Letter

Critical localization with van der Waals interactions

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I discuss the quantum dynamics of strongly disordered quantum systems with critically long range interactions, decaying as $1/r^{2d}$ in d spatial dimensions. I argue that, contrary to expectations, localization in such systems is *stable* at low orders in perturbation theory, giving rise to an unusual "critically many-body localized (MBL) regime." I discuss the phenomenology of this critical MBL regime, which includes distinctive signatures in entanglement, charge statistics, noise, and transport. Experimentally, such a critically localized regime can be realized in three-dimensional systems with van der Waals interactions, such as Rydberg atoms, and in one-dimensional systems with $1/r^2$ interactions, such as trapped ions. I estimate timescales on which high-order perturbative and nonperturbative (avalanche) phenomena may destabilize this critically MBL regime and conclude that the avalanche sets the limiting timescale, in the limit of strong disorder or weak interactions.

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Nonequilibrium many-body quantum dynamics has aroused intense interest over the past decade. A cornerstone of our understanding is the phenomenon of *many-body localization* (MBL) (see Refs. [1,2] for reviews), by which strongly disordered quantum systems can fail to equilibrate and can realize qualitatively new kinds of quantum phases of matter. Most work on MBL has focused on systems with purely *short-range* interactions. However, long-range interactions, which decay as a power law of distance, are ubiquitous in nature, ranging from Coulomb interactions between charges, to dipolar interactions between spins, to van der Waals interactions between molecules and Rydberg atoms. What happens to MBL in the presence of long-range interactions?

The lore on MBL in the presence of long-range interactions is built on three results. Firstly, for *noninteracting* problems with long-range hopping, classic results [3,4] establish that if the hopping matrix element decays at long distance as $1/r^{\alpha}$, then systems can be localized as long as $\alpha > d$, where d is the spatial dimension. For $\alpha < d$ the system is thermal, and the critical case is thermal but not diffusive, supporting instead subdiffusive transport [4]. Secondly, a generalization of this argument to *interacting* systems [5,6] suggests that for systems with long-range two-body interactions decaying as $1/r^{\beta}$, localization is perturbatively stable as long as $\beta > 2d$ but is perturbatively unstable if $\beta < 2d$ (although specific counterexamples are known [7–9]), with explicit relaxation rates having been computed in Ref. [10]. However, what generically happens in the critical case $\beta = 2d$ has never been resolved, and this case is experimentally relevant both to van der Waals interactions in three spatial dimensions and to trapped ions in one spatial dimension, as well as being an important theoretical point of principle. Intuition from the case of long-range hopping and also from studies of (de)localization at critical points [11] would suggest that the critical case should be delocalized but might possibly have some unusual features. Thirdly, nonperturbative arguments [12] suggest that *any* power-law interaction should produce an "avalanche instability." I will make the conservative (and increasingly standard) assumption that this avalanche instability destroys the MBL *phase*. Nevertheless, the timescale associated with the avalanche instability is superpolynomially long in disorder strength [13,14], and up to this long timescale (which could be longer than experimental timescales), the quantum dynamics exhibits an MBL *regime* which will be my focus herein. I emphasize that I do *not* claim that MBL survives in this setting up to infinite times in the thermodynamic limit. Indeed, I expect that beyond the avalanche timescale (which I estimate) this problem is likely thermal. My focus is on the behavior that obtains at timescales that are *short* compared with the avalanche timescale.

In this Research Letter, I show that, contrary to expectations, the case of "critically" long ranged interactions ($\beta = 2d$) admits of a perturbatively stable MBL regime. However, the localization is of an unusual "critical" kind, with sharp few-body resonances uniformly distributed in the logarithmic length scale. I discuss the distinctive phenomenological signatures of this "critically MBL" regime. I also discuss the timescale up to which it is expected to be stable, which (I argue) is set by the avalanche instability.

This Research Letter is structured as follows. I begin by reviewing the basic arguments for critically long range hopping problems [4]. I then generalize this approach to critically long range interactions and demonstrate that the localized phase is (critically) stable at low orders in perturbation theory. I discuss the phenomenology of the resulting critically localized regime, before concluding with a discussion of timescales up to which the regime may be expected to survive, which I argue are set by the avalanche. I work throughout on the lattice, avoiding the complications inherent with analyses in the continuum [15–17].

