Modified multipoles in photonics

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Multipole decomposition method is a promising tool for investigation of radiating or scattering responses of electromagnetic sources or particles. It works even in the case of relatively complicated and compound scatterers like multilayer particles, clusters, or asymmetrical systems. Commonly, the radiation fields of point electric or magnetic sources are decomposed only into electric or magnetic dipole moments, while real sources can be described by a series of multipoles, including higher multipoles and toroidal moments. In this paper, we introduce the concept of modified multipoles describing real sources of electric, magnetic, and toroidal types. Using the analytical expressions of first-order multipoles, we discuss how they depend on the position of the center of radiation, as well as on the shift of the source, relative to the center of coordinates. We present results of multipoles for the sources with defects and asymmetry. The long-awaited question about distinguishing radiation patterns of electric and toroidal dipole moments in a far-field zone is discussed and solved in this paper. We show that radiation patterns of shifted electric and toroidal dipoles can be rotated to different angles relative to each other due to shifting. Moreover, we discuss modified anapoles. Our modified dipole approach will be useful for multipole analysis of complex systems in photonics such as nanoparticle clusters, metamaterials, and nanoantennas, as well as for better understanding issues of toroidal electrodynamics.

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I. INTRODUCTION

Sources of electromagnetic radiation are widely represented in various spectral regions from radio waves to x-ray [1,2]. Electromagnetic systems have been known since the works of Hertz and they are converters of signals supplied to them into radiation energy [3,4]. Generally, characterizations of antennas and scatterers of electromagnetic waves are based on the concept of electric and magnetic dipole moments as well as their modification—the Huygens element that is the simplest type of multipole interaction [5].

Due to the progress in modern nanophotonics, the science of the light interaction with subwavelength metaparticles and metamaterials is based on more complex electromagnetic effects such as Fano resonances [6–8], the effect of electromagnetically induced transparency (EIT) [9–11], Kerker effect [12–16], anapole states [17–23], and bound states in the continuum (BIC) [24] in which electrodynamics is no longer limited to a description of interactions of only electric and magnetic multipoles but also quadrupoles, toroidal interactions, and, even more, moments of mean-square radii [25–28]. Multipole effects pave the way for the design of open high-Q subwavelength resonators, cloaking devices, and invisible nanoparticles, as well as biological and chemical sensors [6,13].

A well-proven tool for describing the electrodynamics of such particles is the method of multipole expansion in spherical and Cartesian harmonics [5]. This theoretical approach was considered by Mie in 1908 [29,30], who introduced the concept of electric and magnetic moments for spherical and cylindrical particles. Moreover, multipole definitions in Taylor multipole expansion are different from the ones considered in the multipole expansion in terms of spherical harmonics. Recently, there has been hot discussion about the relevancy to consider toroidal multipoles as separate members of a multipole series [31–33]. In particular, toroidal multipoles are independent terms in the Cartesian multipole decomposition. However, it is included as part of the electric moment in the spherical expansion. Both methods work well for identical particles. A comprehensive discussion of the topics related to different multipole decompositions for applications in nanophotonics can be found in Ref. [33]. Thus, both methods of multipole decomposition are needed as tools for explaining the radiation/scattering characteristics of objects. In particular, it was demonstrated that in some cases (in long-wavelength approximation) it is convenient to use the Cartesian multipole expansion because it gives fewer terms in the multipole expansion series. Moreover, the spherical multipole expansion is not enough to explain the anapole effects. It does not evaluate the origin of the suppressed scattering of the particle but explains it in a Cartesian decomposition, as a result of destructive interference between electric and toroidal dipole moments [18]. The requirement to take toroidal multipoles into account in many systems has been proven excessively over the years [18,34,35]. From the physical point of view, the introduction of toroidal multipoles allows the prediction and interpretation of unique physical effects like anapole [17,18] and pseudo-anapole states [36], superdipoles [37], and nullifying of multipoles in metamaterials [38]. Commonly, the radiation fields of elementary electric and magnetic sources are decomposed only into these

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moments. Thus, a point electric dipole is described solely by a pure electric dipole moment, while a loop of zero radius with an electric current is described by a magnetic dipole moment, and the rest of the multipoles are supposed to tend to zero.

Meanwhile, in the electrodynamics of metamaterials, artificial negative magnetism is mainly delivered by a split ring resonator (SRR) [39] and, very often in the literature, unconditionally, a SRR is characterized by a pure magnetic moment without paying attention to the contribution of other multipoles to the radiation. However, one can address a valid question on the difference between a closed loop with a current and with a split one. Obviously, a closed loop cannot be characterized by an electric dipole moment at all, but a split ring must acquire it.

The second question that arises on the multipole analysis of metaparticles is the choice of the radiation center of multipoles. Even for a SRR, whose geometric center is no longer coincident with the center of the ring, there is a principal choice of the center affecting the selection of the multipole series. It was shown and thoughtfully discussed [40] that multipole moments, except for zero-order multipoles, depend on the choice of origin in the same way that lower order multipoles define the change of the higher order multipoles. As long as the previous terms in the multipole series are nonzero, they apparently suffer from a shift of the center of mass [41,42].

The obvious consequence is the uncertainty of performing multipole expansion of a compound media as well as the question of placement of a nonpoint source with respect to the origin. The fundamental point distinguishing metamaterials from homogeneous media is that the size of the meta-atoms is proportional to the distance between them, as well as their inclusions being not much less than wavelength. There is no chance to consider a series with only low-order dipole interaction between inclusions, therefore, one should take into account the higher order multipoles [43]. In particular, chiral particles were thoroughly described in Ref. [44]. Their chirality was explained from the point of view of quadrupole moments that were calculated with respect to the center of particles. However, the geometric center of the system was not considered, which led to alternative conclusions about the origin of the chirality of the composite particles, as discussed in Ref. [43] (formula 17, p. 172). In this paper, we argue that the position of the source in its center of mass of the system is crucial for the accurate calculus of multipole moments, especially for the case of secondary multipole decomposition [45].

Recently, such effects as Fano-resonance, the effect of EIT, bound states in the continuum, and anapole modes have a general approach for using hybrid particles for their excitation, consisting of several meta-atoms, some of them being asymmetric, leading to interference between particles [24,35,37,38,46,47]. Thus, the question of the secondary multipole analysis of the multipoles excited in complex metamolecules still remains open. Each element of a hybrid metamolecule is characterized by its own set of multipoles, which are placed relatively to the common center of the metamolecule [45].

The overall scattering of hybrid metamolecules can be defined by their individual responses in long-wavelength

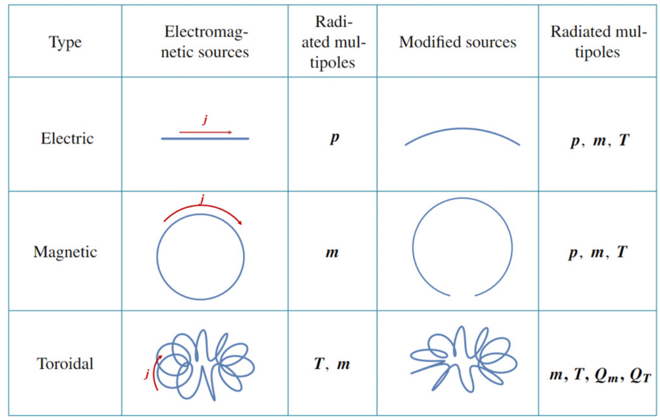


FIG. 1. Radiated electromagnetic dipoles and their multipole contributions.

approximation. For this aim, one can apply the secondary multipole decomposition method introduced by Tuz *et al.* [45]. In particular, they showed how multipole moments of an all-dielectric trimer are related to the multipole moments of individual disks which make up a trimer. As a result, the collective toroidal mode is defined by coupling between magnetic modes of disks.

