Vortices in spin-0 superfluids carry magnetic flux

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(Received 15 December 2021; revised 15 August 2022; accepted 28 November 2022; published 5 January 2023)

Vortices in spin-0 superfluids generically carry magnetic fields inside their cores, so that even neutral superfluid vortices may be thought of as magnetic flux tubes. We give a systematic analysis of this 'vortex magnetic effect' using effective field theory, clarifying earlier literature on the subject. Our analysis shows that in superfluid helium-4 the vortex magnetic effect may be large enough to be experimentally detectable.

DOI: 10.1103/PhysRevB.107.024502

I. INTRODUCTION

Superfluidity is an emergent phenomena observed in numerous many-body systems, and plays a key role in cold nuclear, atomic, and molecular systems. Superfluidity arises due to the formation of a Bose-Einstein condensate (BEC) of bosonic electrically-neutral (quasi)particles at low temperatures, so that a $U(1)$ particle number symmetry is spontaneously broken. The condensing bosons may be individual atoms, as in the atomic superfluids ^{4}He [1,2] or ^{87}Rb [3], or loosely bound neutral Cooper pairs of fermions as in superfluid ^{3}He [4] or dense neutron matter [5]. Superfluidity has many parallels with superconductivity, with the crucial difference that in superconductors the condensing particles are electrically charged.

Although superfluids arise from condensation of electrically neutral particles, in nature these particles always have electrically charged constituents. Novel electromagnetic (EM) properties of a superfluid with scalar condensates (i.e., with vanishing spin S and orbital angular momentum L) have been explored in recent years, starting with the experimental work of Rybalko and collaborators [6,7] in superfluid ^{4}He, which triggered considerable further theoretical and experimental work [8–35]. Previous theoretical analyses have applied a wide variety of phenomenological models to explain the EM properties of liquid ^{4}He and other superfluids.

Our goal in this paper is to provide a systematic description of the EM properties of scalar superfluids, i.e., superfluids with scalar order parameters, in three spatial dimensions using the technique of effective field theory, e.g., see Refs. [36,37]. As an application, we focus on the magnetic properties of superfluid vortices. We will show that generic scalar superfluids embody a "vortex magnetic effect" (VME), namely superfluid vortices carry nonzero magnetic flux. We compare our prediction for the magnitude of the VME in superfluid ^{4}He to prior estimates of this effect.

The claim that superfluid vortices carry magnetic flux might seem surprising. After all, the parallel statement for superconductors holds because the superconducting order parameter is electrically charged, implying that the superflow around a vortex necessarily produces an azimuthal electric current and generates magnetic flux. In contrast, the defining feature of superfluids is that the superfluid order parameter is electrically neutral, so why should a superfluid vortex carry any magnetic flux? For superfluids whose order parameters have nonvanishing spin or orbital angular momentum, standard orbital or spin-orbit interaction terms drive the appearance of magnetic flux [38,39]. But for superfluids associated with scalar order parameters, it is far less obvious why vortices should carry any magnetic flux.

We first discuss the underlying phenomena which produce the dominant contribution to the VME, and other EM properties, in dilute scalar superfluids, following Kosevich [10]. We then present a more general effective field theory (EFT) analysis of the problem. One key result is that the effective Lagrangian of a generic scalar superfluid contains an operator proportional to $\boldsymbol{\omega}\cdot\mathbf{B}$, coupling the fluid vorticity $\boldsymbol{\omega}\equiv\boldsymbol{\nabla}\times\mathbf{u}$ to the magnetic field $\mathbf{B}$. This term leads to magnetic fields localized on superfluid vortices.

A dimensionless diluteness parameter $\gamma\equiv 2\pi n a^3$ will organize contributions in our EFT analysis, where n is the particle density and a characterizes the physical size of the particles comprising the fluid. (For comparison, for close-packed hard spheres of radius a, $\gamma\approx 1.11$.) However, there are multiple measures of particle size, which can differ significantly. For neutral spinless atoms, which are our focus, certain phenomena are sensitive to the s-wave scattering length a_s while other properties are characterized by the van der Waals radius $a_{\rm vdW}$ or related measures. For the electromagnetic effects we focus on here, the most relevant measure of atomic size is the charge radius of the atom, defined as the square root of $\langle r^2\rangle\equiv(Ze)^{-1}\int d^3r\,|\rho(\mathbf{r})|\,r^2$, with Ze the nuclear charge and $\rho(\mathbf{r})$ the total charge density of the atom. In much of the rest of this paper we will denote the charge radius by $a=\langle r^2\rangle^{1/2}$, and use this definition of atomic size to define our EFT parameter γ. In systems of spinless bosons which can be described by EFTs without fine-tuning, these various

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different measures of particle size will typically be comparable in size and so when, e.g., our expansion parameter is tiny ($\gamma = 2\pi n a^3 \ll 1$) so will be other measures of diluteness such as $2\pi n a_{\rm vdW}^3$. In less dilute systems, or systems whose EFT description involves fine-tuning, the differences between alternative measures of the particle size can yield sizable differences in the associated diluteness parameters. We will discuss this issue further at the end of the paper in the context of superfluid ^{4}He. Our EFT analysis will be under full theoretical control for sufficiently dilute scalar superfluids, while in nondilute systems such as liquid ^{4}He the EFT approach may be used to estimate the magnitude of the VME.

