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Letter Editors' Suggestion

Dynamical l-bits and persistent oscillations in Stark many-body localization

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Stark many-body localized (SMBL) systems have been shown both numerically and experimentally to have Bloch many-body oscillations, quantum many-body scars, and fragmentation in the large field tilt limit, but these observations have not been fundamentally understood. We explain and analytically prove all these observations by rigorously perturbatively showing the existence of certain algebraic structures that are exponentially stable in time. In particular, we show that many-body Bloch oscillations persist even at infinite temperature for *exponentially* long times using a type of dynamical algebra which we refer to as *dynamical* l-bits and provide a bound on the tilt strength for this nonergodic transition. We numerically confirm our results by studying the prototypical Stark MBL model of a tilted XXZ spin chain. Our work explains why thermalization was observed in a recent two-dimensional tilted experiment. As dynamical l-bits represent stable, localized, and quantum coherent excitations, our work opens possibilities for quantum information processing in Stark MBL systems even at high temperature.

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Introduction. One of the seminal results of condensed matter physics was Anderson's discovery of localization of free electrons on a lattice [1]. Later it was shown that this localization can possibly persist even when the repulsive interactions between electrons cannot be neglected—a phenomenon dubbed many-body localization (MBL) [2].

One of the main results in MBL is its explanation in terms of exponentially localized intensive conservation laws called *l-bits* [3,4]. The existence of these conservation laws blocks the flow of quantum information through the system. MBL has been numerically argued to lead to logarithmic entanglement growth [5–8] and subdifussive transport [9–14] among other phenomena. Notably, MBL systems should be perfect insulators at any temperature. They have been the subject of huge study in recent decades (e.g., see the review [2]).

However, the existence of disordered MBL has been somewhat controversial [15] despite certain exact [16] and renormalization group [17] results being offered. Rigorous results in many-body localization are therefore very important.

Only very recently has another related form of MBL without disorder been demonstrated both theoretically and experimentally [18–21]. This *Stark* MBL (SMBL) occurs due to an external gradient field being added to an otherwise translationally invariant system. Related to well-known Bloch oscillations of noninteracting electrons, SMBL demonstrates similar oscillations of various many-body observables in both numerics and experiments (e.g., Refs. [19,22–24]). Likewise, recently both Hilbert space fragmentation [23,25] and quantum many-body scars [26] have been numerically observed in these models. Fragmentation means that the Hamiltonian of the systems contains an exponential number of invariant subspaces [27–31] while quantum many-body scars are eigenstates that are equally spaced in energy and have low entanglement. Quantum many-body scars are known to imply oscillations from special initial states [32–38].

In contrast to the aforementioned Bloch oscillations in SMBL, disordered MBL systems do relax to stationary (time independent) states, albeit with a memory of their initial condition given by the l-bits [2,39]. This stationarity of disordered MBL is in particular known to be present when expectation values of observables are averaged across disorder [40,41]. This is puzzling because both forms of localization have an otherwise similar phenomenology [18].

In this Letter we rigorously prove that SMBL models have perturbatively approximate *dynamical* l-bits that are *exponentially* stable in time and are quasilocalized similarly to l-bits explaining all the aforementioned numerical observations. The decay rate of the dynamical l-bits is given by the strength of the field gradient. These algebraic structures are distinct from both standard l-bits of disordered MBL and extensive dynamical symmetries [42]. We show a related type of emergent algebraic structure of the model that implies many-body Bloch oscillations even on the level of correlation functions at *high* temperatures for *exponentially long times* provided that a series converges giving a lower bound for the tilt field strength, implying the existence of a phase transition. The correlation functions persistently oscillate at frequencies given by the dynamical l-bits. We focus on the prototypical example of interacting electrons in an electric field gradient. We numerically confirm our theory by studying the infinite-temperature correlation function and construct the dynamical l-bits. These dynamical l-bits explain the existence of quantum scars and fragmentation in SMBL—both are shown to be consequences of the dynamical l-bits. Importantly, dynamical l-bits are quantum coherent and stable by construction and allow for storing qubits of information. Our work thus opens the possibility of quantum information storage and processing in SMBL systems.