To demonstrate the electrodynamics of modified multipoles, we organize our paper as follows:

(1) We consider modified first-order multipoles—electric, magnetic, toroidal—to show how the deformation of real (nonpoint) dipoles leads to the appearance of the entire family of dipole moments, which must be taken into account for the correct analysis of metamolecules.

(2) We demonstrate a procedure for choosing the center of radiation for calculating the dipole moments of asymmetric structures.

(3) We will show that a source moved from the coordinate center possesses a unique set of multipoles. Moreover, we demonstrate different types of modified anapoles, arising due to the shifting of the source center.

(4) We discuss the long-awaited question of distinguishing toroidal and electric dipole moments in the far field using the example of modified dipole moments.

II. MODIFIED DIPOLES

We depart from the obvious electric, magnetic, and toroidal real sources: electric dipole—straight wire with current; magnetic dipole—closed loop with current; and toroidal dipole—solenoid with current (Fig. 1). The geometry of each source is given by radii vectors obeying the parametric equations of sources [48]:

for electric dipole:

$$r_{\mathrm{el}} = (0,\ 0,\ d_z); \tag{1}$$

for magnetic dipole:

$$r_{\mathrm{mag}} = (R\cos\varphi,\ R\sin\varphi,\ 0); \tag{2}$$

for toroidal dipole:

$$\begin{aligned} r_{\text{tor}} =&((D - R\cos n\varphi)\cos\varphi, \\ &(D - R\cos n\varphi)\sin\varphi, \\ &R\sin n\varphi), \end{aligned} \tag{3}$$

where φ is the angle between x axes and radius vector $\boldsymbol{r}$ projection onto the xy plane ($0 < \varphi < 2\pi$), D—torus radius, R—the radius of its rings, n—number of windings, and d_z is the length of electric dipole.

The dipoles under investigation are assumed to be made of infinitely thin wire. Assuming the uniform current flow I along the wires, we define a current element as $\mathbf{j}dV = Id\mathbf{r}$. Then the current must be introduced into multipole moments equations as $d\mathbf{r} = \frac{d\mathbf{r}}{d\varphi}d\varphi$. We assume that the multipole moments can either be extracted from the induced currents or calculated from the dynamics of charge and current densities anticipated for the sources.

Then, we can write lower order multipoles in parametric form:

electric dipole moment:

$$\boldsymbol{p} = \frac{I}{i\omega}\int_{\varphi_0}^{\varphi_1}\frac{d\boldsymbol{r}}{d\varphi}d\varphi; \tag{4}$$

magnetic dipole moment:

$$\boldsymbol{m} = \frac{I}{2c}\int_{\varphi_0}^{\varphi_1}\left[\frac{d\boldsymbol{r}}{d\varphi}\times\boldsymbol{r}\right]d\varphi; \tag{5}$$

toroidal dipole moment:

$$\boldsymbol{T} = \frac{I}{10c}\int_{\varphi_0}^{\varphi_1}\left[\left(\boldsymbol{r}\cdot\frac{d\boldsymbol{r}}{d\varphi}\right)\boldsymbol{r} - 2\boldsymbol{r}^2\frac{d\boldsymbol{r}}{d\varphi}\right]d\varphi, \tag{6}$$

where φ_0 and φ_1 define the limits of integration. The limits $\varphi_0 = 0$ and $\varphi_1 = 2\pi$ give a closed loop configuration and the finite limits $\varphi_0 = -\varphi_1$ give a split configuration. At small values $\varphi_0 = -\varphi_1$, Eq. (2) of the ring transforms into Eq. (1) and determines a straight line section with current.

The structure of sources ensures some distinct electromagnetic properties. Excitation of their modes is manifested as resonant features in radiating spectra. Obviously, the electric dipole source is characterized by only the electric dipole moment and the magnetic dipole is defined by solely magnetic dipole moment Fig. 1. The toroidal solenoidlike wire configuration is characterized by the toroidal dipole moment, as it would also support a strong magnetic dipole moment because of the helical nature of its windings [48].

We study the far-field radiation of these sources by electric, magnetic, and toroidal multipole families, as demonstrated by Savinov *et al.* ([49], Appendix B).

A. Modified electric and magnetic dipoles

Under this study, we derive modified electric and magnetic dipole moments for current rings and observe the radiation properties due to geometry transforming from magnetic to electric dipole sources, according to Eqs. (1) and (2). We assume the radius of ring $R = \frac{\lambda}{2}$, where λ is the wavelength. Within the limits of integration $\varphi_0 = 0$ and $\varphi_1 = 2\pi$, the

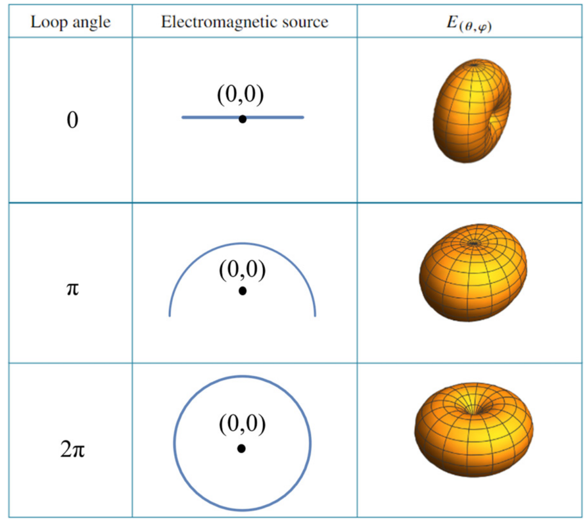


FIG. 2. The modifications of electric/magnetic dipole and their total radiation patterns.

closed ring is characterized only by the magnetic dipole moment, since there is no electric dipole moment for a closed configuration:

$$\begin{aligned} \boldsymbol{p} &= 0, \\ m_x &= 0 \\ m_y &= 0, \\ m_z &= \frac{IR^2\pi}{c}, \\ \boldsymbol{T} &= 0. \end{aligned} \tag{7}$$

However, the contribution of the multipoles changes dramatically when a gap appears in the ring. In this case, the appearance of an electric dipole moment and an accompanying toroidal moment is inevitable. One can imagine that the split ring is being stretched, thus transforming from a magnetic dipole into an electric one. Accordingly, the radiation pattern of the magnetic dipole is turned at a $\pi/2$ angle in the case of an electric dipole and for an intermediate state resembles the radiation pattern of the Huygens element with simultaneously excited electric and magnetic dipole moments, Fig. 2.

We assume that the ring is located in the xy plane. In the case of integration within the limits $-\varphi_0$, φ_0, a nonzero component p_y of the electric dipole moment and T_y component of the toroidal moment appear in the system and they also depends on φ_0:

$$\begin{aligned} p_x &= 0, \\ p_y &= \frac{2IR}{i\omega}\sin\varphi_0, \\ p_z &= 0, \\ m_x &= 0, \\ m_y &= 0, \end{aligned}$$