II. UNDERLYING PHYSICS

Following Kosevich [10], the charge density of any spherically symmetric neutral atom may be expressed as the Laplacian of a radial function with rapid falloff. The Fourier transformed charge density of such an atom has the long wavelength form $\widetilde{\rho}(\mathbf{k}) = Zea^2\mathbf{k}^2/6 + O(\mathbf{k}^4)$, with $-e$ the electron charge and a the above-defined charge radius. This implies that one may write $\rho(\mathbf{r}) = -\nabla^2(\frac{Zea^2}{6} f(\mathbf{r}))$ where $f(\mathbf{r})$ is a rapidly decreasing spherically symmetric function which integrates to unity, or in other words a smeared-out 3D delta function. For hydrogen, $a = \sqrt{3}\, a_{\rm B}$ and $f(\mathbf{r}) = (2\pi)^{-1} a_{\rm B}^{-3}\,(1+a_{\rm B}/|\mathbf{r}|)\, e^{-2|\mathbf{r}|/a_{\rm B}}$ with $a_{\rm B}$ the Bohr radius. Consequently, every atom generates an electrostatic potential proportional to this smeared-out delta function. For an arbitrary collection of widely separated identical atoms at positions $\{\mathbf{x}_i\}$, the net electrostatic potential is thus

$$\Phi(\mathbf{x}) = \frac{Ze\,a^2}{6\epsilon_0} \sum_i f(\mathbf{x}-\mathbf{x}_i) \simeq \frac{Ze\,a^2}{6\epsilon_0}\, n(\mathbf{x})\,. \tag{1}$$

The last form, with $n(\mathbf{x})$ the number density of atoms, is valid whenever the potential is to be integrated against functions slowly varying on the scale of a, so that the atomic scale details of $f(\mathbf{x})$ are irrelevant.

The electric field experienced by a test charge is $-\nabla\Phi$. An inhomogeneous density distribution which is averaged over a region of size $\lambda \gg n^{-1/3} \gg a$ induces a polarization $\mathbf{P} = \frac{1}{6}\, Ze\, a^2\, \nabla n$. This phenomenon has been termed *flexoelectricity* (see, for example, Ref. [9].) If the medium is moving with some velocity $\mathbf{v}$ (small compared to the speed of light c), relativistic invariance of electromagnetism implies that there will be a magnetization $\mathbf{M} = \mathbf{P} \times \mathbf{v}$.

A superfluid vortex directly embodies the above phenomena. For a minimal circulation vortex, the superfluid velocity field $\mathbf{v} = (\hbar/M)\,\hat{\boldsymbol{\theta}}/r$, with r the distance from the vortex core and M the condensing particle mass. [1] This leads to a total magnetic flux

$$\Phi_B \equiv \mu_0 \int d\boldsymbol{\Sigma} \cdot \mathbf{M} = Z\alpha\, \lambda_C\, a^2 \Delta n\, \frac{2}{3}\Phi_0, \tag{2}$$

where $\alpha \equiv e^2/(4\pi\epsilon_0\hbar c)$ is the fine structure constant, $\lambda_C \equiv 2\pi\hbar/(Mc)$ is the Compton wavelength of the fluid particles, $\Delta n \equiv \bar{n} - n(0)$ is the difference of the average particle density $\bar{n}$ and the reduced density $n(0)$ at the vortex center, and $\Phi_0 \equiv \pi\hbar/e$ is the usual magnetic flux quantum.

III. EFFECTIVE FIELD THEORY

There are further mechanisms which can generate nonvanishing magnetic flux in superfluid vortices, including van der Waals induced polarization in the presence of nonuniform density, and inertial effects in accelerating (or rotating) systems. These mechanisms have been discussed and analyzed in various ways in Refs. [8–35]. In dilute systems these other mechanisms lead to effects suppressed by additional factors of the small parameters na^3 and/or m_e/M relative to the flexoelectric mechanism described above. But to be confident one has not neglected some subtle yet important physical effect, it is very helpful to treat the problem systematically, without the need to consider individual microscopic mechanisms in isolation. This is the *raison d'être* of the effective field theory approach.

Consider a translation-invariant system of scalar ($S = L = 0$) electrically neutral nonrelativistic bosons of mass M interacting via short-range interactions. In addition to a conserved particle number, we assume that the interactions are also parity and time reversal invariant. If the system is dilute, meaning that $\gamma \ll 1$, then the effects of interactions can be systematically characterized using effective field theory (EFT).

A complex scalar field ϕ serves as a boson annihilation operator, with the $U(1)$ particle number symmetry acting as $\phi \to e^{i\alpha}\phi$. The electric and magnetic fields are related to the electromagnetic potentials $A_0 \equiv \Phi/c$ and $\mathbf{A}$ in the usual manner, $\mathbf{E} = -\nabla\Phi - \partial_t\mathbf{A}$ and $\mathbf{B} = \nabla \times \mathbf{A}$. On the low energy scales of interest, we assume that the only relevant degrees of freedom are those described by the complex scalar field ϕ together with the electromagnetic field. Consequently, an action built from local gauge-invariant combinations of these fields can provide an effective description of the system.

Since our goal is to understand EM effects, it will be helpful to take into account the constraints of Lorentz invariance within our nonrelativistic EFT. To the order to which we will work, it is sufficient to demand that our EFT be invariant under linearized Lorentz transformations representing boosts by some velocity $v \ll c$. Such transformations act on the fields as $\phi(\mathbf{x},t) \to e^{-iM\mathbf{v}\cdot\mathbf{x}}\,\phi(\mathbf{x}',t')$, $\mathbf{E}(\mathbf{x},t) \to \mathbf{E}(\mathbf{x}',t') + \mathbf{v}\times\mathbf{B}(\mathbf{x}',t')$, and $\mathbf{B}(\mathbf{x},t) \to \mathbf{B}(\mathbf{x}',t') - \mathbf{v}\times\mathbf{E}(\mathbf{x}',t')/c^2$, where $\mathbf{x}' \equiv \mathbf{x} + \mathbf{v}\,t$ and $t' \equiv t + \mathbf{v}\cdot\mathbf{x}/c^2$.