Model. We will focus on the following paradigmatic SMBL Hamiltonian,

$$H = J\sum_{r=1}^{L-1}\left(S_r^xS_{r+1}^x + S_r^yS_{r+1}^y\right) + \Delta\sum_{r=1}^{L-1}S_r^zS_{r+1}^z + \sum_{r=1}^{L}\epsilon_rS_r^z, \qquad (1)$$

where L is the system size, J is the hopping amplitude, Δ is the interaction strength (anisotropy), and ϵ_r is the external magnetic field at site r. This model of SMBL is equivalent to the fermionic model used in Ref. [22], i.e., an interacting Wannier-Stark chain [43–45]. The noninteracting Wannier-Stark chain is well known to feature Bloch oscillations [46–48]. Similarly, in Ref. [18], the field used was given by $\epsilon_r = Wr - \frac{\alpha r^2}{L^2}$ and it was shown that the inclusion of the quadratic component gave rise to a phase with MBL characteristics. It has also been shown that having a linear magnetic field $\epsilon_r = Wr$ also gives rise to an MBL-like phase beyond a critical tilt $W_c \approx 2.2$ in clean systems [20,23]. We are interested in these oscillations and so we study a linear magnetic field for the rest of this Letter. However, we remark on the inclusion of a small quadratic potential in the Supplemental Material [49]. As we will see, the exact form of the potential is not relevant for the qualitative conclusions.

Dynamical l-bits. As discussed previously, the key difference in SMBL is the observation of persistent many-body Bloch oscillations. To capture these, we look to the recently introduced concept of *dynamical symmetries* [42]. These are defined to be *extensive or local* spectrum generating algebras [50] of H, i.e., operators A satisfying the relation [51]

$$[H, A] = \omega A, \qquad (2)$$

where $\omega \neq 0$ is the *frequency* of A.

In this Letter, we extend the notion of dynamical symmetry to include *dynamical l-bits*—these are operators satisfying (2) which are similar to l-bits in MBL in the sense that they are quasilocalized (rather than strictly localized [52]).

Now suppose that our system is initially (at $t = -\infty$) in thermal equilibrium and then locally perturbed suddenly at $t = 0$. This means that the perturbation takes the form $\delta(t)B$ where B giving the new Hamiltonian $H' = H + \delta(t)B$. According to standard results from linear response theory, the resulting deviation of the expectation value of an operator Q at later times from its equilibrium value is given by $\langle Q(t)\rangle_{\text{pert}} - \langle Q\rangle = -i\langle[Q(t), B]\rangle$, where $\langle Q(t)\rangle_{\text{pert}} = \langle e^{iH't}Qe^{-iH't}\rangle$. If $\langle AQ\rangle \neq 0$, $\langle AB\rangle \neq 0$ for some dynamical symmetry A (2), then $\langle Q(t)\rangle_{\text{pert}}$ will oscillate forever with frequency ω [53]. More specifically, a Mazur lower bound on the amplitude of the oscillations exists [54].

We will focus on the infinite-temperature case for which the relevant function from linear response theory to consider is the so-called fluctuation function given by

$$F_{QB}(t) = \tfrac{1}{2}\langle\{Q(t), B\}\rangle = \langle Q(t)B\rangle. \qquad (3)$$

