String-like theory of quantum Hall interfaces

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We derive the effective theories for quantum Hall droplets with attractive interaction among the constituent particles. In the absence of confining potentials, such droplets are defined by their freely moving interfaces (or boundaries) with the vacuum. We demonstrate that the effective theories take forms like string theories. Generalization to interfaces between different quantum Hall liquids is discussed.

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I. INTRODUCTION

The quantum Hall (QH) effect is one of the most active research fields in condensed matter physics [1]. There are many different approaches to describe QH physics [2–8]. In the bulk, the low-energy effective theory of a QH liquid is the Chern-Simons (CS) theory [9]. At the edge, the low-energy effective theory consists of two pieces: The topological and dynamical pieces [9]. The topological piece is dictated by the bulk CS theory (known as bulk-edge correspondence) so that the overall theory describing the bulk and the edge is gauge invariant. Just like the bulk CS theory, it describes the degrees of freedom at the edge (usually chiral bosonic and Majorana fermionic modes) but contains no dynamical information. Dynamics at the edge comes from the external confining potential, which determines the location of the edge and dictates the edge Hamiltonian [1,9].

Recently, one of us considered a different type of boundary [10], namely, *the QH interfaces* that separate different QH phases, whose locations are not determined (or pinned) by any external potential. A special class of such interfaces separates a QH liquid from the vacuum, which (superficially) looks very much like edges. Instead of a repulsive interaction between the electrons in the system, Ref. [10] invoked attractive interactions to hold the QH liquid together, thus eliminating the necessity of the external confining potential. Accordingly, such a QH droplet (QHD) is formed spontaneously and can move freely. In the absence of a confining potential, the interface behaves like a string, whose dynamics is dominated by the string tension [10].

Reference [10] focused on the low-energy excitation spectra of various interfaces, which can be accessed easily by modifying the Hamiltonian of the corresponding edge theories. To describe the general interface dynamics, however, we need to study large distortions of the interface configurations from the ground state. This is the task of this paper. As we will see, this requires a reformulation of the entire theory (including the topological term), which leads to a string-like theory for the interfaces. At low energy, this allows for improvement of the results of Ref. [10] by including curvature effects in a systematic way, which accounts for the (relatively small but definitely noticeable) discrepancy between theory prediction and numerical data. More importantly, we demonstrate that QH interfaces realize certain nonrelativistic versions of both bosonic strings and superstrings; in the latter case, there are Majorana fermion modes that live along the strings. This facilitates synergy between condensed matter and high-energy physics.

The rest of the paper is organized as follows. In Sec. II, we start by including the curvature effects in the energy functional for the interface between a fermionic $\nu = 1$ QH liquid and the vacuum and demonstrate this leads to significant improvement in the comparison between first-principles prediction of the low-energy excitation spectrum and numerical results. In Sec. III, we show similar improvement for the Pfaffian-vacuum interface. Section IV is devoted to the derivation of the topological term in the action when the interface configuration deviates significantly from that of the ground state. Some concluding remarks are offered in Sec. V, where we discuss possible generalization of our approach to interfaces between different QH liquids.

II. INTERFACE BETWEEN FERMIONIC $\nu = 1$ QH LIQUID AND VACUUM

We start with the simplest possible QH interface separating the fermionic $\nu = 1$ QH liquid and the vacuum, with the former stabilized by an attractive Haldane V_1 pseudopotential [1] and the Pauli principle. Due to the simplicity of this state, parameters of the string-like theory can be determined from microscopic calculations.

