Non-Abelian bosonization of topological insulators and superconductors

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Applying the method of [Nucl. Phys. B **972**, 115565 (2021)], which bosonizes *massless* relativistic free fermions, we derive the (non-Abelian) bosonized theory for free fermion topological insulators and superconductors that have, in addition to the U(1) charge, time reversal and charge conjugation symmetries and flavor symmetries. For the case we consider, the flavor symmetries render the topological classification $\mathbb{Z}$. The results are nonlinear σ models with the topological θ term. In addition, we present the theory of a class of bosonic symmetry-protected topological states, whose boundaries are critical spin liquids.

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I. INTRODUCTION

The subject of symmetry-protected topological states (SPTs) is a new frontier in condensed matter physics. SPTs can be divided into two types: the bosonic ones [1] and the fermionic ones [2–4]. A well-known example of the bosonic SPT is the spin 1 chain [5], whose boundary harbors spin 1/2's. This bosonic SPT has been realized experimentally [6,7]. The fermionic SPTs, namely topological insulators and superconductors, are realized by a wide class of quantum materials. Like the bosonic SPT, as long as the protection symmetry is not broken, the boundary of these materials harbors gapless fermion excitations.

Theoretically, the bosonic and fermionic SPTs are described very differently. Nonlinear σ (NLσ) models with the topological θ term are used to describe the bosonic SPTs. In contrast, prototype free fermion SPTs are described by the massive Dirac or Majorana theory. In Ref. [8], the present authors were able to (non-Abelian) bosonize the *massless* free fermions living on the boundary of a class of $\mathbb{Z}$-classified free fermion SPTs. These massless fermions can also be the low-energy quasiparticles of a lattice fermion problem. There the protection symmetry of the SPTs is emergent.

In this paper, we apply the method developed in Ref. [8] to (non-Abelian) bosonize the aforementioned topological insulators and superconductors. One might wonder how a boson theory can capture the fermion excitations. The Bose field in this paper corresponds to the particle-hole excitations (in case of Majorana fermion the hole is the same as the particle) of the fermions. The fermion excitations, as long as they are local, correspond to solitons in the Bose field [8]. However, there is the limitation that we have not been able to write the fermion operator in terms of the boson field. However, this should not impose too strong a constraint because in the Hamiltonian (or the action) only fermion bilinear can occur.

The bosonization theories are NLσ models with the topological θ term. The required fermion symmetries and the bosonized nonlinear σ model are summarized in Tables I and V of the Supplemental Material (SM) [9]. Making an analogy with the spin chains, the topologically trivial fermion SPT is analogous to the spin 0 chain and the topologically nontrivial fermion SPT is analogous to the spin 1 chain. Similar to spin chains, under open boundary conditions the action of the gapless boundary excitations are NLσ models with the Wess-Zumino-Witten term. Of course, here the WZW term manifests the Berry phase of massless fermions instead of spin 1/2's.

In addition to the above, we also present a class of bosonic SPTs, whose boundaries are critical spin liquids. The best-known example of critical spin liquid is the conformal field theory of the antiferromagnetic Heisenberg chain, namely, the $SU(2)$ level-1 WZW theory, or the $O(4)$ NLσ model with WZW term. If we view this critical spin liquid as being protected by the emergent $SU(2)_L \times SU(2)_R$ [or $O(4)$] symmetries, it can be viewed as the boundary theory of a bosonic SPT whose action is a NLσ model with the topological θ terms [10,11]. In two spatial dimensions the $O(5)$ NLσ model with the WZW term is a conformal field theory, which is alleged to describe the "deconfined quantum critical point" [12,13]. If we regard its criticality as being protected by the $O(5)$ symmetry, it too can be realized on the boundary of a bosonic SPT whose action is the $O(5)$ NLσ model with the topological θ term. Thus, after the bosonization, the fermion and bosonic SPTs, and their boundary theories, are unified.

II. TARGETS OF BOSONIZATION: THE TOPOLOGICAL INSULATORS AND SUPERCONDUCTORS

The topological insulators and superconductors we aim to bosonize all have *nonchiral* boundary modes, besides the Q, T, C symmetries.[1]

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[1]Q, T, C stands for charge conservation, time reversal, and charge conjugation considered in the "tenfold way" classification [2,3], where there are flavor symmetries. Moreover, we require the flavor number to be sufficiently large (see later) so that the homotopy group

The Hamiltonian of the fermion SPTs under consideration has the following form:

$$H = \int d^d x\, \psi^\dagger(\boldsymbol{x})\left[-i\sum_{i=1}^{d}\Gamma_i\partial_i - m_0 M_B\right]\psi(\boldsymbol{x}). \quad (1)$$

Here ψ is the Majorana or Dirac fermion field depending on whether we are talking about topological superconductors or insulators. In Eq. (1) and the rest of the paper, we shall use d to denote the space dimension and D to denote the space-time dimensions. Including n flavors, ψ has $2n, 4n, 4n$ components in $d = 1, 2, 3$. In Table I of the SM [9] we list the Γ_i, M_B, and the protection symmetry of these SPTs. In addition, we restrict the number of fermion flavor number, n, to be greater or equal to n_c, so that the WZW term in the bosonized boundary NLσ (see later) has "stabilized" [8]. The values of n_c are given in Table II of the SM [9].