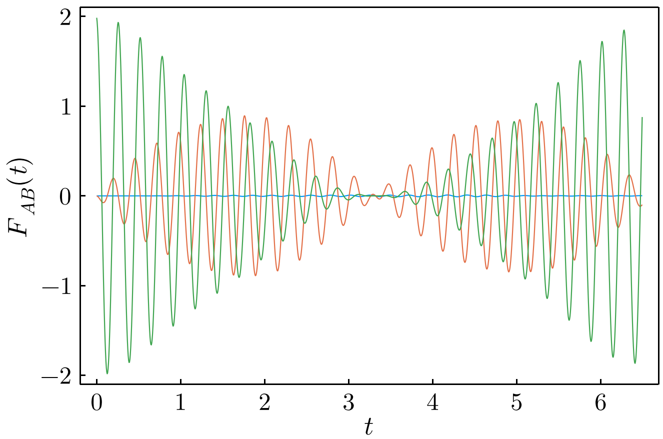

FIG. 1. Plot of the infinite-temperature autocorrelation function in (3) for various choices of local operators. Here, we consider three examples. The parameters used are $L = 16$, $W = 3$, $J = \Delta = 1$. In each case, we take the sudden perturbation at $t = 0$ to be $B = S^x_{L/2}$. The three choices of operators Q are $S^+_{L/2}$ (green), $S^z_{L/2-1}S^+_{L/2}$ (red), and $S^z_{L/2+2}S^+_{L/2+3}$ (blue).

The above form is valid when $Q(0)$ and B are traceless and this is precisely the case which interests us. We now numerically compute the autocorrelation function in (3) for our model Hamiltonian (1) for a few pairs of operators using density matrix renormalization group (DMRG) [55] (Fig. 1). Two cases show persistent many-body Bloch oscillations.

There is also one case where we observe no significant oscillations (the blue curve, which is almost flat). In this case, the time evolved operator acts on sites further away from the perturbation site. Thus, the memory of the perturbation is highly localized. These observations agree in general with how we expect dynamical l-bits would behave in autocorrelation functions, even though the oscillations are not of fixed amplitude. This leads us to theorize that the SMBL Hamiltonian possesses dynamical l-bits and that the operators in the plot which show oscillations have some finite overlap with them in the sense of Ref. [53]. What follows is the main result of this Letter, where we prove that H has exponentially stable dynamical l-bits, at least in the large tilt case.

Exponentially stable dynamical l-bits in SMBL. We begin by noting that our Hamiltonian can be written in the form $H = H_{XX} + H_{ZZ} + M$, where $H_{XX} = J\sum_r(S_r^xS_{r+1}^x + S_r^yS_{r+1}^y)$, $H_{ZZ} = \Delta\sum_r(S_r^zS_{r+1}^z)$, and $M = W\sum_r rS_r^z$. Clearly, the eigenspectrum of M is comprised of the values $\{W, 2W, \ldots, WL(L+1)/2\}$. We now assume that W is large. It directly follows from the work of Abanin *et al.* [56] (see also Ref. [57]) that there exists a quasilocal unitary operator Y close to the identity such that

$$\langle\mathcal{O}(t)\rangle = e^{iY(\hat{D}+M+V)Y^\dagger t}\mathcal{O}e^{-iY(\hat{D}+M+V)Y^\dagger t} \qquad (4)$$

holds for any local operator $\mathcal{O}$, where $[\hat{D}, M] = 0$ and $[M, V] \neq 0$. The error in neglecting V is small in W up to exponentially long times $t^* \propto \exp W$ (the reader is referred to Ref. [56] for a more precise formulation of this statement along with a rigorous proof). In other words, up to exponentially long times, time evolution is governed by the effective Hamiltonian given by $H' = H_{\text{eff}} + O(\frac{J^3}{W^2}) = YHY^\dagger$.

Inspired by Ref. [58], in the Supplemental Material [49] by constructing Y, we show a rather simple form for $H_{\rm eff}$ (cf. Refs. [28,59]),

$$H_{\rm eff} = \sum_r \left(WrS_r^z + \Delta S_r^z S_{r+1}^z\right) + \frac{J^2}{4W}\left(S_L^z - S_1^z\right). \tag{5}$$

It is important to note that the reason the approaches [56,57] are applicable in SMBL, as opposed to disordered MBL, is because the energy of the tilt is equally separated in SMBL, unlike the energy of the disordered field.