The effective action of a droplet with N fermions and radius $r_0 = \sqrt{2N}$ (we set the magnetic length ℓ to unity) in the low-energy (or linear) regime is a slight modification of the edge theory with a single chiral bosonic field [10]:

$$S = -\frac{1}{4\pi}\int dt \int_0^{2\pi r_0} dx \dot{\phi}\phi' - \frac{T_0}{2}\int dt \int_0^{2\pi r_0} dx (u')^2, \quad (1)$$

where T_0 is the string tension, x is the coordinate along the unperturbed string, t is time, prime is ∂_x, dot is ∂_t and u, and the transversal displacement from equilibrium [see Fig. 1(a)]

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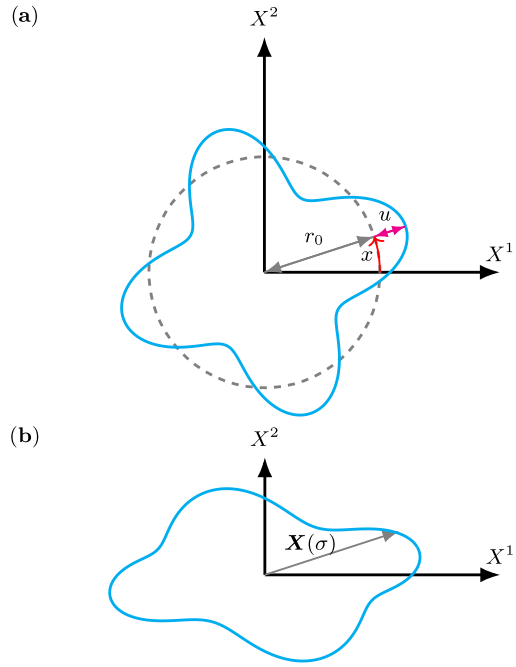

FIG. 1. Illustration of a quantum Hall droplet with emphasis on its interface with the vacuum. (a) A configuration with infinitesimal distortion. Dashed circle with radius r_0 shows the equilibrium configuration, cyan is the distorted configuration, x is the coordinate along the equilibrium, u is the transversal displacement of the interface from the equilibrium, and $X^{1,2}$ are the coordinate system. (b) A configuration with large distortion.

is given by

$$\phi' = u, \tag{2}$$

where ϕ is a compact scalar with identification:

$$\phi \sim \phi + 2\pi. \tag{3}$$

The first term in Eq. (1) is the aforementioned topological piece which emerges from the bulk CS, and the second term, the dynamical piece, is the length [11] times the tension in the limit of $|u'| \ll 1$ and $r_0 \to \infty$. We will use the term *linear regime* for such an approximation, which is valid when the deviation to the ground state configuration is small and slowly varying.

A. Action beyond the linear regime

When we consider larger deformations, higher-order terms in u and its derivatives become important, and Eq. (1) needs to be modified. As a first step, we must replace the approximated arc length in Eq. (1) (that renders it quadratic) with the actual arc length. However, at this point, the energy of the system is completely determined by the length of the interface. Thus, classically, *energy conservation* means conservation of the string length. This together with the area conservation constraint (which comes from the incompressibility of the QHD) implies that the classical equation of motion of the string would be trivial, namely, the only allowed motions are rigid spinning and motion of the center of mass. Thus, as we will see below, such a modified action would be incomplete; it misses the curvature dependence of the local energy density, which has a significant effect on the dynamics.

A convenient way to incorporate the effect of curvature is to make the tension T a function of the curvature κ of the string which is given as [12]

$$\kappa = \frac{2\frac{u'^2}{r_0} - \left(1+\frac{u}{r_0}\right)\left(u'' - \frac{u}{r_0^2} - \frac{1}{r_0}\right)}{\left[\left(\frac{u}{r_0}+1\right)^2 + u'^2\right]^{3/2}}. \tag{4}$$

In this case, the Hamiltonian is no longer just proportional to the string length; thus, energy conservation and area conservation no longer lead to trivial motion. The reason we choose to make T a function of curvature is, for a planar curve (i.e., a curve lying on a plane), arc length and curvature completely determine a curve up to a rigid motion [12,13]. Thus, our more general model that includes the curvature effect is

$$\begin{aligned} S = &-\frac{1}{4\pi}\int d^2x\dot{\phi}\phi' - \int d^2x T(\kappa)\sqrt{\left(\frac{u}{r_0}+1\right)^2 + (u')^2} \\ &+ \int d^2x\lambda(\phi' - u) \\ &+ \int dt\, w\left[\frac{1}{2}\int dx r_0\left(\frac{u}{r_0}+1\right)^2 - \pi r_0^2\right], \end{aligned} \tag{5}$$

where λ is the Lagrangian multiplier field to enforce Eq. (2); w is also a Lagrangian multiplier field which only depends on time, and it enforces area conservation. If we set $r_0 \to \infty$ and expand Eq. (5) up to quadratic order and set $T \to T_0$, we will get Eq. (1) back.

