Domain-wall skyrmions in chiral magnets

Calum Ross [1,2,*] and Muneto Nitta [2,†]

[1] *Department of Mathematics, University College London, London WC1E 6BT, United Kingdom*

[2] *Department of Physics and Research and Education Center for Natural Sciences, Keio University, Hiyoshi 4-1-1, Yokohama, Kanagawa 223-8521, Japan*

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Domain-wall skyrmions are skyrmions trapped inside a domain wall. We investigate domain-wall skyrmions in chiral magnets using a fully analytic approach. Treating the Dzyaloshinskii-Moriya (DM) interaction perturbatively, we construct the low-energy effective theory of a magnetic domain wall in an $O(3)$ sigma model with the DM interaction and an easy-axis potential term, yielding a sine-Gordon model. We then construct domain-wall skyrmions as sine-Gordon solitons along the domain wall. We also construct domain-wall skyrmions on top of a pair of a domain wall and an anti-domain wall. One characteristic feature of domain-wall skyrmions is that both skyrmions and antiskyrmions are equally stable inside the domain wall, unlike the bulk in which only one of them is stable.

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I. INTRODUCTION

Skyrmions are topological solitons proposed by Skyrme as a model of nuclei [1]. They have been shown to be the baryons of large-N_c quantum chromodynamics (QCD) [2] and have been studied extensively [3]. Recently, there has been great interest in magnetic skyrmions [4], which are two-dimensional analogs of skyrmions in chiral magnets, from both fundamental and applied sciences, since they were realized in chiral magnets with the Dzyaloshinskii-Moriya (DM) interaction [5] in laboratory experiments [6] and have been proposed as information carriers in ultradense memory and logic devices with low energy consumption [7]. In a certain parameter region of chiral magnets, a chiral soliton lattice is the ground state [8] where the energy of a single soliton is negative, and one-dimensional modulated states have lower energy than skyrmions. In another parameter region, the ground state is a lattice of skyrmions in which the energy of a single skyrmion becomes negative [9]. Finally, there are ferromagnetic regions of the phase diagram where skyrmions appear as positive energy solitons above the ferromagnetic ground state. In addition, isolated skyrmions are also experimentally observable [10]. Another recent development is observations of skyrmion tubes in three-dimensional (3D) materials [11].

On the other hand, when there is an easy-axis potential term, magnetic domain walls have also been studied for a long time, in particular for their application to magnetic memories [12,13]. It is thus natural to expect, by combining these two objects, that there should be potential applications in constructing further useful nanodevices, such as the domain-wall racetrack memory proposal [12] which has been extended to a proposal using skyrmions [14]. As such a composite object, "domain-wall skyrmions" have been proposed in quantum field theory [15][1] and have been recently observed experimentally in chiral magnets [17–19] (see also Ref. [20]). A first step at treating chiral magnetic domain walls theoretically is given in Refs. [21,22]. In fact, a sine-Gordon-type model of the domain-wall effective energy was first derived in Ref. [22], where a different starting point is taken to that considered here. Composite objects like domain-wall skyrmions also occur as vortex sheets in ^{3}He [23] and in quantum Hall systems [24].

In quantum field theory, domain-wall skyrmions can be described by sine-Gordon kink configurations on top of a domain wall or anti-domain wall [15]: when skyrmions are absorbed into a domain wall, they become sine-Gordon solitons (see Ref. [25] for subsequent studies and Ref. [26] for earlier related works). More precisely, a domain wall in an $O(3)$ sigma model with an easy-axis potential term $V = m^2(1 - n_3^2)$ possesses the moduli space (collective coordinates) $\mathbb{R} \times S^1$ [27]. Then, the low-energy effective theory describing the low-energy dynamics of the domain wall can be constructed by the so-called moduli approximation [28], yielding a nonlinear sigma model with the target space $\mathbb{R} \times S^1$ in this case. When there is a second anisotropy term involving n_1, in addition to the aforementioned potential term V in the $O(3)$ model, it induces a sine-Gordon potential on the S^1 part in the domain-wall effective theory. Then, the domain-wall skyrmion can be expressed as a sine-Gordon soliton in the domain-wall effective theory [15]. In the absence of the second anisotropy n_1, the skyrmion is diluted along the domain wall and eventually disappears, once it is absorbed into the domain wall. In quantum field theory, domain-wall skyrmions

*calum.ross@ucl.ac.uk

†nitta@phys-h.keio.ac.jp

[1] Originally the term "domain-wall skyrmions" was first introduced in Ref. [16] in which Yang-Mills instantons become 3D skyrmions inside a domain wall.

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are generalized in various directions.[2] On the other hand, originally skyrmions were proposed as a model of nuclei and live in three spatial dimensions [1]. In this dimensionality, domain-wall skyrmions are 3D skyrmions absorbed into a domain wall, in which they become two-dimensional (2D) skyrmions (lumps) [34] (see also Ref. [35] for similar configurations). Higher-dimensional domain-wall skyrmions were further proposed [36]. Such domain-wall skyrmions play a key role in understanding relations between topological solitons in quantum field theory as proposed in Ref. [37].

In this paper, we study, in a fully analytic way, domain-wall skyrmions in chiral magnets described by an $O(3)$ sigma model with a DM interaction and an easy-axis potential term. Working in a small coupling regime for the DM term, we construct a domain-wall effective theory and find that it is a sine-Gordon model with the DM term supplying the potential, even in the absence of the second anisotropy term for n_1. We then construct domain-wall skyrmions as sine-Gordon solitons in terms of the domain-wall effective theory. We also construct domain-wall skyrmions on top of a pair of a domain wall and an anti-domain wall, and a domain-wall skyrmion lattice. We find that that both skyrmions and antiskyrmions are equally stable inside the domain wall, in contrast to the bulk where only either of them is stable. The analytic method proposed in this paper should be useful for further studies of domain-wall skyrmions and proposing possible applications to nanotechnology.

