Letter

Exotic thermal transitions with spontaneous symmetry breaking

Hanbit Oh and Eun-Gook Moon*

Department of Physics, Korea Advanced Institute of Science and Technology, Daejeon 305-701, Korea

(Received 29 November 2021; revised 2 June 2022; accepted 4 October 2022; published 17 October 2022)

We show that exotic spontaneous symmetry breaking appears in thermal topological phases by perturbing the exact solutions of *quantum* rotor models coupled to the three-dimensional toric code. The exotic Ising and XY transitions are shown to be in the same universality class, in drastic contrast to the conventional Wilson-Fisher classes without topological orders. Our results indicate that topological orders must be included to pin down the universality classes of thermal transitions in addition to order parameter symmetry and spatial dimension. We evaluate all the critical exponents and find that the exotic universality class is more stable under the couplings to acoustic phonons and disorder. Applying our results to experiments, we provide a plausible scenario in puzzlings of strongly correlated systems, including the absence of a specific heat anomaly in doped $RbFe_2As_2$.

DOI: 10.1103/PhysRevB.106.L140504

Introduction. The phenomenological Landau theory is one of the most successful theories in physics that explain phase transitions by introducing the concept of the order parameter [1,2]. Including fluctuations of order parameters, the universality classes of continuous phase transitions are discovered with the development of renormalization group analysis, so-called Wilson-Fisher (WF) classes [3,4]. The conventional wisdom that the WF classes are solely determined by order parameter symmetry and spatial dimension is established.

Striking and perplexing phenomena are reported even in recent experiments of thermal transitions of high-temperature superconducting materials. The Ising nematicity in layered cuprates, doped $YBa_2Cu_3O_7$ and $HgBa_2CuO_4$, shows a peculiar superlinear onset [5–10]. In doped $RbFe_2As_2$, the diagonal Ising nematicity without an anomaly in heat capacity is observed [11–14]. Conventional WF classes are inapplicable to these experiments, calling for a new theoretical framework. Possibilities of exotic transitions have been suggested by using the ideas of topology, fractionalization, and deconfinement [15–23].

In this Letter, we demonstrate the existence of exotic thermal transitions by investigating spontaneous symmetry breaking transitions in thermal phases with topological orders. Instead of using conventional thermal gauge theories [24–27], we analyze the quantum rotors coupled to qubits of the toric code in three spatial dimensions (3D) [28–30] to explore exotic thermal transitions. Striking characteristics of the exotic thermal transitions are uncovered. The Ising and XY transitions are in the same universality class, in drastic contrast to the WF classes. All critical exponents are evaluated, and their differences from the ones of the WF classes are emphasized.

Our approach has the following advantages. First, there is no imposed gauge invariance in our models and thus no subtlety associated with the origin of the gauge invariance in thermal phases. In fact, we discuss the effects of gauge noninvariant interactions and show their irrelevance to the existence of the exotic universality classes. Second, we utilize the exact solutions of the models and employ a controlled perturbative analysis. Clear-cut conclusions of the existence and properties of exotic thermal transitions are obtained. Couplings to other physical degrees of freedom, such as acoustic phonons or Fermi surfaces, are determined unambiguously, and the stabilities of the thermal transitions are studied. Third, our quantum models naturally provide the relations between the thermal phase transitions and corresponding quantum phase transitions, providing a bird's-eye view of quantum and thermal transitions.

Stability of the WF classes. We revisit the WF universality classes of thermal phase transitions by considering *quantum* rotor models. To be specific, let us consider the $O(2)$ quantum rotor model on a cubic lattice with the conventional hat notation for quantum operators,

$$\hat{H}_R = \sum_j \frac{V_\theta}{2}\hat{n}_j^2 - t_\theta \sum_{\langle i,j\rangle} \cos(\hat{\theta}_i - \hat{\theta}_j),$$

where an angle operator $\hat{\theta}_j$ and its conjugate number operator $\hat{n}_j$ with the commutation relation $[e^{i\hat{\theta}_j}, \hat{n}_j] = e^{i\hat{\theta}_j}$ are introduced. The Hilbert space is a tensor product of local Hilbert spaces,

$$\mathcal{H}_R = \prod_j \otimes \mathcal{H}_j, \quad \mathcal{H}_j = \{|n_j\rangle \mid n_j \in \mathbb{Z}\},$$

