_b\}$ have their support. That is, for random open boundaries of Ω_D, we can partially slice it into $\mathcal{O}(L^{d'})$ nonoverlapping d-dimensional hyperplanes satisfying the conditions of lemma 1. For fermionic (or noncommuting elementary degrees of freedom) systems, if the symmetries involve an even number of operators per site (and/or regions $\{\mathcal{R}_a\}$ and $\{\tilde{\mathcal{R}}_b\}$ having an even number of sites) then the mutually commuting nature of the symmetries on different disjoint regions is, once again, ensured.

The above discussion might seem a bit abstract. To make it more concrete, we will shortly turn (Sec. III) to simple examples. An important ingredient will be the use of general boundary conditions that are different from those of the conventional Born-von Karman form. These general boundary conditions are not merely academic. Indeed, real materials are not wrapped around tori that endow them with periodic boundary conditions nor are they boxed to textbook type square (or hypercubic) boundaries. The boundaries $\partial\Omega_D$ of real physical systems Ω_D are typically very different from these conventions and may generally span many possible geometries. The use of periodic boundaries has indeed largely been a matter of convenience (i.e., since these allow for a Fourier space analysis and have no identifiable boundary sites so as to emulate bulk macroscopic systems in which nearly all real space sites lie far from the boundaries). The use of periodic boundary conditions has, for these and related reasons, become very common by now. In Sec. III D below, we will examine boundaries different from the periodic ones to illustrate how our central theorem 2 mandates an extensive degeneracy of this system. Nearly identical constructs may be introduced for many other theories and disparate boundary conditions different from those that are customarily employed.

Before proceeding further, however, we must reiterate and underscore that our considerations do not apply for many other systems, such as the earlier noted Kitaev's toric code model [47], where local operators are defined on bonds and not sites (vertices). In particular, the ($D = 2$ dimensional) Kitaev toric code model [47] supports its well-known $d = 1$ symmetries [17,47] but our geometric construct will not allow for these to be independent of one another. As we remarked after illustrating lemma 1, only $\mathcal{M} = 2$ sets of these independent symmetries exist for the toric code model. In a related geometrical vein, the symmetries of the Kitaev toric code model are not rigid. These symmetries are associated with the products of Pauli matrices along any of the many contours that wind around one of the two cycles of the torus. An arbitrary smooth deformation of a given initial contour on which a Pauli string product is defined leads to another operator that remains a symmetry. This new string product on the deformed contour is not independent from the symmetry operator associated with the initial contour. In fact, all such deformations lead to symmetry operators that act identically on a given ground state. By contrast, for the compass models that we detail in Sec. III, the symmetries have their support on particular rigid spatial regions. When these regions are disjoint, the associated symmetries are independent of one other.

B. Exact universal exponential degeneracy of hybrids of thermal systems and their environment

We next turn to two exceptionally simple yet general results relating to maxim (b).

We consider a Hamiltonian H_{open} that is a sum of bonds $\{b_\gamma\}$ (Eq. (8)) with each of these bonds being invariant under all of the symmetries $\{U_a\}$ and $\{V_a\}$. For this Hamiltonian, we may establish

Corollary 4. If a degeneracy that is an integer multiple of an exponential in the linear system length (or any other) is established for each level of H_{open} satisfying the requirements of theorem 2 then, a degeneracy of (at least) $2^{\mathcal{M}}$ also follows for the expectation value (as evaluated in eigenstates of H_{open}) in the *same* Hilbert space, of any Hamiltonian H_{sub} that is formed by summing any subsets of the bonds appearing in H_{open}.

Proof. The sequence of steps used to establish theorem 2 can be repeated verbatim here since each of the individual bonds b_γ, whose sum forms H_{sub}, commutes with all the symmetries $\{U_a\}$ and $\{V_a\}$. Thus, for each eigenstate $|\psi\rangle$ of H_{open}, the expectation value $\langle\psi|H_{\text{sub}}|\psi\rangle$ will remain invariant under the change of sign of any of the symmetry eigenvalues $\{\lambda_1\ldots\lambda_a\ldots\lambda_{\mathcal{M}}\}$. Since there are $2^{\mathcal{M}}$ such subsets, the corollary follows. ■

The subsets of the above bonds appearing in H_{open} may tessellate any subregion Ω_{sub} of the domain Ω_D. The above proof is rather general and also holds when subregion Ω_{sub} is not be geometrically contiguous. In what briefly follows, we consider what occurs when Ω_{sub} is not composed of disjoint volumes. Formally, in the thermodynamic limit, we may regard $\Omega_{\text{sub}} \subset \Omega_D$ as the "system bulk" and the remaining $\Omega_{\text{env}} \equiv (\Omega_D - \Omega_{\text{sub}})$ as the thermal reservoir or "environment" with which it interacts and is in equilibrium with. This is so since the combined "system-environment" hybrid is described by H_{open}. If this hybrid is in thermal equilibrium then its state will be a thermal state defined by H_{open}. From theorem 2 and corollary 4, in each of the exponentially many degenerate ground states (which may be regarded as thermal states at temperature $T = 0$) of the environment-system hybrid of H_{open}, the energy of the system alone $\langle H_{\text{sub}}\rangle$ is exactly the same. Identical results trivially hold for all excited states. Thus the logarithm of the number of states of this hybrid having the same system energy $\langle\psi|H_{\text{sub}}|\psi\rangle$ is bounded from below by $\mathcal{O}(L^{d'})$. The linear dimension L of the system is less than or equal to that of the "system-environment" hybrid Ω_D for which theorem 2 applies. We stress that this conclusion holds for all possible Hamiltonians H_{open} satisfying the requirements of theorem 2 that includes the system H_{sub} as a subset. For any such Hamiltonian, we may deform the volume Ω_{env} defining the "environment" and make it arbitrarily small as long as the requirements of theorem 2 are still satisfied. In particular, since $\Omega_{\text{sub}} \subset \Omega_D$, for all such deformations of the environment that geometrically surrounds the system, we will consistently obtain a universal minimal lower bound of $\mathcal{O}(L^{d'})$ on the ground-state entropy. This hints that the introduction of various boundary effects in the thermodynamic limit may still leave the ground-state entropy bounded by $\mathcal{O}(L^{d'})$.

We should perhaps emphasize that, in general, $[H_{\text{open}}, H_{\text{sub}}] \neq 0$. Thus despite the fact that they may share the same eigenvalues of their common d-dimensional symmetry operators when diagonalized separately, H_{open} and H_{sub} do not have identical eigenstates. That is, corollary 4 is not a trivial consequence of enlarging the Hilbert space of the system (on which $H_{\rm sub}$ operates) by the additional degrees of freedom of the environment.

We may further invert the roles of the "system" and "environment" to obtain an additional result. Towards this end, we may let the closed "system-environment" hybrid be a system for which we cannot prove by theorem 2 the exponential degeneracy (i.e., when the conditions of this theorem are not met) and take the "system" to be a theory on a volume $\Omega_{\rm sub}$ for which the conditions of theorem 2 are satisfied. When ensemble equivalence holds (as expected in the thermodynamic $L \to \infty$ limit), this will lead to another very simple consequence.

Corollary 5. Consider a subvolume $\Omega_{\rm sub}$ of linear dimension L for which the conditions of theorem 2 apply. We further assume that an ensemble equivalence holds between the system properties on $\Omega_{\rm sub}$ when it is closed (i.e., within the microcanonical ensemble) and for the system properties on $\Omega_{\rm sub}$ within the (canonical ensemble type) thermally equilibrated system-environment hybrid on Ω_D. Under these conditions, the system's ground-state entropy on $\Omega_{\rm sub}$ will be $\geqslant \mathcal{O}(L^{d'})$.

Proof. The von-Neumann entropy of the "system" on $\Omega_{\rm sub}$ at any given temperature (including the limiting $T = 0$ case associated with the ground-state sector) is

$$S_{\rm sub} = -{\rm Tr}_{\rm sub}\rho_{\rm sub} \log_2 \rho_{\rm sub}, \tag{10}$$

where the reduced density matrix

$$\rho_{\rm sub} = {\rm Tr}_{\rm env}\, \rho_{\text{sub-env}} \tag{11}$$

is formed by a partial trace of the density matrix of the system-environment $\rho_{\text{sub-env}}$ over the Hilbert space of the environment (env). In the microcanonical ensemble, the entropy of the system is given by the logarithm of the number of its ground states. Thus, from theorem 2,

$$S_{\rm sub} \geqslant \mathcal{O}(L^{d'}). \tag{12}$$

We next invoke ensemble equivalence. This equivalence implies that calculations with the reduced density matrix of Eq. (11) reproduce computations with the density matrix of the closed system (i.e., those within the microcanonical ensemble). Considering the ground-state sector of the system-environment hybrid, we see from Eqs. (10) and (12), that whenever the system satisfies the requirements of theorem 2, the partial trace of Eq. (11) yields a reduced density matrix for the system with an entropy that is bounded from below by $\mathcal{O}(L^{d'})$. ■

Similar entropy bounds may be derived on various system-environment hybrids that contain one or more subregions. All such bounds are consistent with known entropy inequalities such as those of Ref. [99]. However, further applying the entropy inequalities of Ref. [99] and subsequent works to the considerations underlying corollary 5 does not rigorously establish exponentially large ground-state degeneracy in *any* theory with higher symmetries on an arbitrary volume Ω_D (i.e., including those for which the conditions of theorem 2 are not satisfied). Returning to the proof of corollary 5, we emphasize that even if the ground states of the system-environment hybrid are not highly degenerate, for all "system" subvolumes $\Omega_{\rm sub}$ for which the conditions of theorem 2 are met, the ground-state entropy bounds of Eq. (12) are satisfied. Notwithstanding special exceptions (see, e.g., Ref. [100]), as is heavily emphasized in statistical mechanics textbooks, ensemble equivalence is quite pervasive in the thermodynamic $L \to \infty$ limit of rather general theories. In the above, we considered general ensembles formed by system-environment hybrids. In the conventional "canonical" setting, the environment is far larger than the system while in the microcanonical ensemble, the system on $\Omega_{\rm sub}$ is closed and there is no environment. Here, ensemble equivalence in the form of the irrelevance of the fine details of the boundary environment with which the system interacts and equilibrates with is conceptually similar to independence of the system behavior from its boundary conditions.

In order to make the content of corollaries 4 and 5 clear, we will discuss their implications for the square lattice compass model at the end of Sec. III D.

In Sec. IV, we will return to a general discussion of maxim (b) as it pertains to the effects of boundary conditions. In the Conclusions (Sec. VI), we will further suggest why an exponential degeneracy may emerge in the thermodynamic limit of rather general systems with rigid higher symmetries irrespective of their particular boundary conditions.

III. EXAMPLES: SQUARE AND CUBIC LATTICE COMPASS MODELS

A. Definition of the 90° square lattice model

As an example in which theorem 2 comes to life, we will first consider the spin $S = 1/2$ square lattice 90° compass model, the simplest of all compass models [5–8,10–14,17–20,72–77].

We start with a brief definition of this model. The 90° square lattice or planar compass model (PCM) [8] on a square lattice of size $N = L \times L$ (with a lattice constant that we set to be unity) is given by a simple bilinear in the Pauli operators,

$$H_{\rm PCM} = -\sum_{r,\mu=x,y} J_\mu\, \sigma_r^\mu \sigma_{r+\mathbf{e}_\mu}^\mu. \tag{13}$$

Interactions (or "bonds") involving the x spin operators $S_r^x = \sigma_r^x/2$ (here and throughout we set $\hbar = 1$) occur only along the spatial x direction of the lattice. Similar spatial direction-dependent spin exchange interactions appear for the y components of the spin. Thus the PCM displays exchange interactions with two Pauli matrix flavors along the x and y square lattice directions, which we henceforth refer to as $\mathbf{e}_\mu$ with $\mu = x, y$ (see Fig. 1). This model may be regarded as a square lattice rendition of Kitaev's honeycomb model [101] (in which all three different Pauli matrices appear along the three honeycomb lattice directions connecting nearest neighbors).

Before proceeding, we provide a quick summary of how compass models such as that of Eq. (13) naturally arise. These models have been used in numerous arenas [14] including frustrated magnetism, cold atoms, and quantum information [75] to name a few. Compass models are best known for describing the real-space directional character of orbitals in transition metal compounds. In these systems, the orbital states dictate the overlaps between electrons on neighboring real space ions. This, in turn, leads to a dependence of the

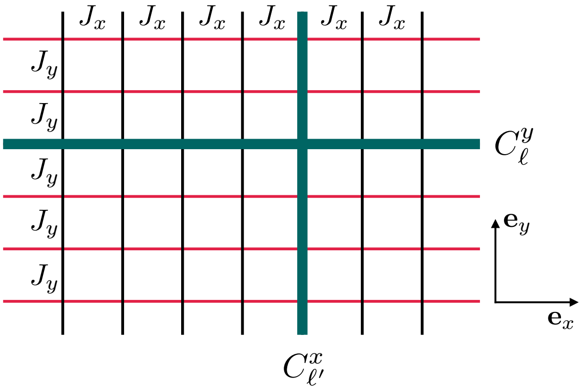

FIG. 1. The PCM, Eq. (13), on an open square lattice. Interactions along the two external Cartesian directions are associated with couplings J_μ, $\mu = x, y$. Corresponding with each of the highlighted contours C_ℓ^x and $C_{\ell'}^y$ are the symmetries of Eq. (15).

interactions in real space on internal-space pseudospin states describing the orbitals. This intricate coupling leads to Hamiltonians describing these systems that, similar to the PCM, exhibit a dependency of real space couplings on the internal pseudospin components. Inasmuch as finite size computations can ascertain [9,10,77], when endowed with periodic boundary conditions, the PCM exhibits a spectral gap between its claimed ground-state sector of an exponentially large number ($\mathcal{O}(2^L)$) of states and the lowest-energy excited states.