B. Extracting the curvature-dependent tension

We can find $T(\kappa)$ by extending the method of Ref. [10] that determined $T(\kappa = 0)$. In units of $|V_1|$ (set to be one from now on), the ground state energy of the droplet in terms of number of electrons was found to be [10]

$$\langle V\rangle = \frac{1}{2}N\left[4 - \frac{4^{2-N}(2N-1)!}{(N-1)!N!}\right]. \tag{6}$$

Expanding at large N yields

$$\langle V\rangle = -2N + \frac{4\sqrt{N}}{\sqrt{\pi}} - \frac{\sqrt{\frac{1}{N}}}{2\sqrt{\pi}} + \mathcal{O}\left[\left(\frac{1}{N}\right)^{3/2}\right]. \tag{7}$$

The first term is the bulk contribution, and the rest is the surface contribution that can be compared with our low energy effective theory in Eq. (5). According to Eqs. (5) and (4), the energy of a perfectly circular interface is just $2\pi r_0 T(1/r_0)$. As we show below, this with Eq. (7) determines $T(\kappa)$ up to third order in κ:

$$T(\kappa) = T_0 + T_1\kappa + T_2\kappa^2 + \mathcal{O}(\kappa^4), \tag{8}$$

where T_0, T_1 and T_2 are constants. Comparing Eqs. (8) and (7) yields

$$T\left(\kappa = \frac{1}{r_0} = \frac{1}{\sqrt{2N}}\right)2\pi\sqrt{2N} = \frac{4\sqrt{N}}{\sqrt{\pi}} - \frac{\sqrt{\frac{1}{N}}}{2\sqrt{\pi}}, \tag{9}$$

which implies $T_1 = 0$ and

$$T_2 = -\frac{T_0}{4} = -\frac{1}{\sqrt{8\pi^3}}, \tag{10}$$

Finally, we can substitute Eq. (10) into Eq. (8) and find, for an arbitrary local curvature κ,

$$T(\kappa) = \frac{4-\kappa^2}{\sqrt{8\pi^3}} + \mathcal{O}(\kappa^4). \tag{11}$$

Note our finding $T_1 = 0$ is not an accident; in fact, all odd terms in the expansion in Eq. (8) are zero because, if we exchange the regions of QH liquid and the vacuum, the sign of the curvature flips, but the surface contribution to the energy remains the same due to the particle-hole symmetry of the two-body Hamiltonian.

C. Effects of the curvature-dependent tension on the low-energy excitation spectrum

We substitute Eq. (8) into Eq. (5) and expand it up to quadratic order in u, which gives

$$\begin{aligned} S = &-\frac{1}{4\pi}\int d^2x \dot{\phi}\phi' + \int d^2x \lambda(\phi' - u) \\ &- \int d^2x \left[T_0 \frac{u'^2}{2} + T_2 \left(\frac{u^2}{r_0^4} + \frac{4uu'' + \frac{3}{2}u'^2}{r_0^2} + u''^2 \right) \right] \\ &+ \int dt\, w \left[\frac{1}{2}\int dx r_0 \left(\frac{u}{r_0} + 1 \right)^2 - \pi r_0^2 \right] + \mathcal{O}(u^3), \end{aligned} \tag{12}$$

where we have dropped unimportant total derivative terms. To obtain the mode spectrum, we first integrate out the λ and then express the action in terms of ϕ in momentum-frequency space:

$$\phi(t,x) \propto \int d\omega \sum_k \exp(-i\omega t + ikx)\phi_{k\omega}, \tag{13}$$

where the discrete momentum (due to the finite size of QHD) is $k = n/r_0$ with $n \in \mathbb{Z}$. The action in momentum-frequency space is thus

$$\begin{aligned} S = &\int_\omega \sum_k |\phi|^2 k \left[\frac{\omega}{4\pi} - T_0 \frac{k^3}{2} - T_2 \left(\frac{k}{r_0^4} - \frac{5}{2r_0^2}k^3 + k^5 \right) \right] \\ &+ \int dt\, w \left[\frac{1}{2}\int dx r_0 \left(\frac{u}{r_0} + 1 \right)^2 - \pi r_0^2 \right]. \end{aligned} \tag{14}$$