with $\hat{n}_j|n_j\rangle = n_j|n_j\rangle$. The integer condition of n_j is associated with the compactification $|\theta_j\rangle = |\theta_j + 2\pi\rangle$. The model enjoys a $U(1)$ symmetry whose action is $\hat{U}(\alpha) = \prod_j e^{i\alpha\hat{n}_j}$ with a real value α. The symmetry is spontaneously broken by varying with V_θ/t_θ. The symmetric phase is adiabatically connected to the ground state at $t_\theta = 0$, $|\mathrm{sym}\rangle = \prod_j |n_j = 0\rangle$, and the symmetry broken phase is adiabatically connected to the ground state at $V_\theta = 0$,

$$|\theta_0\rangle = \prod_j |\theta_j = \theta_0\rangle, \quad \langle\theta_0|e^{i\hat{\theta}_j}|\theta_0\rangle/\langle\theta_0|\theta_0\rangle = e^{i\theta_0},$$

*egmoon@kaist.ac.kr

for $\theta_0 \in (0, 2\pi)$. Finite temperature T drives a thermal phase transition of $U(1)$ symmetry breaking. One direct way to construct the corresponding Landau functional is to introduce a complex order parameter $\psi_j = e^{i\theta_j}$ and perform a coarse graining, which gives

$$\mathcal{Z}_R = \mathrm{Tr}_\theta(e^{-\hat{H}_R/T}) \to \mathcal{Z}_R \simeq \int D\psi\, e^{-\mathcal{F}_R(\psi)},$$

$$\mathcal{F}_R[\psi] = \int_x \left(|\nabla\psi(x)|^2 + r|\psi(x)|^2 + \frac{\lambda}{4}|\psi(x)|^4 \right).$$

Here, $\psi(x)$ is the coarse-grained field and the integration is over a 3D space. Hereafter, we omit the obvious space dependence of the order parameters. This Landau functional describes the WF-XY universality class in 3D and the coefficients (r, λ) are functions of (V_θ, t_θ, T) in addition to a lattice spacing.

It is well known that the WF-XY class is unstable under the Ising potential $V_I = -u\sum_j \cos(\theta_j)^2$, and the presence of a nonzero u breaks the $U(1)$ symmetry down to the Ising one whose order parameter becomes a real variable, $\phi_j = \cos(\theta_j)$. The Ising potential has the form $V_I = -u\sum_j \phi_j^2$, and the Landau functional of the Ising order parameter becomes

$$\mathcal{F}_I[\phi] = \int_x \left(\frac{1}{2}(\nabla\phi)^2 + \frac{r}{2}(\phi)^2 + \frac{\lambda}{4}(\phi)^4 \right).$$

Next, let us investigate how the theory of the order parameter is affected by fermions. The most drastic effects come with the presence of a Fermi surface, so we consider a model Hamiltonian $H = H_I + H_f + H_Y$ and

$$H_f = -t_f \sum_{\langle i,j\rangle} c_i^\dagger c_j - \mu \sum_j c_j^\dagger c_j, \quad H_Y = -g \sum_j \phi_j c_j^\dagger \hat{M} c_j,$$

where $\hat{M}$ is a vertex operator of the Yukawa coupling. After integrating out the electrons, one can easily show that the Ginzburg-Landau coefficient is modified by electrons, $u \to u + \Pi(\mu, t_f)$. The absence of singularities from electrons indicates that the same criticality theory with modified interaction terms describes phase transitions [see Supplemental Material (SM) [31]]. We note that all thermal universality classes are stable under coupling to Fermi surfaces due to thermal fluctuations regardless of the particular choice of symmetry, including the WF-Ising/XY class.

The stability of the WF classes under acoustic phonons/disorder also has been well understood in previous studies [32–35]. Introducing a phenomenological coupling through a strain tensor, the Larkin-Pikin condition is suggested by the lowest-order renormalization group analysis, showing that the Ising (XY) universality becomes unstable (stable) in 3D, respectively. By employing the so-called Harris criterion, one can achieve the same stability behavior under disorder, as in the phonon case. The Ising universality may become either a first-order transition or a different universality class such as the mean-field class.

