Letter

Square-root Floquet topological phases and time crystals

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Periodically driven (Floquet) phases are attractive due to their ability to host unique physical phenomena with no static counterparts. We propose a general approach in nontrivially devising a square-root version of existing Floquet phases, applicable both in noninteracting and in interacting setting. The resulting systems are found to yield richer physics that is otherwise absent in the original counterparts and is robust against parameter imperfection. These include the emergence of Floquet topological superconductors with arbitrarily many zero, π, and $\pi/2$ edge modes, as well as $4T$-period Floquet time crystals in disordered and disorder-free systems (T being the driving period). Remarkably, our approach can be repeated indefinitely to obtain a 2^nth-root version of any periodically driven system, thus, allowing for the discovery and systematic construction of exotic Floquet phases.

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Introduction. It is recently proposed that by simply square rooting an existing topological phase, a completely new material displaying exotic edge states properties is obtained [1]. Inspired by Dirac's idea [2] in treating the Klein-Gordon equation [3,4], such a square-rooting procedure is obtained by enlarging the degrees of freedom of the original system and devising a new Hamiltonian, the square of which yields two copies of the original system's Hamiltonian [1]. In the past few years, various proposals of square-root topological phases have been theoretically made [5–14] and experimentally verified [15–17]. These studies, however, concern only the physics of static systems.

Since the past decade, various phases of matter that can only be found in periodically driven systems (hereafter referred to as Floquet systems) have been identified and gained significant attention [18–31]. Apart from being of fundamental interest, novel Floquet phases have been demonstrated to yield advantages in quantum information processing [32–38]. It is, thus, envisioned that the possibility of square rooting these Floquet systems will lead to even more exotic phases of matter with a significant quantum technological impact. Such square-root Floquet phases are developed here.

A static (Floquet) system is characterized by a Hermitian Hamiltonian H (unitary one-period evolution operator U). This fundamental difference renders any known technique in square-rooting static systems inapplicable for use in Floquet setting. At first glance, square rooting a Floquet system might even appear trivial. Indeed, by writing $U = e^{-iH_{\rm eff}T}$ for some effective Hamiltonian $H_{\rm eff}$, its square root is obtained simply through $H_{\rm eff} \to \frac{H_{\rm eff}}{2}$. However, it is important to note that $H_{\rm eff}$ is generically not physically accessible, especially for Floquet phases that have no static counterpart [39–44]. In this case, simply reducing all system parameters by a half is not equivalent to $H_{\rm eff} \to \frac{H_{\rm eff}}{2}$ and will, thus, not yield the desired square-rooted system.

In this Letter, we propose a general procedure for square rooting a Floquet system in a systematic way, allowing its repetition to further generate any 2^nth-root version of the system. Remarkably, unlike existing square-rooting procedures that typically only work for specific single-particle static topological systems, our proposal is applicable both to noninteracting and interacting Floquet systems as demonstrated in the two explicit systems studied below. These case studies further reveal that such a square-rooting procedure is especially fruitful to yield systems with exotic properties that are otherwise absent in their original counterparts. For these reasons, our proposal opens an exciting opportunity to discover and study a variety of new Floquet phases.

General construction. Note that the one-period evolution operator (hereafter referred to as a Floquet operator) of *any* Floquet system can be written as

$$U = U_2 U_1 = \left[\mathcal{T}\exp\left(-i\int_0^{T/2} H(t+T/2)dt\right)\right] \times \left[\mathcal{T}\exp\left(-i\int_0^{T/2} H(t)dt\right)\right], \quad (1)$$

where $H(t) = H(t+T)$ is the system's Hamiltonian of period T and $\mathcal{T}$ is the time-ordering operator. To obtain its square-root version, we define a two-time-step Hamiltonian,

$$h_{(1/2)}(t) = \begin{cases} H(t)\frac{1+\tau_z}{2} + H\left(t+\frac{T}{2}\right)\frac{1-\tau_z}{2} & \text{for } n < \frac{t}{T} \leqslant n+\frac{1}{2}, \\ M\tau_y & \text{for } n+\frac{1}{2} < \frac{t}{T} \leqslant n+1, \end{cases} \quad (2)$$

where M is a system independent real parameter, $n \in \mathbb{Z}$, and $\tau_{x/y/z}$ are Pauli matrices representing an additional