Although the particles (atoms) created by $\phi^\dagger$ are neutral, they contain charged constituents and interact with EM fields through nonminimal couplings. To enable a systematic treatment we initially assume that the system is dilute, and subsequently discuss implications when extrapolating to a denser fluid like liquid helium. Under these assumptions, the most general effective action will contain a sum of all local

[1] The fluid flow velocity $\mathbf{v}(r) = \mathbf{j}(r)/n(r)$, with $\mathbf{j}$ the particle number current density and n the particle number density. In a mean field treatment of a vortex, the radial dependence of $n(r)$ cancels that of $r|\mathbf{j}(r)|$, leaving $|\mathbf{v}| \propto 1/r$ even inside the vortex core, down to the limit of validity of the mean field treatment set by scale of the interparticle spacing.

terms, consistent with our symmetries, built from ϕ, $\mathbf{E}$, and $\mathbf{B}$ and their spatial derivatives. The result may be expressed as

$$S = S_\phi + S_{\rm EM} + S_{\phi,\,\rm EM}\,, \tag{3}$$

where S_ϕ and $S_{\rm EM}$ contain the free kinetic terms plus self-interactions of the scalar and EM fields, respectively, while $S_{\phi,\,\rm EM}$ describes the couplings between these fields. The EFT [Eq. (3)], correctly constructed, will reproduce physics on sufficiently large spatial and time scales. Taking units where $\hbar = \epsilon_0 = 1$, the spatial scales described by the EFT must be large compared to the atomic size a, or equivalently for spatial momenta small compared to the EFT breakdown scale $\Lambda \sim a^{-1}$. The time scales described by the EFT must be large compared to the inverse of the energy scale $\min(E_\Lambda, E_{\rm bind})$, where $E_\Lambda \equiv \Lambda^2/M$ is the kinetic energy associated with momentum Λ and $E_{\rm bind} \equiv e^2\Lambda/(4\pi)$ is the atomic binding scale. Physically, of course, E_Λ is smaller than $E_{\rm bind}$ by a factor of $m/M \sim 10^{-4}$, where m is the electron mass.

To compare the importance of different terms, we define the scaling dimensions of coordinates and fields as follows:

$$[x] = 1/Q, \quad [t] = M/Q^2, \quad [\phi] = Q^{3/2}, \tag{4a}$$

$$[\mathbf{E}] = [c\mathbf{B}] = M^{-1/2}\,Q^{5/2}, \quad [e^2] = [c] = Q/M\,, \tag{4b}$$

where Q is a characteristic momentum scale. Since S is dimensionless, every term in the Lagrange density must have dimensions of Q^5/M. It is helpful to write each term in the effective Lagrangian in the form $c_i\,\Lambda_i^{\alpha_i}\,E_i^{\beta_i}\,\mathcal{O}_i$, where c_i is an $O(1)$ dimensionless coefficient, $\mathcal{O}_i$ is some combination of fields and their derivatives, Λ_i and E_i are the natural ultraviolet (UV) momentum and energy scales associated with the particular term in question, and the exponents α_i and β_i characterize the sensitivity of the process described by $\mathcal{O}_i$ to the UV spatial momentum and energy scales. For all of the terms that we discuss below, $\Lambda_i = 1/a \equiv \Lambda$ and E_i is either E_Λ or $E_{\rm bind}$.

The part of the action only involving the neutral bosons has the form

$$S_\phi = \int dt\, d^3x \left[\phi^\dagger\left(i\partial_t + \mu + \frac{\nabla^2}{2M}\right)\phi - \frac{f_4\,a}{M}\,|\phi|^4 + \cdots\right]. \tag{5}$$

Here μ is chemical potential for particle number, and the coefficient f_4 in the quartic self-interaction term is a dimensionless $O(1)$ low-energy parameter which is determined by demanding that the quartic interaction correctly reproduce two-particle s-wave scattering. As previously mentioned in the introduction, without fine-tuning the s-wave scattering length a_s, the van der Waals radius $a_{\rm vdW}$ and the charge radius a will generically be roughly comparable in size.

The ellipsis in Eq. (5) represents additional terms involving explicit derivatives and/or higher powers of ϕ, whose coefficients must contain additional powers of a (or $1/\Lambda$) to achieve the correct dimensions. Such higher order terms not explicitly shown in Eq. (5) have negligibly small effects on the long-distance physics in the limit $2\pi n a^3 \ll 1$, making the properties of dilute systems of bosons systematically calculable using the EFT, see, e.g. Ref. [37].

The kinetic terms of the EM fields are contained in $S_{\rm EM}$, which takes the form

$$S_{\rm EM} = \tfrac{1}{2}\int dt\, d^3x\, (\mathbf{E}^2 - c^2\mathbf{B}^2 + \cdots). \tag{6}$$

The ellipsis represents self-interactions of the EM fields induced by radiative effects.