The model. We couple quantum rotors to qubits $\hat{\sigma}_l^{x,y,z}$ at links of a 3D cubic lattice with a periodic boundary condition. The total Hilbert space becomes the tensor product of the ones

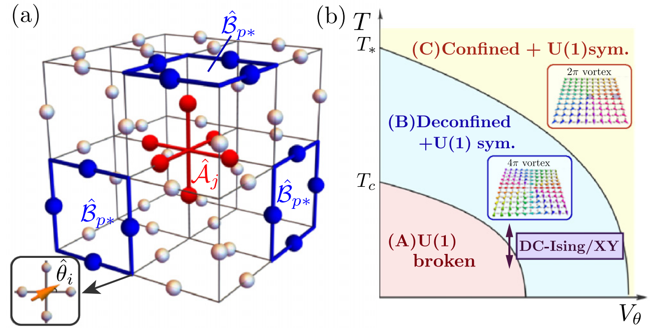

FIG. 1. (a) Interaction terms of $\hat{H}_X$. One of the star operators $\hat{\mathcal{A}}_j$ and three of the plaquette operators $\hat{\mathcal{B}}_{p*}$ of the 3D toric code are denoted with red and blue links, respectively. (b) Schematic phase diagram. Two transitions (T_c, T_*) correspond to a symmetry breaking and deconfinement. The DC-Ising/XY is different from the conventional WF-Ising/XY, whose critical exponents are in Table II.

of quantum rotors and qubits,

$$\mathcal{H}_{\rm tot} = \mathcal{H}_R \otimes \mathcal{H}_Q, \quad \mathcal{H}_Q = \prod_l \otimes\{|\sigma_l^z = \pm 1\rangle\}.$$

The model Hamiltonian is

$$\hat{H}_X = -J\sum_{\langle i,j\rangle} \hat{\sigma}_{ij}^z \cos\left(\frac{\hat{\theta}_i}{2} - \frac{\hat{\theta}_j}{2}\right) - \sum_j e^{i2\pi\hat{n}_j}\hat{\mathcal{A}}_j - \sum_{p^*} \hat{\mathcal{B}}_{p^*}, \tag{1}$$

where the star and plaquette operators, $\hat{\mathcal{A}}_j = (\prod_{l\in j}\hat{\sigma}_l^x)$, $\hat{\mathcal{B}}_{p^*} = (\prod_{l\in p^*}\hat{\sigma}_l^z)$ of the 3D toric code are introduced as shown in Fig. 1(a). The index l and j are for a link and a site, respectively. We note that the original 3D toric code has a thermal phase transition at $T_*^0 = 1.313$, which describes a deconfinement-confinement transition [29].

A few remarks are as follows. The $U(1)$ symmetry of $\hat{H}_X$ is the same as the one in $\hat{H}_R$. The factor $e^{i2\pi\hat{n}_j}$ associated with the star operator is an identity operator due to the integer condition of n_j, still, its presence is useful to check symmetry apparently. Also, the model is exactly solvable because all the terms of $\hat{H}_X$ commute with each other, which is related to the local $\mathbb{Z}_2$ transformation generated by $e^{i2\pi\hat{n}_j}\hat{\mathcal{A}}_j$. The effects of other interactions which break the exact solvability and local $\mathbb{Z}_2$ transformation are discussed below.

One ground state with a quantum number θ_0 is

$$|\tilde{\theta}_0\rangle = \hat{\mathcal{P}}_G\left(|\theta_0\rangle \otimes \prod_l |\sigma_l^z = 1\rangle\right),$$

$$\hat{\mathcal{P}}_G \equiv \prod_j \left(\frac{1 + e^{i2\pi\hat{n}_j}\hat{\mathcal{A}}_j}{2}\right).$$

Here, $\hat{\mathcal{P}}_G$ is a projection operator onto the states with $\mathcal{A}_j = 1$ for all j. We stress that the order parameter is the expectation value of an operator $e^{i\hat{\theta}_j}$, not $e^{i\hat{\theta}_j/2}$, as manifested by

$$\frac{\langle\tilde{\theta}_0|e^{i\hat{\theta}_j}|\tilde{\theta}_0\rangle}{\langle\tilde{\theta}_0|\tilde{\theta}_0\rangle} = e^{i\theta_0}, \quad \frac{\langle\tilde{\theta}_0|e^{i\hat{\theta}_j/2}|\tilde{\theta}_0\rangle}{\langle\tilde{\theta}_0|\tilde{\theta}_0\rangle} = 0. \tag{2}$$

The ground state with a quantum number θ_0 is unique without any degeneracy. For example, we consider the spin-flip

TABLE I. Excitations and their excitation energies of $\hat{H}_X$. The vortex and flux configurations of the rotor and qubit on the lattice are illustrated in Fig. S3 [31]. Three different length scales, system size (L_x), vortex size (R_x), and lattice spacing (a), are used.