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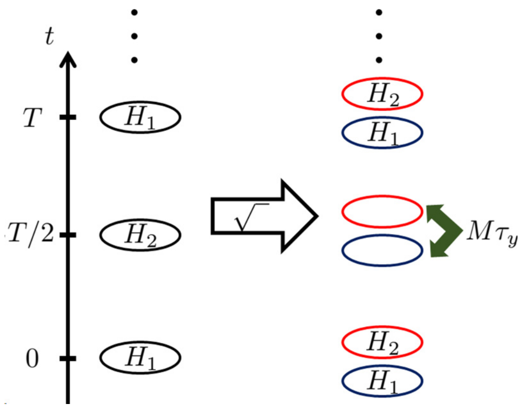

FIG. 1. Construction of a square-root Floquet system from any time-periodic parent Hamiltonian, where $H_1 \equiv H(t=0)$ and $H_2 \equiv H(t=\frac{T}{2})$.

(pseudo)spin-1/2 degree of freedom. Such a square-root Hamiltonian and its original counterpart are schematically shown in Fig. 1. Our construction can be intuitively understood as follows. Consider a particle initially living in the subsystem $\tau_z = +1$. It evolves under $H(t)$ for the first half of the period and then moves to the other subsystem during the second half of the period. As the Hamiltonian repeats, the particle, which is now in the subsystem $\tau_z = -1$, evolves under $H(t + T/2)$ for another half period and moves back to the subsystem $\tau_z = +1$ at the end of the second period. That is, only after two periods will the particle experience the full Floquet operator of the parent system, while remaining on the same subsystem. On the other hand, the particle can exhibit a nontrivial evolution over a period to yield various new physics, some of which are highlighted in the case studies below.

More explicitly, at $MT = \pi$, the Floquet operator associated with Eq. (2) takes the form

$$u_{(1/2)} = e^{-i(\pi/2)\tau_y}\begin{pmatrix} U_1 & \mathbf{0} \\ \mathbf{0} & U_2 \end{pmatrix} = \begin{pmatrix} \mathbf{0} & -U_2 \\ U_1 & \mathbf{0} \end{pmatrix}. \quad (3)$$

In particular, $u^2_{(1/2)} = \text{diag}(-U_2U_1, -U_1U_2)$, thereby reproducing two decoupled copies of the target U (up to a unitary transformation). Its ability to host two different orderings of U is, particularly, fruitful for potential parallel processing applications. For example, topological invariants of one-dimensional (1D) chiral symmetric Floquet topological systems are defined from the winding numbers of Floquet operators at two different orderings [45]. In such cases, the simultaneous realization of both Floquet operators with our construction can offer a significant speed up in detecting their topological invariants.

At $MT \neq \pi$, the diagonal elements of $u_{(1/2)}$ may, in general, become nonzero, whereas its off-diagonal elements are deformed away from U_1 and U_2. This renders $u^2_{(1/2)}$ no longer diagonal and directly related to U. However, by intentionally setting $MT \neq \pi$ in all the numerics below, we find that $u^2_{(1/2)}$ inherits the main physics of the target system. Therefore, Eq. (2) can still be regarded as the same square-root solution of the target model provided $MT - \pi$ is not too large. The insensitivity of our construction to the fine-tuning of M demonstrates the robustness of square-root Floquet phases.

Importantly, our square-rooting procedure is scalable, i.e., it can be applied indefinitely to obtain the 2^nth root of any Floquet system, thus, opening avenues for obtaining a variety of Floquet phases with even more exotic physical properties. In the following, we explicitly apply our procedure on two representative systems. For conciseness, we only focus on their square-root counterparts, emphasizing the unique features not found in the corresponding parent systems. In Ref. [46], the 4th- and 8th-root versions of such systems are presented.