It is now easy to check that for $2 \leqslant r \leqslant L-1$, the four operators given by

$$\begin{aligned}
A_1(r) &= S_r^+ - 4S_{r-1}^z S_r^+ S_{r+1}^z,\\
A_2(r) &= S_{r-1}^z S_r^+ - S_r^+ S_{r+1}^z,\\
A_3(r) &= S_r^+ + 2S_{r-1}^z S_r^+ + 2S_r^+ S_{r+1}^z + 4S_{r-1}^z S_r^+ S_{r+1}^z,\\
A_4(r) &= S_r^+ - 2S_{r-1}^z S_r^+ - 2S_r^+ S_{r+1}^z + 4S_{r-1}^z S_r^+ S_{r+1}^z,
\end{aligned} \tag{6}$$

are exact strictly local dynamical l-bits of $H_{\rm eff}$ with corresponding frequencies

$$\omega_1 = \omega_2 = Wr, \quad \omega_3 = Wr + \Delta, \quad \omega_4 = Wr - \Delta. \tag{7}$$

In the original physical basis the actual dynamical l-bits are $\hat{A}_j(r) = YA_j(r)Y^\dagger$, but since Y is quasilocal, this gives a set of quasilocal dynamical l-bits up to for the original Hamiltonian, up to subleading corrections.

Note that dynamical l-bits imply regular l-bits by the simple identity $[H, [A_j(r), A_j^\dagger(r)]] = 0$, where $Q_j(r) = [A_j(r), A_j^\dagger(r)]$ is an exponentially localized l-bit. Unlike disordered MBL, dephasing here is not possible because, unlike the l-bits of disordered MBL, only four fundamental frequencies contribute to Stark dynamical l-bits, rather than a continuum in disordered MBL [39].

More importantly, we show that [49] V is exponentially small, which implies that the dynamics is dictated up to exponentially long times by $H_{\rm eff} = \hat{D} + M$. This implies exponentially long persistent oscillations in wide classes of observables for both the autocorrelation functions and for quenches from generic initial states. We now introduce an extended dynamical symmetry condition generalizing (2), i.e., an operator $\tilde{A}' = Y^\dagger \tilde{A} Y$ such that

$$[M, \tilde{A}'] = \nu \tilde{A}', \quad [\hat{D}, M] = 0 \tag{8}$$

for some ν with $H_{\rm eff} = \hat{D} + M$. Then we have using $[\hat{D}, M] = 0$,

$$\tilde{A}(t) = Ye^{i\hat{D}t}e^{iMt}Y^\dagger \tilde{A} Y e^{-iMt}e^{-i\hat{D}t}Y^\dagger = e^{i\nu t}e^{i\tilde{H}t}\tilde{A}e^{-i\tilde{H}t}, \tag{9}$$

where $\tilde{H} = Y\hat{D}Y^\dagger$. As Y is quasilocal and close to the identity and $\hat{D}$ is local, $\tilde{H}$ is pseudolocal [60]. Using this and the Mazur bound [61] in the Supplemental Material [49] we show that for generic local operators $\tilde{A}$, we have persistent oscillations at frequency ν both for generic quenches and for autocorrelation functions.

Numerical construction of the dynamical l-bits. Even though we have shown the existence of dynamical l-bits for the full Hamiltonian, we have not found them explicitly since we do not know the operator Y. Our aim in this section is to numerically find these dynamical l-bits. To do this, we note that Y is close to the identity, and so the exact dynamical l-bits of $H'_{\rm eff}$ are still highly relevant, and that the frequencies are unchanged. So we generalize the approaches of Ref. [62] developed for conservation laws and we look at the operator given by

$$\tau = \lim_{T\to\infty} \frac{1}{T}\int_{-T}^{T} dt\, e^{-i\omega t}U^\dagger(t)\mathcal{O}U(t), \tag{10}$$

where $\omega \neq 0$ is a real number, $U(t) = e^{-iHt}$ is the time evolution operator, and $\mathcal{O}$ is some strictly local operator. It can be shown that this satisfies the relation $[H, \tau] = \omega\tau$ regardless of $\mathcal{O}$ [49]. However, we require τ to also be (quasi)localized to be considered a dynamical l-bit and this is not necessarily satisfied for arbitrary local *seed* operators $\mathcal{O}$.