Excitation	Energy cost
A pair of fluxes	$8+2J$
2π vortex	$[\frac{\pi}{4}\log(\frac{R_x}{a})+\frac{R_x}{a}]J\frac{L_x}{a}$
2π vortex + fluxes	$[\frac{\pi}{4}\log(\frac{R_x}{a})J+4]\frac{L_x}{a}$
4π vortex	$[\pi\log(\frac{R_x}{a})]J\frac{L_x}{a}$
Phase fluctuation	$J[1-\cos(\frac{\alpha_j}{2})]$

operator along the crystal plane $\mathcal{L}_{xy}$ of the dual lattice perpendicular to the z direction, $\hat{\mathcal{V}}_{xy}=\prod_{l\in\mathcal{L}_{xy}}\hat{\sigma}_l^x$ (see Fig. S2 [31]). While it connects homologically different ground states in the pure 3D toric code, the operator action in our model gives the energy,

$$\frac{\langle\tilde{\theta}_0|\hat{\mathcal{V}}_{xy}\hat{H}_X\hat{\mathcal{V}}_{xy}|\tilde{\theta}_0\rangle}{\langle\tilde{\theta}_0|\tilde{\theta}_0\rangle}=E_G+2JL_x^2,$$

lifting the degeneracy completely. Here, E_G is for the ground state energy without any spin-flip operator.

The wave functions and excitation energies are summarized in Tables I and S1 (See SM [31]), and a few remarks are as follows. First, the primary topological defect is a 4π vortex line, not a 2π vortex line [36]. Note that the bound state of a 2π vortex line and a flux line has lower energy, but it is still qualitatively bigger than the energy of a 4π vortex line. Second, the energy hierarchy manifests. Only phase fluctuations and the 4π vortex live at low-energy Hilbert space. Hereafter, we consider the case with $J, T \ll T_*^0$, which allows us to focus on the zero-flux sector.

Exotic transitions associated with topological orders. In the zero-flux sector, the effective Hamiltonian is

$$\hat{H}_{\rm eff}=\hat{\mathcal{P}}_0\hat{H}_X\hat{\mathcal{P}}_0\to -J\sum_{\langle i,j\rangle}\cos\left(\frac{\hat{\theta}_i}{2}-\frac{\hat{\theta}_j}{2}\right),$$

where the projection operator onto the zero-flux Hilbert space, $\hat{\mathcal{P}}_0$, is introduced. The right- hand side of the arrow is obtained with the configuration of the zero flux, $\sigma_l^z=1$. We remark that the range of the angle variables is $\theta_j\in(0,2\pi)$, originating from $\mathcal{H}_R$, not the doubled one $(0,4\pi)$.

The form of the effective Hamiltonian indicates that 2π vortex lines are removed from the zero-flux Hilbert space. Thus, the zero-flux Hilbert space can be given by

$$\hat{\mathcal{P}}_0[\mathcal{H}_{\rm tot}]=\mathcal{H}_{0\pi_v}\otimes\mathcal{H}_{4\pi_v}\otimes\mathcal{H}_{8\pi_v}\otimes\cdots,$$

where $\mathcal{H}_{4n\pi_v}$ is for the Hilbert space with $4n\pi$ vortex lines. Then, the corresponding partition function is

$$\mathcal{Z}_{\rm eff}\equiv{\rm Tr}_\theta[e^{-\hat{H}_{\rm eff}/T}]=\mathcal{Z}_{0\pi_v}\mathcal{Z}_{4\pi_v}\cdots,$$

where the subscripts are to specify the topological defects. Then, the remaining step is standard. A topological phase transition associated with 4π vortex lines appears in $\mathcal{Z}_{4\pi_v}$, whose critical temperature is estimated by comparing the energy and entropy of the topological defect, and we find the critical temperature $T_c\sim J$.

We stress that the trace of $\mathcal{Z}_{\rm eff}$ is over $\{|\tilde{\theta}_0\rangle\}$ not over the conventional states $\{|\theta_0\rangle\}$, and there is no ambiguity of the thermal average condition $\langle e^{i\hat{\theta}_j/2}\rangle_T=0$. It is also important to note that the partition function becomes asymptotically exact in the limit $J, T\ll T_*^0$, and the phase transition appears varying with J/T.

TABLE II. Critical exponents of the thermal universality classes associated with Ising and XY symmetries in three spatial dimensions. Using the two critical exponents ν and β, we find the other exponents with the scaling relations $\nu d=2-\alpha$, $\alpha+2\beta+\gamma=2$, $\gamma=\nu(2-\eta)$, and $\beta(\delta+1)=\nu d$ with $d=3$. For comparison, the universality class of the toric code, which is the same as the one of the 3D Ising model, is tabulated and its stability is discussed in SM [31]. As there is no order parameter in thermal topological transitions, any physical quantities out of the correlation functions of the order parameters are ill defined (Denoted as X). Only the specific heat is universally defined without relying on correlation functions, which show definite singular behaviors.