I begin by reviewing at a cartoon level the behavior of noninteracting systems with long-range hopping, since this introduces the basic approach I will employ. I start by switching off the long-range component of the hopping and assume that in this limit the system consists of a set of Anderson localized wave functions $|\alpha\rangle$, with eigenenergies ε_α drawn from a distribution of width W. Now reintroduce the long-range hopping, so that the Hamiltonian becomes $H = \sum_\alpha \varepsilon_\alpha c^\dagger_\alpha c_\alpha + \frac{t}{|r_\alpha - r_\beta|^d} c^\dagger_\alpha c_\beta$. Since we are concerned primarily with the long-range tail of the hopping, the finite size of the wave functions α is not important. I treat the long-range hopping perturbatively, using the method of logarithmic shells employed in Ref. [18]. For a given excitation, and given a length scale R and an integer k, in the logarithmic shell at distance r satisfying $R2^k < r < R2^{k+1}$ there are $\sim R^d$ states to which the excitation can hop, with a minimum level spacing of $\sim W/R^d$. Meanwhile, the matrix element for the hopping is t/R^d. Hopping can be resonant if and only if the matrix element exceeds the level spacing. This ratio is $\sim t/W = \lambda$ which is crucially independent of both R and k. If we assume $\lambda \ll 1$ (strong disorder), it then follows that for the kth logarithmic shell, the probability of a resonant hop is $\sim\lambda \ll 1$. This has two consequences. Firstly, it is very unlikely that one finds a resonance at the same energy scale between n sites with $n > 2$: The most common resonances are between *pairs*, and these resonances are sparse in "logarithmic" space. Secondly, one almost surely finds a resonance at *some* length scale, since the probability of not finding *any* resonance in the first k logarithmic shells falls off exponentially with k. A careful solution involves a renormalization group (RG) treatment of hierarchical resonances [4]; however, the key scaling properties can be read off from the analysis above. In particular, an excitation will almost surely find a site to hop to, ensuring that the state is *delocalized* on long length scales.

Now let us adapt this argument to interacting systems with interactions that decay as $1/r^{2d}$. The prototypical Hamiltonian of interest takes the form

$$\hat{H} = \sum_i \varepsilon_i S^z_i + \sum_{i \neq j} \frac{V}{|r_i - r_j|^{2d}} \left(S^+_i S^-_j + \text{H.c.} + J S^z_i S^z_j\right). \quad (1)$$

Here, the $\mathbf{S}_i$ are spin 1/2 operators, which can be thought of as tracking whether a particular wave function (which is localized at the noninteracting level) is occupied or unoccupied, the ε_i are random numbers drawn from a distribution of width W, and J is an $O(1)$ parameter, the precise value of which is unimportant for the analysis. The $d = 1$ version of this Hamiltonian is relevant for experiments with trapped ions [19], and the $d = 3$ version is relevant for van der Waals interactions, e.g., in three-dimensional Rydberg atom arrays [20]. I have deliberately adopted a notation that parallels that of Ref. [6].

Let us assume that we are working close to a zero-entropy state, with small but nonzero density of excitations ρ. This could correspond to working close to the ground state at low but nonzero temperature, or it could correspond to working close to a "fully polarized" state as in Ref. [21]. The key control parameter for our calculation will be

$$\lambda = V\rho^2/W \ll 1. \quad (2)$$

Clearly, this control parameter can be tuned by changing either interaction strength, disorder strength, or excitation density (which in turn could be altered by tuning temperature, if we were working close to the ground state, or by tuning magnetization, if we were working close to a fully polarized state).

To formalize the calculation, I divide up the Hamiltonian as $\hat{H}_0 + \hat{V}$, where $\hat{H}_0$ is diagonal in the Z basis and $\hat{V}$ is off diagonal. Note that both $\hat{H}_0$ and $\hat{V}$ have a long-range $1/(r^{2d})$ tail. Now consider a pure state (but not necessarily eigenstate) initial condition $|\Psi\rangle$, which has density of excitations ρ, and work in the Schrödinger picture. It is convenient but not essential to consider $|\Psi\rangle$ to be a product state in the Z basis and to adopt a convention whereby "excitations" correspond to the system having local S^z eigenvalue $+1/2$. The $\langle\Psi(t)|S^z_i|\Psi(t)\rangle$ are integrals of motion with respect to the Hamiltonian $\hat{H}_0$, but what happens in the presence of $\hat{V}$? I address this question within perturbation theory in small V.