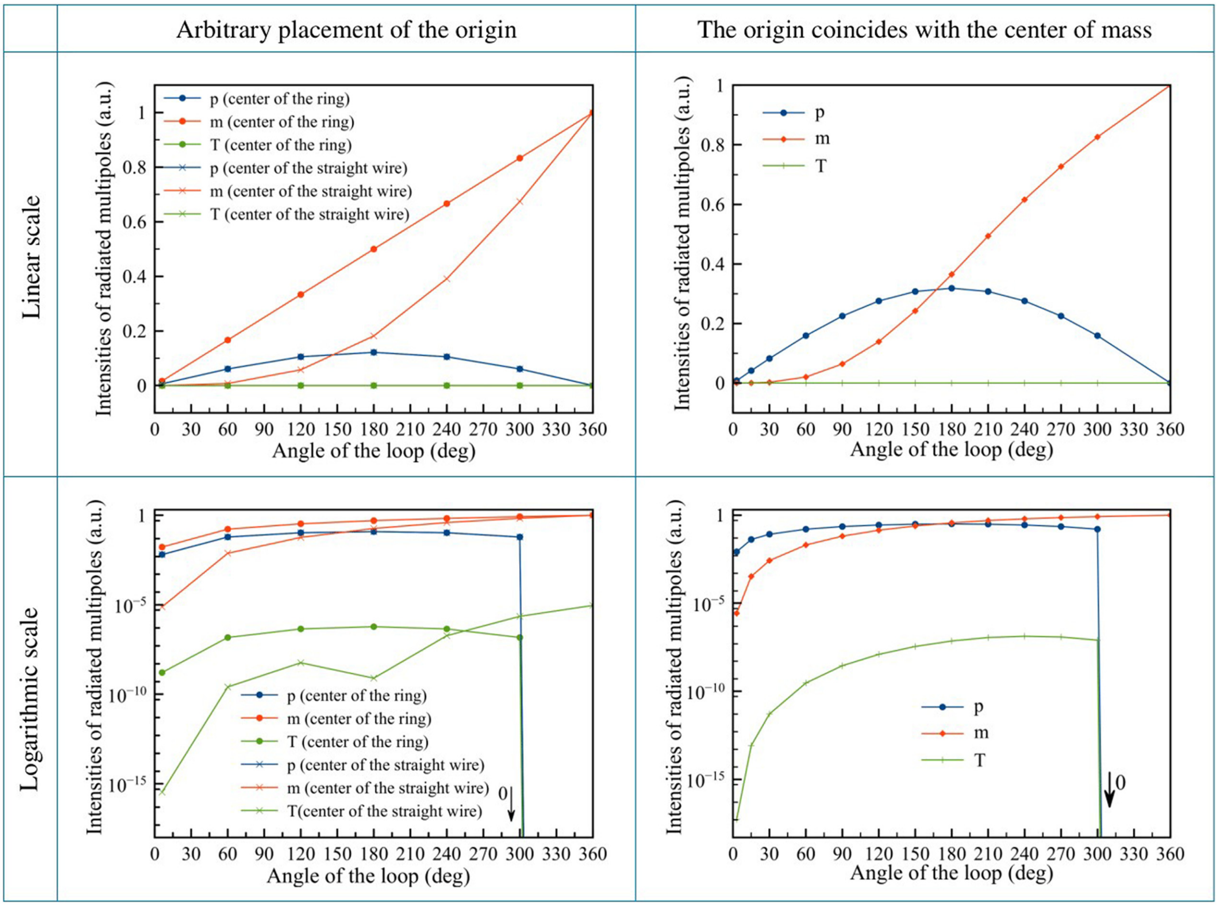


FIG. 3. Dependence of the multipoles intensities over the angle of the curve for different position of radiating center of multipoles.

$$
\begin{aligned}
m_z &= \frac{1}{2c} I R^2 \varphi_0, \\
T_x &= 0, \\
T_y &= -\frac{-2I}{5c} R^2 \sin\varphi_0, \\
T_z &= 0.
\end{aligned} \tag{8}
$$

Here, the m_z component of the magnetic dipole moment depends on φ_0 and tends to zero at small integration angles. Indeed, in this case, the ring equation describes the linear section of the electric dipole. It becomes obvious that only a closed ring with current possesses a pure magnetic dipole. At the same time, only a straight-line section with the current has a pure electric dipole. Within the intermediate limits of integration, the contribution to the response can be defined by all multipoles of electric, magnetic, and toroidal families.

We underline that the dipole moments (except for the electric dipole moment) depend on the choice of the reference point—the radiation center of the system coinciding with the center of mass. In the case of a symmetric system, its geometrical center coincides with the center of radiation, while in the case of asymmetric system, incoincidence of their centers leads to incorrect multipole contributions. The center of radiation must be determined in the same way as the center of mass is chosen in mechanics by integration of particle volume [43,45]:

$$
r_{\mathrm{cm}} = \frac{\int r\varrho(r)dV}{\int \varrho(r)dV}, \tag{9}
$$

where, ϱ is density of wire, in our case, thin wire possesses $\varrho = 1$ in each point of wire.

We note the importance of the setting of the radius vector on the origin of the wire in its center of mass for result correctness as is demonstrated in Fig. 3. In this plot, we present the dependence of multipoles intensities [Eq. (10)] over the angle of the radiating loop.

$$
I_p = \frac{2\omega^4}{3c^3}|\boldsymbol{p}|^2, \quad I_m = \frac{2\omega^4}{3c^3}|\boldsymbol{m}|^2, \quad I_p = \frac{2\omega^6}{3c^5}|\boldsymbol{T}|^2. \tag{10}
$$

One can place the radiation center either in the center of the loop or on the loop—we obtain the correct results only when the radiating center is on the center of mass of the wire. If we set the radiating center on the geometrical center of the loop, we obtain the correct result for the magnetic dipole (i.e., closed loop), however, for the electric dipole (straight wire) we obtain that the magnetic dipole moment is higher than the electric dipole moment. Likewise, when we set the radiating center of the loop on the center of the electric dipole ($|\varphi_1 - \varphi_0| \to 0$), we get the correct results only for case of the electric dipole moment, while for the closed current loop we get a nonzero toroidal dipole moment which is physically wrong. As long as the center of mass of the curve is taken into consideration, the calculation of the multipole moments can be handled regularly. For clarity, we plot Fig. 3 on a logarithmic and linear scale.

Importantly, these results lead to the conclusion that SRRs are not purely magnetic elements, since the toroidal dipole moment is still quite high for the small values of the gap. Moreover, using the case of a SRR, we conclude that splitting of any system always leads to the appearance of an electric dipole moment. On the other hand, the bending of the system is always accompanied by magnetic and toroidal dipole moments due to the excitation of circular and poloidal currents, respectively.

B. Modified toroidal dipole

Meanwhile, a reasonable question remains: Does the deformation of the toroidal dipole source lead to the appearance of an electric dipole moment?

Let us consider the parametric Eq. (3) of a toroidal dipole source and add a small perturbation by the factor $1+\alpha\varphi$ to the z component responsible for the deformation of the toroidal source:

$$\begin{aligned} r_{\text{tor}} = &((D - R\cos n\varphi)\cos\varphi, \\ &(D - R\cos n\varphi)\sin\varphi, \\ &R(1+\alpha\varphi)\sin n\varphi). \end{aligned} \tag{11}$$

If $\alpha = 0$, Eq. (11) turns into the form of an ordinary toroid, Eq. (3). Accordingly, if $\alpha > 0$, then we go to the equation of a toroid stretched on one side; if $\alpha < 0$, we get compressed form.

Direct calculation of the dipole moments by Eqs. (4)–(6) leads to the following expressions for multipole components of modified toroid:

$$\begin{aligned} p_x &= 0, \\ p_y &= 0, \\ p_z &= 0, \end{aligned} \tag{12}$$

$$\begin{aligned} m_x &= \frac{2}{c}\frac{\alpha I n\pi R(-D + 4Dn^2R + R - 3n^2R + 2n^4R)}{1 - 5n^2 + 4n^4}, \\ m_y &= 0, \\ m_z &= \frac{1}{2c} I\pi(2D^2 + R^2), \end{aligned} \tag{13}$$

$$\begin{aligned} T_x &= \frac{20}{c}\frac{\alpha I n^2\pi(1+\alpha\pi)R^2[D(-1+10n^2-9n^4) + R(1-7n^2+12n^4)]}{-1+14n^2-49n^4+36n^6}, \\ T_y &= -\frac{20}{c}\frac{\alpha^2 I n^2\pi R^2[D(-3+4n^2)(1-10n^2+9n^4)^2 + R(1-4n^2)^2(3-22n^2+51n^4)]}{(1-14n^2+49n^4-36n^6)^2}, \\ T_z &= \frac{1}{2c} D I R^2 n\pi(1+\alpha\pi). \end{aligned} \tag{14}$$