Obviously, models exactly dual to the PCM will share identical spectra [19,20]. For instance, a simple duality taking $J_x \leftrightarrow J_{\rm P}$ and $J_y \leftrightarrow h$ [8,20] maps the PCM of Eq. (13) onto a model first introduced by Xu and Moore [24] for describing Josephson coupled superconducting arrays,

$$H_{\rm XM} = -J_{\rm P} \sum_{\square} \prod_{r\in\square} \sigma_r^z - h \sum_r \sigma_r^x. \tag{14}$$

In Eq. (14), $\prod_{r\in\square}\sigma_r^z$ and $\sum_\square$ denote, respectively, the product of the four spins lying at the vertices of minimal square lattice plaquettes $\square$ and the sum over all such plaquettes $\square$ in the square lattice. As discussed elsewhere [8,24], $H_{\rm XM}$ (in a finite square lattic) displays $\mathcal{O}(L)$ mutually commuting $d = 1$ non-independent symmetries. In the next sections, we turn to the investigation of the symmetries of the PCM on systems with conventional and modified boundary conditions. A proof of a high degeneracy of the PCM (employing its mutually noncommuting symmetries on systems with modified boundary conditions) allows a demonstration of a similar degeneracy for $H_{\rm XM}$.

B. $d = 1$ Gaugelike symmetries of the square lattice 90° compass model

The PCM is invariant under the following $\mathbb{Z}_2$ symmetries [8–11,13]

$$\hat{O}_\ell^\mu = \prod_{r\in C_\ell^\mu} i\sigma_r^\mu \text{ , for } \mu = x, y \tag{15}$$

with $C_\ell^\mu \perp \mathbf{e}_\mu$ axis (see Fig. 1). We briefly review some of the properties of these symmetries. Our notation will follow that of [8,11,13], while the results pertaining to the commutation and anti-commutation of these symmetry operators amongst themselves were first reported in [9,10]. All of the symmetries associated with lines C_ℓ^μ, that are all parallel to the same direction μ, commute amongst themselves,

$$[\hat{O}_\ell^\mu, \hat{O}_{\ell'}^\mu] = 0. \tag{16}$$

By contrast, operators of the type of Eq. (15) that are related to orthogonal lines anticommute with one another,

$$\{\hat{O}_\ell^x, \hat{O}_{\ell'}^y\} = 0. \tag{17}$$

The above anticommutativity is apparent since any two orthogonal lines ℓ and ℓ' intersect and share one common lattice site r'. Associated with this point of intersection, $\hat{O}_\ell^x$ has a single factor of $\sigma_{r'}^x$ in the string product of Eq. (15). Similarly, $\hat{O}_{\ell'}^y$ has, in the string product of Eq. (15) that defines it, a factor of $\sigma_{r'}^y$. The anticommutator $\{\sigma_{r'}^x, \sigma_{r'}^y\} = 0$ implies the anticommutation relation of Eq. (17). For general boundary conditions discussed in the next subsections [102], different independent composites of the two dual sets of $d = 1$ symmetries $\{\hat{O}_{\ell'}^y\}$ and $\{\hat{O}_\ell^x\}$ on horizontal and vertical lines correspondingly relate, in the general setting of our central theorem 2, to the two respective sets of operators $\{U_a\}$ and $\{V_a\}$.

When $J_x = J_y$, a global reflection symmetry augments the symmetries of Eq. (15) [17]. We will, however, focus on the anisotropic model of Eq. (13) with $J_x \neq J_y$. As $[H_{\rm PCM}, \hat{O}_\ell^\mu] = 0$, similar to Eq. (5), we can label, with some abuse of notation, the eigenstates of the Hamiltonian $H_{\rm PCM}$ by

$$|\psi\rangle = |\lambda_{\mu;1} \ldots \lambda_{\mu;L}, \{\nu\}\rangle, \tag{18}$$

with $\lambda_{\mu;\ell} = \pm 1$ an eigenvalue associated with the mutually commuting symmetries $\hat{O}_\ell^\mu$ (all operators with the same value of μ) and $\{\nu\}$ an additional label for all states within a given sector of $\{\lambda_{\mu;\ell}\}_{\ell=1}^L$. This latter label may mark the energies of these states and, when additional degeneracies appear, any other remaining quantum numbers.

The symmetries $\hat{O}_\ell^\mu$ of Eq. (15) have exceptionally simple pictorial representations. For all sites r lying along columns/rows, these symmetries respectively correspond to reflections about internal spin directions μ. Specifically, given a site r, the operators $\hat{O}_\ell^\mu$ either implement

(i) for $\mu = x$, the transformation $S_r^x \to S_r^x$ and $S_r^y \to -S_r^y$ for any vertical line ℓ containing r, or (ii) for $\mu = y$, the transformation $S_r^x \to -S_r^x$ and $S_r^y \to S_r^y$ for any horizontal line ℓ' containing r.

In the large spin S limit of $H_{\rm PCM}$, the model transforms into a classical spin model with classical XY spins of unit norm $\vec{S}_r = (S_r^x, S_r^y)$ replacing the Pauli operators in Eq. (13). It is readily verified that the above two operations (i) and (ii) remain symmetries of this large S rendition of the $H_{\rm PCM}$ Hamiltonian. As we now explain, the symmetries (i) and (ii) imply that the classical ($S \to \infty$) limit of the PCM trivially exhibits a degeneracy that is exponential in the linear system length L. Towards this end, envision classical XY spins at any lattice site r. Clearly, any such classical spin $\vec{S}_r$ cannot be parallel to both its internal vertical and horizontal spin directions. Thus any such classical spin will change under the application of (at least) one of the two symmetries $\hat{O}_\ell^x$ and $\hat{O}_{\ell'}^y$ associated with the vertical and horizontal lines that pass through the site r. With this simple observation made, proving the exponential degeneracy of the classical system is rather immediate. Towards this end, consider traversing the real space square lattice along one of its main diagonals (comprised of L sites). In what follows, we sequentially label the L sites along such a given diagonal by $r_{\rm diag}$. Consider any $r_{\rm diag}$ that lies at the intersection of the vertical line ℓ and the horizontal line ℓ'. The classical spin at $r_{\rm diag}$ (as well as the spin at any other site) might be invariant under a reflection about its internal x direction or its internal y axis. However, as just noted above and as we reiterate again, any such classical spin cannot possibly be simultaneously invariant under reflections about both its internal x and y directions. Thus by applying, given any site $r_{\rm diag}$, at least one of the two reflection operations $\hat{O}^x_\ell$ or $\hat{O}^y_{\ell'}$ will generate a new degenerate classical spin configuration. For each of the L sites lying along the diagonal, let us define an associated reflection symmetry $\hat{O}_{r_{\rm diag}}$ [of either type (i) or (ii)] that is guaranteed to change its configuration. As we traverse the square lattice along one of its main diagonals, we may turn "on" or "off" these L reflection symmetries $\hat{O}_{r_{\rm diag}}$. Following this recipe, we may generate 2^L states (i.e., XY spin configurations) that are degenerate with any initially given classical spin state. This is an example of the rather universal exact degeneracies of the classical model (which appear for general systems independent of their size or boundary conditions) that were discussed in the Introduction. In the context of that earlier discussion, in the PCM, the number of independent classical rigid higher symmetries, $\mathcal{M}_{\rm cl} = L$.

Complementing these above exact $d = 1$ reflection symmetries, the classical system further exhibits an emergent continuous rotational symmetry that appears in its ground-state sector. For instance, for positive coupling constants $J_x = J_y = J$, it is indeed readily established that all uniform (i.e., ferromagnetic) states are ground states of the Hamiltonian with continuous global rotation connecting these ground states [5,6,14]. These global emergent symmetries remain unchanged in the classical limit for states formed by the application of any combination of the $d = 1$ symmetries on the classical ferromagnetic states. Similar to anomalies in quantum field theories, this continuous symmetry of the classical system in its ground state is no longer a symmetry of the quantum theory in its ground-state sector. In this sense, this is an *emergent anomaly*.

In what follows, we focus our attention on the spin $S = 1/2$ quantum model of Eq. (13).

C. Open square lattice with boundaries parallel to the Cartesian directions

We first briefly summarize the standard open boundary condition realization of the $S = 1/2$ PCM. We will then turn to consider other boundary conditions and illustrate how they imply an exponential degeneracy of the spectrum. Our bound on the degeneracy applies for *each level of the spectrum, not only the ground-state sector*.

When the square lattice is aligned along the Cartesian x and y directions, then given any eigenstate of the form of Eq. (18) with μ fixed to be either x or y, we can apply a symmetry operator of the type $\hat{O}^{\mu'\neq\mu}_{\ell'}$ to $|\psi\rangle$. This leads to a new eigenstate of $H_{\rm PCM}$,

$$\begin{aligned}\hat{O}^{\mu'\neq\mu}_{\ell'}|\lambda_{\mu;1}\dots\lambda_{\mu;L},\{\nu\}\rangle &= |-\lambda_{\mu;1}\cdots-\lambda_{\mu;L},\{\nu\}\rangle\\ &\equiv|\psi'\rangle\end{aligned} \tag{19}$$

that has the same energy as the original state $|\psi\rangle$. The result of Eq. (19) is the same for all lines $C^{\mu'}_{\ell'}$ with $1 \leqslant \ell' \leqslant L$. Thus, given any initial eigenstate $|\psi\rangle$, we see that we can construct a second eigenstate that has the same energy. A two-fold degeneracy [10] thus follows from the existence of the noncommuting symmetries of Eqs. (15) and (17). An identical effect and conclusion trivially follow from time-reversal symmetry (and Kramers degeneracy) as applied to square lattices with an odd $L \times L$ size [13]. On a square lattice having its edges parallel to the Cartesian directions, no additional degeneracy follows from symmetries. Indeed, numerically only a two-fold degeneracy is observed on finite size square lattices [9,10]. Indeed, as our proofs make clear (requiring the need for the conditions underlying theorem 2 to be satisfied), *the existence of higher symmetries does not imply an exponentially large degeneracy of all finite size systems*. A curious numerical observation [10,77] of the PCM on finite size square lattice (insofar as finite size calculations can suggest) is that as the system size becomes progressively larger, multiplets of $\mathcal{O}(2^L)$ states each seem to become degenerate as $L \to \infty$ (with a gap between these multiplets that decays algebraically with system size). The results of our work rationalize how such a degeneracy that is exponential in the system perimeter may arise. We first explain why in open systems with various boundaries, the symmetries of Eq. (15) mandate a degeneracy of each level that is exponential in the perimeter.

D. "Cylindrical cuts," "toroidal cuts," "dilute vacancies," and other boundary conditions or changes of internal geometry for which exponential degeneracy follows from symmetries

We next consider, see Fig. 2, a finite $L \times L$ square lattice which lies inside a parallelogram of an angle θ relative to the x axis [with the x and y coordinates defining the nearest-neighbor compass interactions of Eq. (13)]. This lattice can be viewed as that generated by "cutting" and "opening up" an $L \times L$ square lattice with cylindrical boundary conditions. If the cut were performed along a Cartesian direction ($\theta = \pi/2$ in Fig. 2), the result would be the standard square lattice with open boundary conditions. If, however, the cut is performed at another general angle θ, the resulting parallelogram will define the lattice that we focus on now. This parallelogram will be composed of L horizontal lines (each of length L). In Fig. 2, we show a parallelogram formed by choosing $\theta = \pi/4$. In traversing the lattice by a distance of order unity along the vertical direction, these lines are horizontally displaced relative to one another by one lattice constant. Thus, overall, the horizontal span of the lattice is equal to $(2L)$ lattice constants. We will denote the Hamiltonian of Eq. (13) on the so-generated open cylindrical cut by $H_{\rm open}$. As earlier, the operators of Eq. (15) are symmetries for lines C^μ_ℓ orthogonal to the μ direction. Now, however, we have the lines $\{C^x_\ell\}_{\ell=1}^{2L}$ and $\{C^y_{\ell'}\}_{\ell'=1}^{L}$. Clearly, if two eigenstates have a differing set of

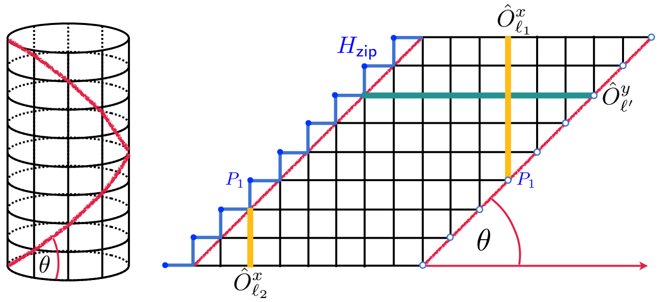

FIG. 2. The system of Eq. (13) on a square ($L = 8$) lattice with cylindrical boundary conditions (black grid, left panel) and its cut along a slanted line (highlighted in red). The cut leads to the system shown on the right. On the original cylinder, the two (yellow) lines marked by P_1 were one and the same. The nearest-neighbor PCM Hamiltonian on the open parallelogram on the right can be made to agree with that on the cylinder by inserting additional interactions (staircase blue line) that become nonlocal on the parallelogram. The sum of these additional $2L - 1$ interaction terms defines the "zipper Hamiltonian" $H_{\rm zip}$. See Sec. IV and, in particular, Eq. (28) for a discussion of the $H_{\rm zip}$ that captures the blue staircase bonds needed to seal the open parallelogram on the right, and transform it (as a closing a zipper) to the closed cylinder on the left. Both the cylindrical system on the left and the open parallelogram on the right are invariant under $d = 1$ symmetries of the type of Eq. (15). Both, open and cylindrical, systems are invariant under the symmetries $\hat{O}^y_{\ell'}$ associated with the horizontal lines $C^y_{\ell'}$ (marked in green). The open system is invariant under the symmetries $\hat{O}^x_\ell$ affiliated with all vertical lines C^x_ℓ connecting one side of the open system to the other. However, since the vertical boundaries of the cylindrical system differ from those of the open system, its symmetries differ. For instance, individually, while the two operators $\hat{O}^x_{\ell_2=3}$ and $\hat{O}^x_{\ell_1=11}$ (marked in yellow) are exact symmetries of the open system Hamiltonian $H_{\rm open}$ they are no longer symmetries of the cylindrical system. However, the product $\hat{O}^x_3\hat{O}^x_{11}$ is that of Eq. (15) associated with a vertical generator of the cylinder and, as such, is a symmetry of the cylindrical system.