We can neglect terms of the order of $1/r_0$ and higher for sufficiently large N. The quadratic action is thus (back to space-time domain)

$$\begin{aligned} S = &-\frac{1}{4\pi}\int d^2x \dot{\phi}\phi' - \int d^2x \left(T_0 \frac{u'^2}{2} + T_2 u''^2 \right) \\ &+ \int d^2x \lambda(\phi' - u) + \mathcal{O}(u^3) + \mathcal{O}\left(\frac{1}{r_0} \right). \end{aligned} \tag{15}$$

Comparing with Eq. (1), we find a correction due to T_2 in this limit. The chiral mode dispersion relation can be easily extracted:

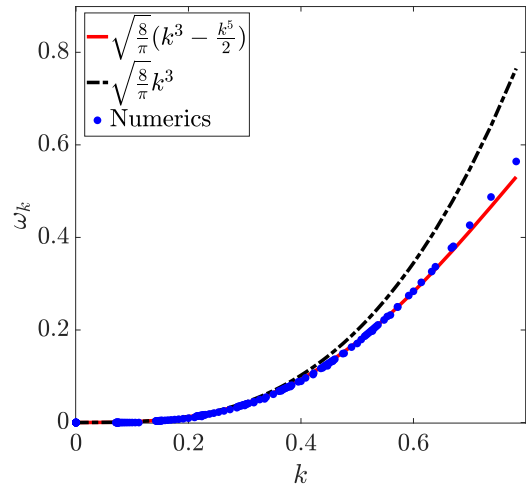

FIG. 2. Comparison between the bosonic spectrum (with and without curvature corrections) and numerical results of Ref. [10], for the interface between a $\nu = 1$ fermionic integer quantum Hall liquid and the vacuum. The agreement is much better when the curvature effects are included (red solid line) than that without the curvature effects (black dashed line).

$$\omega = \sqrt{\frac{8}{\pi}}\left(k^3 - \frac{k^5}{2} \right), \tag{16}$$

which compares extremely well with numerical results (see Fig. 2); the agreement is significantly better than the spectrum without considering the curvature effect. We note that, if we had included in Eq. (8) higher-order terms, their contribution would vanish at the present level of approximation (i.e., keeping quadratic terms only and taking the $r_0 \to \infty$ limit). This is clearer if we just express the curvature in the linear regime as

$$\kappa = u'' + \mathcal{O}(u^2) + \mathcal{O}\left(\frac{1}{r_0} \right). \tag{17}$$

This means the spectrum Eq. (16) is exact in the large N or r_0 limit; any further deviation with numerics (see Fig. 2) must be accounted for by nonlinearity (or interaction among the bosons) and/or finite-sized effect.

D. General form of string action

Our discussion thus far has assumed the deviation of the interface from its ground state (circular) configuration is (still) small. As a result, our action in Eq. (5) is an expansion of the deviation parameterized by u. However, since there is no confinement potential, such a deviation is unbounded in the present case of the interface (and unlike the edge), rendering u hard to define, let alone using it for expansion in general [see Fig. 1(b)]. In the general case, we should instead work directly with the world sheet of the interface $\mathbf{X}(t,\sigma)$, where σ parameterizes a point on the interface [see Fig. 1(b)]. The general form of the interface action is a functional of the world sheet $\mathbf{X}(t,\sigma)$ and is a sum of a topological

(or Wess-Zumino-like) term and an energy term:

$$S = S_{\text{top}}[\mathbf{X}(t,\sigma)] - \int dt E[\mathbf{X}(t,\sigma)], \tag{18}$$

where, based on our analysis above,

$$E[\mathbf{X}(t,\sigma)] = \int d\sigma T[K(\sigma)] \left| \frac{\partial \mathbf{X}}{\partial \sigma} \right|. \tag{19}$$

The derivation of $S_{\text{top}}[\mathbf{X}(t,\sigma)]$ is subtler and will be the subject of Sec. IV. What should be clear at this point is the action in Eq. (18) takes the form of some kind of (bosonic) string theory [14].