If $\alpha = 0$, multipoles correspond to the dipole moments of the pure toroid obtained earlier by Afanasiev [50] and Marinov *et al.* [48]:

$$\begin{aligned} p_x &= 0, \\ p_y &= 0, \\ p_z &= 0, \\ m_x &= 0, \\ m_y &= 0, \\ m_z &= \frac{1}{2c} I\pi(2D^2 + R^2), \\ T_x &= 0, \\ T_y &= 0, \\ T_z &= \frac{1}{2c} D I R^2 n\pi. \end{aligned} \tag{15}$$

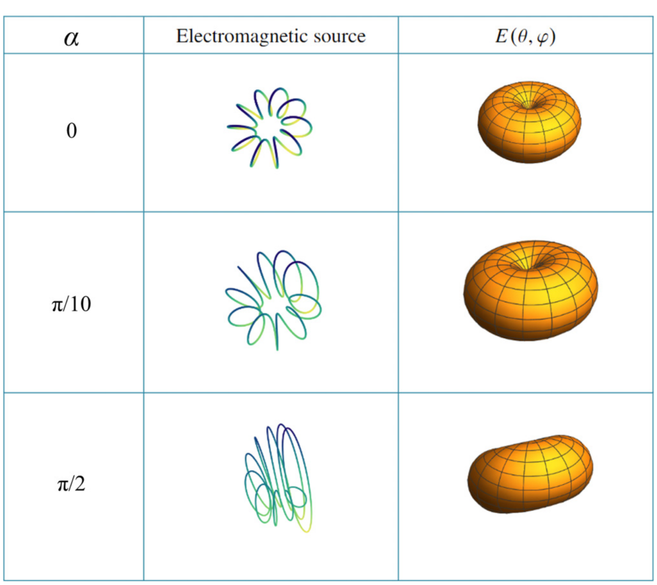


FIG. 4. The modified toroidal dipole sources and their total radiation patterns.

Note that a pure toroid ($\alpha = 0$) is characterized by zero electric dipole moment p. This is quite obvious, since the system is closed. At the same time, the m_z component of the magnetic dipole moment appears and, moreover, is independent on the number of turns n. Toroidal moment component T_z is also aroused in the system and possesses a symmetric radiation pattern (Fig. 4) that resembles the radiation pattern of the electric dipole, however, the current distribution of the toroidal dipole is different from the electric one.

As far as we know at this moment, there is no method allowing detection of whether the electric or toroidal source is radiating in the far-field zone [19,34]. However, in this part of the paper, we pay attention to the fact that with the

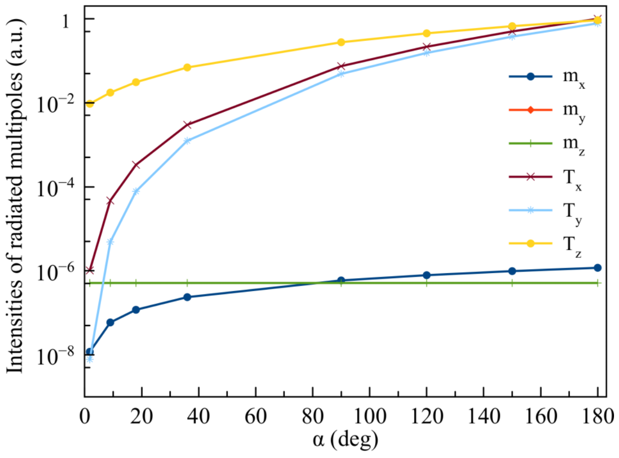


FIG. 5. Multipole components intensities [Eq. (10)] of modified toroid over α.

modification of the geometry of the toroidal source (without breaking the solenoid), only the toroidal moment components are changing, while the electric dipole moment remains zero as for pure toroid.

In particular, in addition to the z component of the magnetic moment, an x component arises and depends on the number of windings n. At large values of $\alpha \to \pi$, the radiation pattern of the magnetic moment is turned.

It is noteworthy that the toroidal moment of the modified solenoid has all three components. For lower values of α, T_x and T_y components are negligible, however, the are increasing significantly for large α (Fig. 5), so toroidal moment radiation pattern acquires a pronounced asymmetry (Fig. 4), namely, the donut shape of its radiation pattern is stretched and tilting. Thus, the total radiation pattern of the modified toroidal dipole is mainly determined by the contribution of the toroidal moment, while the magnitude of the magnetic moment is small and does not contribute to the entire radiation. The total radiation pattern (Fig. 4) is also asymmetric at higher values of α. At the same time, the electric dipole moment is still zero.

Let us note below the main features of the modified dipole radiation.

(1) An electric dipole moment is always present in a radiating system that has a split.

(2) A closed system of currents is always characterized by a magnetic dipole moment. If the number of turns is more than one, $n > 1$, inside the source, it leads to toroidal moment manifestation.

(3) Modification of the toroidal source affects only toroidal moment components leading to asymmetry of the radiation pattern. Meanwhile, the electric dipole moment in the system is equal to zero.

(4) Due to such asymmetry of the radiation pattern, it is possible to detect the difference between the radiation pattern of the electric and toroidal moment in the far field zone, if one modifies the source and knows *a priori* that the source is closed.

(5) Modification of an electric dipole does not lead to the asymmetry of its radiation pattern.

III. COMPENSATION OF THE TRANSLATION DEPENDENCE

Generally, definitions for dipole moments are used for sources located at the center of radiation. In this case, the center of mass of the system (the center of radiation) coincides with the geometrical center of the source. However, in the case of a source moving from the coordinate center due to experimental or fabrication errors, as well as compound sources consisting of several elements located at a distance from each other, this approach is not always correct. This is especially typical for the study of clusters of metamolecules.

For this aim, we consider sources characterized by a set of multipoles. We analyze the definitions for electric, magnetic, and toroidal dipole moments for a source shifted from the coordinate center and consider it by translation vector $\boldsymbol{d}$. For this purpose, radius vector $\boldsymbol{r}$ in the equations for multipoles, Eq. (16) is replaced by $\boldsymbol{r}-\boldsymbol{d}$. Indeed, the electric dipole moment is not changed due to independence from radius vector $\boldsymbol{r}$. However, the magnetic and toroidal dipole moments are changed dramatically since extra multipoles arose in the system.

Indeed, a shift in the system of multipoles leads to the appearance of additional multipole terms of another family in multipole decomposition, which we denote by symbol $\sim$. We also add common formulas for electric and magnetic modified quadrupoles:

$$\tilde{p}_i = \frac{1}{i\omega}\int j_i d^3r = p_i,$$

$$\tilde{m}_i = \frac{1}{2c}\int [(\boldsymbol{r}-\boldsymbol{d})\times\boldsymbol{j}]_i d^3r = m_i - \frac{i\omega}{2c}(\boldsymbol{d}\times\boldsymbol{p})_i,$$

$$\tilde{T}_i = \frac{1}{10c}\int [(\boldsymbol{r}-\boldsymbol{d})\cdot\boldsymbol{j}](r-d)_i - 2(r-d)^2 j_i d^3r$$

$$= T_i - \frac{i\omega}{5c}p_i d^2 + \frac{i\omega}{10c}p_i d_i^2 + \frac{i\omega}{10c}d_i d_j p_j + \frac{i\omega}{10c}d_i d_k p_k + \frac{3}{10}[\boldsymbol{m}\times\boldsymbol{d}]_i + \frac{\omega}{10ci}d_j Q^e_{ij} + \frac{\omega}{10ci}d_i Q^e_{ik} + \frac{3\omega}{20ci}d_k Q^e_{ii},$$

$$\tilde{Q}^e_{ij} = \frac{i}{\omega}\int [(r_j-d_j)j_i + (r_i-d_i)j_j) - \frac{2}{3}\delta((\boldsymbol{r}-\boldsymbol{d})\cdot\boldsymbol{j})]d^3r$$

$$= Q^e_{ij} + d_j p_i + d_i p_j - \frac{2}{3}\delta(\boldsymbol{d}\cdot\boldsymbol{p}),$$

$$\tilde{Q}^m_{ij} = \frac{1}{3}\int [(\boldsymbol{r}-\boldsymbol{d})\times\boldsymbol{j}]_i(\boldsymbol{r}-\boldsymbol{d})_j + (\boldsymbol{r}-\boldsymbol{d})_i[(\boldsymbol{r}-\boldsymbol{d})\times\boldsymbol{j}]_j d^3r$$
$$= Q^m_{ij} + \frac{1}{3c}i\omega d_i d_k p_i - \frac{1}{3c}i\omega d_j d_k p_j + \frac{i\omega}{3c}\left(d_j^2 - d_i^2\right)p_k - d_j m_i - \frac{1}{3}d_i m_j - \frac{\omega}{6ic}d_j Q^e_{jk} + \frac{\omega}{6ic}d_i Q^e_{ik} + \frac{\omega}{6ic}d_k\left(Q^e_{jj} - Q^e_{ii}\right). \tag{16}$$