The perturbative analysis follows Ref. [3]. The perturbation theory is structured in terms of matrix element numerators and energy denominators. When the numerator is small compared with the denominator (off-resonance), this corresponds to *virtual* hopping and does not transport excitations. In contrast, *resonant* hops, where the matrix element equals or exceeds the energy denominator, *do* move excitations around. In principle, resonant hops can arise at any level in perturbation theory. However, it is by now well established that (for our model) in the limit $\rho \to 0$, when excitations are so dilute as to be effectively noninteracting, resonances are rare and do not percolate and the problem is well localized at strong disorder [3]. In contrast, at $\rho \neq 0$, a process first identified by Burin [5] guarantees percolation of "resonant" rearrangements if the interactions fall off more slowly than $1/r^{2d}$. Our interactions, however, fall off *exactly* as $1/r^{2d}$ and are thus marginal with respect to the Burin criterion.

Our analysis of marginally long range interactions proceeds by generalizing the analysis of Ref. [18] for the noninteracting problem. Consider a logarithmic shell of radius $2^k R < r < 2^{k+1} R$. The volume of the kth region is $V_k \sim (2^k R)^d$. This contains ρV_k excitations. The characteristic level spacing for two particle states in this volume is $\Delta_k = W/(\rho V_k)^2$. This sets the size of the typical energy denominators in perturbation theory. Meanwhile, the matrix element on this length scale is $V/(2^k R)^{2d}$. The probability of a resonance in the kth logarithmic shell is thus equal to P_k, given by

$$P_k = \frac{V/(2^k R)^{2d}}{W/(\rho(2^k R)^d)^2} = \frac{V\rho^2}{W} = \lambda \ll 1, \quad (3)$$

and is independent of R and k. Now the probability of no resonance for any k up to some macroscopic N is $(1-\lambda)^N \to 0$, so resonances almost surely exist, with a broad distribution of length scales. When a resonance exists, we should rediagonalize the problem to obtain new effective eigenstates, which will be bilocalized on two sites with separation $2^k R$ and with level splitting on the order of the matrix element, $V/(2^k R)^{2d}$. However, the existence of a sparse set of long-range resonances is not in itself sufficient to produce delocalization: The resonances need to percolate [3,18], and I now argue that in our problem of interest, they do *not* percolate.

To understand the failure of resonances to percolate, suppose you have m resonances. The probability that they all have the same length scale up to a factor of 2 is $\lambda^{m-1} \ll 1$. Thus triples (and higher-order resonances) are *rare* when $\lambda \ll 1$: Given two resonances, one of them will have a much smaller length scale than the other. The resonance with the smaller length scale will then develop a splitting that will be *large* compared with the matrix element of the longer-ranged resonance. It follows from the above that resonances cannot percolate. More carefully, suppose we are doing real-space RG and have coarse grained up to a scale R, whereupon a resonance first appears. By inspection of Eq. (1) it is clear that $\hat{V}$ only acts nontrivially in the subspace where the two sites in question have total $S^z = 0$. In this subspace, and in the basis of states $|\uparrow\downarrow\rangle$ and $|\downarrow\uparrow\rangle$, the effective Hamiltonian takes the form

$$H_{\text{eff}} = \begin{pmatrix} -\varepsilon & V/R^{2d} \\ V/R^{2d} & \varepsilon \end{pmatrix}, \tag{4}$$

where we require the energy splitting $2\varepsilon \leqslant V/R^{2d}$ in order for this to be a resonance, and where we rely on the rareness of n-tuple resonances with $n > 2$ to justify the restriction to a two-site effective Hamiltonian at scale R. Now, diagonalizing the Hamiltonian above, we straightforwardly obtain an energy splitting (for this resonance) of $2\Delta E(R)$, where

$$\Delta E(R) = \sqrt{\varepsilon^2 + V^2/R^{4d}} \geqslant V/R^{2d}. \tag{5}$$

The occupation numbers of the eigenstates of the effective Hamiltonian equation (4) are integrals of motion at this stage of the RG and are bilocalized on the two sites hosting the resonance. Now, at later RG scales we will be perturbing with an off-diagonal perturbation of strength $V/(R')^{2d}$, where $R' > R$. Since $V/(R')^{2d} < \Delta E(R)$, the resonance will not be involved in additional resonances at later stages of the RG. Note also that trying to "pair up" this resonance with another distant resonance with energy splitting $\Delta E(R)$, so as to make a higher-body resonance at scale $R' > R$, will typically not work, because for this given resonance, the phase space for other resonances to pair with will only grow as R'^d, whereas the matrix element will fall off faster, as R'^{-2d} [22]. We therefore conclude that the scale R resonance above will with high probability be an isolated resonance. Thus, for critically long range interactions (in contrast to the case of critically long range hopping), *we do not expect resonances to percolate*. Resonances do indeed form at all scales, as in the case of critically long range hopping, but this time they are all *isolated* resonances, not percolating resonances, and thus they do not form a heat bath. As such, we expect the system to be *localized*, but critically so insofar as there exist isolated few-body resonances on all length scales. This situation is illustrated in Fig. 1.