III. PFAFFIAN-VACUUM INTERFACE

In this section, we consider a more complicated and interesting interface: That between a Moore-Read (Pfaffian) state formed by bosons of filling factor $\nu = 1$ with a trivial vacuum. Here, again, the $\nu = 1$ boson droplet is formed spontaneously by an attractive V_0 pseudopotential, supplemented by an infinitely strong three-body interaction that prevents the bosons from collapsing, which also makes the Moore-Read (Pfaffian) wave function the exact ground state [10]. The edge theory of the regular Moore-Read state consists of a chiral Majorana mode and a chiral bosonic mode which are decoupled, and both have linear dispersions. In the present interface case, however, the dispersion of the bosonic mode was found to have a similar form as in the previous section, while the dispersion of the Majorana mode remains linear [10]. We can still expect them to decouple since, by a simple dimensional analysis (or power counting), there is no relevant interaction term between the bosonic and fermionic modes. While we cannot extract parameters from microscopics in this case as in the previous section, we expect the curvature effects to be present and affect the spectrum of the bosonic mode, which we now analyze.

We write the effective edge theory for such a system as

$$S = \int d^2x (i\psi\dot{\psi} - v i\psi\psi') + S_B, \tag{20}$$

where S_B is the action given in Eq. (5), with different coefficients for T_0, etc., which will be estimated by fitting to numerical data, Fig. 3. Accordingly, we find the dispersion relation for pure bosonic excitations to be

$$\omega \approxeq 0.2k^3 - 0.04k^5, \tag{21}$$

from which we extract $T_0 \approxeq 0.031$ and $T_2 \approxeq -0.0031$. The fermionic excitation dispersion relation remains linear with $v \approxeq 0.98$, all in units of $|V_0|$. Since there is no particle-hole symmetry for bosons, T_1 is not expected to be zero in this case. However, it does not enter the quadratic action in Eq. (12), as a result of which it has no effect on the dispersion. It does, however, have effects on the action beyond the quadratic level, which gives rise to boson-boson interaction.

Due to the presence of the fermionic mode, the interface here is more like a superstring [14], although there is no supersymmetry here. It has been shown [15], however, the Moore-Read (Pfaffian) edge can be made supersymmetric, and this supersymmetry can even be broken spontaneously, resulting in a Goldstino mode at low energy. It is thus worthwhile to explore the analogy between the dynamics of this interface and superstring theory in future work.

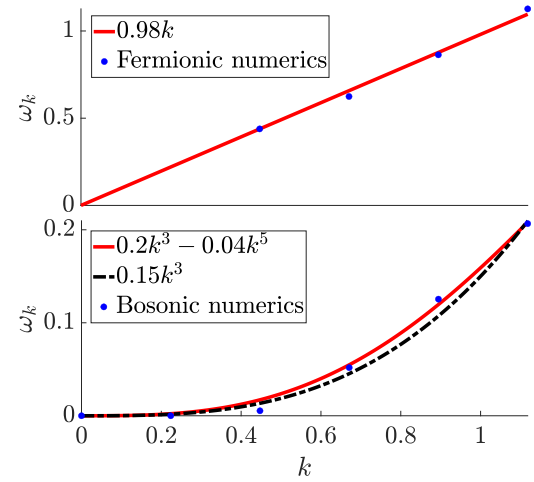

FIG. 3. Same as Fig. 2, for the Pfaffian-vacuum interface with $N = 10$ bosons. The influence of the curvature corrections here is not as drastic for the bosonic mode as in Fig. 2, but we still see better agreement with the numerical result. The fermionic mode spectrum (included here for reference) is not affected.