A. Shifted electric dipole

Accordingly, an electric dipole shifted to distance $\boldsymbol{d}$ from the coordinate center of the system leads to the appearance of additional nonzero magnetic dipole and toroidal moments. Importantly, the shift of the system results in unique multipoles, which are determined by the multipoles of the unshifted source. Hence, if the source response is determined only by the electric dipole moment, then only its contribution establishes other dipole moments of the shifted source: electric, magnetic, and toroidal due to the terms, including the electric dipole moment as well as other terms in Eq. (16), should be zero. The multipoles of the shifted electric dipole are the following:

$$\begin{aligned}
\tilde{p}_i &= p_i,\\
\tilde{m}_i &= -\frac{i\omega}{2c}(\boldsymbol{d}\times\boldsymbol{p})_i,\\
\tilde{T}_i &= -\frac{i\omega}{5c}p_i d^2 + \frac{i\omega}{10c}p_i d_i^2 + \frac{i\omega}{10c}d_i d_j p_j + \frac{i\omega}{10c}d_i d_k p_k + \frac{\omega}{10ci}d_j Q^e_{ij} + \frac{\omega}{10ci}d_i Q^e_{ik} + \frac{3\omega}{20ci}d_k Q^e_{ii},\\
\tilde{Q}^e_{ij} &= Q^e_{ij} + d_j p_i + d_i p_j - \frac{2}{3}\delta(\boldsymbol{d}\cdot\boldsymbol{p}),\\
\tilde{Q}^m_{ij} &= \frac{1}{3c}i\omega d_i d_k p_i - \frac{1}{3c}i\omega d_j d_k p_j + \frac{i\omega}{3c}\left(d_j^2 - d_i^2\right)p_k - \frac{\omega}{6ic}d_j Q^e_{jk} + \frac{\omega}{6ic}d_i Q^e_{ik} + \frac{\omega}{6ic}d_k\left(Q^e_{jj} - Q^e_{ii}\right).
\end{aligned} \tag{17}$$

The presence of magnetic and toroidal moments in the multipole contribution of the shifted electric dipole leads to tilting and stretching of the radiation pattern of the source via the appearance of x and y components of the magnetic and toroidal moments (Fig. 6).

B. Shifted magnetic dipole

On the one hand, it is clear that absence of electric multipoles in the source does not lead to multipole terms when the system is shifted. Thus, in the case of a magnetic dipole (ring with electric current) shifted to distance d from the coordinate center, the multipole response is characterized by the magnetic dipole moment and toroidal dipole moment:

$$\begin{aligned}
\tilde{p}_i &= 0,\\
\tilde{m}_i &= m_i,\\
\tilde{T}_i &= \tfrac{3}{10}[\boldsymbol{m}\times\boldsymbol{d}]_i,\\
\tilde{Q}^e_{ij} &= 0,\\
\tilde{Q}^m_{ij} &= Q^m_{ij} - d_j m_i - \tfrac{1}{3}d_i m_j.
\end{aligned} \tag{18}$$

Indeed, results become obvious if we imaginary rotate a ring with current around the center of radiation, thereby producing virtual poloidal currents. This is a reason for the occurrence of a toroidal moment in the system. Moreover, due to the closed currents in the ring, the electric dipole moment disappears even when the ring is shifted. Using the magnetic dipole (current ring) as an example, we demonstrate that the magnetic dipole moment does not change when the ring is translated by the vector $\boldsymbol{d}$. The presence of a z component of the magnetic dipole moment leads to excitation of x and y components of the toroidal dipole moment, resulting in an asymmetrical radiation pattern as shown in Fig. 6.

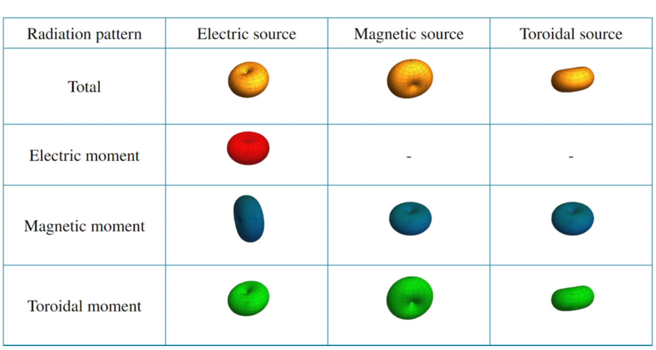


FIG. 6. Radiation patterns of shifted sources.

C. Shifted toroidal dipole

In the case of a shifted toroidal dipole, the magnetic dipole moment also remains the same as in the system without an electric dipole moment and there is only the first term in Eq. (16). On the other hand, the toroidal dipole changes with the occurrence of an extra component proportional to the magnetic dipole moment. Thus, the radiation pattern of the shifted toroidal source (Fig. 6) becomes asymmetric due to the occurrence of components proportional to the magnetic dipole moment, as in the following from Eq. (19):

$$\begin{aligned} \tilde{p}_i &= 0, \\ \tilde{m}_i &= m_i, \\ \tilde{T}_i &= T_i + \tfrac{3}{10}[\boldsymbol{m} \times \boldsymbol{d}]_i, \\ \tilde{Q}^e_{ij} &= 0, \\ \tilde{Q}^m_{ij} &= Q^m_{ij} - d_j m_i - \tfrac{1}{3} d_i m_j. \end{aligned} \tag{19}$$

For a toroidal dipole (solenoid) shifted from the origin, the magnetic dipole moment is the same as for a system without an electric dipole moment whenever the toroidal dipole is different and its radiation is responsible for the asymmetry of the radiation pattern changing, as shown in Fig. 6.

Moreover, comparison of the total radiation patterns of the modified electric and toroidal sources may cause confusion: Why are they rotated by $\pi/2$ relative to each other? On the one hand, with a small shift of the toroidal source by $d = \lambda/2$, equal to half the outer radius of the toroid, it becomes obvious that the distribution of the near zone of the source plays a role in the formation of radiation. In the toroid, the currents are concentrated in the volume, which contrasts to an electric dipole. When comparing the formulas for the dipole moments of shifted sources, one can note the difference in the modified toroidal moments of the electric dipole source, Eq. (20), and toroidal dipole sources, Eq. (21). For example, when the sources are oriented along the z axis, the T_x component of the toroidal source is negative, Eq. (21) and it leads to the rotation of the pattern, that is, in contrast with an electric source. This is an important fundamental difference between the far-field radiation of toroidal and electric dipoles, which can be revealed in future experiments.