I clarify that the argument I have presented only establishes stability of localization at low orders in perturbation theory. Furthermore, it relies on arguments based on "typical" matrix elements and energy denominators and does not take into account distributions and rare events. Localization could still

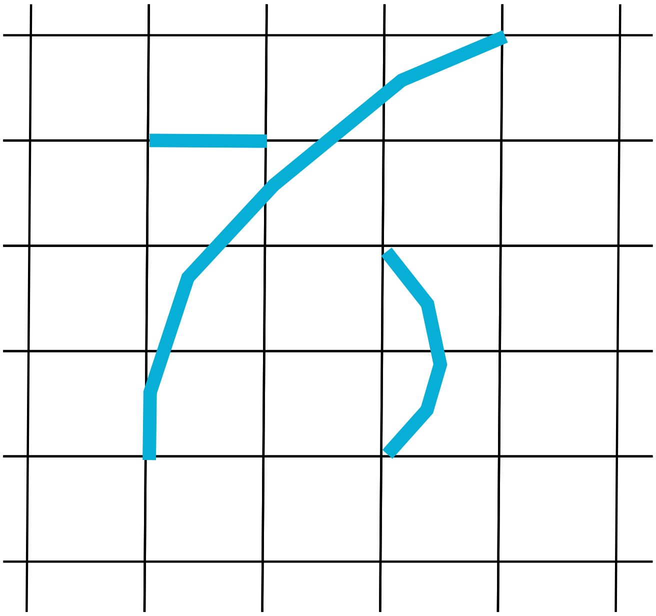

FIG. 1. A cartoon illustrating the structure of resonances in the critically localized phase. Spins live on every vertex of the lattice shown above. Thick blue lines connect spins that are in resonance. Resonances exist on all length scales but do not percolate.

be destabilized at high orders or by fully nonperturbative rare-event effects such as the avalanche. Indeed, I believe it *will* be so destabilized, and I will discuss the associated timescales in due course. However, first I would like to discuss the phenomenology of the "critically localized" regime that obtains on intermediate timescales, before rare events have a chance to come into play and when arguments based on "typical" matrix elements and energy denominators are expected to be accurate.

I start by discussing some properties of the MBL regime which are common to all systems with interactions that decay as $1/r^\beta$ with $\beta \geqslant 2d$. These are summarized in Table I. These properties rely on the state being stable at low orders in perturbation theory but do *not* rely on the stability being critical (i.e., $\beta = 2d$). I start by noting that *most* sites are not part of any resonance. Thus one could imagine "projecting" out the (nonpercolating!) resonances above some cutoff length scale to obtain an effectively localized problem [23]. Since most sites do not participate in any resonance, autocorrelation functions for a typical site should look indistinguishable from those of a conventional short-range interacting localized phase [24]. Meanwhile, the resonances that *do* exist are isolated and thus should manifest as *sharp* spectral lines, e.g., in nonlinear spectroscopy experiments [10,25] that couple to the relevant flip-flop process. Since resonances exist on all length scales (and, thus, on all energy scales), nonlinear spectroscopy should see sharp resonant spectral lines all the way down to zero frequency, inside a localized phase. This is similar to the behavior that obtains in short-range interacting systems. However, the existence of power laws also produces notable deviations from "short-range interacting behavior." Consider the dynamics of entanglement starting from unentangled (noneigenstate) initial conditions. Entanglement

TABLE I. Summary of some of the properties of the MBL regime in three scenarios: (1) short-range interactions (including exponential decay), (2) power-law interactions that decay as $1/r^{\beta}$ with $\beta > 2d$ (supercritical), and (3) power-law interactions that decay as $1/r^{2d}$ (critical: the focus of this Research Letter). All discussions pertain to the MBL *regime* at intermediate timescales, when "typical" arguments apply and rare events have not yet had time to come into play. For subcritical power laws (slower than $1/r^{2d}$) there is typically no MBL regime, except in certain special cases [7] which we do not discuss here. For details on the above, see the text.