This paper is organized as follows. In Sec. II, we give our model and discuss the Derrick scaling argument [38] for stability of topological solitons in chiral magnets. In Sec. III, we construct the low-energy effective action of a single domain wall and present the domain-wall skyrmion solutions as well as a configuration where a lattice of sine-Gordon kinks is placed on top of a domain wall. Section IV is devoted to a summary and discussion.

II. THE MODEL

In this section, we describe the model that we consider in this paper and apply the Derrick scaling argument to chiral magnets.

A. Energy functionals

The static energy of a chiral magnet is described by an energy functional in terms of the magnetization vector field $\vec{n} = (n_1, n_2, n_3) : \mathbb{R}^2 \to S^2$, a normalized three-vector. Considering a specific model where there is both a DM term and an easy-axis anisotropy term in the z direction leads to an $O(3)$ model:

$$\mathcal{E} = \tfrac{1}{2}\nabla\vec{n}\cdot\nabla\vec{n} + \kappa\,\vec{n}\cdot(\nabla_{-\alpha}\times\vec{n}) + m^2\big(1-n_3^2\big). \tag{1}$$

The rotated gradient $\nabla_{-\alpha}$ with $\alpha \in (0, 2\pi]$ shows that we have a one-parameter family of models as discussed in Ref. [39]. Changing α changes the type of DM term with $\alpha = 0$ corresponding to a Bloch DM term and $\alpha = \frac{\pi}{2}$ a Néel DM term.

A convenient representation of the energy uses a complex stereographic coordinate $u \in \mathbb{C}$ related to the magnetization vector field through $u = \frac{n_1+in_2}{1+n_3}$. In this coordinate, the static energy can be rewritten in the form of the $\mathbb{C}P^1$ model as

$$\begin{aligned} E[u] &= 2\int d^2x\left[\frac{|\nabla u|^2 + 2\kappa\,\mathrm{Im}(e^{i\alpha}[\partial_z u + u^2\partial_z\bar{u}]) + 2m^2|u|^2}{(1+|u|^2)^2}\right] \\ &= E_H + E_{\mathrm{DM}} + E_0, \end{aligned} \tag{2}$$

$$E_H = 2\int d^2x\frac{|\nabla u|^2}{(1+|u|^2)^2}, \tag{3}$$

$$E_{\mathrm{DM}} = 2\int d^2x\frac{2\kappa\,\mathrm{Im}(e^{i\alpha}[\partial_z u + u^2\partial_z\bar{u}])}{(1+|u|^2)^2}, \tag{4}$$

$$E_0 = 2\int d^2x\frac{2m^2|u|^2}{(1+|u|^2)^2}, \tag{5}$$

where E_H is the Heisenberg, or Dirichlet, term in the energy involving two derivatives, E_{DM} is the DM interaction term, and E_0 is the anisotropy term.

In terms of minimizing the energy, it is well known that the three terms in Eq. (2) all want different things: The Heisenberg term wants the magnetization vector field to align at nearby points, the potential term is easy-axis anisotropy and so wants the magnetization to point in the third direction $\vec{n} = \pm\hat{e}_3$, and finally the DM term wants the magnetization vector to twist with the nature of the twist determined by the material parameter α. In particular as pointed out in Ref. [39] when $\alpha = 0$ the DM is of Bloch type and describes materials supporting Bloch-type skyrmions, while $\alpha = \frac{\pi}{2}$ gives a Néel-type DM term and describes materials supporting Néel-type skyrmions.

B. Stability and Derrick scaling

Given a static energy functional such as Eq. (2), it is natural to ask if energy-minimizing solutions exist. The standard approach is to use a Derrick scaling argument [3, 38] to show that static solutions exist. This scaling argument is naturally dimension dependent and here we want to keep track of both the two-dimensional and one-dimensional (1D) cases, relevant for skyrmions and domain walls, respectively.

Let $u_\lambda = u(\lambda x)$ be a scaled field and then treat the 1D and 2D cases separately.

One-dimensional case. In one dimension, the DM term is scale invariant since it involves one integral and one derivative, while the potential and Heisenberg terms scale oppositely to each other:

$$E[u_\lambda] = \lambda E_H[u] + E_{\mathrm{DM}}[u] + \lambda^{-1}E_0[u]. \tag{6}$$

[2]Some generalizations in field theory are the following: It was generalized to the $\mathbb{C}P^{N-1}$ model in Ref. [29], where $\mathbb{C}P^{N-1} = SU(N)/[SU(N-1)\times U(1)]$ is a complex projective space of complex dimension $N-1$. The $\mathbb{C}P^{N-1}$ model also admits skyrmions due to $\pi_2[\mathbb{C}P^{N-1}] \simeq \mathbb{Z}$, and $N-1$ parallel domain walls in the presence of a suitable potential term as a generalization of the easy-axis potential [30]. Then, domain-wall skyrmions in the $\mathbb{C}P^{N-1}$ model are $U(1)^{N-1}$ coupled sine-Gordon solitons inside $\mathbb{C}P^{N-1}$ domain walls [29]. It was also shown in Refs. [31] that skyrmions in the Grassmannian sigma model become non-Abelian sine-Gordon solitons [32] inside a non-Abelian domain wall [16,33].

There are stable static solutions if this has a stationary point as a function of λ, in other words, when $\frac{dE}{d\lambda}|_{\lambda=1} = 0$. This happens when

$$E_H[u] = E_0[u]. \tag{7}$$

Thus, the DM term is not needed for stable domain-wall solutions to exist.