Universality class	α	β	γ	ν	η	δ
WF-Ising	0.11	0.33	1.24	0.63	0.036	4.79
WF-XY	−0.015	0.35	1.32	0.67	0.038	4.78
Toric code [39,40]	0.11	X	X	X	X	X
DC-Ising/XY	−0.015	0.83	0.35	0.67	1.47	1.43

The coarse-grained Landau functional is obtained by introducing the complex variable $\psi_j=e^{i\theta_j/2}$,

$$\mathcal{F}_{\rm DC}[\psi]=\int_x\left(|\nabla\psi|^2+r|\psi|^2+\frac{\lambda}{4}|\psi|^4\right).$$

The universality class of $\mathcal{F}_{\rm DC}[\psi]$ is different from the one of $\mathcal{F}_R[\psi]$ even though they have the same form because the variable ψ_j is not an order parameter. Instead, a secondary operator of the variable, ψ_j^2, is an order parameter as shown in Eq. (2). To determine the universality class, one may study how the coupling with toric code J in Eq. (1) is renormalized to understand the interplay physics between the symmetry order parameter and the topological order. We here adopt the analogous strategy of the nice work by Haldane, where the renormalization scheme of the sine-Gorden model in two spatial dimensions is used [37]. Then, we employ thorough investigations by Vicari and collaborators to numerically analyze the model in three spatial dimensions and determine all critical exponents [38].

The correlation length critical exponent with $\xi\sim(T_c-T)^{-\nu_{\rm DC}}$ is estimated as $\nu_{\rm DC}=0.67$, and the order parameter scaling dimension with $\langle(\psi)^2+(\psi^\dagger)^2\rangle_T\sim(T_c-T)^{\beta_{\rm DC}}$ is estimated by the scaling dimension of the secondary operator, $\beta_{\rm DC}=0.83$. With the two independent critical exponents, all other critical exponents are obtained by the scaling relations, which are summarized in Table II.

Let us consider the Ising potential described by ψ_j,

$$V_I=-u\sum_j\cos(\theta_j)^2=-u\sum_j\left[\psi_j^2+(\psi_j^\dagger)^2\right]^2,$$

and add $\int_x[\psi^2+(\psi^\dagger)^2]^2$ to $\mathcal{F}_{\rm DC}(\psi)$ up to coupling constant renormalization. The scaling dimension of u is estimated numerically and shown to be negative [38]. Thus, the Ising potential is irrelevant at T_c, and the universality class and the critical exponents are the same in both cases with the Ising

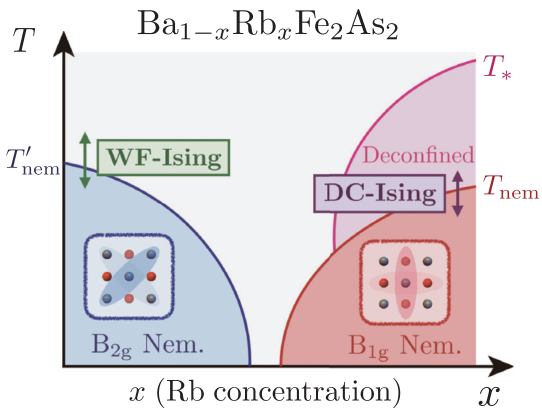

FIG. 2. Proposed phase diagram of $\mathrm{Ba}_{1-x}\mathrm{Rb}_x\mathrm{Fe}_2\mathrm{As}_2$ with doping concentration x and temperature T. T_{nem} ($T_{\mathrm{nem}'}$) is a critical temperature of B_{1g} (B_{2g}) nematicity. Our deconfined thermal transition scenario predicts the existence of a thermal topological transition at T_* much larger than T_c. The absence of a specific heat anomaly is observed at $x \simeq 1$, which may be better explained by $\alpha_{\text{DC-Ising}} < 0$ than $\alpha_{\text{WF-Ising}} > 0$.

and XY symmetries. The universality class of the critical point is dubbed deconfined Ising/XY (DC-Ising/XY) class. We emphasize that the irrelevance of $\mathbb{Z}_2$ anisotropy gives a direct route to see the interplay physics between the topological order and Ising order parameter, which is mainly discussed in the next section.

Based on our analysis, we provide a schematic phase diagram in Fig. 1(b), expecting that the phase diagram is asymptotically exact if T_*^0 is much bigger than the other energy scales. Generalization to different symmetry groups is straightforward. For example, we compare our results with previous literature uncovering Heisenberg symmetry with thermal gauge theories in SM [26,27,31].