Property	Short range	Supercritical	Critical
Entanglement growth from unentangled initial conditions	$\log t$	$t^{1/\beta}$	$t^{1/2d}$
Zone of disturbance in nonlinear response	$\log t$	$t^{1/\beta}$	$t^{1/2d}$
Entanglement entropy of approximate eigenstates	Area	Area	Area $\times \log L$
Standard deviation of charge in region of linear size L	L^{d-1}	L^{d-1}	$L^{d-1} \log L$

entropy will grow with time as $t^{1/\beta}$ instead of the conventional logarithmic-in-time growth for short-range interacting systems, for reasons anticipated in Refs. [26,27], and will presumably saturate to a volume law, much as it does in short-range interacting localized systems. The "zone of disturbance" in the nonlinear response [28] (i.e., the size of the region of space that is affected by a local perturbation that ramps up on timescale τ) will likewise scale as $\sim\tau^{1/\beta}$ with the timescale τ on which the system is perturbed, instead of the logarithmic scaling obtained for short-range interacting systems. The general theme is that various "logarithms" (in the short-range interacting problem) get turned into power laws, which have the advantage of being easier to observe experimentally. These features are common to all power laws with $\beta \geqslant 2d$. If $\beta < 2d$, then localization is destabilized already at low orders in perturbation theory, and there is no MBL regime.

I now turn to features that are particular to the "critical" power law ($\beta = 2d$) that we have discussed herein. These properties will rely on the fact that low-order resonances exist at *all* length scales and, moreover, the density of resonances on length scale R decays precisely as R^{-d}, so that resonances are equiprobable in *every logarithmic shell*. This is in contrast to conventional localized states, where the probability of resonances decays rapidly as one moves to larger logarithmic shells. The critical nature of the localization has distinctive signatures in the entanglement entropy of (approximate [29]) eigenstates. To zeroth order, this should be an area law (as in a typical localized state), but the long-range resonances that straddle the entanglement cut will enhance the entanglement entropy. Since the resonances are uniformly and sparsely distributed in logarithmic distance, we will end up with an eigenstate entanglement entropy of $A \log L$, where A is the area of the entanglement cut and L is the linear size of the smaller of the partitioned subregions in the direction normal to the cut. Such a logarithmic correction to the area law is familiar in, e.g., Fermi liquid systems [30,31], but here has completely distinct origins.

A similar logarithm will arise in the statistics of the conserved charge (S^z), without the requirement to prepare the system in an (approximate) eigenstate. If one takes a critically localized system, not necessarily in an eigenstate, bipartitions the system, and measures the charge in one-half of the system (e.g., in a quantum gas microscope), then the measured charge will have a quantum uncertainty associated with all resonances that straddle the boundary of the subregion, insofar as the charge measurement can "collapse" the resonance such that the charge shows up either inside or outside the subregion being probed. The number of such boundary-straddling resonances will scale as $A \log L$, and so if the charge measurement is repeated multiple times to generate statistics, then the measured charge will be drawn from a distribution with standard deviation scaling as $A \log L$, and this logarithmic scaling provides an experimentally accessible signature of critical localization which does not require the ability to prepare the system in eigenstates.

Signatures also arise in frequency domain measurements of charge fluctuations. The conserved charge in a subregion will fluctuate due to resonances that span the boundaries of the subregion, and the *timescale* for the fluctuations will be determined by the characteristic energy scales of the resonances, which in turn are related to the length scales by $E \sim 1/R^{2d}$. Since the resonances are uniformly distributed in a logarithmic length scale, they are also uniformly distributed in a logarithmic energy scale. On converting to a linear energy scale, we find that the probability distribution of resonances scales with frequency f as $1/f$ and thus may be expected to produce $1/f$ noise, as another signature of the "critical" nature of the localization.

Finally, energy transport will be subdiffusive. This last result follows from an argument similar to that employed in Ref. [4]: Resonances exist at all length scales, and energy can be transported on a length scale L through a pairwise resonance on that length scale; however, the matrix element falls off as $M_{fi} \sim 1/L^{2d}$, and the timescale may be extracted from $M_{fi}t \approx 1$ to yield the strongly subdiffusive scaling $L \approx t^{1/2d}$.