Two-dimensional case. In this case the Heisenberg term is scale invariant and the energy scales as

$$E[u_\lambda] = E_H[u] + \lambda^{-1}E_{\rm DM}[u] + \lambda^{-2}E_0[u]. \tag{8}$$

Here $\frac{dE}{d\lambda}|_{\lambda=1} = 0$ implies that

$$E_{\rm DM} = -2E_0, \tag{9}$$

which is possible since the DM term can give a negative contribution to the energy. Due to the chirality of the DM term, for a given material either skyrmions or antiskyrmions will have negative DM energy. The other will have positive DM energy and is thus unstable. So either skyrmions or antiskyrmions are stable in a given material but the other is unstable.[3]

III. DOMAIN-WALL SKYRMIONS FROM LOW-ENERGY EFFECTIVE THEORY

A. Domain-wall solutions in the absence of the DM term

From Refs. [15,27] it is known that in the absence of the DM term there are static domain-wall solutions. Using the stereographic coordinate u, the domain-wall and anti-domain-wall configurations are

$$u_{dw} = e^{m(x_1 - X)+i\varphi}, \quad u_{adw} = e^{-m(x_1-X)+i\varphi}, \tag{10}$$

with X and φ moduli parameters corresponding to translation and phase of the wall, respectively. Both of these solutions have the same energy when $\kappa = 0$; the energy is the domain-wall tension,

$$E[u_{dw}] = |T| = \left| \frac{m}{2}\left[\frac{1-|u|^2}{1+|u|^2}\right]_{x=-\infty}^{x=+\infty} \right| = m. \tag{11}$$

By using the moduli approximation [28], we promote the moduli X, φ fields on the wall to be functions of the coordinates orthogonal to the wall; for our two-dimensional model this is the x_2 direction. From Ref. [15], the effective energy along the domain wall for the Heisenberg and anisotropy terms is

$$E_{\rm eff} = \frac{1}{2m}(m^2\partial_2 X\partial^2 X + \partial_2\varphi\partial^2\varphi), \tag{12}$$

where a constant term $-m$ has been subtracted from the effective energy.

B. Effective energy for small κ and kinks solutions

Turning on a small DM term perturbatively, $\kappa < 1$, we can still take the domain-wall and anti-domain-wall solutions as valid configurations and we can consider fluctuations in X and φ around the domain-wall and anti-domain-wall solutions. Substituting in the (anti-)domain-wall profile from Eq. (10) to the DM term in Eq. (4), it becomes

$$\begin{aligned} E_{\rm DM}[u] = \int_{-\infty}^{\infty} dx_1 \Bigg[&\frac{2\kappa}{(1+|u|^2)^2}|u|\left(\pm m\sin(\varphi+\alpha) \mp m\partial_2 X\cos(\varphi+\alpha) - \frac{d}{dx_2}(\cos(\varphi+\alpha))\right) \\ &+ \frac{2\kappa}{(1+|u|^2)^2}|u|^3\left(\pm m\sin(\varphi+\alpha) \mp m\partial_2 X\cos(\varphi+\alpha) + \frac{d}{dx_2}(\cos(\varphi+\alpha))\right)\Bigg]. \end{aligned} \tag{13}$$

The two integrals are the same,

$$\int_{-\infty}^{\infty}\frac{|u|^3}{(1+|u|^2)^2}dx_1 = \frac{\pi}{4m} = \int_{-\infty}^{\infty}\frac{|u|}{(1+|u|^2)^2}dx_1, \tag{14}$$

which means that the boundary terms in the effective energy cancel and the contribution due to the DM term is

$$E_{\rm DM} = \pm\pi\kappa[\sin(\varphi+\alpha) - (\partial_2 X)\cos(\varphi+\alpha)]. \tag{15}$$

Adding this to the known effective energy from Ref. [15] gives

$$\begin{aligned} E_{\rm eff} = \frac{1}{2m}(&m^2\partial_2 X\partial^2 X + \partial_2\varphi\partial^2\varphi \pm \tilde{\kappa}[\sin(\varphi+\alpha) \\ &- \partial_2 X\cos(\varphi+\alpha)]) \end{aligned} \tag{16}$$

with $\tilde{\kappa} = 2m\pi\kappa$.

This derivation is for a domain wall perpendicular to the x_1 direction. However, if the domain wall is instead perpendicular to $\tilde{x} = x_1\cos\theta + x_2\sin\theta$ then the (anti-)domain-wall solution becomes

$$u = e^{\pm m(\tilde{x}-X)+i\varphi}, \tag{17}$$

and the only term which changes is the DM term, where $\varphi \to \varphi + \theta$. Thus, the effective energy density along the (anti-)domain wall is

$$\begin{aligned} \mathcal{E}[u_{dw}] = \frac{1}{2m}\Bigg[&(\partial_2 X)^2 + (\partial_2\varphi)^2 \pm \tilde{\kappa}\sin(\alpha+\theta+\varphi) \\ &\pm 2\frac{\tilde{\kappa}}{m}\partial_2 X\cos(\alpha+\theta+\varphi)\Bigg]. \end{aligned} \tag{18}$$

When the translation modulus is constant, $\partial_2 X = 0$, the domain wall is straight, and this reduces to the sine-Gordon model,

$$\mathcal{E}[u_{dw}] = \frac{1}{2m}[(\partial_2\varphi)^2 \pm \tilde{\kappa}\sin(\alpha+\theta+\varphi)], \tag{19}$$

[3]An exception to this is the solvable model of Ref. [39] where both skyrmion, $Q = -1$, and antiskyrmion, $Q = 1$, configurations are explicitly constructed. The resolution is that in the solvable model the $Q = 1$ configurations are not true antiskyrmions, but rather a superposition of a skyrmion and two antiskyrmions, which is stable.