$\lambda_{y;\ell'}$ eigenvalues then they will be orthogonal to one another,

$$\langle\lambda_{y;1}\dots\lambda_{y;L},\{\nu\}|\lambda'_{y;1}\dots\lambda'_{y;L},\{\nu\}\rangle = \prod_{\ell'=1}^{L}\delta_{\lambda_{y;\ell'},\lambda'_{y;\ell'}}. \quad (20)$$

From the symmetry operators of Eq. (15), we can construct the subset of composite symmetry operators

$$\hat{O}^x_{\mathcal{S}} = \prod_{\ell\in\mathcal{S}}\hat{O}^x_\ell, \quad (21)$$

where $\mathcal{S}$ is any of the possible subsets of the integers $\{1, 2, 3, \dots, L\}$ labeling the L leftmost vertical lines. Note that here we allow for the action of the independent vertical symmetry operators $\{\hat{O}^x_\ell\}$ only on the L leftmost vertical lines out of the $(2L)$ vertical lines. The effect of higher $\ell > L$ vertical symmetry operators $\hat{O}^x_\ell$ on the eigenvalues $\lambda_{y;1}\dots\lambda_{y;L}$ can be expressed in terms of that of lower ℓ operators $\{\hat{O}^x_\ell\}_{\ell=1}^L$. That is, with reference to Fig. 2, for all $1 \leqslant \ell < L$, the result of applying $\hat{O}^x_\ell\hat{O}^x_{L+\ell}$ on the set of eigenvalues $\lambda_{y;1}\dots\lambda_{y;L}$ is the same as that of applying $\hat{O}^x_L$ (which simply flips the sign of all of these eigenvalues). There is an exponentially large number (i.e., $\sum_{k=0}^L\binom{L}{k} = 2^L$) of such independent subsets of products amongst the vertical symmetry operators. On the square lattice of Sec. III C, the application of these symmetry operators in this subset can lead to one of two outcomes: if the set $\mathcal{S}$ contains an even number of integers then given any eigenstate of the form of Eq. (18), the product $\hat{O}^x_{\mathcal{S}}|\psi\rangle$ will give back the same full set of symmetry eigenvalues $\lambda_{y;\ell'}$. By contrast, if the set $\mathcal{S}$ contains an odd number of integers $1 \leqslant \ell \leqslant L$ then $\hat{O}^x_{\mathcal{S}}|\psi\rangle = |\psi'\rangle$ with $|\psi'\rangle$ being the state defined in Eq. (19). Now, here is one of trivial yet nonetheless crucial points that we wish to bring to the fore: The application of the symmetry operations of Eqs. (21) on an initial state $|\psi\rangle$ will give rise to an exponential number of orthogonal states,

$$\hat{O}^x_{\mathcal{S}}|\psi\rangle = \left|\lambda^{\mathcal{S}}_{y;1}\dots\lambda^{\mathcal{S}}_{y;L},\{\nu\}\right\rangle \equiv |\psi^{\mathcal{S}}\rangle. \quad (22)$$

Here, $\lambda^{\mathcal{S}}_{y;\ell'} = \eta_{\ell'}\lambda_{y;\ell'}$. For a given ℓ', the Ising variable $\eta_{\ell'} = \pm 1$ denotes the even/odd parity of the number of vertical lines C^x_ℓ associated with the set $\mathcal{S}$ (i.e., $\ell \in \mathcal{S}$) that intercept the line $C^y_{\ell'}$. Each of the possible different general choices for the set $\mathcal{S}$ will uniquely lead to a different binary string $\mathcal{B} = (\lambda^{\mathcal{S}}_{y;1}\dots\lambda^{\mathcal{S}}_{y;L})$. By the orthogonality relation of Eq. (20), this implies an exponentially large number of degenerate orthogonal eigenstates. In particular, since each amongst the possible different choices of the strings $(\lambda_{y;1} = \pm 1, \dots, \lambda_{y;L} = \pm 1)$ can be achieved then the system will have a degeneracy which is (at least) of size 2^L. More generally, the degeneracy as associated with the $\lambda_{y;\ell}$ eigenvalues alone is bounded by the cardinality $\mathcal{M} = |\mathcal{B}| = |\mathcal{S}|$ of the set of attainable binary strings by applying the different operators $\{\hat{O}^x_{\mathcal{S}}\}$ as in Eq. (22). For general θ, the number of such binary strings scales as $|\mathcal{B}| = 2^L$. When $\theta = \pi/2$, the number of different obtainable binary strings (and thus a lower bound on the degeneracy) is $|\mathcal{B}| = 2$ (as in Sec. III C). If, in approaching the thermodynamic limit, the same degeneracy is found irrespective of the tilt angle θ then we see how a degeneracy of 2^L must appear for a square lattice oriented along the Cartesian axes.

In fact, this is an example of the consequences of our general theorem 2. Towards this end, we may identify $U_1 = \hat{O}^y_{\ell'=1}$, $V_1 = \hat{O}^x_{\ell=1}$, $U_2 = \hat{O}^y_{\ell'=2}$, $V_2 = (\hat{O}^x_{\ell=1}\hat{O}^x_{\ell=2})$, $\dots, U_{1<a\leqslant L} = \hat{O}^y_{\ell'=a}$, $V_{1<a\leqslant L} = (\hat{O}^x_{\ell=a-1}\hat{O}^x_{\ell=a})$, $\dots$ so that $\mathcal{M} = 2^L$. (See Ref. [102] for further comparison to the case of conventional boundary conditions.)

Interestingly, our open parallelogram is a bipartite lattice with sublattices Λ_A and Λ_B of equal cardinality. One can construct the unitary operator

$$\mathcal{U}_{\rm ch} = \prod_{r\in\Lambda_A}\sigma^z_r, \quad (23)$$

such that $\{H_{\rm open}, \mathcal{U}_{\rm ch}\} = 0$, meaning that the spectrum of $H_{\rm open}$ is symmetric with respect to zero and, thus, $\mathcal{U}_{\rm ch}$ is a chiral symmetry. This result is valid for any bipartite lattice.

Thus far, in this subsection, we considered a cut of the cylinder of Fig. 2 to generate the open system of $H_{\rm open}$ for which we can establish the exponential degeneracy. Along nearly identical lines, we may similarly examine a torus that is cut a general angle $\theta \neq \pi/2$ and opened up to as produce an oblique cylinder. For such general angles θ, theorem 2 will then imply the existence of an exponential ground-state degeneracy on the resultant oblique cylinders.

Similar constructs may be devised for many other boundary geometries. For instance, placing the square lattice on a closed cone (with the generating line of the cone defining the y axis and its base parallel to the x axis) instead of the cut cylinder of Fig. 2, would, for general opening angles of the cone, enable a proof a degeneracy that would scale, once again, exponentially in the linear system dimension. One may similarly examine a hybrid of two such cones sharing the same base and in an analogous fashion also for a trapezoid. A particular realization of these other geometries is that of the "Aztec diamond" lattice (a square lattice rotated at 45° so as to have a diamond shaped boundary) [103–105]. More generally, a repetition of the above steps for generic geometries in which the top and bottom boundaries (or, similarly, left and right boundaries) are shifted relative to one another by a distance that is $\mathcal{O}(L)$, will yield a lower bound of the logarithm of the degeneracy that is of order $\mathcal{O}(L)$. Further yet, numerous other open boundaries exist other than the above boundary conditions that correspond to different cuts of cylindrical or toroidal systems. As the reader can indeed see, the boundaries of generic open two-dimensional systems of linear size L will allow for $\mathcal{O}(L)$ independent $d=1$ symmetries. Thus *the logarithm of degeneracy of general physical realizations of two-dimensional systems will scale as* $\mathcal{O}(L)$. The same considerations can be extended to arbitrary dimensions.

In addition to modifying boundary conditions to go to the open parallelogram or other systems for which we can prove our theorem, we can also modify the *internal geometry* by inserting *dilute vacancies* such that the conditions of theorem 2 are satisfied. For the PCM on an $L \times L$ square lattice, we can remove a single site from every column at a different height, i.e., we insert L vacancies. Then, an exponential degeneracy is assured by theorem 2.

Along related lines, we may arrive at similar conclusions by noting that there is, in the conventional square lattice boundary condition geometry of Section III C, a two-dimensional representation of all of the algebraic relations, Eqs. (16) and (17), and thus a required two-dimensional degeneracy of each level yet not beyond that. That is, for the conventional boundary conditions, we may represent each of these horizontal/vertical $d=1$ symmetries as, respectively, the same single Pauli y or same Pauli x operator while still maintaining all of the algebraic relations. We cannot do this for the modified boundary conditions discussed in this subsection since, by our construction, the algebraic relations between the operators of Eq. (15) are no longer uniform. Depending on the lines on which they act, the operators $\hat{O}^{\mu}_{\ell}$ will commute or anticommute with different other subsets of these $d=1$ symmetries. Then, this increase of dimensionality of the smallest irreducible representation capturing the algebra (a representation of minimal size 2^L) mandates that each of the levels is exponentially degenerate.

Before concluding this subsection, we briefly return to corollaries 4 and 5. In the context of corollary 4, we note that if the region Ω_D is a parallelogram (as it is in the current example of the PCM) then we may take its subset, for instance, to be a square that can be inscribed in the parallelogram Ω_D. For situations in which we can prove exponential degeneracies for $H_{\rm open}$ augmented by a number of additional boundary terms (e.g., any number of horizontal bonds in our example of the PCM and a reduction by a factor of a half in our lower bound on the degeneracy when any vertical bond was added), we may similarly remove any subset of the bonds to establish, at least, the same degeneracies. With reference to corollary 5, we may take Ω_D to be of any shape (including the open square lattice of Sec. III C) and $\Omega_{\rm sub} \subset \Omega_D$ to be a parallelogram of linear scale L inscribed within it. $\Omega_{\rm sub}$ may similarly be any other subvolume of Ω_D such that the Hamiltonian $H_{\rm open}$ on it satisfies the requirements of theorem 2. Corollary 5 then asserts that if ensemble equivalence holds (as expected in the thermodynamic limit), given the $T=0$ ground-state sector density matrix on Ω_D, the reduced density matrix of Eq. (11) on $\Omega_{\rm sub}$ yields an entropy satisfying Eq. (12).

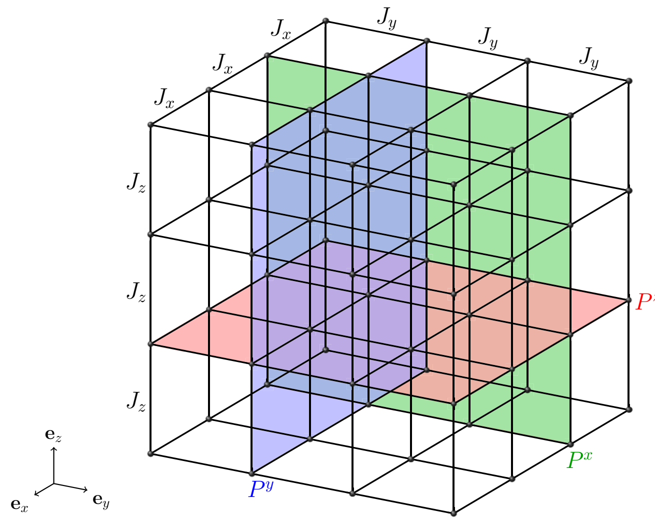

FIG. 3. The cubic lattice compass model and its symmetries of Eq. (24) (see text).

E. The cubic lattice 90° compass model

Complementing Kitaev's honeycomb model [101], the above square lattice compass model may be trivially extended to other geometries [14]. In particular, the cubic ($L \times L \times L$) lattice 90° compass model Hamiltonian $H_{\rm CCM}$ is given by the righthand side of Eq. (13) with the sum over μ now spanning the three external spatial Cartesian directions $\mu = x, y, z$ and the three associated internal spin components (see Fig. 3). This model features $d=2$ symmetries given by Eq. (15) where, on the cubic lattice, $\mu = x, y,$ and z and $C^{\mu}_{\ell} \perp \mathbf{e}_{\mu}$ now become $d=2$ dimensional planes $P \perp \mathbf{e}_{\mu}$. That is, rather explicitly,

$$\hat{O}^{\mu}_{P} = \prod_{r \in P} i\sigma^{\mu}_{r} \text{ , for } \mu = x, y, z. \tag{24}$$

Similar to the discussion in Sec. III B, in the large spin classical ($S \to \infty$) limit, the $d=2$ symmetries that this system hosts become reflections about internal spin directions. The exponential degeneracy in the number of planes L similarly follows. We now discuss the $S=1/2$ quantum model.

With the trivial replacement of the $d=1$ dimensional line label ℓ by that of the $d=2$ dimensional plane P, the symmetries $\hat{O}^{\mu}_{P}$ satisfy the commutation relations of Eq. (16), i.e., $[\hat{O}^{\mu}_{P}, \hat{O}^{\mu}_{P'}] = 0$. Whenever there are, at the intersection of two orthogonal planes $P \perp \mathbf{e}_\mu$ and $P' \perp \mathbf{e}_{\mu' \neq \mu}$, an odd number of sites, we will find the $D = 3$ dimensional analog of Eq. (17), i.e., the anticommutator

$$\left\{\hat{O}_P^\mu, \hat{O}_{P'}^{\mu'}\right\} = 0. \tag{25}$$

As in the square lattice compass model, the eigenstates of the Hamiltonian $H_{\rm CCM}$ will be of the form of Eq. (18), $|\psi\rangle = |\lambda_{\mu;1} \ldots \lambda_{\mu;L}, \{v\}\rangle$ with $\{\lambda_{\mu_1}, \lambda_{\mu_2}, \ldots \lambda_{\mu_L}\}$ now being the set of eigenvalues $\{\lambda_{\mu_P}\}$ of the symmetries $\{\hat{O}_P^\mu\}$ associated with the L parallel planes $\{P\}$ that are orthogonal to $\mathbf{e}_\mu$. So long as we ensure that intersecting orthogonal planes share an odd number of common sites so that Eq. (25) is satisfied, all of the previous considerations that detailed above for the square lattice may be repeated for the cubic lattice. In particular, if we consider boundary conditions satisfying this intersection property then we may examine three-dimensional variants of the deformed boundary conditions of Sec. III D. Given the L symmetry eigenvalues $\{\lambda_{\mu_P}\}$, repeating the considerations for the square lattice compass model for the cubic compass model, we find that *each energy eigenvalue* (whether that of a ground state or an arbitrary excited state) enjoys, once again, a degeneracy that is an integer multiple of 2^L.