By our previous arguments, we will use $\mathcal{O} = A_j(r)$ as the seed operators, along with their corresponding frequencies, and time evolve them to determine their locality in Fig. 2(a).

These results make it clear that τ is quasilocal for these choices of seed operator, which agrees with our hypothesis.

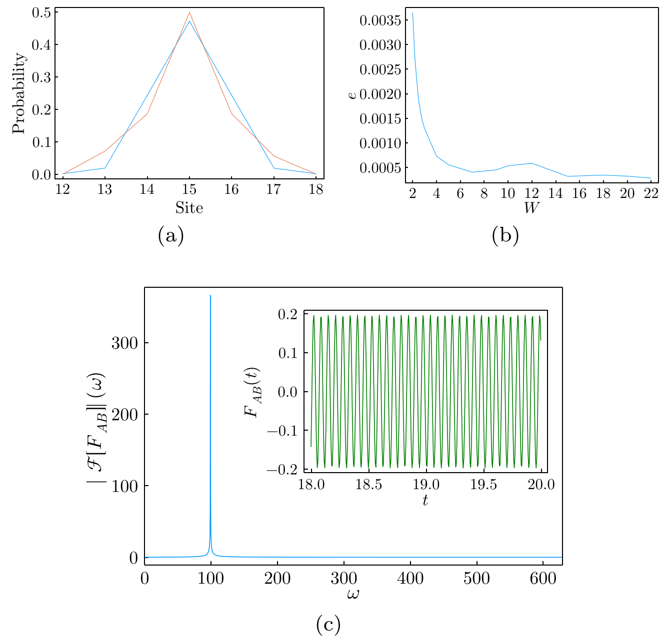

FIG. 2. Results of the numerics on the dynamical l-bit candidate τ. The parameters used throughout are $J = \Delta = 1$ and $W = 10$ together with a total time $T = 350$ to simulate the limit $T \to \infty$ and time step $dt = 0.005$ for the integral in (10). The same time step is also used to calculate the autocorrelation function. (a) Plots demonstrating locality of the operator τ. The system size used is $L = 30$. The seed operators considered are $A_2(L/2)$ (blue) and $A_4(L/2)$ (red). The probabilities give a measure of how much of τ lives on each site. As we can see, almost all of the operator is always concentrated on the central five sites, and the shape of the plot is not affected by the system size. (b) A plot showing the variation of the error $e = \frac{\|[H,\tau]-\omega\tau\|}{\|[H,\tau]\|}$ with tilt strength W. We can see a high error for smaller values of W, where we naturally expect a lot of entanglement and thus larger errors due to truncation in DMRG. (c) Plot of the infinite-temperature autocorrelation function (inset) F_{AB} from (3) with $A = S_{L/2}^x$ and $B = \tau$, and its Fourier transform for $L = 20$. The seed operator used was $A_4(L/2)$. The single spike in the Fourier transform confirms that τ is indeed a dynamical l-bit.

Thus we have numerically found quasilocal dynamical l-bits of the full Hamiltonian H. We also study how the error behaves when we change W in Fig. 2(b) and we can see that it decays with increasing W, as expected.

Furthermore, we can look at the autocorrelation function given by (3), with local operator $A = \tau$ and perturbation $B = S^x_{L/2}$ similar to what we did before. This is plotted in the inset of Fig. 2(c), where we can see clear, uniform oscillations at one frequency, which show that we have now found an accurate dynamical l-bit. The dynamical l-bit thus explains the existence of oscillations in Fig. 1 at the appropriate frequency. This is confirmed by the Fourier transform in Fig. 2(c) and it demonstrates that the frequency of the oscillations is indeed the corresponding ω (which is 99 in this case). Moreover, we see that the operators which caused the nonuniform oscillations in Fig. 1 do in fact overlap with the operators A_i as conjectured previously. Furthermore, their support is indeed quasilocalized and close to the original dynamical l-bits as expected by the quasilocality of Y.