IV. TOPOLOGICAL TERM OF THE STRING THEORY VIA CS THEORY WITH FLUCTUATING BOUNDARY

So far, we have treated interfaces as string-like objects such that the energy of the system is a functional of its length and local curvature. We showed that this identification enabled us to derive an effective low-energy interface theory that gives a low-energy dispersion relation that agrees with the numerical calculations of the microscopic models extremely well. We have achieved this success by combining the energy functional with a topological term in the action that was derived earlier for edge theory which, as we discussed in Sec. II D, is only appropriate for small fluctuations of the interface. In this section, we attempt to derive the topological action that is appropriate for large fluctuations, for the simplest case of interface between the fermionic $\nu = 1$ QHD and the vacuum.

Let us recall how the topological term in the edge theory, namely, the first term of Eq. (1), was obtained [9]. One starts with the bulk CS theory and places it on a two-dimensional (2D) manifold with a fixed (or time-independent) boundary. One finds that the CS theory is not gauge invariant, as a gauge transformation generates an additional boundary term to the action. This indicates there must be boundary degrees of freedom not captured by the bulk CS theory, which are precisely the edge states. To describe such edge states, one imposes a certain gauge fixing condition at the boundary, which turns a certain gauge degree of freedom there to a physical degree of freedom, whose dynamics is actually sensitive to the gauge fixing condition. One thus has to carefully choose the gauge fixing condition (in an *ad hoc* way) to get the correct edge dynamics.

The situation we are facing here has three important differences: (i) The interface, as parameterized by $\mathbf{X}(t,\sigma)$, is fluctuating and time dependent. (ii) $\mathbf{X}(t,\sigma)$ is the explicit

physical boundary degree of freedom, so no new degree of freedom needs to be introduced. (iii) Perhaps most importantly, the dynamics of $\mathbf{X}(t,\sigma)$ comes from the energy functional $E[\mathbf{X}(t,\sigma)]$ in Eq. (18). We thus should not get an additional dynamical term from gauge-fixing the CS theory. As we show below, these allow us to derive the appropriate topological term in the action in Eq. (18).

To proceed, we go back to the bulk CS theory and now explicitly assume a kinematically time-dependent boundary and discuss its gauge properties. The theory is given as

$$S_{\rm CS} = \int_{-\infty}^{\infty} dt \int_{D(t)} d^2x \times \left[-\frac{1}{4\pi}\epsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda + \frac{1}{2\pi}\epsilon^{\mu\nu\lambda} A_\mu \partial_\nu a_\lambda \right], \quad (22)$$

where A is the background gauge field, and D is the time-dependent region in which the droplet lives. Under the gauge transformation $a_\mu \to a_\mu + \partial_\mu \Lambda$, the total change of the action is

$$\begin{aligned}\Delta S_{\rm CS} &= \int_R dt \int_{D(t)} d^2x \epsilon^{ij} \partial_t (\Lambda \partial_i a_j) \\ &\quad + \int_R dt \int_{\partial D(t)} dx^j \Lambda(-\partial_t a_j + \partial_j a_t) \\ &= \int_R dt \int_{\partial D(t)} [dx^j(-\partial_t a_j + \partial_j a_t)\Lambda \\ &\quad - dx^\ell v^k \epsilon_{k\ell}\epsilon^{ij}(\Lambda \partial_i a_j)], \end{aligned} \quad (23)$$

where we have used the Reynolds-Leibniz rule for differentiation under the integral [16], R is $[-\infty,\infty]$, and $\mathbf{v}(\mathbf{X},t)$ is the velocity vector of the boundary at point $\mathbf{X}$. Note, in earlier work considering a fixed boundary, $\mathbf{v}=0$ and only the first term above is present, and it is the existence of such a boundary term that ruins the gauge invariance of the CS theory in manifolds with boundaries. We note, however, in the present case, we can make $\Delta S_{\rm CS}$ zero and hence the theory gauge invariant, if we impose the following local boundary condition:

$$J_n(\mathbf{X},t) = J_0 v_n(\mathbf{X},t), \quad (24)$$

where we used the fact that the bulk current [9]:

$$J^\mu = \frac{1}{2\pi}\epsilon^{\mu\nu\lambda}\partial_\nu a_\lambda, \quad (25)$$

and the subscript n stands for the normal component of a (spatial) vector at the boundary point $\mathbf{X}$. Specifically, $J_0 = \frac{1}{2\pi}$ is the density of the $\nu=1$ QH liquid which is a constant when the background magnetic field $B = \epsilon^{ij}\partial_i A_j$ is uniform, although everything we say applies to nonuniform B as well.