To construct interaction terms coupling ϕ to the EM fields, let $\mathbf{j} \equiv \frac{i}{2M}\,((\nabla\phi^\dagger)\phi - \phi^\dagger\nabla\phi)$ denote the conserved particle number current density and $n \equiv \phi^\dagger\phi$ the particle number density. We also define the density gradient $\boldsymbol{\rho} \equiv \nabla n$ and vorticity $\boldsymbol{\omega} \equiv \nabla \times \mathbf{j}$. The operators $\boldsymbol{\rho}$ and $\boldsymbol{\omega}$ will play roles analogous to electric and magnetic dipole moment densities, respectively. The fields $\mathbf{E}$ and $\boldsymbol{\rho}$ are parity odd while $\mathbf{B}$, $\boldsymbol{\omega}$, and ϕ are parity even. Under time reversal, $\mathbf{B}$ and $\boldsymbol{\omega}$ are odd, $\mathbf{E}$ and $\boldsymbol{\rho}$ are even, and $\phi \leftrightarrow \phi^\dagger$.

Now we can discuss the leading interaction terms in $S_{\phi,\,\rm EM}$. For our purposes, it will suffice to write out all symmetry-allowed terms with two powers of the scalar field, up to two powers of the EM fields, and at most two spatial derivatives. There are three such terms,[2]

$$\begin{aligned} S_{\phi,\,\rm EM} = \int dt\, d^3x \Big[& b\,e\,a^2\,(\boldsymbol{\rho}\cdot\mathbf{E} + \boldsymbol{\omega}\cdot\mathbf{B}) \\ & + \frac{1}{2}c_E\,a^3\,(n\,\mathbf{E}^2 - 2\,\mathbf{j}\cdot(\mathbf{E}\times\mathbf{B})) \\ & - \frac{1}{2}c_M\,e^4a^3\left(n\,\mathbf{B}^2 - \frac{2}{c^2}\,\mathbf{j}\cdot(\mathbf{E}\times\mathbf{B})\right) + \cdots\Big], \end{aligned} \tag{7}$$

where the ellipsis stands for terms with higher powers of fields and/or explicit time or space derivatives. Note that terms proportional to $\boldsymbol{\rho}\cdot\mathbf{B}$, $\boldsymbol{\omega}\cdot\mathbf{E}$, and $n\,\mathbf{E}\cdot\mathbf{B}$ are ruled out by our discrete symmetries.

We have organized the terms appearing in interaction action Eq. (7) so that each line is invariant under linearized Lorentz boosts, up to residuals suppressed by quadratic combinations of boost velocity over c and/or field time derivatives over Mc^2 (such residual terms may be canceled by systematically adding yet higher order terms to the action). Imposing boost invariance reduces the set of independent dimensionless parameters (or "low-energy constants") characterizing the EFT, at this order, to three: b, c_E, and c_M, all of which will generically be $O(1)$ unless the interactions in the underlying microscopic theory are deliberately fine-tuned.[3]

[2]One could eliminate the b term by performing the electromagnetic field redefinition, $A_0 \to A_0 + \frac{1}{2}b\,ea^2\phi^\dagger\phi/(1 + c_E a^3 n)$ and $\mathbf{A} \to \mathbf{A} + \frac{1}{2}b\,ea^2\mathbf{j}/(c^2 + c_M e^4 a^3 n)$, but this would change the physical meaning of the $\mathbf{E}$ and $\mathbf{B}$ fields in an unhelpful manner while not, of course, affecting any observable effects. We prefer to use the standard electric and magnetic fields, and hence choose to work with the action Eq. (7) in which the b term appears explicitly.

[3]If the interactions were to preserve a discrete symmetry $\mathcal{S}$ which flips the EM potentials $\Phi \to -\Phi$ and $\mathbf{A} \to -\mathbf{A}$ while leaving the neutral scalar ϕ unchanged, then $b = 0$. Such a putative symmetry is not charge conjugation, which would also conjugate ϕ, and is not a symmetry of the nonrelativistic action [Eq. (5)]. But if the field ϕ represents some composite particle built from oppositely-charged but otherwise identical constituents, then $\mathcal{S}$ would correspond to a

The factors of e and a shown explicitly in the above three terms of $S_{\phi,\mathrm{EM}}$ serve to render the coefficients b, c_E, and c_M dimensionless. But since e^2/c and e^2Ma are dimensionless combinations, these factors are not solely determined by dimensional analysis. The given prefactors correspond to the statement that the relevant UV energy scale for these EM interaction terms is E_{bind}. (Equivalently, these factors are also determined by noting that, in a homogeneous medium at rest, these interaction terms should be unaffected by sending $c \to \infty$ and $M \to \infty$.)

The c_E and c_B terms in $S_{\phi,\mathrm{EM}}$, which are quadratic in EM fields, characterize the dielectric and diamagnetic linear response of the medium, so that $\epsilon/\epsilon_0 = 1 + c_E\, a^3\, \bar{n}$ and $\mu_0/\mu = 1 + c_M\, (e^2/c)^2 a^3\, \bar{n}$. We will show that the b term generates the rotation-induced polarization and magnetization effects described in the introduction.