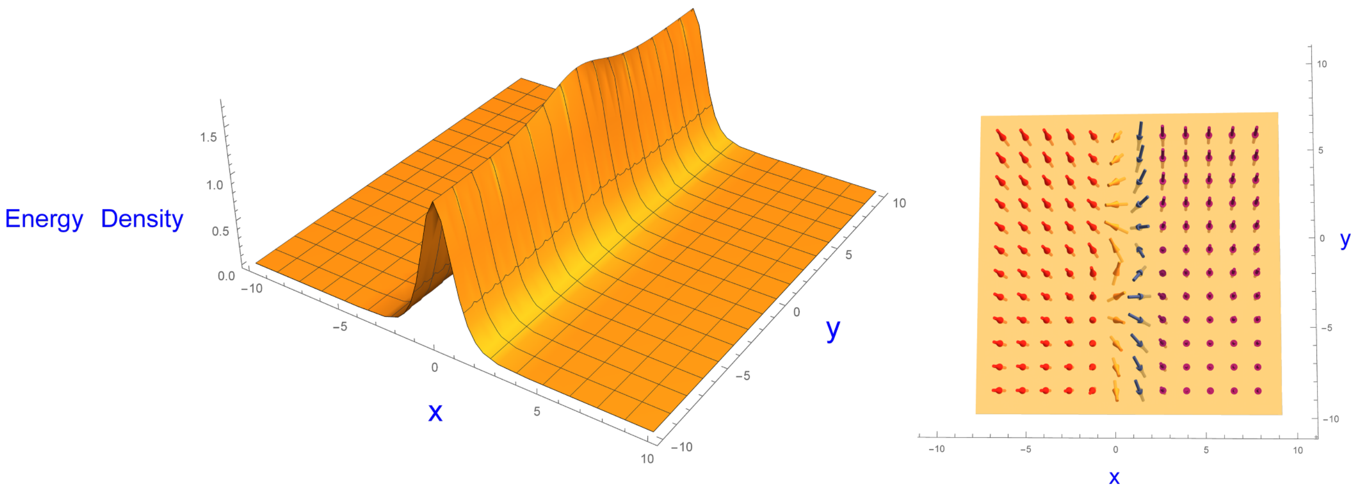

FIG. 1. The energy density and magnetization plots for a domain wall with a kink on it. For simplicity we have chosen $\tilde{\kappa} = \frac{1}{5}$, $\alpha = \theta = X = c = 0$, to give a kink centered at zero on the domain wall.

with potential term $U(\varphi) = \pm\tilde{\kappa}\sin(\alpha + \theta + \varphi)$. This sine-Gordon-type expression for the effective energy on the domain wall was also found in Ref. [22], where a different starting point was taken. There the authors started from the domain-wall solution of the critically coupled model [39] which contains both an anisotropy and a Zeeman term, and then considered an x_2-dependent phase.

We said above that the value of α dictates the type of DM term. It also differentiates between Bloch- and Néel-type domain walls; this is because the value of α determines the ground state of the effective energy in Eq. (18).

The kink solutions are found by solving the first-order equation

$$\frac{d\varphi}{dx_2} = \pm\sqrt{U(\varphi) + C_0} \tag{20}$$

when $C_0 = \tilde{\kappa}$ is the first integral of the system, if we think of x_2 as the analog of the time coordinate. A derivation of this equation and its solutions is given in Appendix A. The kink and antikink solutions are phase-shifted versions of the familiar sine-Gordon kinks:

$$\varphi = 4\arctan\left(e^{\pm\sqrt{\frac{\tilde{\kappa}}{2}}x_2}\right) - \alpha - \theta - \frac{\pi}{2} \tag{21}$$

on u_{dw}, and

$$\varphi = 4\arctan\left(e^{\pm\sqrt{\frac{\tilde{\kappa}}{2}}x_2}\right) - \alpha - \theta + \frac{\pi}{2} \tag{22}$$

on u_{adw}. The energy density and magnetization for a kink on a domain wall are shown in Fig. 1.

TABLE I. The topological charges for the four basic combinations of domain-wall skyrmions.

	Kink	Antikink
Domain wall	$Q = 1$	$Q = -1$
Anti-domain wall	$Q = -1$	$Q = 1$

The topological charges (skyrmion numbers) are given in Table I. The topological charges are computed from

$$Q[u] = \frac{i}{2\pi}\int \frac{\partial_1 u \partial_2 \bar{u} - \partial_2 u \partial_1 \bar{u}}{(1 + |u|^2)^2} d^2x, \tag{23}$$

and for a domain-wall skyrmion this becomes

$$Q[u] = \begin{cases} k & \text{for a domain wall} \\ -k & \text{for an anti-domain wall,} \end{cases} \tag{24}$$

with k the winding number of the sine-Gordon kink. Since in the chiral magnets literature the $Q = -1$ configuration is known as a skyrmion we should call an antikink on a domain wall a domain-wall skyrmion.

One of the remarkable features of the domain-wall skyrmions is that both skyrmion and antiskyrmion are stable on the domain wall, unlike in the bulk where only either skyrmions or antiskyrmions are stable and the others are unstable. This fact can be understood from the Derrick scaling argument since both the domain wall and kink are separately stable configurations for their respective energies. However, we need to be careful when applying a Derrick scaling argument here. An x_1-dependent configuration such as a domain wall will have infinite energy when considered within a 2D configuration as it only varies in one of the directions. We have avoided this difficulty here by considering the domain wall in one dimension, applying the scaling argument, then integrating over x_1 to find the effective energy for $X(x_2)$, $\varphi(x_2)$ and applying the scaling argument for x_2 here.