Applications to experiments. Our theoretical proposal is relevant to the experiments on strongly correlated systems, including iron-based superconductors. One important example is an exotic nematic transition observed in doped $\mathrm{RbFe}_2\mathrm{As}_2$ compounds [13,14]. The recent experiments have found two Ising nematicities with different irreducible representations (B_{1g}, B_{2g}) in $\mathrm{Ba}_{1-x}\mathrm{Rb}_x\mathrm{Fe}_2\mathrm{As}_2$, whose schematic phase diagram is illustrated in Fig. 2. Remarkably, the temperature dependence of specific heat of the B_{1g} nematicity exhibits no discernible anomaly, in sharp contrast to that of B_{2g} nematicity. The origin of the B_{1g} nematic order has been considered in several theoretical works [41,42], but there are still puzzlings in the experiments such as the absence of specific heat anomaly.

Based on the deconfined thermal transitions, we predict the scaling relation of the nematic susceptibility $\chi_{\mathrm{nem}}[h_{\mathrm{nem}}, T]$ in an extension of the previous experiments,

$$\frac{\chi_{\mathrm{nem}}[h_{\mathrm{nem}}, T]}{\chi_{\mathrm{nem}}[h_{\mathrm{nem}}, T_{\mathrm{ref}}]} = \frac{C_1}{|T - T_{\mathrm{nem}}|^{\gamma_{\mathrm{DC}}}} \mathcal{F}\left[\frac{C_2 h_{\mathrm{nem}}}{(T_{\mathrm{nem}} - T)^{\beta_{\mathrm{DC}}\delta_{\mathrm{DC}}}}\right],$$

with an external field h_{nem} and a dimensionless scaling function $\mathcal{F}$. Two dimensionful parameters (C_1, C_2) depend on the reference temperature T_{ref} as well as the microscopic details. The nematic susceptibility without knowing the critical temperature T_{nem} was already reported in a previous work [13], while T_{nem} was recently uncovered by a Mössbauer experiment in Ref. [14].

The scenario provides the following additional predictions. First, there is an additional thermal transition at $T = O(T_*)$ whose scale is much higher than the onset temperature of the B_{1g} nematicity. The energy scale of T_* is expected to depend on microscopic interactions such as the Coulomb interaction and Hund coupling. Specific heat measurements at higher temperatures would be useful to estimate the scale of T_*. Second, the milder singularity of the specific heat jump in the B_{1g} nematicity may be a manifestation of the negative value of $\alpha_{\text{DC-Ising}} = -0.015$. We show the absence of the specific heat jump never occurs in the WF classes, even with fermionic excitations (see SM [31]). The presence of other degrees of freedom, such as phonons, would make the anomaly of the specific heat invisible in experiments. Third, the negative value of $\alpha_{\text{DC-Ising}}$ further indicates that the DC-Ising class is much more stable under disorder and acoustic phonon couplings than the Ising class based on the Harris criterion [43] and the Larkin-Pikin criterion [32]. Thus, the deconfined universality class for thermal transitions can bypass decoherence issues in lattice vibrations or disorder, which makes it more reliable to be found experimentally. Fourth, the exponent of the susceptibility (order parameter onset) $\gamma_{\text{DC-Ising}}$ ($\beta_{\text{DC-Ising}}$) is much smaller (larger) than the WF universality classes, which should be tested in experiments.

Discussion and conclusion. Our analysis with the exactly solvable model of $\hat{H}_X$ allows us to include additional microscopic quantum interactions such as $\frac{V_\theta}{2}\sum_j \hat{n}_j^2$ or $h_z \sum_l \hat{\sigma}_l^z$. Note that such inclusions are tricky in the conventional effective thermal gauge theories in the sense that precise couplings are not directly determined. Performing perturbative calculations with the other interactions, we check that that the charging effect term with V_θ induces quantum fluctuations of the rotors, similar to the conventional Bose-Hubbard model. Also, the local gauge invariance breaking term with h_z is shown to be irrelevant for $h_z \ll J$ (see SM [31]). Thus, we argue that the exotic thermal transitions are intact under the other interactions which break the exact solvability and local $\mathbb{Z}_2$ transformation.