Quantum many-body scars and fragmentation. Quantum many-body scars [32–38,63,64] are (at least) an extensive number of eigenstates that have low entanglement. They imply oscillations from special initial states. These states have recently been numerically identified in Stark MBL models [26]. Our analytically proven dynamical l-bits directly imply quantum scars [34,35,65,66] (see also the Supplemental Material of Ref. [42] for an earlier proof implying scarring and that includes dissipation). In our case, the existence of scars follows from the fact that dynamical l-bit when acting on the (product) ground state $|0\rangle$ will create eigenstates with low entanglement that are equally separated in energy, e.g.,

$$H\left(\prod_{\{r_i\}} A_j(r_i)\,|0\rangle\right) = d(j)\omega_j\left(\prod_{\{r_i\}} A_j(r_i)\,|0\rangle\right), \qquad (11)$$

where $[H, A_j(r_i)] = \omega_j A_j(r_i)$, and $d(j)$ is the number of $A_j(r_i)$ that appear in the product. We set $H|0\rangle = 0$ for simplicity. The entanglement of the state is guaranteed to be low due to the localized structure of $A_j(r_i)$.

We note that dynamical l-bits imply quantum scars, but not the other way around. More specifically, models with quantum many-body scars have oscillations only for very special initial states, whereas dynamical l-bits imply oscillations generically and even at infinite temperature as shown here.

Fragmentation follows immediately from the dynamical l-bits by the arguments presented in Ref. [67]. This is likewise consistent with the results of Ref. [25]. In fact this means that Stark MBL fragmentation is not true fragmentation [27,28,68], but rather *local fragmentation* as defined by Ref. [67].

Conclusion. In this Letter we developed a theoretical explanation of SMBL and explained the origin of persistent oscillations which have been previously numerically and experimentally observed in SMBL [18,20,22,24]. To achieve this we have shown the existence of dynamical l-bits, which should be contrasted with the standard l-bits of disordered MBL [4]. We then proved that in the large tilt case, the SMBL Hamiltonian can be reduced to an effective Hamiltonian up to exponentially long times using a theorem from Ref. [2] and further showed the existence of an exact and complete dynamical l-bit basis of this effective Hamiltonian up to subleading corrections. We likewise have proven that generic observables oscillate for exponentially long times by introducing a type of extended dynamical symmetry algebra. Thereafter, we numerically constructed dynamical l-bits of the full Hamiltonian with excellent accuracy. A similar effective Hamiltonian was obtained in a very recent work (up to a rotating wave basis transform) in Ref. [59] where transport was studied (cf. also Ref. [28]). However, here we focused on oscillations, scars, and fragmentation, as well as found the complete l-bit basis. We have proven that dynamical l-bits imply persistent oscillations in the autocorrelation function for exponentially long times, even at infinite temperature, quantum many-body scars, and Hilbert space fragmentation, as well as l-bits. The fact that the dynamical l-bits of SMBL have only four fundamental frequencies explains why SMBL has many-body Bloch oscillations, unlike disordered MBL [39]. Even though the XXZ model we studied is distinct from the tilted Fermi-Hubbard models with scars and fragmentation, our results indicate that these models likewise have dynamical l-bits. Note that dynamical l-bits immediately imply many-body flat bands [69] in the tilted XXZ spin chain [52]. Our approach relies on the tilt being single body and therefore a tilt with two-body terms is expected to thermalize, which is fully consistent with the two-dimensional (2D) experiment of Ref. [70]. Moreover, we predict that putting a two-body tilt in both directions will not cause an absence of thermalization, in contrast to the proposal in Ref. [70].

Our work opens many avenues for future work. In future work we will study in Stark MBL models possible realizations of time crystals in both driven (discrete) [71–80] (cf. also Ref. [81]) and dissipative models [42,82–95], synchronization [96–98], and other possible kinds of nonstationary dynamics [99–101,101–112]. Likewise, connection with large-tilt and large interactions limits in 1D models will be explored [113–115]. Most intriguing, however, is the fact that a dynamical l-bit is local coherent excitation that can store a qubit. This hints that Stark MBL systems could have the potential for robust quantum information storage and processing.

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