We can also ask about the stability of a domain-wall skyrmion versus the stability of a skyrmion and a domain wall as separate objects. One way to compare these is to consider a superposition of a skyrmion and a domain wall and ask how they interact: For example, is there an attractive force between them where we would expect them to merge into a domain-wall skyrmion, or is there a repulsive force where we expect them to stay as separate objects? To study this analytically we consider a well-separated configuration of a skyrmion and a domain wall; this is so we can approximate the skyrmion configuration by its far-field profile. Then we can appeal to the results of Ref. [26] where the interaction of domain walls and

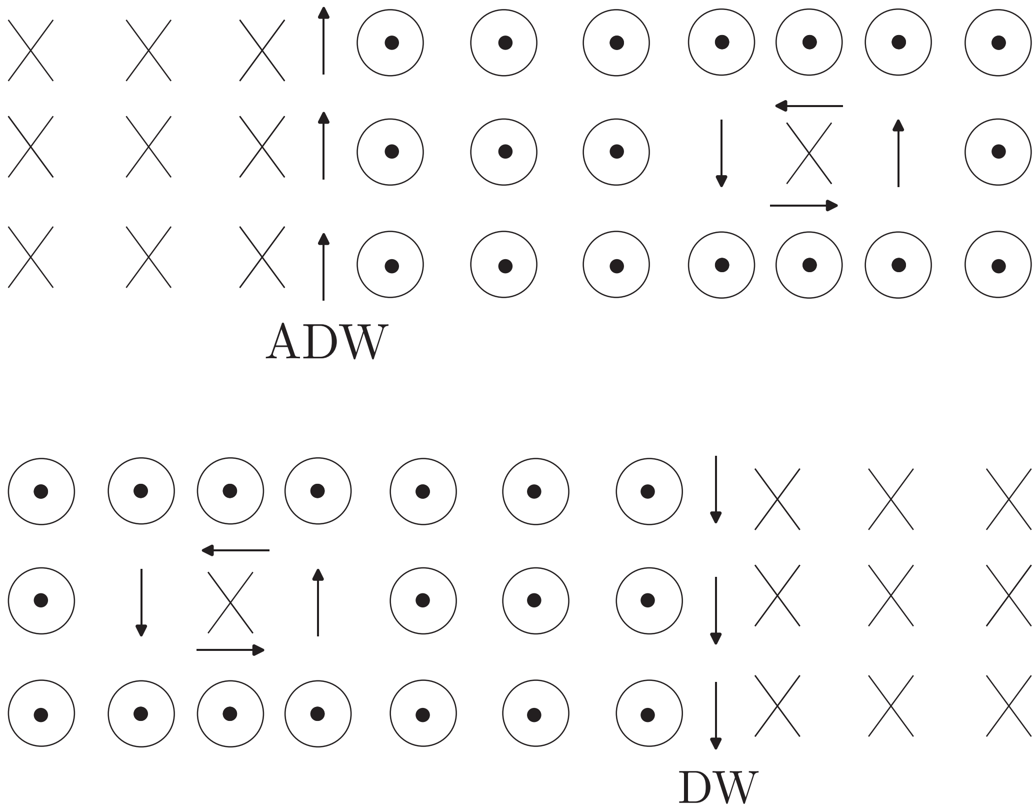

FIG. 2. Schematic figures of superpositions of skyrmions and domain walls to demonstrate the relative orientation; these are plotted for $\theta = \alpha = 0$. The upper figure is an anti-domain-wall skyrmion superposition while the lower figure is a skyrmion domain-wall superposition. In these schematics the cross signifies a downward-pointing magnetization $n_3 = -1$ while the circle with a dot in the middle signals an upward-pointing arrow $n_3 = 1$, e.g., one that is pointing out of the page. There are two further pictures that could be made where the skyrmion is on the opposite side of the domain wall; these also showcase an attractive interaction between the skyrmion and the domain wall.

skyrmions was studied and it was found that it is the relative orientation which is important for deciding if they attract or repel. In Ref. [26] the relative phase is a pseudo-zero mode; here it is fixed by the DM term. The complex functions for the domain wall and skyrmion fields are

$$u_{(\mathrm{a})\mathrm{dw}} = e^{\pm m(\tilde{x}-X)+i\left(-\alpha-\theta\mp\frac{\pi}{2}\right)}, \tag{25}$$

$$u_{\mathrm{sk}} = \tan\left(\frac{f(r)}{2}\right)e^{i\left(\vartheta-\alpha-\frac{\pi}{2}\right)}, \tag{26}$$

where $f(r) = \frac{1}{\sqrt{r}}K_1(mr)$ is far-field profile function of the skyrmion given in terms of the modified Bessel function [9], and ϑ is the plane polar angle centered at the center of the skyrmion. Looking at the superposition

$$u_{\mathrm{sup}} = u_{(\mathrm{a})\mathrm{dw}} + u_{\mathrm{sk}}, \tag{27}$$

we see that the relative phase is fixed as π, due to the presence of the DM interaction; i.e., the domain wall and the skyrmion have opposite orientation and are thus expected to attract one another. The same is true for an anti-domain wall and a skyrmion. This argument is only valid when the configurations are well separated, but as the relative phases are fixed the type of interaction is not expected to change. To fully understand what happens when the domain wall and the skyrmion are close together would require a full numerical study. This gives further evidence that domain-wall skyrmions and domain-wall antiskyrmions should be stable as a skyrmion is attracted to both the domain wall and the anti-domain wall. Schematic figures showing the relative orientation of an anti-domain wall and a skyrmion are shown in Fig. 2.