We also remark on the subtlety of quantum-classical (QC) mapping in our model. Employing the conventional QC correspondence, our thermal DC-XY universality class could be mapped to the quantum-XY* universality class in two spatial dimensions [44–46]. The reported critical exponent of the quantum-XY* class, $\eta_{\mathrm{XY}^*} = 1.49$, is quite similar to the one of our deconfined thermal transitions, which is a strong indication of holding the QC correspondence [44]. However, the QC mapping no longer holds once order parameters couple to fermions with Fermi surfaces. This is because the singularities from the Fermi surfaces are suppressed by thermal fluctuations, in the case of deconfined thermal transitions, while the Fermi surfaces in quantum systems destabilize an order parameter, as in the pioneering work, the so-called Hertz-Millis theory [47,48]. Therefore, the QC correspondence between our DC-Ising/XY and quantum-XY* becomes elusive in metallic systems.

In conclusion, we show the existence of exotic thermal transitions with spontaneous symmetry breaking from topological orders. All critical exponents of the exotic universality classes are evaluated, and differences from the conventional mean-field and WF classes are emphasized. We provide smoking-gun experiments to test the exotic thermal transitions in plausible connections with doped $Ba_{1-x}Rb_xFe_2As_2$.

Acknowledgements. The authors thank H. Kontani and T. Shibauchi for invaluable discussions and comments. We are especially grateful to T. Shibauchi for communications about their experimental results. This work is supported by the National Research Foundation of Korea (NRF) funded by the Ministry of Science and ICT (No. 2019M3E4A1080411, No. 2020R1A4A3079707, No. 2021R1A2C4001847, No. 2022M3H4A1A04074153) and National Measurement Standard Services and Technical Services for SME funded by Korea Research Institute of Standards and Science (KRISS – 2022 – GP2022-0014).