IV. MAGNETIC STRUCTURE OF ROTATING SUPERFLUIDS

The b term of $S_{\phi,\mathrm{EM}}$, which is linear in $\mathbf{E}$ and $\mathbf{B}$, vanishes in any homogeneous, nonrotating system, but generates novel effects in inhomogeneous systems such as, in particular, low temperature superfluids with vortices. To show this, we first consider a nonrotating zero-temperature system described by the EFT [Eq. (3)], with repulsive self-interactions and a chemical potential driving boson condensation, $f_4 > 0$ and $\mu > 0$. The field ϕ acquires a nonvanishing expectation value with squared magnitude

$$\bar{\phi}^2 \equiv |\langle\phi\rangle|^2 = \frac{M\mu}{2\,a f_4}\,, \tag{8}$$

to leading order in the EFT expansion. This indicates a superfluid state with spontaneously broken $U(1)$ particle number symmetry. In dilute superfluids, $\bar{\phi}^2$ is close to (but less than) the total particle density $\bar{n} = \langle n\rangle$ in the interacting ground state.

Rotating a superfluid sample induces a nonzero superflow from variation in the condensate phase $\mathbf{v}_\mathrm{s} \equiv \mathbf{j}_\mathrm{s}/|\langle\phi\rangle|^2 = -\boldsymbol{\nabla}(\arg\langle\phi\rangle)/M$, where one should remember that in a rotating system $\langle\phi\rangle$ can depend on position. Vortices are characterized by a quantized circulation arising from nontrivial winding of the phase, $\mathcal{C} \equiv \oint d\boldsymbol{\ell}\cdot\mathbf{v}_\mathrm{s}(\mathbf{x}) = 2\pi\nu/M$, where $\nu \in \mathbb{Z}$. Equivalently, the vorticity $\boldsymbol{\omega}$ is nonzero, with surface integrals of vorticity counting the number of vortices piercing the surface, $\int_\mathcal{S} d\boldsymbol{\Sigma}\cdot\boldsymbol{\omega} = (2\pi/M)\,\nu$.

Due to the $\boldsymbol{\omega}\cdot\mathbf{B}$ interaction term of the EFT, nonvanishing vorticity acts as a bias which drives a shift in the magnetic field minimizing the energy. Indeed, we can write $bea^2\,\boldsymbol{\omega}\cdot\mathbf{B} = bea^2[(\boldsymbol{\nabla}\times\boldsymbol{\omega})\cdot\mathbf{A} - \boldsymbol{\nabla}\cdot(\boldsymbol{\omega}\times\mathbf{A})]$. Integrating this term over some volume $\mathcal{V}$ with boundary $\mathcal{S} \equiv \partial\mathcal{V}$ gives $\int_\mathcal{V}\mathbf{J}_\mathcal{V}\cdot\mathbf{A} + \int_\mathcal{S}\mathbf{J}_\mathcal{S}\cdot\mathbf{A}$, with $\mathbf{J}_\mathcal{V} \equiv bea^2(\boldsymbol{\nabla}\times\boldsymbol{\omega})$, $\mathbf{J}_\mathcal{S} \equiv bea^2\,\boldsymbol{\omega}\times\hat{\mathbf{n}}$, and $\hat{\mathbf{n}}$ an outward normal to the boundary. This shows that in a rotating sample with boundary $\mathcal{S}$, there is a volume EM current density $\mathbf{J}_\mathcal{V}$ proportional to the curl of vorticity plus a surface EM current density $\mathbf{J}_\mathcal{S}$ directly proportional to vorticity.

V. ANALYSIS OF VME

A straight superfluid vortex with minimal winding is a field configuration of the form $\langle\phi(\mathbf{x})\rangle = \bar{\phi}f(r)e^{i\theta}$, using cylindrical coordinates $\mathbf{x} = (r,\theta,z)$ centered on and aligned with the vortex. Configurations with nonminimal winding in simply-connected regions typically resemble lattices built from minimal-winding vortices. The radial function $f(r)$ is determined, to leading order in density, by solving the classical equations of motion. It varies smoothly from 0 to 1 as r ranges from 0 to ∞, with asymptotics $f(r) \sim r/\zeta$ as $r \to 0$ and $f(r) \sim 1 - \xi^2/r^2$ as $r \to \infty$. Here $\xi \equiv (4M\mu)^{-1/2} = a(8f_4\bar{n}\,a^3)^{-1/2}$ is the "healing length" of the condensate. In the low-density limit the core size ζ is proportional to ξ, with $g \equiv \xi/\zeta = 0.412(4)$.

The vorticity of a minimal vortex is given by $\boldsymbol{\omega} = 2\hbar\bar{\phi}^2\,\hat{\mathbf{z}}\,f(r)f'(r)/(Mr)$, and its curl gives

$$\mathbf{J}_\mathcal{V} = -\frac{2\hbar bea^2\bar{\phi}^2}{M}\left(\frac{f(r)f'(r)}{r}\right)'\hat{\boldsymbol{\theta}}\,. \tag{9}$$

Since $\boldsymbol{\omega}$ falls as $O(r^{-5})$ at large r, the surface current $\mathbf{j}$ is negligible. A straightforward application of Ampère's law gives the magnetic field associated to a single minimal vortex, $\mathbf{B}_\mathrm{s}(\mathbf{x}) = 2\mu_0\hbar\frac{be\,a^2\bar{\phi}^2}{M}\frac{f(r)f'(r)}{r}\,\hat{\mathbf{z}}$. The magnitude of $\mathbf{B}_\mathrm{s}$ approaches $2\mu_0\hbar\,be\,a^2\bar{\phi}^2/(M\zeta^2)$ at the center of the vortex, and falls as $O(r^{-4})$ at large distance. The resulting magnetic flux is

$$\Phi_B^{\nu=1} = 2\pi\,\mu_0\hbar\,\bar{\phi}^2\,\frac{be\,a^2}{M}\,. \tag{10}$$

At least to this order in the EFT analysis, the magnetic flux is completely independent of the internal structure of the superfluid vortex. More generally, any configuration with the topology of a superfluid vortex will necessarily carry a nonvanishing magnetic flux. A configuration of ν well-separated vortices will have total flux which is just ν times the minimal value [Eq. (10)].