F. Continuous symmetry compass type models

All of the symmetries in the compass model examples discussed thus far were of a discrete ($\mathbb{Z}_2$) nature. A simple model harboring a continuous U(1) symmetry is given by the cubic lattice Hamiltonian

$$H_{\rm U(1)} = -J \sum_{r, \nu \neq x} \left(\sigma_r^y \sigma_{r+\mathbf{e}_\nu}^y + \sigma_r^z \sigma_{r+\mathbf{e}_\nu}^z\right) + H_I\left(\left\{\sigma_r^x\right\}\right), \tag{26}$$

where $H_I(\{\sigma_r^x\})$ is any (Ising type) Hamiltonian in the operators $\{\sigma_r^x\}$. The Hamiltonian of Eq. (26) is invariant under continuous rotations in the planes $P \perp \mathbf{e}_x$,

$$\prod_{r \in C_P^x} e^{i\theta\sigma_r^x/2}. \tag{27}$$

In the taxonomy of Refs. [11–14,17–20], the operators of Eq. (27) are $d = 2$ dimensional (since the planes P are two-dimensional) U(1) symmetries. On an $L \times L \times L$ lattice, there are L such planes P and thus L symmetries of the type of Eq. (27).

A more realistic and much more isotropic compass type system displaying continuous higher SU(2) symmetries is that of the Kugel-Khomskii model [14,72], that we briefly alluded to in the Introduction. The three-dimensional Kugel-Khomskii model was first introduced [72] to describe super-exchange in transition metal systems. This model features planar SU(2) symmetries associated with each of the individual ($d = 2$) planes [14,21].

From a generalization of Elitzur's theorem [11,12] that invokes an extension [106] of the Mermin-Wagner-Coleman theorem [107,108] to zero temperature gapped systems, continuous $d \leqslant 2$ symmetries in systems having a spectral gap cannot be broken even at zero temperature nor can they be broken at finite temperatures (i.e., at positive energy density above the ground state) in both gapped and gapless systems. Various U(1) (and SU(2)) symmetric systems harbor a spectral gap, e.g., Ref. [109] (including the AKLT spin chain [110]). In the context of the focus of the current paper, it follows that in gapped renditions of the above and similar systems, two-dimensional (and/or other continuous) U(1) symmetries cannot be broken. The above U(1) symmetric model illustrates how continuous higher symmetries may appear in lattice systems.

In what follows, when referring to examples, the canonical model that we will refer to will largely be that of the spin $S = 1/2$ PCM of the earlier sections.

IV. EFFECTS OF BOUNDARY CONDITIONS ON SPECTRAL DEGENERACIES

We next turn to the influence of the boundaries on spectral degeneracies. We will do so by amending the open system Hamiltonian $H_{\rm open}$ on the volume Ω_D by additional terms (whose sum is the "zipper Hamiltonian" $H_{\rm zip}$) that will capture the effects of changing the boundary conditions. In effect, we will "stitch" back the open system to form a closed surface. Intuitively, one may anticipate $H_{\rm zip}$ to not radically influence the spectrum of the bulk system. As we will argue, the effect of $H_{\rm zip}$ on the system degeneracy may indeed be rather modest (vanishing in the thermodynamic limit) in certain situations. Augmenting corollaries 4 and 5, this will strengthen the plausibility of maxim **(b)** of Introduction as it pertains to boundary effects. To make our discussion clear, we first explain what $H_{\rm zip}$ is for the PCM of Eq. (13) and then outline results for general systems.

We proceed with the "surgery" outlined in Fig. 2. Following the incision of the lattice on the cylinder (left panel of that figure) and the "flattening" of the square lattice leading to a open parallelogram boundary (right panel), we may "sew" back the missing cut links associated with the lattice sites that lie on the boundaries of the open square lattice. We do so as to exactly reproduce the Hamiltonian of Eq. (13) on the original square lattice on the cylinder.

Within the coordinate frame of the open square lattice, all of the interactions contained in $H_{\rm zip}$ (linking one boundary site on one side of the system to another on the opposite side) are long ranged (of spatial separation that is of the order of the system size $\mathcal{O}(L) \to \infty$ in the thermodynamic limit). Referring to Fig. 2, we may write

$$\begin{aligned} H_{\rm zip} &\equiv -\sum_{r \in B_L} \left(J_x\, \sigma_r^x \sigma_{r+(L-1)\mathbf{e}_x}^x + J_y\, \sigma_r^y \sigma_{r+(L-1)\mathbf{e}_x+\mathbf{e}_y}^y\right) \\ &\equiv \sum_{\partial\gamma} b_{\partial\gamma}. \end{aligned} \tag{28}$$

The unitary operator $\mathcal{U}_{\rm ch}$ of Eq. (23) anticommutes with $H_{\rm zip}$ implying that like $H_{\rm open}$ it, too, trivially has a spectrum that is symmetric about zero. For the PCM, the index $\partial\gamma$ in Eq. (28) labeling the boundary bonds forming $H_{\rm zip}$ is comprised of the spatial location $r \in B_L$ marking the line of sites lying to the left of the "cylindrical" cut and the index $\mu = x, y$ denoting the internal components of the spins appearing in the first equality of Eq. (28). In the (uncut) cylindrical system on the left panel of Fig. 2, these long range terms $b_{\partial\gamma}$ correspond to nearest neighbor interactions (see also the staircase on the righthand panel of that figure), and $H_{\rm zip}$ is a one-dimensional Hamiltonian having a rather trivial spectrum (Appendix A).

The complication arises for the full spectral problem posed by the cylindrical Hamiltonian

$$H_{\rm cyl} = H_{\rm open} + H_{\rm zip}. \tag{29}$$

In the zipper Hamiltonian bonds associated with interactions between spins that lie on the same row commute with all of the symmetries $\hat{O}^y_{\ell'}$ and thus do not lift the degeneracy of the states of Eq. (21) associated with their eigenvalues. By contrast, the vertical bonds in $H_{\rm zip}$ do not commute with these symmetries and thus may lift the earlier found exponential degeneracy.

Indeed, both the open cylindrical cut Hamiltonian $H_{\rm open}$ and the closed cylinder Hamiltonian $H_{\rm cyl}$ commute with the horizontal line symmetries $\{\hat{O}^\mu_{\ell'}\}$ of Eq. (15) (and thus, trivially, with their products). However, it is only for $H_{\rm open}$ that we are able to rigorously establish the existence of an exponential degeneracy for arbitrary linear system size L. In the PCM, if we add (i.e., "stitch back") a subset of $L^{\rm vert}_{\rm zip}$ vertical bonds that appear in Eq. (28) and further similarly add (or "sew back") all of the horizontal bonds appearing in Eq. (28) to $H_{\rm open}$ then we will be left with $(L - L^{\rm vert}_{\rm zip})$ horizontal symmetries $\{\hat{O}^\mu_{\ell'}\}$. Repeating *mutatis mutandis* the steps of Sec. III, we will then be able to prove that each of the eigenstates has a degeneracy that is an integer multiple of $2^{L-L^{\rm vert}_{\rm zip}}$.

In the common eigenbasis of the symmetries $\{U_a\}$ (i.e., the symmetries $\hat{O}^y_{\ell'}$ commuting with $H_{\rm cyl}$) and $H_{\rm cyl}$, the effect of $\hat{O}^x_\ell$ is to flip the sign of the single vertical line C^x_ℓ that has P as one of its endpoints. Furthermore, the open string $\hat{O}^x_\ell$ flips the sign of all of the $\hat{O}^y_{\ell'}$ eigenvalues $\{\lambda_1, \dots, \lambda_L\}$ associated with horizontal lines $C^y_{\ell'}$ that intersect with C^x_ℓ. In the ground state, one would expect all of the vertical links to have a negative contribution to the energy.

Although the first equality of Eq. (28) is specific to the PCM, the more abstract second equality may apply to any system. In all such systems, when summed over all $\partial\gamma$, the individual long range boundary bond operators will form the relevant zipper Hamiltonian. In general, the boundary bonds $b_{\partial\gamma}$ are functions of local operators (fields) ϕ^μ_r associated with the boundary $r \in B_L$. Whenever we change the boundary conditions for an arbitrary local Hamiltonian (that may differ from that of the PCM), we will be able to express $H_{\rm zip}$ as a sum of such local terms. Similarly, instead of sewing the boundaries of the open parallelogram to form a cylinder, one may repopulate the *dilute vacancies* discussed towards the end of Sec. III D to form other zipper type Hamiltonians (in which the Hilbert space will now be increased) that are a sum of local terms.

Generically, the full Hamiltonian involving $H_{\rm zip}$ does not commute with $H_{\rm open}$. This noncommutativity makes the analysis using symmetries less obvious. Nonetheless, rather broad conclusions may still be drawn. In what follows, we outline the general structure of the exact eigenstates when the effects of the zipper Hamiltonian $H_{\rm zip}$ are included. We then discuss the diagonal matrix elements of $H_{\rm zip}$ in the eigenbasis of $H_{\rm open}$ in systems with a spectral gap. The latter result shows that, to lowest order in perturbation theory, exponential degeneracy remains asymptotically unchanged in the thermodynamic limit. Henceforth, our discussion will become more qualitative.

We now return to the general problem (not focusing our discussions to the example of the PCM) and reformulate some of our considerations more broadly. *Since the symmetries $\{U_a\}$ are not lifted by $H_{\rm zip}$*, it follows that the eigenstates of $H_{\rm cyl}$ may be expressed as a linear superposition of eigenstates of $H_{\rm open}$, in general of different energy [Eq. (5)], with the same symmetry eigenvalues, i.e., as

$$|\lambda_1\lambda_2\dots\lambda_L, \{\nu_{\rm cyl}\}\rangle = \sum_{\{\nu\}} c_{\lambda_1\lambda_2\dots\lambda_L,\{\nu\}} |\lambda_1\lambda_2\dots\lambda_{\mathcal{M}}, \{\nu\}\rangle. \tag{30}$$

We underscore that in the above sum, sectors of differing quantum numbers $\{\nu\}$ of the open system eigenstates may be mixed. By contrast, different eigenvalues of the symmetry operators U_a *cannot* be superposed. Thus, rather trivially, $H_{\rm zip}$ cannot directly lift the degeneracy associated with the symmetries $\{U_a\}$ by mixing states with different eigenvalues of these symmetries. In a related vein, since $H_{\rm cyl}$ commutes with the symmetries $\{U_a\}$, it follows that all of the projected Hamiltonians $P_{\{\nu\}} H_{\rm cyl} P_{\{\nu'\}}$ are block diagonal in the eigenspace of $H_{\rm open}$. This structure of Hamiltonian (and ensuing form of its eigenstates) is reminiscent of that in textbook type [16] Bloch systems wherein mixing may occur only between states, differing by reciprocal lattice vectors $\vec{K}$, in different Brillouin zones (playing the role of $\{\nu\}$) yet not between states belonging to the same Brillouin zone (with crystal momentum labels $\vec{k}$ in the first Brillouin zone replaced here by the eigenvalues $\lambda_1\lambda_2\dots\lambda_L$). Augmenting Eq. (30), as the commutation relations $[H_{\rm cyl}, U_a] = 0$, for all $a = 1, 2, \dots, \mathcal{M}$, further make clear, symmetry considerations trivially reduce the eigenvalue problem of $H_{\rm cyl}$ to that in the $2^{\mathcal{M}}$ decoupled sectors labeled by different eigenvalue strings $\{\lambda_1 \dots \lambda_{\mathcal{M}}\}$. In each such sector, we need to diagonalize the projected Hamiltonian $P_{\lambda_1\dots\lambda_{\mathcal{M}}} H_{\rm cyl} P_{\lambda_1\dots\lambda_{\mathcal{M}}}$ with $P_{\lambda_1\dots\lambda_{\mathcal{M}}}$ denoting the projection operator to a set of fixed higher symmetry eigenvalues. The union of the eigenvalues of these projected Hamiltonians over all $\{\lambda_1 \dots \lambda_{\mathcal{M}}\}$ trivially forms the complete spectrum of $H_{\rm cyl}$. Since $[H_{\rm open}, V_a] = 0$, the eigenvalues of the projected Hamiltonians $P_{\lambda_1\dots\lambda_{\mathcal{M}}} H_{\rm open} P_{\lambda_1\dots\lambda_{\mathcal{M}}}$ are the same in different sectors $\{\lambda_a\}_{a=1}^{\mathcal{M}}$, this is, in essence, the content of lemma 1. As we noted previously, for the very same reasons, the demonstration of exponential degeneracy can be further extended to the eigenstates of $H^c_{\rm cyl} \equiv H_{\rm open} + H^c_{\rm zip}$ where $H^c_{\rm zip}$ is the sum of all of the bonds appearing in $H_{\rm zip}$ that commute with all of the operators $\{V_a\}_{a=1}^{\mathcal{M}}$. In the context of the PCM, $H^c_{\rm zip}$ is the sum of all horizontal boundary terms appearing in Fig. 2 connecting sites that lie along opposite sides of the cylindrical cut. However, as we further underscored for the full the zipper Hamiltonian $H_{\rm zip}$ (not the sum $H^c_{\rm zip}$ of the subset of bonds in $H_{\rm zip}$ that commute with $\{V_a\}_{a=1}^{\mathcal{M}}$), due to Eq. (29) the commutator $[H_{\rm zip}, V_a] = [H_{\rm cyl}, V_a] \neq 0$ for general V_a. As a consequence of this nonvanishing commutator, the eigenvalue problem of $P_{\lambda_1\dots\lambda_{\mathcal{M}}} H_{\rm cyl} P_{\lambda_1\dots\lambda_{\mathcal{M}}}$ in different sectors $\lambda_1 \dots \lambda_{\mathcal{M}}$ is not identically the same. This discrepancy led to the aforementioned possible removal of the spectral degeneracy. This lifting of the degeneracy is associated with the mixing of states of different quantum numbers $\{\nu_{\rm cyl}\}$ in Eq. (30) (as noted, the latter labels include the respective eigenvalues of $H_{\rm open}$).

As we elaborate in Appendix B, the Wigner-Eckart theorem may be naturally extended for higher symmetries. This trivial generalization leads to selection rules on nonvanishing matrix elements of general operators (including the zipper Hamiltonian) in the common eigenbasis of H_{open} and its symmetries.

The expectation values of H_{zip} in the cylindrical states $|\lambda_1\lambda_2\ldots\lambda_L, \{\nu_{\text{cyl}}\}\rangle$ of Eq. (30) are trivially equal to the energies associated with H_{zip} on these states on the closed cylinder. Since H_{zip} is a sum of short range terms on the cylinder, the expectation values in these cylindrical states are, generally, finite (also within the thermodynamic $L \to \infty$ limit). For the translationally invariant ground states on the cylinder, these expectation value will equal to the average expectation per bond within the global ground state of H_{cyl} times the number of bonds appearing in H_{zip}.