The boundary condition [Eq. (24)] has a very clear physical meaning: When there is a bulk current J_n flowing toward the boundary, the boundary must recede with speed v_n to accommodate the excess charge. As a result, charge conservation is preserved, hence gauge invariant. This contrasts with the usual approach to the edge theory which assumes the boundary is fixed; as a result, charge would not be conserved (hence lacking gauge invariance, or charge conservation becomes anomalous at the boundary) without the additional degree of freedom u introduced by hand. It should be clear from Fig. 1 that u is nothing but the fluctuation of the boundary. Our discussion above thus provides a perspective of the traditional edge theory. A bonus of Eq. (24) is it also enforces area conservation (which is nothing but charge conservation for an incompressible liquid).

The topological term of the full string theory $S_{\rm top}$ of Eq. (18) is nothing but the CS action in Eq. (22) subject to the constraint, Eq. (24), which is an (implicit) functional of $\mathbf{X}(t,\sigma)$. In other words, $S_{\rm top}$ can be obtained from integrating over the gauge fields a in Eq. (22), subject to the constraints of Eqs. (24) and (25).

Armed with Eq. (24) which couples the boundary (interface) with the bulk CS field a, we are ready to derive the topological action $S_{\rm top}$ in Eq. (18). To do that, we adopt the gauge-fixing condition $a_0(\mathbf{X})=0$ at the interface, which is known not to generate (additional) dynamical terms [17]. Following the standard procedure, we introduce a scalar field ϕ such that

$$a_j = A_j + \partial_j \phi \quad (26)$$

in the bulk, which solves the bulk constraint $\epsilon^{ij}\partial_i a_j = \epsilon^{ij}\partial_i A_j$ obtained from integrating over a_0. With Eq. (26), $S_{\rm CS}$ reduces to a boundary term:

$$S_{\rm top} = -\frac{1}{4\pi}\int dt \int d\sigma \dot{\phi}\phi', \quad (27)$$

which is essentially the first term of Eq. (1) but with arbitrary parametrization along the boundary $\mathbf{X}(t,\sigma)$, and now $\phi' = \frac{\partial\phi}{\partial\sigma}$, so $S_{\rm top}$ as well as the entire action in Eq. (18) is explicitly invariant under reparameterization. Most crucially, however, ϕ is constrained by Eq. (24), via the identification:

$$J_n(\mathbf{X},t) = \frac{1}{2\pi}\left(\frac{\partial\dot{\phi}}{\partial\sigma}\right)\Big/\left|\frac{\partial\mathbf{X}}{\partial\sigma}\right|. \quad (28)$$

As a result, $S_{\rm top}$ is an (implicit) functional of $\mathbf{X}(t,\sigma)$.

V. CONCLUDING REMARKS

In this paper, we have studied string-like theories for free interfaces separating QHDs from the vacuum. The two cases we studied correspond to bosonic strings and superstrings, respectively, in terms of the degrees of freedom associated with their interfaces. While they may seem like the usual QH edges, we demonstrated their dynamics is very different when such interfaces are not pinned by confining potentials and requires string-like theories to describe. We derived such a theory for the case of the bosonic string and leave such a derivation for the superstring to future work.

Physically, more interesting cases are interfaces between different QH liquids. A particularly interesting example is the $\nu=\frac{5}{2}$ QH liquid, where multiple energetically competitive states are present and may form various interfaces (see

Ref. [18] for a recent review). The methods we developed in this paper can be generalized to such interfaces as well. The energy term in Eq. (18) should be present in the generic case. The topological term, on the other hand, needs to be derived by combining the bulk CS-like theories on both sides of the interface, supplemented by appropriate boundary conditions like Eq. (24).

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