Comparing our EFT result [Eq. (10)] and the result [Eq. (2)] of our earlier discussion based on the flexoelectric effect, one sees that the two expressions (for dilute systems where $\Delta n \approx \bar{n}$) coincide when the undetermined $O(1)$ EFT coefficient b has the explicit value

$$b = Z/6\,. \tag{11}$$

VI. OUTLOOK

Numerical estimates suggest that the VME may be experimentally observable. First, consider a superfluid composed of bosonic atoms with atomic number A, so that $m_e/M = 5.4\times10^{-4}A^{-1}$. Expressing the flux carried by a unit-winding superfluid vortex in terms of the superconducting flux quantum $\Phi_0 = \pi\hbar/e$ and the Bohr radius $a_\mathrm{B} = \hbar/(\alpha m_e c)$ (with

charged constituent permutation and could in principle be a symmetry of the long-distance EFT (3). However, in physical systems of interest there is, of course, no such symmetry interchanging electrons and ions. Therefore the b term is not symmetry forbidden, and one should expect the coefficient b to be $O(1)$; we show this explicitly in a toy model calculation in the Supplementary Materials.

fine structure constant $\alpha \equiv e^2/(4\pi\epsilon_0\hbar c) \approx 1/137$), we have

$$\frac{\Phi_B}{\Phi_0} = 8\pi\,\alpha^2\, b\,\bar{n}\,a^2 a_{\rm B}\,\frac{m_e}{M}$$
$$= 7.2\times 10^{-7}\,\frac{b}{A}\Big(\frac{a_{\rm B}}{a}\Big)(\bar{n}\,a^3). \qquad (12)$$

Note that we have replaced a factor of $\bar{\phi}^2$ by $\bar{n}$. The difference between these quantities is suppressed by a positive power of $\bar{n}a^3$, and we have only worked to leading order in $\bar{n}a^3$, so at the level of precision of our analysis we would not be justified in making a distinction between $\bar{\phi}^2$ and $\bar{n}$.

To maximize the magnetic flux carried by a vortex and make experimental detection of the VME easier, we now estimate the size of the VME in nondilute superfluids. The paradigmatic example of a nondilute spin-0 superfluid is superfluid Helium-4. The helium charge radius $a \approx 1.1\,a_{\rm B} = 0.58\,$Å (see, e.g., p. 325 in Ref. [40])[4] and the van der Waals radius $a_{\rm vdW} \approx 1.4\,$Å [42]. The ^{4}He dimer binding energy is measured to be 151.9 ± 13.3 neV [43,44], corresponding to a very large s-wave scattering length $a_s \approx 83\,$Å. The large hierarchy between the s-wave scattering length a_s and other measures of helium atom size indicates that any EFT description of helium would need to involve fine-tuning of the quartic interaction strength. We emphasize that the resulting EFT would accurately describe phenomena at distances which are either large or small compared to the scattering length a_s, as long as relevant distances are large compared to the charge radius a (or $a_{\rm vdW}$), and so long as the diluteness parameters $2\pi\bar{n}a_{\rm vdW}^3$, $2\pi\bar{n}a^3$ are both sufficiently small.

However, the particle density of superfluid ^{4}He is $\bar{n} \approx 0.022\,$Å^{-3} [45], which leads to a charge radius based diluteness parameter $\gamma \approx 0.03$, while the somewhat larger van der Waals radius of helium gives an $\mathcal{O}(1)$ value for the corresponding diluteness parameter $2\pi\bar{n}a_{\rm vdW}^3 \approx 0.4$. This indicates that superfluid helium cannot be viewed as highly dilute, and so applying our EFT results to helium necessarily involves an extrapolation with considerable uncertainty.

Inserting the values above into Eq. (12), and setting $Z = 2$ so that $b \approx 1/3$, leads to the estimate

$$\frac{\Phi_B}{\Phi_0} \approx 2\times 10^{-10}\,. \quad \text{(Superfluid Helium).} \qquad (13)$$

This estimate, based on an extrapolation of results from our dilute system analysis, should be conservatively viewed as subject to an order of magnitude theoretical uncertainty. Nevertheless, our result for the magnetic flux of a superfluid ^{4}He vortex is two orders of magnitude larger than the earlier prediction of Ref. [16]. The primary reason for the discrepancy is that Ref. [16] assumed that the flexoelectric effect arises primarily from Van der Waals interactions between atoms, rather than just the nonuniform spatial distribution of the individual atoms. Our EFT analysis gives a systematic demonstration that the latter physics drives the leading-order vortex magnetic effect, at least in a limit where systematic analytic calculations are possible.

One plausible approach to experimentally measuring the magnetic properties of ^{4}He superfluid vortices would involve using SQUIDs (superconducting quantum interference devices) to detect the vortex magnetic flux. Quantum-limited SQUIDs of radius $1\,\mu$m and a noise in the range of $\sim 45\times 10^{-9}\,\Phi_0/\sqrt{Hz}$ were reported in Ref. [46]. Using a SQUID with this performance in an experiment with a measurement time of several days should enable one to measure directly the vortex magnetic effect from a single superfluid ^{4}He vortex.