We now briefly turn to a somewhat more specific discussion of theories (dependent on the local operators $\{\phi_r^\mu\}$) that exhibit higher symmetries. In the eigenbasis of H_{open}, the diagonal matrix elements of H_{zip} are the expectation values of the sum of these interactions ($H_{\text{zip}} = \sum_{\partial\gamma} b_{\partial\gamma}(\{\phi_r^\mu\})$) in the eigenstates of H_{open}. Note that in any eigenstate $\langle\psi|b_{\partial\gamma}|\psi\rangle$ is a sum of correlation functions involving the boundary local operators $\{\phi_r^\mu\}$. If the connected correlation function between local observables in eigenstates of H_{open} decays exponentially in distance between these observables then we may state another simple lemma.

Lemma 6. Consider the open system Hamiltonian H_{open} to be a sum of *interactions of finite range* and strength (Eq. (8)) such that the eigenstates of H_{open} do not support, in any eigenstate, infinite range nor algebraic correlations between local boundary fields ϕ_r^μ (i.e., for asymptotically large separation distance L, the correlation functions [111] are given by $C + \mathcal{O}(e^{-L/\xi})$ with C being a constant that depends only on the energy eigenvalue of H_{open}, and ξ a characteristic correlation length). Under these conditions, the diagonal elements of H_{zip} in the eigenbasis of H_{open} (i.e., $\langle\psi|H_{\text{zip}}|\psi\rangle = \sum_{\partial\gamma}\langle\psi|b_{\partial\gamma}|\psi\rangle$ with $|\psi\rangle$ an eigenstate of H_{open}) must tend to a uniform constant in the $L \to \infty$ limit.

Proof. H_{zip} is a sum of, at most, $\mathcal{O}(L^{D-1})$ boundary interactions between local boundary operators ϕ_r^μ that are a distance $\mathcal{O}(L)$ apart. Given the assumption above, each of the interaction terms $b_{\partial\gamma}$ formed by their products has, in a sector of fixed eigenvalue of H_{open}, a constant expectation value of C up to exponentially small corrections in L. This implies that the diagonal matrix elements of H_{zip} are a sum of $\mathcal{O}(L^{D-1})$ individual expectation values $\langle b_{\partial\gamma}\rangle$ that, up to the above stated uniform shift, are each bounded by decaying exponential in L. Such a sum is bounded from above by a number of order $\mathcal{O}(L^{D-1}e^{-L/\xi})$ and thus tends to zero in the $L \to \infty$ limit. ■

Thus, in the thermodynamic limit, the expectation values $\langle H_{\text{cyl}}\rangle$ in each of the exponentially many degenerate eigenstates of H_{open} are the same. We reiterate that, as our proof of lemma 6 illustrates, this property *emerges only in the asymptotic $L \to \infty$ limit*. For finite L, there are additional deviations $\mathcal{O}(L^{D-1}e^{-L/\xi})$ about the asymptotic uniform constant value of $\langle H_{\text{cyl}}\rangle$.

In those models in which H_{zip} is a positive semidefinite operator and $C = 0$ in the ground-state manifold of H_{open}, there is an (asymptotic) exponentially large degeneracy in the ground state of H_{cyl}. The proof of the latter assertion is nearly immediate as we now explain. In such all such models, each of the ground states of H_{open} of which, by theorem 2, there are exponentially many, may be taken as variational ground states for H_{cyl}. In each of these (linearly independent) variational states [112,113], lemma 6 asserts that the expectation value of H_{zip} tends, in the thermodynamic limit, to zero. On the other hand, positive semidefinite Hamiltonians H_{zip} imply that the ground states of H_{open} provide lower bounds to H_{cyl}. It follows that H_{cyl} enjoys precisely the same (asymptotic) exponentially large (in $L^{d'}$) ground-state degeneracy in the thermodynamic limit.

In Appendix C, we will review general arguments for the exponential decay of correlations in the ground states of Lorentz invariant gapped systems that admit a Wick rotation. If the thermal averages may be replaced by eigenstate averages (in the spirit of the Eigenstate Thermalization Hypothesis [114–117]) then it follows that all diagonal matrix elements of H_{zip} vanish in the thermodynamic limit.

In Appendix D, we specifically demonstrate that lemma 6 mandates that to lowest order in degenerate perturbation theory (in the perturbation H_{zip} that alters the system boundary conditions), the exponential degeneracy of H_{open} is not lifted.

In Appendix E, we review and extend the discussion of perturbation theory of a different sort. For the particular case of the PCM, such perturbative considerations complemented by numerical results for the isotropic model with periodic boundary conditions [9,10,77] suggest that, within its ground-state sector, the PCM may preferably order along the x or y spin directions. Thus a natural set of variational ground states for the PCM is given by the 2^L product states of spins along different columns (or rows) that are all positively or all negatively oriented along the x (or y) and superpositions thereof.

V. THEORIES WITH UV/IR MIXING DISPLAYING CONVENTIONAL IR BEHAVIOR

We now discuss the physical consequences of the degeneracies that we rigorously established for various boundary conditions and suggested for others. As we reviewed in the Introduction, the application of higher symmetries naturally leads to a mixing of IR and UV modes. This mixing is apparent in the ground states. Already at the classical level, the low-energy field configurations involve Fourier modes of both very low and very high wavenumbers. Similar to ($d = 0$) gauge symmetries, also $d = 1$ discrete symmetries or $d = 2$ continuous symmetries that give rise to low-energy short wavelength variations of the fields *cannot*, by the generalized Elitzur theorem [11,12], be spontaneously broken. Just as in gauge theories in which the local symmetries cannot be spontaneously broken, these low d-dimensional gaugelike symmetries do not preclude the existence of usual thermodynamic transitions. Indeed, just as the two-dimensional Ising model displays spontaneous symmetry breaking, general systems with higher $d \geqslant 2$ discrete symmetries (either exact or emergent) may be broken spontaneously [11,12].

We first briefly discuss the case of exact higher symmetries in which our earlier results establish a degeneracy of each energy level. Lemma 1 implies that if all quantum numbers $\{\nu\}$ in Eq. (5) are eigenvalues of mutually commuting operators that also commute with all of the symmetries $\{U_a\}$ and $\{V_a\}$ then the finite temperature partition function of the full system $Z_{\text{open}} = \mathrm{Tr}(e^{-\beta H_{\text{open}}})$ is trivially an integer multiple ($2^{\mathcal{M}}$) of the partition function associated with the trace of $e^{-\beta H_{\rm open}}$ over a restricted Hilbert space in which all eigenvalues λ_a (with $a = 1, 2, \ldots, \mathcal{M}$) are of a uniform sign. This, e.g., implies that if the higher symmetries are of the $\mathbb{Z}_2$ type (as in the PCM) then we may "fold" the system back onto the sector in which the symmetry eigenvalues are uniform and do not change when traversing the lattice. That is, the contribution to the partition function from states associated with a "UV" type rapid change of the symmetry eigenvalues $\{\lambda_a\}$ is identical to that from either of the uniform "IR" sectors of these symmetry eigenvalues in which they all obtain a uniform value, $\lambda_1 = \ldots \lambda_{\mathcal{M}} = \pm 1$. For more general higher symmetries, we may examine low temperature fluctuations about any of the exponentially many ground states. The contributions to the partition functions from those eigenstates associated with low-energy fluctuations about uniform states (states $\{|\psi_{\rm low,IR}\rangle\}$) are identical to those associated with their counterparts that are related by applying the symmetry operators V_a on these states $\{|\psi_{\rm low,IR}\rangle\}$. Notwithstanding an exponential degeneracy, the free energy density will thus be equal to that of the "IR" system that one may focus on.

We note (as was briefly mentioned in the Introduction) that in several theories featuring higher symmetries, an exact dimensional reduction may occur as may be established via dualities, e.g., Refs. [12,14,51,52]. The associated dual lower dimensional models [12,14,51,52] may belong to universality classes with conventional IR behaviors. These dualities have been used to establish exponential degeneracies in other systems, such as those trivially encountered when mapping spins to Majorana fermions in high dimensional interacting (Hubbard type and other) systems, as in, e.g., Ref. [118].

As we underscored earlier, some systems have symmetries that are not exact but rather only emerge at low energies as symmetries of the projected ground-state subspace, e.g., the classical 120° compass model that exhibits emergent discrete $d = 2$ symmetries in its ground-state sector [5,6,14]. In this and other theories with such higher symmetries, entropic fluctuations may stabilize ordering about a uniform state [5,6,14] or those with other symmetry allowed orders [11,12]. Where rigorous results exist, it is seen that at positive temperatures, whenever spontaneous symmetry can occur, UV/IR mixing may be lifted [5,6,14] and, at low temperatures, the system will display long wavelength fluctuations about the low-energy, higher symmetry allowed [11,12] orders. Such finite temperature entropic "order by disorder" stabilization effects appear in numerous other systems [14,86,119–121]. As we alluded to in Introduction, such symmetries may lead to a proliferation of minimizing modes on d-dimensional ("flat band" type) surfaces in **k** space [14]. Such numerous low-energy states can also lead to glassy dynamics and rich spatial structures [79,122,123].

VI. CONCLUSIONS

Our central result is that of theorem 2. Systems displaying spatially rigid higher (independent) symmetries that are embedded on general open geometries display an exponentially large spectral degeneracy. The situation for the conventional textbook Born-von Karman (i.e., periodic) boundary conditions is more subtle, since those symmetries may not be independent. We demonstrated that by modifying the boundary conditions, the independent higher symmetries may be guaranteed to transform given eigenstates differently. By the modification of the boundary conditions (or, as discussed in Sec. III D, internal geometry), the number of such independent symmetries can be made to be vary from finite values to a number that is exponentially large in the system size. This independence allows for a spectral degeneracy that is exponential in the number of independent symmetries. If this result does not depend on the boundary conditions then the degeneracy holds rather universally. We discussed specific conditions under which this degeneracy may persist asymptotically when the geometry is deformed to be that of the Born-von Karman or other types by the addition of external environments with which the system interacts or by the insertion of a "zipper Hamiltonian."

Our formal theorem for the *exponential degeneracy of systems with rigid higher symmetries may be rationalized more intuitively whenever the correlations are local*. Indeed, as noted in the Introduction and Sec. III B, in the classical analogs of the systems discussed, the degeneracy is exponentially large. This occurs since the localized rigid higher symmetry operations that act on different regions of space are, in the product type states appearing in the classical limit, independent of each other. The number $\mathcal{M}_{\rm cl}$ of these higher symmetries in the classical limit scales as $L^{d'}$, where L is the linear extent of the system residing in D spatial dimensions. Here, $d' = D - d$ with d characterizing the dimensionality of the higher symmetries (these symmetries act on volumes of size L^d). The degeneracy of each configuration of the classical system is, at least, exponential in $\mathcal{M}_{\rm cl}$. The subtlety that triggered the current paper originates from nontrivial correlations between the fields/spins. These correlations enable the entire quantum state of the system to transform as a singlet (or doublet, etc.) under different rigid higher symmetries. In other words, correlations allow for the situation in which a new independent quantum state does not appear whenever any of the numerous independent rigid higher symmetry operations (whose number we denoted by $\mathcal{M}$) is applied. However, if the correlations between the disparate independent symmetry operators that have their support on different spatial regions decay asymptotically with the distance between the regions of their support (say with a finite correlation length R) then the classical arguments as to why the logarithm of the degeneracy should scale with the system size might still be repeated verbatim for $\mathcal{M} = \mathcal{O}((L/R)^{d'})$ independent symmetries.

The above intuition can further rationalize why, for the modified boundary conditions that we employed in the current work, the exponential degeneracy may also rigorously persist also for finite size systems. Indeed, by comparison to the more conventional case, when our boundary conditions are employed, the fields/spins along the boundaries are attached to fewer other fields/spins and are thus less correlated with these. This, in turn, may allow for a larger independence between the higher symmetries enabling a higher degeneracy.

ACKNOWLEDGMENTS

We are extremely grateful to L. Pusey-Nazzaro and M. Oshikawa for numerous discussions, critical reading of the work, and comments. We are also grateful for discussions with A. Abanov, S. Nussinov, and G. Tarjus and to a question posed by K. Shtengel. Part of this work was performed at the Aspen Center for Physics (where some of our results were also presented) which is supported by National Science Foundation Grant No. PHY-1607611. G.O. acknowledges support from the US Department of Energy Grant No. DE-SC0020343.

APPENDIX A: SPECTRUM OF THE COMPASS MODEL ZIPPER HAMILTONIAN H_{zip}

Much of our discussion focused on the projected H_{zip} in the common eigenbasis of H_{open} and the symmetries in a sector fixed $\{\nu\}$ and $\{\nu'\}$. We may also readily compute the eigenvalues of H_{zip} sans such a projection. Towards that end, we simply view H_{zip} as a one-dimensional Hamiltonian on a $2L$ spin chain formed by the "zipper" Hamiltonian of Eq. (28) which we rewrite as

$$H_{\mathrm{zip}} = -J_x \sum_{j=1}^{L} \sigma_{2j-1}^x \sigma_{2j}^x - J_y \sum_{j=1}^{L-1} \sigma_{2j}^y \sigma_{2j+1}^y, \tag{A1}$$

with open boundary conditions. This Hamiltonian can be diagonalized analytically by using the techniques developed in Ref. [124]. Each eigenvalue is at least 2^L-fold degenerate. If we examine our H_{zip} on the original Hilbert space of the N spins lying on the cylinder of Fig. 2, each eigenvalue is, at least, 2^{N-L}-fold degenerate. In addition, the spectrum of H_{zip} is symmetric with respect to zero, since there exists a chiral operator equivalent to the $\mathcal{U}_{\mathrm{ch}}$ of Eq. (23) that satisfies $\{H_{\mathrm{zip}}, \mathcal{U}_{\mathrm{ch}}\} = 0$.