For a shifted electric dipole oriented along z:

$$\begin{aligned} \tilde{T}_x &= 0, \\ \tilde{T}_y &= 0, \\ \tilde{T}_z &= \frac{i\omega}{10c}[(d_x d_z) p_z]. \end{aligned} \tag{20}$$

For a shifted toroidal dipole oriented along z:

$$\begin{aligned} \tilde{T}_x &= -\tfrac{3}{10} m_z d_y, \\ \tilde{T}_y &= \tfrac{3}{10} m_z d_x, \\ \tilde{T}_z &= T_z. \end{aligned} \tag{21}$$

D. Difference between radiation of electric and toroidal dipole sources

Indeed, the radiation patterns of electric and toroidal dipole moments are identical in the far-field zone. Moreover, three multipole families (electric, magnetic, toroidal) are needed to describe the source, and only two (electric and magnetic spherical multipoles) are needed to describe the fields radiated by them [31]. Moreover, Fernandez-Corbaton *et al.* established that the electric and toroidal parts cannot be separately determined by measuring the electromagnetic fields produced by the source outside its region or by measuring its coupling to external electromagnetic waves. Indeed, electric and toroidal dipole moments radiate with an electrical-type pattern; similarly, it is also true that they radiate with a toroidal-type radiation pattern. This is especially correct in the case of a solenoid whose electric dipole moment is exactly zero.

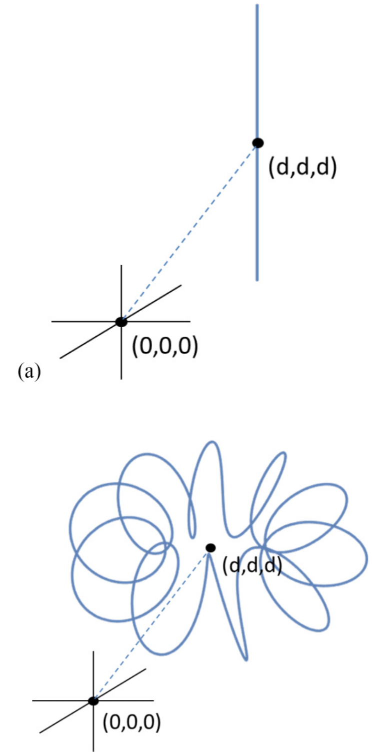


FIG. 7. Shifted (a) electric and (b) toroidal sources.

Therefore, this statement works only if we compare two point sources placed in a coordinate center, both electric and toroidal sources have the same x, y, or z components and both of them placed in vacuum. Moreover, the intensity of the multipoles are different if we place the sources in a medium with refractive index n [34,51]. Thus, if one would like to observe differences in toroidal and electric multipoles radiation, the modification of the source (position, transformation, surrounding media) should be performed.

For a demonstration of our approach, we consider two sources as electric dipole and toroidal dipole, represented as a thin wire with current and solenoid, respectively (Fig. 7). Electric dipole has a λ length. We study the evolution of the total radiation pattern and radiation of each multipole via shifting of the source from the coordinate center by translation vector d. If the dipole is placed in the coordinate center $d = (0, 0, 0)$, the radiation pattern has a symmetrical shape and is defined by electric dipole moment radiation. A small shift, $d = (\lambda/10, \lambda/10, \lambda/10)$, is accompanied by an increase of the intensity of the toroidal, magnetic dipole moments, and quadropoles (Fig. 10). However, the shape of their radiation patterns changes slightly with increasing shift d (Fig. 8).

The shape of the toroidal dipole radiation acquires asymmetry. Nevertheless, we should note that the electric dipole moment is unchangeable due to shifting d. It retains its shape and intensity that follows from Eq. (17) for an electric dipole moment.

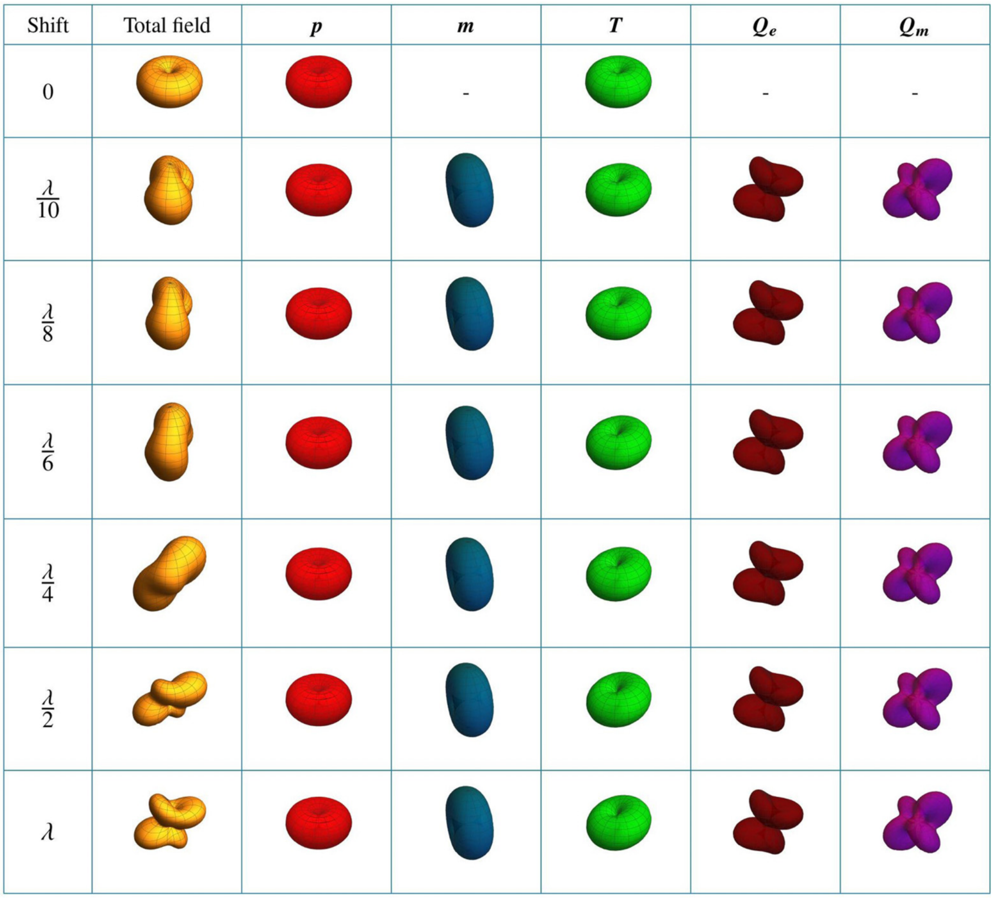


FIG. 8. Radiation patterns of shifted electric source.

On the contrary, the shifting of the toroidal dipole from the coordinate center is different. We consider a toroidal source,—its shape is defined by Eq. (3)—with $n = 10$, $R = \frac{1}{3}\lambda$, and $D = \frac{2}{3}\lambda$. The multipole decomposition is still devoid of the contribution of the electric dipole moment and electric quadropole. For small shifting, the total radiation pattern is defined by asymmetric radiation of the toroidal dipole moment, which is accompanied by its additional components (Fig. 11). The shape of the magnetic dipole moment and quadropoles is independent from shift d. However, intensities of radiated quadropoles are changed with d and remain unchanged for the magnetic moment, accordingly with Eq. (19) (Fig. 9).

Moreover, we highlight differences between radiation of shifted electric and toroidal sources. An electric dipole is more susceptible to shift. The high contribution of the electric dipole moment is preserved only for low $d < 0.2\lambda$ and accompanied by toroidal, magnetic moments, and quadropoles. In the case of a toroidal source, the dominance of the toroidal dipole is kept for the broadband of d, as well as magnetic quadropole is growing for large d.

Moreover, the total radiation patterns of shifted electric and toroidal sources are rotated by $\pi/2$ relative to each other, giving us a chance to observe their differences in far-field zone due to shifting. Thus, the long-awaited question of distinguishing toroidal and electric dipole moment radiation in the far field can be resolved if we perform the real experiment and measure radiation patterns of both sources shifted from the coordinate center by translation vector d. This difference in the radiation patterns of an electric and toroidal source is a direct confirmation of the physical meaning of the toroidal moment. We note that the statement about the identity of the radiation patterns of the toroidal and electric moments is correct, although it is necessary to make a clarification that it works only for sources placed in the center of coordinates and if the sources have the same components x, y or z of moments. In our case of shifted from coordinate center dipoles, we observe a difference in radiation patterns measured, however, relative to coordinate center.