It is important at this point to note that "MBL" gets used two different ways in the literature. There are discussions of the MBL *phase*, concerning what happens at infinite times in the thermodynamic limit, and there are discussions of the MBL *regime*, concerning what happens at the kinds of intermediate timescales relevant for most experiments in the field (e.g., Refs. [25,32–36]). The former sort of question requires consideration of broad distributions and rare events, and the answers to many questions of this type are still controversial. However, the latter kind of question is not controversial and is expected to be well described by analyses of "typical" matrix elements and energy denominators. The discussion above pertains to the MBL *regime* (intermediate times) that is stable at low orders in perturbation theory. We now discuss whether we expect the MBL regime to extend to a true MBL *phase* (infinite times) and argue that we do *not* expect this. We estimate the timescales on which we expect the MBL regime to end.

Firstly, we note that the MBL regime must be destabilized at high orders in perturbation theory. The key argument is adapted from Ref. [37] and is based on mapping the problem to Ref. [38]. The mapping proceeds as follows: On a length scale L, there are $\rho^2 L^{2d}$ possible transitions that can be made, each with a matrix element V/L^{-2d}. One can then map this to a hopping problem on a tree [38], with parameters

$$K = \rho^2 L^{2d}, \quad Z = \frac{\rho^2}{\lambda} L^{2d}, \tag{6}$$

where I have adopted the same notation as Ref. [38]. Here, K is the branching number in Fock space (i.e., the number of many-body states to which a given state can "connect" via a single application of the Hamiltonian on scale L), and Z is the ratio of energy detuning to matrix element for a typical such transition. An inspection of Eq. (12) from Ref. [38] then leads to the conclusion that localization will necessarily be destabilized at order n in perturbation theory if

$$f_n \approx \frac{\lambda}{\sqrt{n}} \left[\lambda \log \left(\frac{\rho^2 L^{2d}}{\lambda n} \right) \right]^{n-1} \geqslant 1, \tag{7}$$

with the estimate being valid for $n \gg 1$. Now, at fixed large n this leads to destabilizing nth-order resonances at a length scale L_n satisfying

$$L_n^{2d} \approx \frac{\lambda n}{\rho^2} \exp\left(c_1 \frac{n^{1/(2(n-1))}}{\lambda^{n/(n-1)}} \right), \tag{8}$$

where c_1 is an $O(1)$ constant. This then yields a matrix element

$$M \approx \lambda^n L_n^{-2dn}, \tag{9}$$

where L_n is given by Eq. (8). Optimizing with respect to n, I obtain, at small λ, $n \approx \log(1/\lambda) \gg 1$, consistent with our prior assumption that n is large. Substituting back into Eq. (9), I obtain (at small λ) the asymptotic timescale

$$t_{agkl} \approx \exp\left(\tilde{c}_1 \frac{1}{\lambda} \log(1/\lambda) \right), \tag{10}$$

where $\tilde{c}_1$ is another $O(1)$ numerical constant. This is the timescale on which effects at high order in perturbation theory may be expected to destabilize the localized regime. It should be contrasted with the avalanche timescale, which was estimated in Ref. [14] [see their Eq. (18)] as

$$t_{\text{avalanche}} \approx \exp\left[c_2 \log^2(\lambda)\right], \tag{11}$$

where c_2 is an undetermined numerical constant. This latter timescale is set by the density of rare thermal "inclusions" that destabilize the localized phase. It is manifestly clear that at small λ the avalanche timescale is parametrically *shorter* than the timescale from high-order breakdown, and thus it is the avalanche timescale that sets the limit of our MBL regime.

To conclude, I have discussed the case of strongly disordered systems with critically long range two-body interactions falling off with distance as $1/r^{2d}$ in d spatial dimensions. This problem is relevant to a range of experimental platforms, including van der Waals interactions in three space dimensions and trapped ions in one space dimension. I have argued that, contrary to expectations, localization in such systems is perturbatively *stable* but of critical character, with (nonpercolating) resonances uniformly distributed in the logarithmic length scale. This leads to a localized state with a distinctive phenomenology, including a "logarithmic correction" to the entanglement entropy of approximate eigenstates and also a logarithmic scaling of the charge fluctuations (not necessarily in an exact eigenstate). This last provides an experimentally accessible diagnostic which could be used to identify the critically localized regime in experiments. I have estimated the timescale on which this critically MBL regime is destabilized and have argued that this is set by the avalanche timescale.

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