APPENDIX B: WIGNER-ECKART-TYPE SELECTION RULES AND SPARSITY OF THE ZIPPER HAMILTONIAN

Independent bounds, fortifying the considerations underlying our results, arise from the Wigner-Eckart theorem [91,93,125] as in its common applications to SU(2) and other symmetry groups [126]. Here, we extend these considerations to the matrix elements of the few body operators $b_{\partial\gamma}$ [Eq. (28)] in the higher ($d \geqslant 1$ dimensional) symmetry operator eigenbasis [17]. The arising symmetry constraints will, in particular, demand that each of the operators $b_{\partial\gamma}$ is a sparse matrix when written in the eigenbasis of H_{open} spanned by the states of Eq. (5). To see why this is so, we first consider as a general illustrative example, the textbook situation of an SU(2) symmetry eigenbasis labeled by the eigenvalues of the total squared angular momentum J^2 [eigenvalue $\mathrm{j}(\mathrm{j}+1)$] and J_z (eigenvalue m). Specifically, for the $\mathcal{M}$ independent higher form symmetries, we consider situations in which we express the zipper Hamiltonian H_{zip} as a sum of products of spherical tensors $\{T_\ell^{\mathrm{k}_\ell \mathrm{q}_\ell}\}_{\ell=1}^{\mathcal{M}}$ that transform irreducibly under each of these higher symmetries. With the full eigenvalue spectrum of H_{open} labeled by the higher symmetry eigenvalues along with any additional multiplet index $\{\nu\}$ (that includes the energy eigenvalue), the Wigner-Eckart theorem states that for any irreducible operator $T_\ell^{\mathrm{k}_\ell \mathrm{q}_\ell}$,

$$\begin{aligned}&\left\langle \mathrm{j'}_1\mathrm{m'}_1\ldots\mathrm{j'}_\ell\mathrm{m'}_\ell\ldots\mathrm{j'}_{\mathcal{M}}\mathrm{m'}_{\mathcal{M}}, \{\nu'\}\right| T_\ell^{\mathrm{k}_\ell\mathrm{q}_\ell} \left| \mathrm{j}_1\mathrm{m}_1\ldots\mathrm{j}_\ell\mathrm{m}_\ell\ldots\ldots\mathrm{j}_{\mathcal{M}}\mathrm{m}_{\mathcal{M}}, \{\nu\}\right\rangle \\ &= \frac{1}{\sqrt{2\mathrm{j}_\ell+1}} \left\langle \mathrm{j'}_1\ldots\mathrm{j'}_\ell\ldots\mathrm{j'}_{\mathcal{M}}, \{\nu'\}\right| T_\ell^{\mathrm{k}_\ell} \left| \mathrm{j}_1\ldots\mathrm{j}_\ell\ldots\mathrm{j}_{\mathcal{M}}, \{\nu\}\right\rangle \langle \mathrm{j'}_\ell\mathrm{m'}_\ell; \mathrm{k}_\ell\mathrm{q}_\ell | \mathrm{j}_\ell\mathrm{m}_\ell\rangle.\end{aligned} \tag{B1}$$

Here, $\langle \mathrm{j}'_\ell\mathrm{m}'_\ell; \mathrm{k}_\ell\mathrm{q}_\ell|\mathrm{j}_\ell\mathrm{m}_\ell\rangle$ denote the Clebsch-Gordan coefficients associated with the d-dimensional gaugelike symmetry of the ℓ-th layer. Thus, for any Hamiltonian H_{zip} that spans, at most, the spatial support of R independent d-dimensional symmetries disjoint regions, the matrix element between two orthogonal eigenstates of the $d \geqslant 1$-dimensional symmetry operator

$$\left\langle \mathrm{j'}_1\mathrm{m'}_1\ldots\mathrm{j'}_\ell\mathrm{m'}_\ell\ldots\mathrm{j'}_{\mathcal{M}}\mathrm{m'}_{\mathcal{M}}, \{\nu'\}\right| T_\ell^{\mathrm{k}_\ell\mathrm{q}_\ell} \left| \mathrm{j}_1\mathrm{m}_1\ldots\mathrm{j}_\ell\mathrm{m}_\ell\ldots\ldots\mathrm{j}_{\mathcal{M}}\mathrm{m}_{\mathcal{M}}, \{\nu\}\right\rangle = 0 \tag{B2}$$

in numerous instances. Given its above definition, R may be viewed as a measure of the "range of the interaction." If when contrasting the ket and bra of Eq. (B2), more than R different eigenvalue pairs $(\mathrm{j}_\ell\mathrm{m}_\ell)$ differ from one another then the associated matrix element of $T_\ell^{\mathrm{k}_\ell\mathrm{q}_\ell}$ will vanish. Furthermore, for those few symmetry eigenvalues that do differ from one another (i.e., those marking off-diagonal elements in the symmetry eigenbasis), additional constraints will appear if the tensors $T_\ell^{\mathrm{k}_\ell\mathrm{q}_\ell}$ are products of a finite number of single body operators (each of finite maximal angular momentum). Thus, when the off-diagonal matrix elements are nonzero, since the total (d-dimensional) angular momentum that a local operator can carry is finite, the eigenvalue differences $|\mathrm{j}_\ell - \mathrm{j'}_\ell|$ and $|\mathrm{m}_\ell - \mathrm{m'}_\ell|$ can assume, at most, system size independent values of order unity. Such constraints can indeed be readily extended for symmetries other than SU(2). Applying these symmetry selection rules illustrates that, for the compass and other local models, the matrix elements of various local operators in the eigenbasis of H_{open} of Eq. (5) vanish. Indeed, H_{zip} is diagonal in the symmetry projected eigenbasis of H_{open}. The application of the higher symmetry variant of the Wigner-Eckart theorem to other local operators generally illustrates that these may only have sparse nonvanishing matrix elements.

APPENDIX C: EXPONENTIAL DECAY OF THE MATRIX ELEMENTS OF THE PROJECTED H_{zip} IN THE EIGENBASIS OF GENERAL GAPPED SYSTEMS AS SUGGESTED BY WICK ROTATIONS IN LORENTZ INVARIANT THEORIES

We next briefly review considerations for exponential decay of correlations in gapped systems and extensions thereof to off-diagonal matrix elements. An insightful approach [127] for establishing the exponential decay of spatial correlations relies on quasi-adiabatic processes and the Lieb-Robinson bounds [128]. In what follows, we discuss and extend an earlier method that relates correlations and evolution in time to those in space. Specifically, we will briefly touch on an analytic continuation (specifically a Wick rotation) of Lorentz invariant theories or their effective nonrelativistic limit. This will allow us to relate temporal correlations (that are naturally associated with spectral gaps) to spatial correlations. We will extend the dynamics provided by a general Hamiltonian $H_{\rm open}$ with eigenvectors $\{|\psi_m\rangle\}$ to the Euclidean domain and illustrate that when the respective energy difference of the associated eigenstates is finite then the diagonal matrix elements of the additional bonds ($H_{\rm zip}$) that appear when the boundary conditions are changed,

$$\lim_{L\to\infty} \langle \lambda_{\mu;1}\dots\lambda_{\mu;L},\{\nu'\}|H_{\rm zip}|\lambda_{\mu;1}\dots\lambda_{\mu;L},\{\nu\}\rangle = 0. \quad \text{(C1)}$$

To establish Eq. (C1), we trivially apply standard Euclidean space demonstrations of the decay of correlations in the presence of a spectral gap (i.e., that of the diagonal component of the field bilinears) as in, e.g., Ref. [129]. We will consider, for a general Hamiltonian, equal time matrix elements of the product of two fields (spins). We define

$$F^\mu(r,r') \equiv \langle\psi_0|\phi^\mu_r\phi^\mu_{r'}|\psi_0\rangle = \sum_m \langle\psi_0|\phi^\mu_r|\psi_m\rangle\langle\psi_m|\phi^\mu_{r'}|\psi_0\rangle \quad \text{(C2)}$$

with $|\psi_0\rangle$ denoting a ground state and $|\psi_m\rangle$, $m\neq 0$, an excited eigenstate of $H_{\rm open}$. We next focus on each of the matrix elements in the sum of Eq. (C2) and evaluate these by analytic continuation to the field theory to Euclidean space where we can invoke rotational invariance. In the original theory, the two fields ϕ^μ_r and $\phi^\mu_{r'}$ at the sites r and r' lie on the same time slice. With some of abuse of notation with r and r' next denoting the corresponding Euclidean space vectors that include temporal coordinates, the separation $(r'-r)$ is, obviously, spacelike. We can express the two fields in terms of displacements from the field at the origin,

$$\phi^\mu_r = e^{-ipr}\,\phi^\mu_0\,e^{ipr},$$
$$\phi^\mu_{r'} = e^{-ipr'}\,\phi^\mu_0\,e^{ipr'}. \quad \text{(C3)}$$

On inserting the complete set of eigenstates of the Hamiltonian $H_{\rm open}$, Eq. (C2) becomes

$$F^\mu(r,r') = \sum_m \langle\psi_0|e^{-ipr}\,\phi^\mu_0\,e^{ipr}|\psi_m\rangle\langle\psi_m|e^{-ipr'}\,\phi^\mu_0\,e^{ipr'}|\psi_0\rangle. \quad \text{(C4)}$$

We next use the aforementioned translational and rotational invariance of the Euclidean theory to evaluate $F^\mu(r,r')$. With the aid of these symmetries, we may translate and perform a Wick rotation so as to arrive at $r\to(0,\vec{0})$ and $r'\to(\pm|t'-t|,\vec{0})$ [130]. The resulting vectors lie along the time axis of the Euclidean theory with $|t'-t|=|r'-r|$. With p denoting the energy-momentum 4-vector, this yields $e^{-ipr'}\,\phi^\mu_0\,e^{ipr'}\to e^{\mp H(t'-t)}\,\phi^\mu_0\,e^{\pm H(t'-t)}$. Here, the sign in the exponential is chosen so as to ensure a well-defined analytic continuation. This yields

$$F^\mu(r,r') = \sum_m |\langle\psi_0|\phi^\mu_0|\psi_m\rangle|^2 e^{-(E_m-E_0)|r'-r|}. \quad \text{(C5)}$$

To obtain the connected correlation function, i.e., the contribution to $F^\mu(r,r')$ from the excited states, we subtract from Eq. (C5) the ground-state products $\langle\psi_0|\phi^\mu_0|\psi_0\rangle^2$. The exponential factor in Eq. (C5) then suggests that, in gapped systems, the constant time connected correlation function must decay exponentially in the spatial separation $|r-r'|$. Equation (C5) therefore illustrates that if $H_{\rm zip}$ contains $\mathcal{O}(L)$ terms that each connect sites separated by a distance $|r-r'|=\mathcal{O}(L)$ then, in the presence of finite gaps, $E_m-E_0>0$, the diagonal matrix elements of $H_{\rm zip}$ in the eigenbasis of $H_{\rm open}$ will vanish in the $L\to\infty$ limit. Although, one may expect individual off-diagonal matrix elements of bounded local operators to decay with increasing system size (as in, e.g., the Eigenstate Thermalization Hypothesis [114–117]), the full contribution of the (exponential in size) number of off-diagonal matrix elements to the energy eigenvalues is more complex. Clearly not all off-diagonal matrix elements (i.e., those between different excited states $|\psi_m\rangle$) can be uniformly bounded such their sum vanishes exponentially with the distance $|r-r'|$ when the square norm $(\phi^\mu_0\phi^\mu_r)^\dagger(\phi^\mu_0\phi^\mu_r)$ is a constant and thus so is its expectation value in any state, e.g., $\langle\psi_m|(\phi^\mu_0\phi^\mu_r)^\dagger(\phi^\mu_0\phi^\mu_r)|\psi_m\rangle = \sum_n |\langle\psi_m|\phi^\mu_0\phi^\mu_{\bar r}|\psi_n\rangle|^2 = \text{const}$. An example of a situation in which such a square norm is constant is that of the Pauli bilinears appearing in the zipper Hamiltonian of the compass model [Eq. (28)]. When ϕ^μ_r are Pauli operators, the latter "const" is equal to one.

APPENDIX D: LOWEST-ORDER DEGENERATE PERTURBATION THEORY

In this short Appendix, we establish a simple theorem:

Theorem 7. Given the assumptions of lemma 6, in the thermodynamic $L\to\infty$ limit, to lowest order in degenerate perturbation theory, the exponential degeneracy of $H_{\rm open}$ will not be lifted by the perturbation $H_{\rm zip}$.

Proof. To lowest order in degenerate perturbation theory, we need to diagonalize $H_{\rm zip}$ in a projected fixed energy eigenbasis of $H_{\rm open}$. In general, $H_{\rm zip}$ contains off-diagonal matrix elements that may mix the different degenerate states of $H_{\rm open}$ that have different eigenvalues of the symmetry operators. The diagonalization of $H_{\rm zip}$ in this exponentially large space of degenerate eigenstates of $H_{\rm open}$ is not at all trivial. However, the assumption of lemma 6 simplifies the problem considerably and immediately leads to the above stated result. The diagonalization of $H_{\rm zip}$ in the projected subspace of degenerate states of $H_{\rm open}$ will trivially lead to states that are still eigenstates of $H_{\rm open}$ having the same fixed eigenvalue of $H_{\rm open}$. By lemma 6, given *any* such eigenstate $|\psi\rangle$ of $H_{\rm open}$, the corresponding expectation value associated the correlator between the boundary fields that are a distance L apart,

$$\langle\psi|b_{\partial\gamma}(\{\phi^\mu_r\})|\psi\rangle = C + \mathcal{O}(e^{-L/\xi}), \quad \text{(D1)}$$

with C a fixed constant independent of the state $|\psi\rangle$ (dependent only on the energy eigenvalue of $H_{\rm open}$), and ξ a correlation length.

Thus the proof of lemma 6 demonstrates that, in the thermodynamic $L\to\infty$ limit, the expectation value of any of the terms appearing in $H_{\rm zip}$ will tend to a uniform constant value. As we emphasized above, since the condition underlying lemma 6 is made for *any* eigenstate of $H_{\rm open}$, it will, in particular, include also those specific eigenstates that diagonalize $H_{\rm zip}$ in the projected fixed energy eigenbasis of $H_{\rm open}$. These latter states are the zeroth order eigenstates in degenerate perturbation theory. The eigenvalues of these states yield the energy to first order in degenerate perturbation theory. We thus see that, to this order in perturbation theory, all corrections due to $H_{\rm zip}$ correspond to a uniform energy shift. Thus, *to first order in perturbation theory*, when $L \to \infty$, *the, at least,* $2^{\mathcal{M}}$*-fold degeneracy of* $H_{\rm open}$ *is not lifted by* $H_{\rm zip}$. ■

We briefly remark in [131] on our PCM example where $H_{\rm zip}$ is a bilinear in the boundary spins and $\langle H_{\rm zip}\rangle$ becomes a sum of two-point correlators. In systems with a spectral gap (as further discussed in Appendix C for Euclidean theories), the ground-state correlation functions may indeed decay exponentially with the distance between the local fields.