E. Modified anapoles

The presence of extra terms in multipole coefficients leads to unique kinds of anapoles via destructive interference between parts of shifted multipoles. We call them modified anapoles of magnetic type [following from Eq. (18)] and toroidal type [following from Eq. (19)]. These solutions nullify the magnetic and toroidal moments of the sources via shifting, along with the usual anapole, that is, destructive interference between electric and toroidal moments.

In principle, modified anapoles also enable suppressing radiation of electric or magnetic types. In particular, an electric anapole is excited due to the shifted electric dipole. In this case, the source only possesses a modified magnetic dipole moment, while electric multipoles are suppressed due

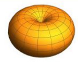
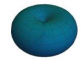
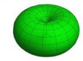
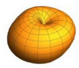
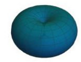
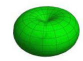
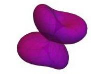
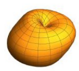
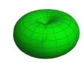
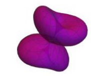
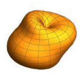
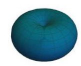
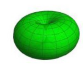
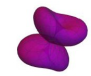
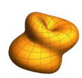
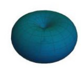
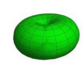
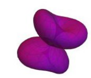
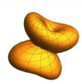
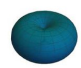
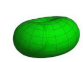
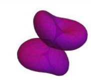
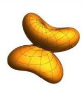
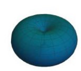
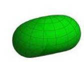
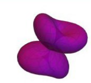
Shift	Total field	$\boldsymbol{p}$	$\boldsymbol{m}$	$\boldsymbol{T}$	$\boldsymbol{Q_e}$	$\boldsymbol{Q_m}$
0		-			-	-
$\frac{\lambda}{10}$		-			-	
$\frac{\lambda}{8}$		-			-	
$\frac{\lambda}{6}$		-			-	
$\frac{\lambda}{4}$		-			-	
$\frac{\lambda}{2}$		-			-	
λ		-			-	

FIG. 9. Radiation patterns of shifted toroidal source.

to a common electric anapole, Table I. Similarly, a modified toroidal anapole suppresses electric-type radiation, resulting in only a magnetic dipole moment in the system. Thus, shifted electromagnetic sources relative to the center of radiation can acquire the properties of multipoles of other families due to the appearance of anapoles of the common and modified types suppressing their proper type of radiation.

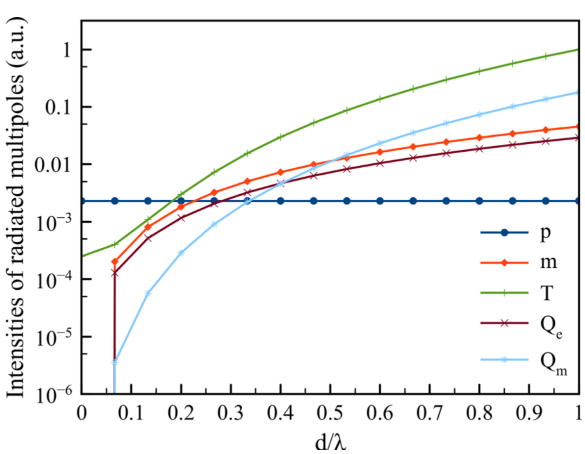


FIG. 10. Intensities of radiated dipoles of the shifted electric source over its shift.

F. Independence multipoles contributions on the translation of the source

Furthermore, a reasonable question is about whether there is a system comprising a set of all multipoles regardless of the shift of the system center, so modified multipoles correspond to multipoles of unshifted ones. For this aim, we consider a double solenoid system independent of the placement of the system center up to quadrupole moments. Marinov *et al.* [48] proposed a configuration that allows one to suppress the magnetic dipole moment and quadrupole in a solenoid of two wires wound in opposite directions, as described by the radii vectors in Eq. (22):

$$\begin{aligned} r_{\mathrm{tor}} =&((D - R\cos n\varphi)\sin\varphi, \\ &(D - R\cos n\varphi)\cos\varphi, \\ &R\sin n\varphi), \\ r_{\mathrm{tor}}^{\mathrm{rev}} =&((D - R\cos n\varphi)\cos\varphi, \\ &(D - R\cos n\varphi)\sin\varphi, \\ &R\sin n\varphi). \end{aligned} \tag{22}$$

Toroidal, magnetic dipole, and magnetic quadrupole moments are now calculated as $\mathbf{T}_1 = \mathbf{T}_2$, $\mathbf{m}_1 = -\mathbf{m}_2$, and $\mathbf{Q}_{m1} = -\mathbf{Q}_{m2}$. Thus, a double-wound toroidal solenoid has a zero net magnetic dipole and magnetic quadrupole moments whereas the toroidal moment of the structure is twice that of a single-

TABLE I. Modified anapoles and their general and common definitions.

Anapoles	Common electric anapole	Modified magnetic anapole	Modified toroidal anapole
General expression	$\tilde{\boldsymbol{p}} = -ik\tilde{\boldsymbol{T}}$	$\boldsymbol{m} = \frac{i\omega}{2c}(\boldsymbol{d} \times \boldsymbol{p})$	$T_i = \frac{i\omega}{5c}p_i d^2 - \frac{i\omega}{10c}p_i d_i^2 - \frac{i\omega}{10c}d_i d_j p_j - \frac{i\omega}{10c}d_i d_k p_k - \frac{3}{10}[\boldsymbol{m} \times \boldsymbol{d}]_i - \frac{\omega}{10ci}d_j Q^e_{ij} - \frac{\omega}{10ci}d_i Q^e_{ik} - \frac{3\omega}{20ci}d_k Q^e_{ii}$
Electric source	$\tilde{\boldsymbol{p}} = -ik\tilde{\boldsymbol{T}}$		$\frac{i\omega}{5c}p_i d^2 - \frac{i\omega}{10c}p_i d_i^2 - \frac{i\omega}{10c}d_i d_j p_j - \frac{i\omega}{10c}d_i d_k p_k - \frac{\omega}{10ci}d_j Q^e_{ij} - \frac{\omega}{10ci}d_i Q^e_{ik} - \frac{3\omega}{20ci}d_k Q^e_{ii} = 0$
Magnetic source			-
Toroidal source			$\mathrm{T}_i = -\frac{3}{10}[\boldsymbol{m} \times \boldsymbol{d}]_i$

wound torus: $2T$. In the case of shifting of the double-toroidal source, its multipole contribution is described only by the toroidal moment and radiation pattern of the double-toroidal source shifted by d, similar to the system positioned in the coordinate center, up to higher multipoles.

Here, we summarize and highlight the following mechanisms of the shifting of dipole sources from the coordinate center of the system:

(1) The shifting of the electric dipole source is accompanied by the occurrence of magnetic and toroidal dipoles proportional to the electric dipole moment of the source. However, the electric dipole moment remains unperturbed in comparison with the electric dipole placed in the center of radiation.

(2) The shifting of the magnetic dipole source is accompanied by a magnetic dipole moment and occurance of a toroidal dipole moment proportional to the magnetic dipole moment of source.

(3) The shift of the toroidal dipole source is accompanied by a magnetic dipole moment equaling the magnetic moment of the centered torus. However, the toroidal dipole moment consists of a toroidal dipole moment of the centered torus and proportional term to the magnetic moment.

(4) Indeed, all sources with shifted centers suffer from excitation of extra magnetic and toriodal dipole moments. However, only electric dipole source possesses an electric dipole moment. The main difference between shifted magnetic and toroidal sources is the presence of a toroidal term, an unshifted toroidal dipole moment.

(5) The radiation patterns of the electric and toroidal dipoles can be shifted by $\pi/2$ relative to each other, which is a way to distinguish them in the far-field zone.