Square-root Floquet topological superconductor with arbitrarily many edge modes. A remarkable feature of Floquet topological phases is their possibility to support any number of edge modes through appropriate choice of system parameters [48–54]. To demonstrate our square-rooting procedure at work, we consider the Floquet topological superconducting model introduced in Ref. [53], which is described by a two-time-step Hamiltonian switching between H_1 and H_2 at every half period (two continuous variations of such a model are further considered in Ref. [46]). There,

$$H_\ell = \sum_{j=1}^{N} \mu_\ell c_j^\dagger c_j + \sum_{j=1}^{N-1}(-J_\ell c_j^\dagger c_{j+1} + \Delta_\ell c_j^\dagger c_{j+1}^\dagger + \text{H.c.}), \quad (4)$$

where μ_ℓ, J_ℓ, and Δ_ℓ are, respectively, the chemical potentials, hopping amplitudes, and pairing strengths, $\ell = 1, 2$, c_j is the fermionic operator at site j, and N is the system size. In particular, at $\mu_2 = -2J_2 = -2\Delta_2 = m\mu_1 = 2mJ_1 = 2m\Delta_1$ with $m \in \mathbb{R}$, such a system supports n pairs of Majorana zero modes (MZMs) and Majorana π modes (MPMs) for $n\pi < m < (n+\frac{1}{2})\pi$ [53]. Here, MZMs and MPMs are topologically protected Hermitian edge-localized operators which, respectively, commute and anticommute with the system's Floquet operator [32–34]. Even at the special parameter values above, such a system has a very complex and unphysical effective Hamiltonian of the form $H_{\text{eff}} \propto \arccos\{\cos[x\cos(k/2)]\cos[mx\cos(k/2)]\}$, where x is a constant and k is the quasimomentum. Therefore, the trivial square-root procedure $H_{\text{eff}} \to \frac{H_{\text{eff}}}{2}$ is indeed infeasible.

Following our general construction, the corresponding square-root system is obtained as a two-time-step Hamiltonian $h^{(FTSC)}_{(1/2)}(t)$ which switches between $h^{(FTSC)}_{(1/2),1} = h_1 + h_2$ and $h^{(FTSC)}_{(1/2),2} = \sum_{j=1}^{N} Mic^\dagger_{1,j}c_{2,j} + \text{H.c.}$ after every $T/2$ time interval, where h_ℓ with $\ell = 1, 2$ take the form of Eq. (4) with $c_j \to c_{\ell,j}$. Physically, such a system represents a pair of p-wave superconductors with interchain hopping applied during the second half of the period. It can, in principle, be realized by proximitizing two chains of semiconducting wires with an s-wave superconductor such that h_1 and h_2 are achieved through the same mechanism as that in the realization of Kitaev chain [55,56]. The interchain hopping is further obtained and controlled by modulating the separation between the two chains.

The Bogoliubov–de Gennes Floquet operator $\mathcal{U}$, which is related to the actual Floquet operator via $u = \frac{1}{2}\Psi^\dagger\mathcal{U}\Psi$, can be explicitly obtained, and its exact form is detailed in Ref. [46]. The system's quasienergy (ε) excitation spectrum is then