More generally we can consider higher-energy solutions with a spiral on a domain wall or anti-domain wall. The derivation of these spiral solutions is given in Appendix B, and the energy density for a spiral (sine-Gordon lattice) on a domain wall and a plot of the corresponding magnetization is given in Fig. 3. The spiral configurations are described in terms of the elliptic modulus $k^2 = \frac{2\tilde{\kappa}}{C_0+\tilde{\kappa}}$, where C_0 is the conserved first integral of Eq. (A1). These spirals on domain-wall configurations have topological charge $Q = \pm k$, where k is the number of domain walls in the spiral, and they have a higher energy than the single-domain-wall skyrmion. These spiral configurations are sometimes known as the chiral soliton lattice [8], since they can be viewed as a lattice configuration of sine-Gordon kinks.

C. Superposition configurations

For "well-separated" domain-wall–anti-domain-wall pairs, we can consider the configuration

$$u_{\mathrm{dw\text{-}adw}} = e^{-m(x-X_1)+i\varphi_1} + e^{m(x-X_2)+i\varphi_2}. \tag{28}$$

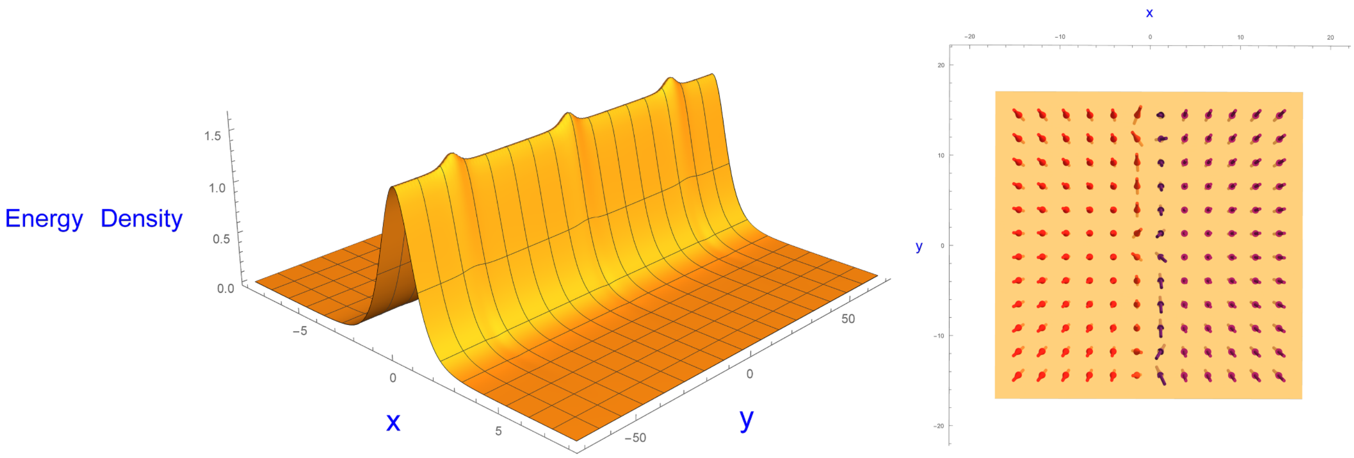

FIG. 3. Left: Energy density of a spiral configuration with $\tilde{\kappa} = \frac{1}{5}$. Right: A magnetization plot of the $k = \frac{99998}{100000}$ spiral on a domain wall. Comparing the magnetization plot in Fig. 1 we see that the spiral involves multiple kinks along the domain wall.

The case of constant phases $\varphi_1 = \varphi_2 + \pi$ was considered in the absence of a DM term in Ref. [40]. For well-separated domain walls, $|X_2 - X_1| \gg \frac{1}{m}$, this superposition is a valid configuration, assuming also that $X_2 > X_1$.

There are two separations that we can vary: one is $R = X_2 - X_1$, the separation between the domain wall and the anti-domain wall; the other is when φ_i is a kink or antikink and we can vary the x_2 separation of the kink positions. Computing how the energy of this superposition varies with the separation of the domain wall and the anti-domain wall we see that this replicates the expected result that the domain wall and anti-domain wall have an attractive interaction. See Fig. 4 for a plot of the energy of the superposition against the domain-wall separation R.

Turning to the question of how the energy density varies with the x_2 separation of the kinks, we do not get such clear results. The energy density and a magnetization plot for the case of a kink centered at $x_2 = -2$ on the domain wall and an antikink centered at $x_2 = 2$ on the anti-domain wall is shown in Fig. 5. As we can vary the relative positions of the center of the kink along the walls, we can compute the energy and find that there is no change. This suggests that the kinks can be thought of as being free to move along the wall. More sophisticated numerical studies are needed to show if they are actually free or if the interaction is just very small.

IV. SUMMARY AND OUTLOOK

In this work, we have studied the effective energy of straight domain walls in chiral magnets and explored its relationship to the sine-Gordon model. This extends the work of Ref. [15] on domain-wall skyrmions by including a DM term. The advantage of this is that the domain-wall effective energy becomes the sine-Gordon model without need of the extra