ACKNOWLEDGMENTS

We are grateful to F. Burnell for discussions and collaboration at the beginning of this project, and thank A. Kamenev, M. Pospelov, S. I. Shevchenko, B. Shkolovskii, M. Shifman, and B. Spivak for discussions, as well as J. Kas and J. Rehr for directing us to useful helium structure calculations. We also thank A. Andreev, B. Svistunov, G. Volovik, and an anonymous referee for drawing our attention to an error in an earlier version of this paper, and acknowledge support from U. Minn (A.C.), Grant No. DE-SC0011637 (L.G.Y.) and the DOE Nuclear Physics Quantum Horizons Program through the Early Career Award No. DE-SC0021892 (S.S.).

APPENDIX: MICROSCOPIC TOY MODEL

We consider a microscopic toy model describing neutral atoms comprised of oppositely charged constituents, which at low energies is described by an effective field theory analogous to (7). The discussion here supports the more general power-counting arguments we used to determine the combinations of scales appearing in the EFT describing the leading electromagnetic interactions of neutral atomic superfluids.

Suppose that a neutral atom field ϕ represents a bound state of a positively charged heavy nucleus of mass m_+ and a single electron of mass m_-, described by the fields ϕ_+ and ϕ_- respectively. For our purposes the statistics of the charged particles are irrelevant, and we take them to be bosons for simplicity. The corresponding Lagrangian is given by

$$\mathcal{L} = -\frac{1}{4}F^2 + \phi^\dagger\Big(i\partial_t - 2\mu + \frac{\boldsymbol{\nabla}^2}{2M}\Big)\phi$$
$$+ \sum_\pm \phi_\pm^\dagger\Big(i\partial_t - \mu \mp eA_0 + \frac{(\boldsymbol{\nabla} \mp \frac{ie}{c}\boldsymbol{A})^2}{2m_\pm}\Big)\phi_\pm$$
$$- \epsilon(\phi\,\phi_+^\dagger\phi_-^\dagger + \phi^\dagger\phi_+\phi_-)\,, \qquad (A1)$$

where $M = m_+ + m_-$ and μ is the chemical potential for the single $U(1)$ global symmetry under which ϕ_+, ϕ_- have charge 1 and ϕ has charge 2. The ϵ coupling reflects the microscopic picture of the neutral atom as a bound state of charged constituents, see e.g. Ref. [47] for a more general discussion of this sort of approach in EFT. We will use Eq. (A1) to evaluate the amplitude for scattering of bound states due to a classical

[4]This value is from a Dirac-Fock calculation of ground state helium. A more accurate multi-configuration calculation [41] produces a value of $\langle r\rangle$ for helium which is only 0.2% larger than the corresponding Dirac-Fock value, so the result of Ref. [40] for the helium charge radius is surely accurate to 1% or better. Physically, it is clear that the charge radius of ground state $Z = 2$ helium must be smaller than the hydrogenic value of $\sqrt{3}\,a_{\rm B}$.

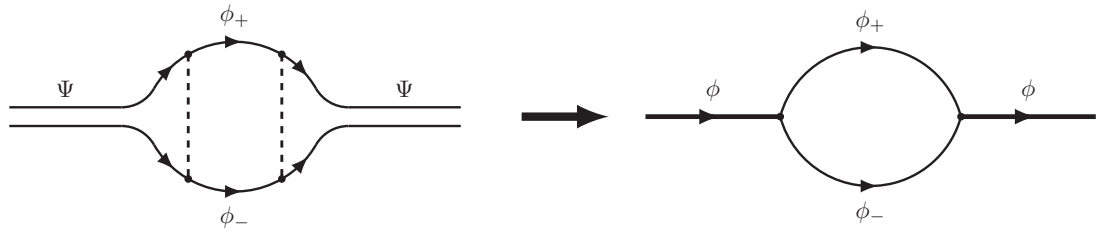

FIG. 1. One-loop diagrams with and without an explicit ϕ field. The dashed line represents Coulomb exchange, and the double line Ψ represents the result of taking all Coulomb ladder diagrams into account to produce the neutral external states. The top and bottom solid lines denote ϕ_+ and ϕ_- respectively.

external electromagnetic field, and match the result to the same amplitude as computed in the EFT in Eq. (7).

To begin, we observe that the ϵ coupling relates the atomic bound state ϕ to its constituents ϕ_+ and ϕ_-, and can be understood as a contact interaction which approximates the re-summed ladder of Coulomb exchanges between ϕ_+ and ϕ_-. This means that ϵ can be related to the microscopic parameters a, e, m_-, m_+.

As depicted in Fig. 1, in the microscopic theory the basic bubble self-energy diagram is approximately

$$\sim \int \prod_{i=1}^{4} d^4x_i \langle \Psi_{\mathbf{p}'}|\phi_+^\dagger(x_3)\phi_-^\dagger(x_4)\frac{e^2\delta(t_4-t_3)}{4\pi|\mathbf{x}_4-\mathbf{x}_3|}iG^-_{4,2}iG^+_{3,1} \times \frac{e^2\delta(t_2-t_1)}{4\pi|\mathbf{x}_2-\mathbf{x}_1|}\phi_+(x_1)\phi_-(x_2)|\Psi_{\mathbf{p}}\rangle\,, \tag{A2}$$

where

$$G^\pm_{i,j} = \int \frac{d^4k}{(2\pi)^4}\frac{e^{-ik\cdot(x_i-x_j)}}{k^0-\frac{\mathbf{k}^2}{2m_\pm}+\mu+i\epsilon} \tag{A3}$$

are the non-relativistic propagators for $\phi_\pm$. To get the amplitude we are interested in, we need to contract the external legs, taking into account that the external states are bound states rather than free particle states. This can be done by writing