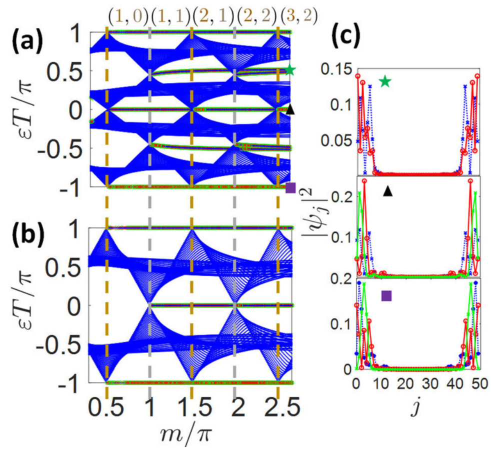

FIG. 2. Quasienergy excitation spectrum of the (a) square-root and (b) original Floquet topological superconductor of Ref. [54]. The vertical lines mark the topological phase-transition points, whereas (ν_π, ν_0) is a pair of topological invariants determining the number of MPMs and MZMs in the original system. (c) Typical wave-function profiles of the system's edge modes. The system parameters are chosen as $\mu_2 T = m\mu_1 T = 2m$, $-J_2 T = mJ_1 T = 1.05m$, $-\Delta_2 T = m\Delta_1 T = 0.95m$, $MT = 0.9\pi$, and $N = 50$.

obtained from the eigenvalues $e^{-i\varepsilon T}$ of $\mathcal{U}$ and is summarized in Fig. 2(a). The associated quasienergy excitation spectrum of the original system is plotted in Fig. 2(b) for reference. Note, in particular, that both systems share the same topological phase transition points, marked by parameter values at which gap closing exists. MZMs and MPMs are associated with quasienergy zero and π/T solutions, respectively, in Figs. 2(a) and 2(b). Moreover, the following two features are clearly observed.

First, the presence of MPMs in the original system leads to the simultaneous presence of MZMs and MPMs in the square-root system. This feature can further be analytically proven by computing a pair of topological invariants (ν_0, ν_π) and $(\nu_0^{(1/2)}, \nu_\pi^{(1/2)})$ for the original and square-root system, respectively, [46,53]. In particular, ν_0 $(\nu_0^{(1/2)})$ and ν_π $(\nu_\pi^{(1/2)})$, respectively, determine the number of pairs of MZMs and MPMs in the original (square-root) system. By leaving the technical detail in Ref. [46], we indeed find that $\nu_\pi = \nu_0^{(1/2)} = \nu_\pi^{(1/2)}$, thus, confirming our observation above.

Second, the presence of MZMs in the original system leads to the emergence of edge modes at $\approx\pi/(2T)$ quasienergy. Recently, it was shown that such $\pi/2$ modes may become parafermions [57–59] in the presence of interaction [60]. These $\pi/2$ modes are, however, not as ubiquitous as MZMs and MPMs, and their construction previously involves a rather elaborate driving scheme [60]. With our square-rooting procedure, such $\pi/2$ modes can be systematically generated, and their origin traced back from the topology of the squared model. That is, whereas a topological invariant characterizing these $\pi/2$ modes in the square-root system directly is presently unknown to us, the presence of $\pi/2$ modes can still be inferred from the invariant ν_0 defined on the squared system.

As elaborated in Ref. [46], the presented square-root Floquet topological superconductor inherits the chiral symmetry of its parent system. This chiral symmetry is responsible for protecting MZMs and MPMs in the system. Indeed, even at imperfect square-root parameter $MT \neq \pi$, MZMs and MPMs remain pinned at 0 and π/T quasienergies, respectively, [see Fig. 2(a)]. By contrast, the observed $\pi/2$ modes slightly deviate from the expected $\pi/(2T)$ quasienergy due to the absence of symmetry protection. It remains to be seen if the presence of interaction that promotes these $\pi/2$ modes to Z_4 parafermions in the ideal limit, such as via the mechanism elucidated in Ref. [60], may render them more robust against such an imperfection effect.

Square-root Floquet time crystals. Our construction is not limited to single-particle systems. Indeed, it can be applied to square-root a Floquet time crystal (FTC), i.e., a many-body phase of matter characterized by robust subharmonic observable dynamics [61–93]. Focusing first on the many-body localization (MBL) protected FTC model of Ref. [62], its square-root is obtained by plugging in