APPENDIX E: ANOTHER (HIGHER-ORDER) PERTURBATION THEORY

We now sketch a different perturbative approach that generalizes those introduced in [9,10] for the PCM. The perturbation theory that we consider is that for rather general systems that exhibit higher symmetries. Specifically, we consider systems in D spatial dimensions displaying d-dimensional symmetries ($d < D$) for which the Hamiltonian can be expressed as

$$H = H_0 + H_{\rm pert}. \tag{E1}$$

Here, H_0 is a Hamiltonian that has its support on $\mathcal{M} = \mathcal{O}(L^{d'})$ decoupled d-dimensional regions $\{\mathcal{R}_a\}$ where the d-dimensional higher symmetries operate. Here, as throughout, $d' = D - d$ (Eq. (2)). The perturbative "interaction" $H_{\rm pert}$ couples these $\mathcal{M}$ regions each of dimension d to one another. In what follows, we will ask what transpires when $H_{\rm pert}$ is a sum of local operators. Following the steps that led to lemma 1, we see that, in the absence of the interaction term $H_{\rm pert}$, each level has a degeneracy that is an integer multiple of $2^{\mathcal{M}}$ (for spin-1/2 systems). The corresponding basis set of eigenstates can be expressed as the tensor product

$$\begin{aligned} &|\lambda_1\lambda_2\ldots\lambda_{\mathcal{M}}, \{\nu\}\rangle \\ &\quad = |\lambda_1, \{\nu_1\}\rangle \otimes |\lambda_2, \{\nu_2\}\rangle \otimes \ldots \otimes |\lambda_{\mathcal{M}}, \{\nu_{\mathcal{M}}\}\rangle. \end{aligned} \tag{E2}$$

Here, each of the wave functions $|\lambda_a, \{\nu_a\}\rangle$ has its support on the d-dimensional volume $\mathcal{R}_a$. As throughout, λ_a mark the eigenvalues of the symmetry operators U_a. Since $H_{\rm pert}$ is local, only terms that are of sufficiently high order $\mathcal{O}(||\mathcal{R}_a||) = \mathcal{O}(L^d)$ in perturbation theory might lift the degeneracy of the eigenstates of Eq. (E2). Thus, in the thermodynamic $L \to \infty$ limit, the exponential degeneracy may remain unchanged to all orders in perturbation theory. In [136], we briefly review and write this perturbation theory for the PCM which suggests that these simple product states as good variational ground states (which may become better with increasing system size L).

[1] E. Lieb, T. Schultz, and D. Mattis, Ann. Phys. **16**, 407 (1961).
[2] M. Oshikawa, Phys. Rev. Lett. **84**, 1535 (2000).
[3] M. B. Hastings, Phys. Rev. B **69**, 104431 (2004).
[4] O. Dubinkin, J. May-Mann, and T. L. Hughes, Phys. Rev. B **103**, 125133 (2021).
[5] Z. Nussinov, M. Biskup, L. Chayes, and J. van den Brink, Europhys. Lett. **67**, 990 (2004).
[6] M. Biskup, L. Chayes, and Z. Nussinov, Commun. Math. Phys. **255**, 253 (2005).
[7] A. Mishra, M. Ma, F.-C. Zhang, S. Guertler, L.-H. Tang, and S. Wan, Phys. Rev. Lett. **93**, 207201 (2004).
[8] Z. Nussinov and E. Fradkin, Phys. Rev. B **71**, 195120 (2005).
[9] B. Douçot, M. V. Feigelman, L. B. Ioffe, and A. S. Ioselevich, Phys. Rev. B **71**, 024505 (2005).
[10] J. Dorier, F. Becca, and F. Mila, Phys. Rev. B **72**, 024448 (2005).
[11] C. D. Batista and Z. Nussinov, Phys. Rev. B **72**, 045137 (2005).
[12] Z. Nussinov, G. Ortiz, and E. Cobanera, Ann. Phys. **327**, 2491 (2012).
[13] Z. Nussinov and G. Ortiz, Phys. Rev. B **77**, 064302 (2008).
[14] Z. Nussinov and J. van den Brink, Rev. Mod. Phys. **87**, 1 (2015).
[15] G. Ortiz and R. M. Martin, Phys. Rev. B **49**, 14202 (1994).
[16] N. W. Ashcroft and N. D. Mermin, *Solid State Physics* (Saunders College Publishing, Philadelphia, 1976).
[17] Z. Nussinov and G. Ortiz, Ann. Phys. **324**, 977 (2009).
[18] Z. Nussinov and G. Ortiz, Proc. Natl. Acad. Sci. U.S.A. **106**, 16944 (2009).
[19] E. Cobanera, G. Ortiz, and Z. Nussinov, Phys. Rev. Lett. **104**, 020402 (2010).
[20] E. Cobanera, G. Ortiz, and Z. Nussinov, Adv. Phys. **60**, 679 (2011).
[21] A. B. Harris, T. Yildirim, A. Aharony, O. Entin-Wohlman, and I. Ya. Korenblit, Phys. Rev. Lett. **91**, 087206 (2003).
[22] A. Paramekanti, L. Balents, and M. P. A. Fisher, Phys. Rev. B **66**, 054526 (2002).
[23] M. J. Lawler and E. Fradkin, Phys. Rev. B **70**, 165310 (2004).
[24] C. Xu and J. E. Moore, Phys. Rev. Lett. **93**, 047003 (2004).
[25] J. McGreevy, Annu. Rev. Condens. Matter Phys. **14**, 57 (2023).
[26] D. Gaiotto, A. Kapustin, N. Seiberg, and B. Willett, J. High Energy Phys. 02 (2015) 172.
[27] T. Banks and N. Seiberg, Phys. Rev. D **83**, 084019 (2011).
[28] M. Qi, L. Radzihovsky, and M. Hermele, Ann. Phys. **424**, 168360 (2021).
[29] Y.-H. Du, U. Mehta, D. X. Nguyen, and D. T. Son, SciPost Phys. **12**, 050 (2022).
[30] N. Seiberg and S.-H. Shao, SciPost Phys. **10**, 027 (2021).
[31] N. Iqbal and J. McGreevy, SciPost Phys. **13**, 114 (2022).
[32] F. Benini, C. Cordova, and P.-S. Hsin, J. High Energy Phys. 03 (2019) 118.
[33] C. Cordova, T. T. Dumitrescu, and K. Intriligator, J. High Energy Phys. 02 (2019) 184.
[34] K. Slagle and Y. B. Kim, Phys. Rev. B **96**, 195139 (2017).

[35] Y. You, T. Devakul, F. J. Burnell, and S. L. Sondhi, Phys. Rev. B **98**, 035112 (2018).
[36] D. T. Stephen, J. Garre-Rubio, A. Dua, and D. J. Williamson, Phys. Rev. Res. **2**, 033331 (2020).
[37] D. J. Williamson, A. Dua, and M. Cheng, Phys. Rev. Lett. **122**, 140506 (2019).
[38] W. Shirley, K. Slagle, and X. Chen, SciPost Phys. **6**, 041 (2019).
[39] T. Devakul, W. Shirley, and J. Wang, Phys. Rev. Res. **2**, 012059(R) (2020).
[40] H. Katsura and Y. Nakayama, Journal of High Energy Phys. 08 (2022) 072.
[41] W. Cao, M. Yamazaki, and Y. Zheng, Phys. Rev. B **106**, 075150 (2022).
[42] B. Heidenreich, J. McNamara, M. Montero, M. Reece, T. Rudelius, and I. Valenzuela, J. High Energy Phys. 09 (2021) 203.
[43] G. Arias-Tamargo and D. Rodriguez-Gomez, arXiv:2204.07523.
[44] Y. Choi, H. T. Lam, and S.-H. Shao, Phys. Rev. Lett. **129**, 161601 (2022).
[45] L. Bhardwaj, S. Schafer-Nameki, and J. Wu, Fortschr. Phys. **70**, 2200143 (2022).
[46] S. Elitzur, Phys. Rev. D **12**, 3978 (1975).
[47] A. Kitaev, Ann. Phys. **303**, 2 (2003).
[48] C. Chamon, Phys. Rev. Lett. **94**, 040402 (2005).
[49] J. Haah, Phys. Rev. A **83**, 042330 (2011).
[50] S. Vijay, J. Haah, and L. Fu, Phys. Rev. B **94**, 235157 (2016).
[51] Z. Weinstein, E. Cobanera, G. Ortiz, and Z. Nussinov, Ann. Phys. **412**, 168018 (2020).
[52] Z. Weinstein, G. Ortiz, and Z. Nussinov, Phys. Rev. Lett. **123**, 230503 (2019).
[53] The equations of motion remain invariant also when incorporating external thermal white noise in the two dual systems (having delta function inner products between the external noise sources at different points in space and time). This is so since (i) the delta function inner products between the external noise sources remain invariant under general unitary transformations [12] and (ii) dualities are unitary maps [19,20]. In the thermal equilibria of these dual systems, the dynamics are identical to those in conventional thermal low dimensional short range systems. Since the finite temperature autocorrelation times in these low dimensional systems, are finite and do not scale with the system size, it follows that due to the dimensional reduction (that is not only that associated with bounds but can also be made exact via dualities), many of the above stabilizer systems might not be immune to thermal fluctuations [13,51,52]. Thus some of these exactly solvable systems may not display glassy dynamics with long time correlations. Other theories, however, may exhibit glassy dynamics. Indeed, qualitatively, higher symmetries may naturally lead to a multitude of numerous degenerate and metastable states leading to nonuniform spatial structures and glassy dynamics. Proving the existence of such numerous degenerate states is indeed the central result of the current work.
[54] While the dynamics of the Chamon, Haah, and similar models that are dual to simple classical Ising systems may be trivial [13,51,52], we note that if one places additional restrictions on the allowed dynamics then glassy dynamics may arise. For instance, if in the classical plaquette models [132–134], only single spin flips are allowed then glassy dynamics will appear. The pattern of low-energy single defect motion in the plaquette models is very similar to those in the models of Chamon and Haah's code. In the triangular plaquette models, each defect leads to "Pascal's triangle" of flipped spins in its wake and these triangles can be an integer power of two. That the dynamics of the classical plaquette models is glassy is a consequence of constraining the system so as to only allow single spin flips. With open boundaries, the classical plaquette models [132–134] also have a number of independent spins (once a duality mapping is performed to the plaquette variables) that is linear in the system perimeter. This linear scaling is similar to the number of independent symmetries in the PCM of Sec. III. Such single spin flip dynamics are allowed by the transverse field (h) term in the Xu-Moore plaquette model [24] of Eq. (14) in its σ^z basis. Since the Xu-Moore model is dual [8,19,20] to the PCM, it exhibits thermodynamic and dynamic behaviors identical to those of the PCM (including its phase transition) [14]. See Ref. [135] for insightful detailed analysis of quantum variants of classical plaquette models from a different perspective.
[55] N. Seiberg and S.-H. Shao, SciPost Phys. **10**, 003 (2021).
[56] P. Gorantla, H. T. Lam, N. Seiberg, and S.-H. Shao, Phys. Rev. B **103**, 205116 (2021).
[57] S. Bravyi, B. Leemhuis, and B. M. Terhal, Ann. Phys. **326**, 839 (2011).
[58] W. Shirley, K. Slagle, and X. Chen, Phys. Rev. B **99**, 115123 (2019).
[59] A. Prem, J. Haah, and R. Nandkishore, Phys. Rev. B **95**, 155133 (2017).
[60] R. Nandkishore and M. Hermele, Annu. Rev. Condens. Matter Phys. **10**, 295 (2019).
[61] G. B. Halasz, T. H. Hsieh, and L. Balents, Phys. Rev. Lett. **119**, 257202 (2017).
[62] W. Shirley, K. Slagle, and X. Chen, Ann. Phys. **410**, 167922 (2019).
[63] W. Shirley, K. Slagle, Z. Wang, and X. Chen, Phys. Rev. X **8**, 031051 (2018).
[64] W. Shirley, K. Slagle, and X. Chen, SciPost Phys. **6**, 015 (2019).
[65] K. Slagle and Y. B. Kim, Phys. Rev. B **97**, 165106 (2018).
[66] M. Pretko and L. Radzihovsky, Phys. Rev. Lett. **120**, 195301 (2018).
[67] H. Ma, A. T. Schmitz, S. A. Parameswaran, M. Hermele, and R. M. Nandkishore, Phys. Rev. B **97**, 125101 (2018).
[68] T. Devakul, S. A. Parameswaran, and S. L. Sondhi, Phys. Rev. B **97**, 041110(R) (2018).
[69] N. Tantivasadakarn, W. Ji, and S. Vijay, Phys. Rev. B **103**, 245136 (2021).
[70] P.-S. Hsin and K. Slagle, SciPost Phys. **11**, 032 (2021).
[71] P. Sala, T. Rakovszky, R. Verresen, M. Knap, and F. Pollmann, Phys. Rev. X **10**, 011047 (2020).
[72] K. I. Kugel and D. I. Khomskii, Sov. Phys. JETP **37**, 725 (1973); Sov. Phys. Usp. **25**, 231 (1982).
[73] L. F. Feiner, A. M. Oles, and J. Zaanen, Phys. Rev. Lett. **78**, 2799 (1997).
[74] J. van den Brink, P. Horsch, F. Mack, and A. M. Oles, Phys. Rev. B **59**, 6795 (1999).
[75] D. Bacon, Phys. Rev. A **73**, 012340 (2006).