IV. SECONDARY MULTIPOLE DECOMPOSITION

Here we demonstrate several examples of a modified multipole approach to explain effects in compound systems. Recently, Tuz *et al.* [45] proposed secondary multipole decomposition for explanation of interference effects in clusters of metaparticles. Metaparticle clusters are the building blocks for demonstrating such high Q-factor effects as Fano-resonance, EIT, lattice Kerker effects, etc. From multipole analysis of such clusters, it must be taken into account that the contribution of the modified multipoles will explain the contribution of the entire system.

A. Toroidal dipole excitation due to modified dipoles

Let us consider a system consisting of four magnetic dipole moments placed in an xy plane distanced from the center at a distance d (Fig. 12).

We use our modified multipole approach and definitions for the magnetic dipole shift and we get modified multipoles for each of the four dipoles.

Dipole 1 :

$$\begin{aligned} \tilde{p}_i &= 0, \\ \tilde{m}_i &= m_x, \\ \tilde{T}_z &= \tfrac{3}{10}[m_x d_y]. \end{aligned} \tag{23}$$

Dipole 2 :

$$\begin{aligned} \tilde{p}_i &= 0, \\ \tilde{m}_i &= -m_y, \\ \tilde{T}_z &= \tfrac{3}{10}[m_y d_x]. \end{aligned} \tag{24}$$

Dipole 3 :

$$\begin{aligned} \tilde{p}_i &= 0, \\ \tilde{m}_i &= -m_x, \\ \tilde{T}_z &= \tfrac{3}{10}[m_x d_y]. \end{aligned} \tag{25}$$

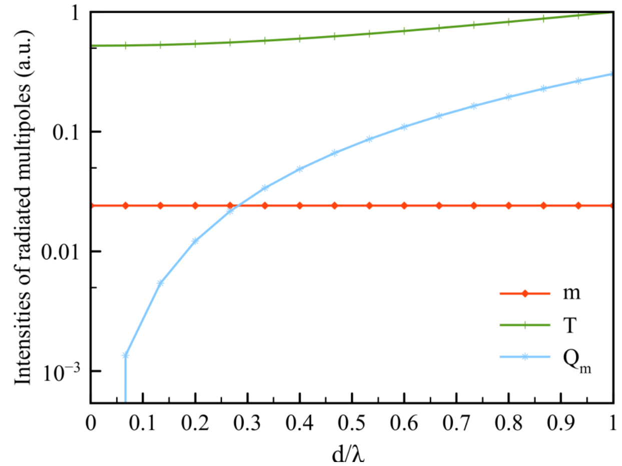


FIG. 11. Intensities of radiated dipoles of the shifted toroidal source over its shift.

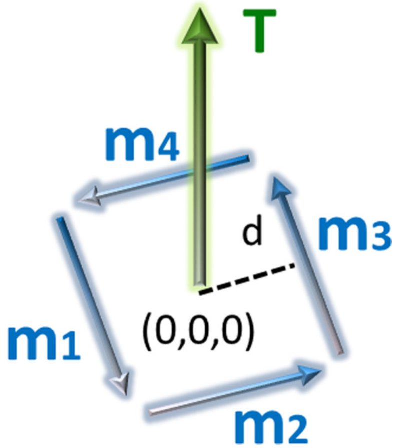


FIG. 12. Toroidal dipole excitation due to magnetic dipoles interference.

Dipole 4 :

$$\tilde{p}_i = 0,\quad \tilde{m}_i = m_y,\quad \tilde{T}_z = \tfrac{3}{10}[m_y d_x], \tag{26}$$

And for the total multipoles we can write

$$\tilde{p}_i = 0,\quad \tilde{m}_i = 0,\quad \tilde{T}_z = 4\tfrac{3}{10}md. \tag{27}$$

Obviously, each of shifted dipole moments is characterized by a zero electric moment as well as magnetic and toroidal moments excited due to shift. Magnetic dipole moments have x or y components, while toroidal moments have only a z component. As a result of interference, the total multipole contribution is determined solely by the toroidal moment, four times surpassing magnetic moment per distance d.

B. Broadband nonradiating system

The double solenoid system is characterized only by toroidal dipole moment T that can be easily suppressed by flattening the structure, as shown in Fig. 13. However, the flat double wound toroidal structure can be considered as a broadband nonradiating source. We consider that a system of two flat helices organized as currents in each helix are directed oppositely.

Direct multipole decomposition reveals that all moments in a system are zero due to nullifying multipoles up to higher multipoles, which can arise due to near-field coupling. Obviously, closed current loops of each helix cannot be characterized by an electric moment and its higher multipoles. Magnetic moments of double wound helices are compensated by each other. Toroidal moments are also zero because of poloidal currents compensating each other.

Thus, one can be confused because the law of energy conversation is violated. However, separated multipole decomposition of each helix demonstrates magnetic moments with opposite signs, so their coupling leads to the zero magnetic moment in a system and the radiated electric field in far-field zone is zero ([49], Appendix B):

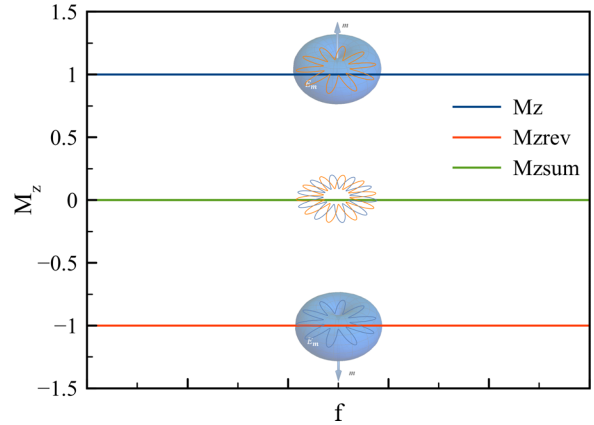


FIG. 13. Broadband nonradiating system based on two flat helixes.

$$\begin{aligned} \boldsymbol{E}_{l=1} &= \boldsymbol{E}_{l=1}^{\text{loop1}} + \boldsymbol{E}_{l=1}^{\text{loop2}} \\ &\approx \frac{\mu_0 c^2}{3\sqrt{2\pi}} \frac{\exp(-ikr)}{r} \Bigg[\sum_{m=1,\pm1} i\sqrt{3}k^2 |M_{1,m}| \times \boldsymbol{Y}_{1,1,m} \\ &\quad + \sum_{m=1,\pm1} i\sqrt{3}k^2 (-|M_{1,m}|) \times \boldsymbol{Y}_{1,1,m} \Bigg] \approx 0. \end{aligned} \tag{28}$$

Indeed, if excitation of a double helix is due to one channel (cable), than this system is unphysical. In our case, excitation should be organized by two independent cables for each helix. Such a nonradiating system, unlike an anapole, does not depend on the frequency, as a magnetic dipole, unlike an electric one, does not. Actually, this system can be realized experimentally for demonstration of wide band anapole excitation in planar structures. Thus, the secondary multipole decomposition can be crucial for explanation of hybrid system, where direct decomposition gives unclear results.

V. CONCLUSION

In conclusion, we have proposed and examined theoretically modified multipoles. We introduce the multipole decomposition for explaining radiating effects in real (nonpoint) systems like sources and particles, as well as clusters of particles. The shift from the coordinate center or asymmetrical sources should be described by multipoles, taking into account modified multipoles of new families and calculated by taking into account the center of mass of the system. Moreover, we introduced the modified anapoles which arise due to shifting. We found a long-awaited answer about distinguishing electric and toroidal dipole far-field radiation. We demonstrated that the total radiation patterns of shifted electric and toroidal sources are rotated by $\pi/2$ relative to each other, which gives us a chance to observe their differences in a far-field zone due to shifting. Our approach of modified multipoles can be useful in multipole analysis of complicated metaparticles and metamaterials, Fano systems, and nonradiating systems in photonics.

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