$$H(t) = h_j X_j, \quad H(t + T/2) = J_j Z_j Z_{j+1} + h_j^Z Z_j, \tag{5}$$

to Eq. (2). There, $P_j = J_j, h_j, M$, and h_j^Z are each randomly taken from a uniform set $[\bar{P} - \Delta P, \bar{P} + \Delta P]$, X_j, and Z_j are Pauli matrices associated with the jth site in the 1D lattice, and $\tau_{x/y/z}$ are additional Pauli matrices. As a rather unrealistic interpretation of the system, it describes a single spin $(\tau_{x/y/z})$ interacting with a 1D Ising model. In a more physical setting, it can be effectively and more robustly realized with two interacting and periodically driven spin-1/2 chains. As detailed in Ref. [46], this is achieved by replacing $Z_j \to Z_{j,A}$, $X_j \to X_{j,A}X_{j,B}$, $\tau_z \to Z_{j,A}Z_{j,B}$, and $\tau_y \to \sum_{j=1}^N X_{j,B}$ in Eq. (5), where A, B label the two chains. The resulting system, which involves at most nearest-neighbor two-body interactions, can, in turn, be implemented with FTC successful trapped ions [64,65] and superconducting circuit [71] platforms [91].

In Figs. 3(a) and 3(b), we plot the stroboscopic magnetization dynamics, i.e., $\langle S_z\rangle = \frac{1}{N}\sum_{j=1}^N \langle Z_j\rangle$, and its associated power spectrum, i.e., $\langle\tilde{S}_z\rangle = |\frac{1}{L}\sum_{m=\ell}^{L}\langle S_z\rangle e^{-(\ell\Omega T)/L}|$, under the square root of Eq. (5) and a generic initial-state $|\psi(0)\rangle = \prod_{j=1}^N e^{-i(\pi/8)Y_j}|00\cdots 0\rangle$. Despite considerable deviation from the ideal values $MT = h_j T = \pi$, a robust $4T$ rather than $2T$ periodicity is observed, thus, highlighting the system's nature as a square-root version of Ref. [62]. Moreover, Fig. 3(b) reveals that this subharmonic behavior improves with the system size. These features imply that such a square-root model is indeed a genuine FTC. It is also worth noting that this $4T$-period FTC is physically different from that proposed in Ref. [91] due to the absence of the $\Omega = \frac{\pi}{T}$ peak in Fig. 3(b) (cf. Fig. 2 of Ref. [91]).

The advantage of our square-rooting procedure over the FTC construction proposed in Ref. [91] is the possibility to devise large-period FTCs without MBL. This can be accomplished, e.g., by square rooting the disorder-free continuously driven Lipkin-Meshkov-Glick (LMG) model. The latter is a variation of the kicked LMG model of Ref. [76] in which the Dirac-δ driving is replaced by a harmonic driving. It is,

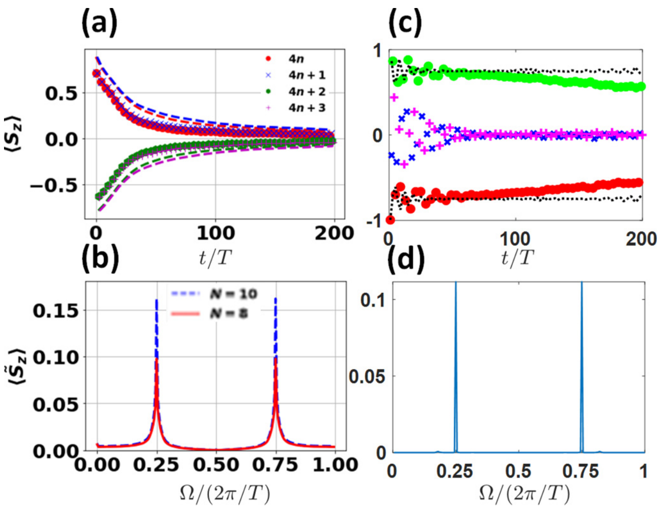

FIG. 3. Stroboscopic magnetization profile (a) and (c) and its associated power spectrum (b) and (d) under square-root of (a) and (b) Eq. (5) after averaging over 500 disorder realizations for a system of eight (solid marks) and ten (dashed lines) particles, (c) and (d) Eq. (6) for a system of 100 particles. In panel (c), the black dotted lines show the corresponding profile under its parent system. The system parameters are chosen as (a) and (b) $\bar{h}T = 0.92\pi$, $\bar{J}T = 2$, $\bar{h}^Z T = 0.3$, $\bar{M}T = 0.95\pi$, $\Delta hT = 0.05\pi$, $\Delta JT = 1$, $\Delta h^Z T = 0.3$, and $\Delta MT = 0.05\pi$, (c) and (d) $JT = 1$, $hT = 0.1$, $\phi T = \frac{0.9\pi}{2}$, and $MT = 0.98\pi$. Note that in all panels, the interaction strength J is comparable with that used in previous studies of FTCs, such as Refs. [62,83].