[76] A. van Rynbach, S. Todo, and S. Trebst, Phys. Rev. Lett. **105**, 146402 (2010).
[77] W. Brzezicki and A. M. Oles, Phys. Rev. B **87**, 214421 (2013).
[78] M.-S. Vaezi, G. Ortiz, and Z. Nussinov, Phys. Rev. B **93**, 205112 (2016).
[79] Z. Nussinov, Phys. Rev. B **69**, 014208 (2004).
[80] S. Sachdev and D. R. Nelson, Phys. Rev. B **32**, 1480 (1985).
[81] Z. Nussinov, arXiv:cond-mat/0105253.
[82] H. Yan and J. Reuther, Phys. Rev. Res. **4**, 023175 (2022).
[83] V. Cvetkovic, Z. Nussinov, and J. Zaanen, Philos. Mag. **86**, 2995 (2006).
[84] A. J. Beekman, J. Nissinen, K. Wu, K. Liu, R.-J. Slager, Z. Nussinov, V. Cvetkovic, and J. Zaanen, Phys. Rep. **683**, 1 (2017).
[85] E. Rastelli and A. Tassi, J. Phys. C: Solid State Phys. **20**, L303 (1987).
[86] C. L. Henley, Phys. Rev. Lett. **62**, 2056 (1989).
[87] A. Chubukov, Phys. Rev. Lett. **69**, 832 (1992).
[88] U. Hizi, P. Sharma, and C. L. Henley, Phys. Rev. Lett. **95**, 167203 (2005).
[89] U. Hizi and C. L. Henley, Phys. Rev. B **73**, 054403 (2006).
[90] I. Klich, S.-H. Lee, and K. Iida, Nat. Commun. **5**, 3497 (2014).
[91] C. Eckart, Rev. Mod. Phys. **2**, 305 (1930).
[92] H. Weyl, *The Theory of Groups and Quantum Mechanics* (Dover, New York, 1950).
[93] E. Wigner, *Group Theory and Its Application to the Quantum Mechanics of Atomic Spectra* (Academic Press, New York, 1959).
[94] M. Hamermesh, *Group Theory and its Application to Physical Problems* (Dover, New York, 1989).
[95] M. Tinkham, *Group Theory and Quantum Mechanics* (Dover, New York, 2003).
[96] E. Sagi, Y. Oreg, A. Stern, and B. I. Halperin, Phys. Rev. B **91**, 245144 (2015).
[97] We emphasize that we are discussing the exact degeneracy $g(E)$ associated with a precise given energy E of the full many-body system (a macroscopic quantity scaling with the system size) not the energy density (e.g., the energy per particle or energy per unit volume). Throughout the current work, we will spend considerable time discussing boundary effects. While boundary effects trivially do not change the energy density they may, a priori, significantly change the energy E itself.
[98] C. D. Batista and G. Ortiz, Adv. Phys. **53**, 1 (2004).
[99] H. Araki and E. Lieb, Commun. Math. Phys. **18**, 160 (1970).
[100] J. Barre, D. Mukamel, and S. Ruffo, Phys. Rev. Lett. **87**, 030601 (2001).
[101] A. Kitaev, Ann. Phys. **321**, 2 (2006).
[102] Specifically, if a vertical line ℓ and a horizontal line ℓ' intersect at a point then we may set, in the convention of Section II, for such a *single* vertical line ℓ, a unique symmetry operator to be $U_{a=1} = \hat{O}^x_\ell$ and its dual to be $V_{a=1} = \hat{O}^y_{\ell'}$. These two dual symmetry operators trivially satisfy both conditions of lemma 1 for this single ($\mathcal{M} = 1$) unique pair of dual symmetries. Indeed, for conventional boundary conditions applied to finite size systems [10], the PCM exhibits only a $2^{\mathcal{M}} = 2$ fold degeneracy. For the "cylindrical cut" and a large set of other boundary conditions, we may define additional symmetries U_a and V_a, $a > 1$, satisfying conditions (1) and (2) with $\mathcal{M} = \mathcal{O}(L)$. For standard open rectangular and periodic boundary conditions, the operators $\hat{O}^x_{\ell_1}$ for any vertical line ℓ_1 and the product of the symmetries $\hat{O}^y_{\ell'_1}\hat{O}^y_{\ell'_2}$ for any two horizontal lines ℓ'_1 and ℓ'_2 trivially commute with one another (satisfying the second relation of Eq. (4) for *all* ℓ_1, ℓ'_1, and ℓ'_2) while failing to generate a single nontrivial commutator [the first equality appearing in Eq. (4)]. This commutation relation follows since at a single common site r the Pauli operators σ^x_r and σ^y_r anticommute. Thus, as is the case for conventional boundary conditions, any pair of horizontal lines will intersect a vertical line at two common sites. Hence for these boundary conditions, $\mathcal{M} = 1$. Here, one may not set the operators U_a and V_a to be single line operators for a larger value of $\mathcal{M}$. This is so since any vertical line and any horizontal line intersect at a single point and thus lead to Eq. (17) making it impossible to satisfy the second equality in Eq. (4) for $\mathcal{M} > 1$.
[103] N. Elkies, G. Kuperberg, M. Larsen, and J. Propp, J. Algebr. Comb. **1**, 111 (1992).
[104] W. Jockusch, J. Propp, and P. Shor, arXiv:math/9801068.
[105] D. Romik, Ann. Probab. **40**, 611 (2012).
[106] A. Auerbach, *Interacting Electrons and Quantum Magnetism* (Springer-Verlag, New York, 1994), see, in particular, Sec. 6.3 on Quantum Disorder at $T = 0$.
[107] N. D. Mermin and H. Wagner, Phys. Rev. Lett. **17**, 1133 (1966).
[108] S. Coleman, Commun. Math. Phys. **31**, 259 (1973).
[109] F. D. M. Haldane, Phys. Lett. A **93**, 464 (1983); Phys. Rev. Lett. **50**, 1153 (1983); J. Appl. Phys. **57**, 3359 (1985).
[110] I. Affleck, T. Kennedy, E. H. Lieb, and H. Tasaki, Commun. Math. Phys. **115**, 477 (1988).
[111] We briefly provide the rationale as to why we generally expect this condition to hold in physical theories. In systems with local interactions, the terms $\{b_{\partial\gamma}\}$ in H_{zip} involve products of a finite number of fields $\{\phi^\mu_r\}$ that lie on the boundary. An expectation value $\langle b_{\partial\gamma}\rangle$ evaluated within such an eigenstate of H_{open} thus defines a correlation function amongst those particular local fields ϕ^μ_r that form $b_{\partial\gamma}$. Similar to standard procedure elsewhere, the latter correlation functions forming $\langle b_{\partial\gamma}\rangle$ may be expressed as products of the individual single site expectation values $\langle\phi^\mu_r\rangle$ of these local fields augmented, by definition, by a remaining connected correlation function. In theories in which H_{open} has a spectral gap, when evaluated in the ground-state sector of H_{open}, the latter connected correlation functions generally decay exponentially in the spatial separation between the local fields $\{\phi^\mu_r\}$. In excited states of H_{open} whose expectation values emulate finite temperature equilibrium averages in a thermal system defined by H_{open} (as in systems obeying the Eigenstate Thermalization Hypothesis [114–117]), these connected correlation functions must similarly decay with distance so long as the system is not critical. By the construction of the zipper Hamiltonian H_{zip}, at least two of the local boundary fields ϕ^μ_r appearing in $b_{\partial\gamma}$ must be a distance L away from each other. This leads to an exponentially small connected correlation function contribution to the expectation value $\langle b_{\partial\gamma}\rangle$. This exponential decay is captured by the $\mathcal{O}(e^{-L/\xi})$ term appearing in this condition in lemma 6. As we next explain, the products of the individual averages $\langle\phi^\mu_r\rangle$ of the boundary fields ϕ^μ_r that form $b_{\partial\gamma}$ are

anticipated to be a constant C that depends only on the energy eigenvalues of H_{open}. To appreciate why this is so, we note that the single site expectation values $\langle \phi_r^\mu \rangle$ as computed in an eigenstate of H_{open} of a specific eigenvalue E_{open} will, in systems satisfying the Eigenstate Thermalization Hypothesis, be equal to thermal averages of ϕ_r^μ at a temperature T for which the internal energy associated with H_{open} is equal to this energy eigenvalue E_{open}. In the absence of spontaneous symmetry breaking in the system defined by H_{open}, the latter averages of the local fields $\langle \phi_r^\mu \rangle$ will be of the same value for all eigenstates sharing the same energy E_{open}. This is why as in this condition underlying lemma 6, in physical theories, C is indeed expected to be a constant that depends only on the energy eigenvalue E_{open} of H_{open}. In some theories such as that of the PCM, use of the generalized Elitzur's theorem [11,12] illustrates that the individual fields appearing in $b_{\partial\gamma}$ cannot obtain finite expectation values. Consequently, in such systems, since the expectation values of each of the single local fields may vanish ($\langle \phi_r^\mu \rangle = 0$), the constant C formed by products of these single field expectation values may be zero as well.

[112] J. K. L. MacDonald, Phys. Rev. **43**, 830 (1933).

[113] M. D. Jones, G. Ortiz, and D. M. Ceperley, Phys. Rev. E **55**, 6202 (1997).

[114] J. M. Deutsch, Phys. Rev. A **43**, 2046 (1991).

[115] M. Srednicki, Phys. Rev. E **50**, 888 (1994).

[116] M. Rigol, V. Dunjko, and M. Olshanii, Nature **452**, 854 (2008).

[117] F. Borgonovi, F. M. Izrailev, L. F. Santos, and V. G. Zelevinsky, Phys. Rep. **626**, 1 (2016); W. Beugeling, R. Moessner, and M. Haque, Phys. Rev. E **91**, 012144 (2015).

[118] Z. Nussinov, G. Ortiz, and E. Cobanera, Phys. Rev. B **86**, 085415 (2012).

[119] J. Villain, R. Bidaux, J. P. Carton, and R. Conte, J. Phys. France **41**, 1263 (1980).

[120] E. F. Shender, Sov. Phys. JETP **56**, 178 (1982).

[121] R. Moessner, Can. J. Phys. **79**, 1283 (2001).

[122] J. Schmalian and P. G. Wolynes, Phys. Rev. Lett. **85**, 836 (2000).

[123] Z. Nussinov, I. Vekhter, and A. V. Balatsky, Phys. Rev. B **79**, 165122 (2009).

[124] A. Alase, E. Cobanera, G. Ortiz, and L. Viola, Phys. Rev. B **96**, 195133 (2017).

[125] E. P. Wigner, Am. J. Math. **63**, 57 (1941).

[126] V. K. Agrawala, J. Math. Phys. **21**, 1562 (1980).

[127] M. B. Hastings, arXiv:2111.01854v1.

[128] E. H. Lieb and D. W. Robinson, Commun. Math. Phys. **28**, 251 (1972).

[129] S. Nussinov and M. A. Lampert, Phys. Rep. **362**, 193 (2002).

[130] The first coordinates in r and r' are those of the times following the translation and Wick rotation while $\vec{0}$ pertains to the spatial location at the origin following this Wick rotation.

[131] To provide some physical intuition, we regress to the compass models. For asymptotically large spatial separation $|r - r'|$ between local operators that are coupled in a bilinear form in H_{zip}, the expectation values

$$\langle \phi_r^\mu \phi_{r'}^\nu \rangle \to C^{\mu\nu} + \mathcal{O}(e^{-|r-r'|/\xi}). \quad \text{(E3)}$$

Here, $C^{\mu\nu}$ is a constant and Eq. (E3) applies to states that lie in a sector of fixed energy. Given Eq. (E3), as $L \to \infty$, the spatial distance $|r - r'|$ between any spins that are coupled to each other in Eq. (28) diverges in the open lattice geometry of Fig. 2.

[132] M. E. J. Newman and C. Moore, Phys. Rev. E **60**, 5068 (1999).

[133] J. P. Garrahan and M. E. J. Newman, Phys. Rev. E **62**, 7670 (2000).

[134] A. Lipowski, J. Phys. A: Math. Gen. **30**, 7365 (1997).

[135] Z. Zhou, X.-F. Zhang, F. Pollmann, and Y. You, arXiv:2105.05851.

[136] Such an occurrence indeed happens in the PCM [9,10] where the ground states of the decoupled horizontal Ising chain Hamiltonian

$$H_0 = -\sum_r J_x \sigma_r^x \sigma_{r+\mathbf{e}_x}^x \quad \text{(E4)}$$

correspond to spins that are polarized along a uniform (i.e., the x) direction [9,10]. Since there are L decoupled Ising chains, the ground-state degeneracy of H_0 is 2^L. The local perturbation

$$H_{\text{pert}} = -\sum_r J_y \sigma_r^y \sigma_{r+\mathbf{e}_y}^y \quad \text{(E5)}$$

flips pairs of spins on neighboring horizontal chains. Only if $\mathcal{O}(L)$ such spins are flipped may one ground state of H_0 be linked to another. Indeed, for the PCM, the perturbative corrections due to H_{pert} are exponentially small in L [9,10]. We may consider the Hamiltonian of Eq. (E1) to be defined on a closed cylinder (so that it becomes the cylindrical Hamiltonian H_{cyl}). In this case, perturbation theory would suggest that, in the common eigenstates of the Hamiltonian H_0 and all of the higher symmetry operators, the expectation values of H_{pert} (and any of the local terms forming H_{pert}) monotonically decrease with L. In the $H_{\text{pert}} \to 0$ limit of the PCM, common eigenstates of the Hamiltonian $H = H_0$ and of all of the symmetries $\{\hat{O}_{C_\ell}^y\}$ are rather trivial to write down. These are formed by having, along each row, one of the two equal amplitude (i.e., symmetric or antisymmetric) superpositions of all of the spins fully polarized to the right or to the left, i.e., by direct products of states of the form $\frac{1}{\sqrt{2}}(|\rightarrow\rightarrow \dots \rightarrow\rangle \pm |\leftarrow\leftarrow \dots \leftarrow\rangle)$ along each one of the horizontal rows. Since there is a binary ($\pm$) choice for each for each row, there are 2^L such product states. By being fully polarized to the right or to the left, we allude here to all the spins being in the $(+1)$ eigenstate ($|\rightarrow\rightarrow \dots \rightarrow\rangle$) or all of the spins being in the (-1) eigenstate ($|\leftarrow\leftarrow \dots \leftarrow\rangle$) of all local Pauli operators σ_r^x that are associated with sites r belonging to that row. These states are eigenstates of each of the σ_r^x operators. Since the perturbative corrections decrease with the system size, these common eigenstates of H_0 and all of the symmetry operators might constitute good variational ground states. The above perturbative considerations suggest that the matrix elements of H_{pert} vanish as $L \to \infty$ and thus equal norm superpositions of all spins pointing to the right or left along different rows may be simply replaced by the 2^L product states of $|\rightarrow\rightarrow \dots \rightarrow\rangle$ along any one of the L rows or by $|\leftarrow\leftarrow \dots \leftarrow\rangle$ alone (and general linear combinations thereof).