[1] E. M. Lifshitz and L. P. Pitaevskii, *Statistical Physics: Theory of the Condensed State*, Vol. 9 (Elsevier, Amsterdam, 2013).
[2] S. Sachdev, *Quantum Phase Transitions* (Cambridge University Press, Cambridge, U.K., 2011).
[3] K. G. Wilson and J. Kogut, Phys. Rep. **12**, 75 (1974).
[4] M. E. Fisher, Rev. Mod. Phys. **46**, 597 (1974).
[5] J. Xia, E. Schemm, G. Deutscher, S. A. Kivelson, D. A. Bonn, W. N. Hardy, R. Liang, W. Siemons, G. Koster, M. M. Fejer, and A. Kapitulnik, Phys. Rev. Lett. **100**, 127002 (2008).
[6] T. Wu, H. Mayaffre, S. Krämer, M. Horvatić, C. Berthier, W. N. Hardy, R. Liang, D. A. Bonn, and M.-H. Julien, Nat. Commun. **6**, 6438 (2015).
[7] S. Badoux, W. Tabis, F. Laliberté, G. Grissonnanche, B. Vignolle, D. Vignolles, J. Béard, D. A. Bonn, W. N. Hardy, R. Liang, N. Doiron-Leyraud, L. Taillefer, and C. Proust, Nature (London) **531**, 210 (2016).
[8] R. Daou, J. Chang, D. LeBoeuf, O. Cyr-Choinière, F. Laliberté, N. Doiron-Leyraud, B. J. Ramshaw, R. Liang, D. A. Bonn, W. N. Hardy, and L. Taillefer, Nature (London) **463**, 519 (2010).
[9] Y. Sato, S. Kasahara, H. Murayama, Y. Kasahara, E.-G. Moon, T. Nishizaki, T. Loew, J. Porras, B. Keimer, T. Shibauchi, and Y. Matsuda, Nat. Phys. **13**, 1074 (2017).
[10] H. Murayama, Y. Sato, R. Kurihara, S. Kasahara, Y. Mizukami, Y. Kasahara, H. Uchiyama, A. Yamamoto, E.-G. Moon, J. Cai, J. Freyermuth, M. Greven, T. Shibauchi, and Y. Matsuda, Nat. Commun. **10**, 3282 (2019).
[11] F. Eilers, K. Grube, D. A. Zocco, T. Wolf, M. Merz, P. Schweiss, R. Heid, R. Eder, R. Yu, J.-X. Zhu, Q. Si, T. Shibauchi, and H. v. Löhneysen, Phys. Rev. Lett. **116**, 237003 (2016).
[12] X. Liu, R. Tao, M. Ren, W. Chen, Q. Yao, T. Wolf, Y. Yan, T. Zhang, and D. Feng, Nat. Commun. **10**, 1039 (2019).
[13] K. Ishida, M. Tsujii, S. Hosoi, Y. Mizukami, S. Ishida, A. Iyo, H. Eisaki, T. Wolf, K. Grube, H. v. Löhneysen, R. M. Fernandes, and T. Shibauchi, Proc. Natl. Acad. Sci. USA **117**, 6424 (2020).
[14] Y. Mizukami, O. Tanaka, K. Ishida, M. Tsujii, T. Mitsui, S. Kitao, M. Kurokuzu, M. Seto, S. Ishida, A. Iyo *et al.*, arXiv:2108.13081.
[15] J. M. Kosterlitz and D. J. Thouless, J. Phys. C: Solid State Phys. **6**, 1181 (1973).
[16] X.-G. Wen, Int. J. Mod. Phys. B **04**, 239 (1990).
[17] Y. Zhou, K. Kanoda, and T.-K. Ng, Rev. Mod. Phys. **89**, 025003 (2017).
[18] L. Savary and L. Balents, Rep. Prog. Phys. **80**, 016502 (2017).
[19] T. Senthil, A. Vishwanath, L. Balents, S. Sachdev, and M. P. A. Fisher, Science **303**, 1490 (2004).
[20] T. Senthil, L. Balents, S. Sachdev, A. Vishwanath, and M. P. A. Fisher, Phys. Rev. B **70**, 144407 (2004).
[21] Z. Bi and T. Senthil, Phys. Rev. X **9**, 021034 (2019).
[22] S. Lee, J. Jung, A. Go, and E.-G. Moon, arXiv:1803.00578.
[23] H. Oh, S. Lee, Y. B. Kim, and E.-G. Moon, Phys. Rev. Lett. **122**, 167201 (2019).
[24] J. B. Kogut, Rev. Mod. Phys. **51**, 659 (1979).
[25] E. Fradkin and S. H. Shenker, Phys. Rev. D **19**, 3682 (1979).
[26] P. E. Lammert, D. S. Rokhsar, and J. Toner, Phys. Rev. Lett. **70**, 1650 (1993).
[27] P. E. Lammert, D. S. Rokhsar, and J. Toner, Phys. Rev. E **52**, 1778 (1995).
[28] A. Kitaev, Ann. Phys. **303**, 2 (2003).
[29] A. Hamma, P. Zanardi, and X.-G. Wen, Phys. Rev. B **72**, 035307 (2005).
[30] C. Castelnovo and C. Chamon, Phys. Rev. B **78**, 155120 (2008).
[31] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevB.106.L140504 for the details on the effects of the coupling with fermions, excitations and perturbation calculations of H_X, and comparison with previous works, which includes Refs. [25–27,36,39,40,49,50].
[32] A. Larkin and S. Pikin, Sov. Phys. JETP **29**, 891 (1969).
[33] D. J. Bergman and B. I. Halperin, Phys. Rev. B **13**, 2145 (1976).
[34] S. E. Han, J. Lee, and E.-G. Moon, Phys. Rev. B **103**, 014435 (2021).
[35] U. Karahasanovic and J. Schmalian, Phys. Rev. B **93**, 064520 (2016).
[36] S. Sachdev, Rep. Prog. Phys. **82**, 014001 (2019).
[37] F. Haldane, arXiv:1612.00076.
[38] M. Hasenbusch and E. Vicari, Phys. Rev. B **84**, 125136 (2011).
[39] Z. Nussinov and G. Ortiz, Phys. Rev. B **77**, 064302 (2008).
[40] Z. Weinstein, G. Ortiz, and Z. Nussinov, Phys. Rev. Lett. **123**, 230503 (2019).
[41] Y. Wang, W. Hu, R. Yu, and Q. Si, Phys. Rev. B **100**, 100502(R) (2019).
[42] S. Onari and H. Kontani, Phys. Rev. B **100**, 020507(R) (2019).
[43] A. B. Harris, J. Phys. C: Solid State Phys. **7**, 1671 (1974).
[44] S. V. Isakov, R. G. Melko, and M. B. Hastings, Science **335**, 193 (2012).
[45] S. V. Isakov, M. B. Hastings, and R. G. Melko, Nat. Phys. **7**, 772 (2011).
[46] T. Senthil and O. Motrunich, Phys. Rev. B **66**, 205104 (2002).
[47] J. A. Hertz, Phys. Rev. B **14**, 1165 (1976).
[48] A. J. Millis, Phys. Rev. B **48**, 7183 (1993).
[49] S. Raghu, X.-L. Qi, C.-X. Liu, D. J. Scalapino, and S.-C. Zhang, Phys. Rev. B **77**, 220503(R) (2008).
[50] N. Nagaosa, *Quantum Field Theory in Condensed Matter Physics* (Springer, Berlin, 1999).