thus, related to the model of Ref. [83]. Specifically, the parent Hamiltonian to be inserted in Eq. (2) takes the form

$$H(t) = \left(\sum_{i,j} \frac{J}{2N} Z_i Z_j + \sum_i h X_i\right)[1 + \cos(\omega t)] + \sum_i \phi X_i. \tag{6}$$

Since such a system preserves the total spin $\mathcal{S}^2 = \sum_{i,j}(X_i X_j + Y_i Y_j + Z_i Z_j)$, numerical studies of very large system sizes are accessible via exact diagonalization. In Figs. 3(c) and 3(d), a clear and long-lived $4T$ oscillation profile is observed from $\langle S_z(t)\rangle$ and its power spectrum, taking $|\psi(0)\rangle = |00\cdots 0\rangle$ at 100 particles as the initial state. By implementing the system via two interacting LMG chains as detailed in Ref. [46], a long-lasting $4T$ oscillation profile is observable with merely ∼10 particles, a much smaller system size than that required for observing a similar feature in Ref. [83] (see Ref. [46] for the underlying mechanism).

By repeating the square-root procedure to the above systems, disordered and clean $2^n T$-period FTCs can, respectively, be obtained (see, e.g., Ref. [46]). Importantly, they are also observable at system sizes accessible with current technology [64,65,71], thus, paving the way for experimentally exploring FTCs beyond their subharmonic signatures, e.g., confirmation of condensed-matter phenomena in the time domain [89,90].

Concluding remarks. We have proposed a systematic and general construction of square-root Floquet phases exhibiting exotic properties not found in the parent systems. We explicitly applied our procedure to obtain Floquet topological superconductors with arbitrarily many MZMs, MPMs, and the elusive $\pi/2$ modes, as well as FTCs beyond period doubling. As a primary advantage of our construction, it amounts to coupling two copies of the parent systems and can, thus, be realized in the same platform as the latter, inheriting their feasibility. Indeed, our square-root FTCs can be directly implemented in the platforms of Refs. [64,65,71] under the available resources [46]. While the parent model of our square-root topological superconductor has not been experimentally realized, the predicted $\pi/2$ modes can arise from square rooting a simpler model. In an upcoming work [94], we will experimentally demonstrate the signature of $\pi/2$ modes in an acoustic square-root topological insulator.

Our procedure can generate nontrivial square-root Floquet systems even if their parent systems have static counterparts. For example, a Hamiltonian of the form $H(t) = H_0[1 + \sin(\omega t)]$ has a simple Floquet operator $U = e^{-iH_0 T}$ that can be trivially square rooted through $H_0 \to \frac{H_0}{2}$. However, by explicitly plugging in $H(t)$ to Eq. (2), the resulting Floquet operator instead yields a complex effective Hamiltonian that has no static counterpart [46].

The above example demonstrates that a square-root Floquet system is not unique. Consequently, the proposed approach is not the only means of generating a square-root Floquet system, but it provides a motivation and inspiration for devising other square-rooting schemes. In particular, an alternative procedure involving a smooth driving protocol rather than the two-time-step drive proposed here would be desirable. Another possible improvement of the current scheme is to exploit the time degree of freedom to replace the ancillary Pauli matrices used in our construction. This may be achieved by adapting the technique proposed in Ref. [44]. Finally, in Ref. [95], we extend the present approach to generate a nth-root Floquet phase, where n is any arbitrary integer.

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