$$\phi_+(x_1)\phi_-(x_2)|\Psi_{\mathbf{p}}\rangle = e^{-ip^0t}\phi_{\mathbf{p}}(\mathbf{R}_1)\,e^{-iE_bt}\psi(\mathbf{r}_1), \tag{A4}$$

where $\mathbf{R}_1 = \frac{m_+\mathbf{x}_1+m_-\mathbf{x}_2}{m_++m_-}$ and $\mathbf{r}_1 = \mathbf{x}_2-\mathbf{x}_1$ are the center of mass and relative coordinates of the particles $\phi_\pm$, p^0 and $\mathbf{p}$ are the energy and momentum of the center of mass, and E_b is the binding energy. The center of mass motion is described by a plane wave $\phi_{\mathbf{p}}(\mathbf{R}_1) = e^{i\mathbf{p}\cdot\mathbf{R}_1}$. For the motion with respect to the relative coordinates, we use the S-wave Coulomb bound state wave function of size a, $\psi(\mathbf{r}_1) \sim \frac{1}{a^{3/2}}e^{-|\mathbf{r}_1|/a}$. Analogous forms hold for the final state contraction. The contraction with the external bound states introduces a screening length a for the Coulomb interaction. Matching the momentum space loop integral to the analogous result in the low energy effective theory amounts to identifying

$$\epsilon^2 \longleftrightarrow \frac{\alpha^2c^2}{a^3}\frac{1}{\left(\left(\mathbf{k}-\frac{m_r}{m_+}\mathbf{p}\right)^2+a^{-2}\right)^2}, \tag{A5}$$

where $\mathbf{k}$ is the loop momentum and $m_r = m_+m_-/(m_++m_-)$ is the reduced mass. The loop integral arising from Eq. (A2) is dominated by momentum of the order of the external momentum, so in the Coulomb propagators we can expand in both $\mathbf{p}^2$ and $\mathbf{k}^2$ and we obtain the relation

$$\epsilon^2 \approx \alpha^2c^2a \approx \frac{1}{m_-^2a}. \tag{A6}$$

where to get to the final expression of the right we took $m_- \ll m_+$.

We now evaluate the amplitude for scattering of neutral bound states in a classical EM field, see Fig. 2. More specifically, we expand the S-matrix element in the external momenta and concentrate on the leading non-zero contributions, corresponding to the leading terms in the effective Lagrangian (7). It suffices to consider scattering from a static electric field,

$$S_{fi} = (-i)^3e\epsilon^2\int d^4x_1d^4x_2d^4x_3\langle i| \times [\phi^\dagger(x_2)iG^+_{2,3}iG^+_{3,1}iG^-_{2,1}A_0(x_3)\phi(x_1)]|f\rangle - (+\leftrightarrow -)\,. \tag{A7}$$

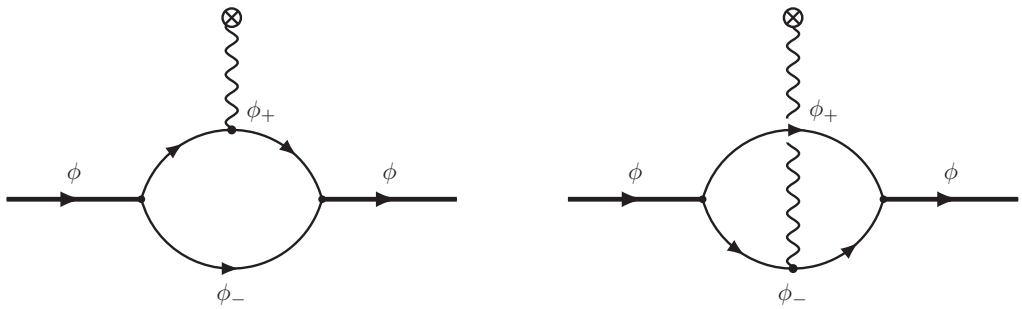

FIG. 2. One-loop diagrams giving rise to the couplings between $\mathbf{E}$, $\mathbf{B}$ and $\boldsymbol{\rho}$, $\boldsymbol{\omega}$ defined in Eq. (7). The bold line denotes ϕ and the top and bottom solid lines denote ϕ_+ and ϕ_- respectively.

The initial and final states are single-particle ϕ states with momenta p and p'. Expanding in the external momenta, the leading contribution to the amplitude is

$$\mathcal{A} \sim e\epsilon^2 \frac{\Delta m}{m_+ m_-} \left(\frac{m_r}{p^0 + 2\mu} \right)^{3/2} (\mathbf{p} - \mathbf{p}')^2 A_0(\mathbf{p}' - \mathbf{p}), \quad \text{(A8)}$$

where $\Delta m = m_+ - m_-$. In the case of interest $m_- \ll m_+$, and to leading order in the momentum expansion we can let $p^0 = E_b = -\frac{1}{2m_- a^2}$ equal the binding energy. Furthermore, since the chemical potential μ is suppressed relative to E_b by a factor of m_-/m_+, we obtain the final expression

$$\mathcal{A} \sim ea^2 (\mathbf{p}' - \mathbf{p})^2 A_0(\mathbf{p}' - \mathbf{p}), \quad \text{(A9)}$$

neglecting $O(m_-/m_+)$ corrections and using Eq. (A6). This coincides with the same scattering amplitude as computed using the effective field theory in Eq. (7) with an $O(1)$ value for the coefficient b.

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