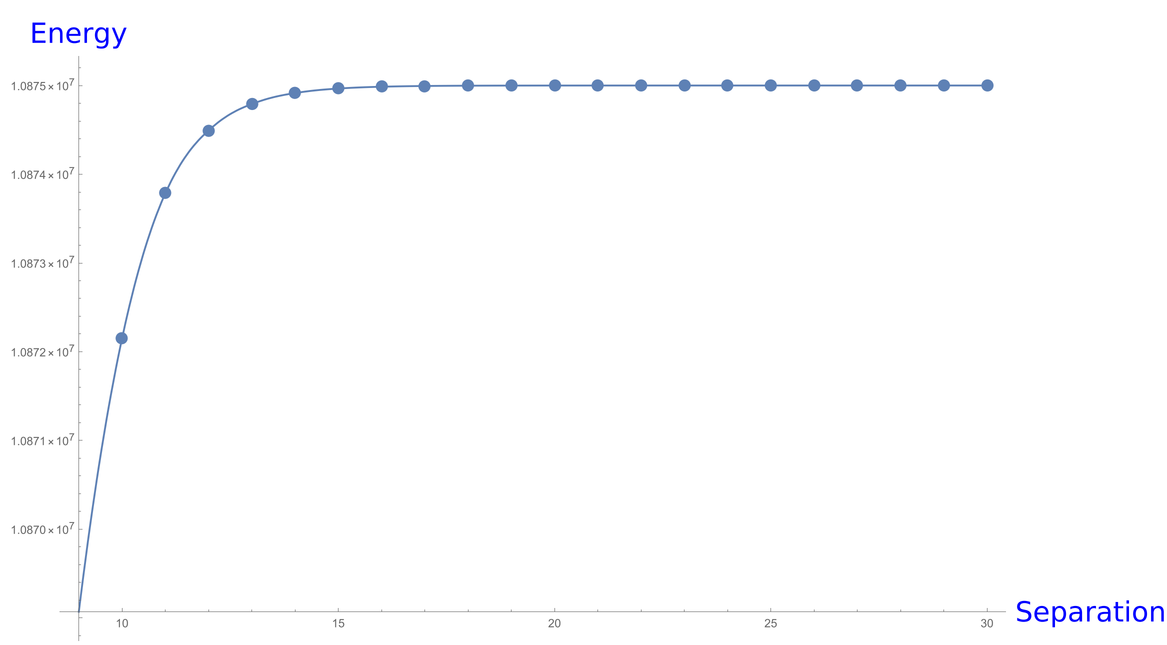

FIG. 4. The energy for the superposition against the domain-wall separation for the case of $m = 1$.

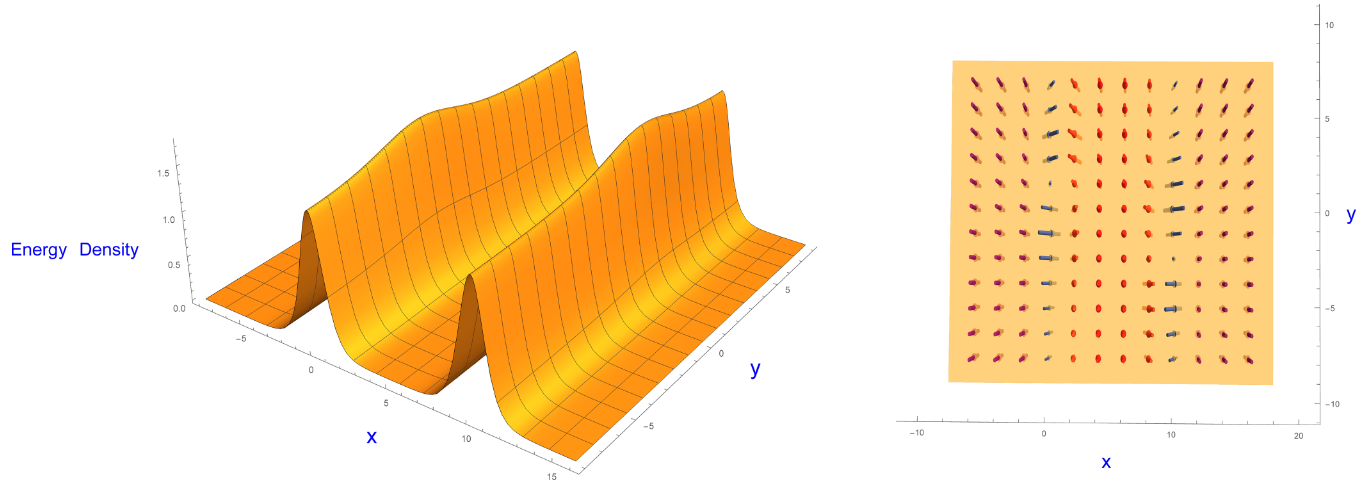

FIG. 5. The energy density and magnetization plots for the domain-wall–anti-domain-wall superposition of Eq. (28) with $X_1 = 0$, $X_2 = 10$, φ_1 (φ_2) is a kink (antikink) centered at $x_2 = \pm 2$.

anisotropy term used in Ref. [15]. We have found that both skyrmions and antiskyrmions are stable on the domain wall, in contrast to the bulk where only either skyrmions or antiskyrmions are stable and the others are unstable. We also have constructed a domain-wall skyrmion lattice. We then have considered superpositions of domain walls and anti-domain walls with kinks on each of them and found that these kinks are approximately free.

When addressing the question of stability we have used the results of Ref. [26]. These apply to baby skyrmions where there is a Skyrme term in the energy rather than a DM term; however, the linearized equations of motion are the same. So the well-separated superposition of a skyrmion and a domain wall is the same in a chiral magnet as in the baby Skyrme model. Care does need to be taken as the baby Skyrme model has second-order time dynamics, while the time dynamics of a chiral ferromagnet are governed by the Landau-Lifshitz-Gilbert (LLG) equation. Thus, to completely understand the interactions between a domain wall and a skyrmion requires a full numerical study using LLG dynamics. However, the results of Ref. [26] tell us that the interaction energy must be negative, which corresponds to an attractive interaction.

It is known that an applied electric field deflects a free skyrmion [41]; thus it can be expected that a suitably high electric field will overcome the attractive interaction between the skyrmion and the domain wall and cause them to separate. Fully understanding the effect of an applied electric field would require a full numerical study of the LLG equation for this system.

Possible applications to nanodevices will be the most important future problem. For instance, the position of a domain wall can be fixed by pinning on impurities. Then, the domain wall may play the role of a guide or a rail for skyrmions. The fact that both skyrmions and antiskyrmions are stable on the domain wall may be useful. As a store of skyrmions, one can reserve either skyrmions or antiskyrmions on the domain wall, which is impossible in the bulk outside the domain wall. For instance, if we put both skyrmions and antiskyrmions, they can pair-annihilate each other. Along the domain wall one can avoid pair annihilation since the sine-Gordon model is known to admit a breather solution in which a soliton and antisoliton pair oscillates without annihilation. As for another application, a junction of multiple domain walls may be useful.

As already pointed out in Ref. [15], domain-wall skyrmions are equivalent to Josephson vortices in a Josephson junction of two superconductors. In this case, the difference between phases of two superconductors yields a dynamical degree of freedom which is described by the sine-Gordon model [42]. These Josephson vortices are nothing but Abrikosov vortices in superconductors. Therefore, technologies developed in Josephson vortices can be imported to domain-wall skyrmions in magnets.

On the more theoretical side, the $O(3)$ model is equivalent to the $\mathbb{C}P^1$ model, which can be generalized to the $\mathbb{C}P^{N-1}$ model, admitting $\mathbb{C}P^{N-1}$ skyrmions. The $\mathbb{C}P^{N-1}$ model with a potential term that is a generalization of the easy-axis potential also admits N parallel domain walls [30]. With a potential term generalizing the second isotropy term, $\mathbb{C}P^{N-1}$ skyrmions become $U(1)^{N-1}$ coupled sine-Gordon solitons inside domain walls [29]. On the other hand, $\mathbb{C}P^{N-1}$ skyrmions were also discussed in a $\mathbb{C}P^{N-1}$ model with a generalized DM term [43]. Thus, the $\mathbb{C}P^{N-1}$ model with the generalized DM term and a generalized easy-axis potential should admit $\mathbb{C}P^{N-1}$ domain-wall skyrmions. A junction of domain walls can also stably exist in the $\mathbb{C}P^{N-1}$ model with a more general extension of the easy-axis potential [44]. Domain-wall skyrmions on a domain-wall junction should be useful for nanotechnology.

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APPENDIX A: DERIVATION OF THE KINK SOLUTIONS

To derive the kink solutions recall that for a one-dimensional problem there is a conserved first integral of motion

$$\left(\frac{d\varphi}{dx_2}\right)^2 - U(\varphi) = C_0 = \text{const.} \tag{A1}$$

This first integral reduces to the first-order equation

$$\frac{d\varphi}{dx_2} = \pm\sqrt{U(\varphi)+C_0}. \tag{A2}$$

Here $U(\varphi) = \pm\tilde{\kappa}[\sin(\varphi+\alpha) - \partial_2 X \cos(\varphi+\alpha)]$ is the potential energy piece of the effective energy. Focusing on the case of a kink on a domain wall with nonconstant translation modulus, $\partial_2 X = \gamma$, when the translation modulus is linear in x_2 the potential is

$$\begin{aligned} U(\varphi) &= \tilde{\kappa}[\sin(\varphi+\alpha) - \gamma\cos(\varphi+\alpha)] \\ &= \tilde{\kappa}(1 - \sqrt{1+\gamma^2}\sin(\varphi+\alpha+\theta-\delta)), \end{aligned} \tag{A3}$$

with $\tan\delta = \gamma$. If the constant C_0 is taken to be $C_0 = \tilde{\kappa}(\sqrt{1+\gamma^2}-1)$ the first-order equation becomes

$$\begin{aligned} \frac{d\varphi}{dx_2} &= \sqrt{\tilde{\kappa}(1-\sqrt{1+\gamma^2}\sin(\varphi+\alpha+\theta-\delta)) + \tilde{\kappa}(\sqrt{1+\gamma^2}-1)} \\ &= \sqrt{\tilde{\kappa}\sqrt{1+\gamma^2}(1-\sin(\varphi+\alpha+\theta-\delta))}. \end{aligned} \tag{A4}$$

This is solved by making the substitution $\Phi = \varphi+\alpha+\theta-\delta-\frac{\pi}{2}$ and then directly integrating

$$\frac{d\Phi}{\sqrt{1-\cos\Phi}} = \pm\sqrt{\tilde{\kappa}\sqrt{1+\gamma^2}}dx_2, \tag{A5}$$

to give a kink or antikink depending on the $\pm$ centered at $x_2 = c$:

$$\Phi = 4\arctan\left(\exp\left(\pm(x_2-c)\sqrt{\frac{\tilde{\kappa}}{2}\sqrt{1+\gamma^2}}\right)\right). \tag{A6}$$

For the anti-domain wall, the only change is that the $\frac{\pi}{2}$ is added rather than subtracted when going from φ to Φ.

APPENDIX B: DERIVATION OF THE MULTIKINK SOLUTIONS

The single kink is not the only solution for the sine-Gordon model. If we return to the first-order equation

$$\frac{d\varphi}{dx_2} = \pm\sqrt{U(\varphi)+C_0}, \tag{B1}$$

then there are more general spiral solutions. To see these take

$$\Phi = \alpha+\theta+\varphi\pm\frac{\pi}{2}, \tag{B2}$$

as above. The first-order equation then becomes

$$\frac{d\Phi}{dx_2} = \pm\sqrt{C_0+\tilde{\kappa}}\sqrt{1-k^2\cos^2\left(\frac{\Phi}{2}\right)}, \tag{B3}$$

where $k^2 = \frac{2\tilde{\kappa}}{C_0+\tilde{\kappa}}$ is the elliptic modulus. This has a solution in terms of the elliptic Jacobi amplitude as

$$\Phi = \pm\text{am}\left(\frac{\sqrt{C_0+\tilde{\kappa}}}{2}x_2, k\right) - \pi. \tag{B4}$$

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