III. BOUNDARY GAPLESS FERMION MODES

The boundary of the fermion SPTs in Eq. (1) can be modeled by a domain wall where m_0 changes sign. The Hamiltonian is given by Eq. (1) except m_0 is replaced with $m_0 f(x_d)$, where x_d is, say, the last spatial coordinate. The function $f(x)$ is given by, say, $\tanh(x/\lambda)$, where λ is the width of the domain wall. The Hamiltonian of the gapless fermions on the domain wall is given by [14]

$$H_{\rm dw} = \int d^{d-1}x\, \chi^\dagger(\boldsymbol{x})\left[-i\sum_{i=1}^{d-1}\gamma_i\partial_i\right]\chi(\boldsymbol{x}). \quad (2)$$

In the above $\gamma_i = \mathcal{P}\Gamma_i\mathcal{P}$, $\chi = \mathcal{P}\psi$, where the projection operator is given by

$$\mathcal{P} = (i\Gamma_d M_B + I)/2.$$

The boundary symmetry generators are equal to $\mathcal{P}$ (bulk symmetry generators) $\mathcal{P}$. Because $(i\Gamma_d M_B)^2 = I$ and $\mathrm{Tr}[i\Gamma_d M_B] = 0$, the number of components in χ is half of that in ψ. The boundary γ matrices and symmetry generators are summarized in Table III of the SM [9].

IV. BOSONIZED EQ. (2)

Applying the method in Ref. [8], the non-Abelian bosonized action of Eq. (2) is the NLσ model with the level-1 WZW term. The action and the OPM of such NLσ models are summarized in Table IV of the SM [9].

Here the boundary space-time manifold is S^D, and $\mathcal{D}^{D+1}$ is a $D+1$ dimensional disk such that $\partial(\mathcal{D}^{D+1}) = S^D$. The order parameter $Q^{\mathbb{C},\mathbb{R}}$ appearing in the NLσ model are matrices in the OPM. In the following we explain the meaning of the WZW term.

corresponding to the topological terms has stabilized. Due to the flavor symmetries the topological classification of the fermion model we consider is always $\mathbb{Z}$. In reality most systems have flavor symmetries. For example, graphene has spin and valley flavors, bilayer graphene has spin, valley, and layer flavors, and in many condensed matter systems there is often the orbital flavor.

For example, the WZW term for the $D = (2+1)$ dimensional boundary of a topological insulator is given by

$$W_{\rm WZW}[\tilde{Q}^{\mathbb{C}}] = \frac{2\pi i}{256\pi^2}\int_{\mathcal{D}^4} \mathrm{tr}[\tilde{Q}^{\mathbb{C}}\,(d\tilde{Q}^{\mathbb{C}})^4]. \quad (3)$$

Here $Q^{\mathbb{C}}, \tilde{Q}^{\mathbb{C}}$ are matrices in $\frac{U(n)}{U(n/2)\times U(n/2)}$. In Eq. (3) $\tilde{Q}^{\mathbb{C}}(\tau, x, y, u)$ is a smooth one-parameter extension of the space-time configuration $Q^{\mathbb{C}}(\tau, x, y)$. At $u = 0$, $\tilde{Q}^{\mathbb{C}}(\tau, x, y, 0)$ is a constant matrix independent of τ, x, y.[2] At $u = 1$, $\tilde{Q}^{\mathbb{C}}(\tau, x, y, 1) = Q^{\mathbb{C}}(\tau, x, y)$. Moreover, in Eq. (3)

$$\int_{\mathcal{D}^4} \to \int_0^1 du \int_{S^3} d^3x,$$
$$\tilde{Q}^{\mathbb{C}}\,(d\tilde{Q}^{\mathbb{C}})^4 = \epsilon^{\mu\nu\rho\lambda}\tilde{Q}^{\mathbb{C}}\partial_\mu\tilde{Q}^{\mathbb{C}}\partial_\nu\tilde{Q}^{\mathbb{C}}\partial_\rho\tilde{Q}^{\mathbb{C}}\partial_\lambda\tilde{Q}^{\mathbb{C}}, \quad (4)$$

where $\mu, \nu, \rho, \lambda \in \{\tau, x, y, u\}$. Physically the WZW term is the accumulated Berry phase during the adiabatic evolution (as a function of u) from $\tilde{Q}^{\mathbb{C}}(\tau, x, y, 0)$ to $\tilde{Q}^{\mathbb{C}}(\tau, x, y, 1)$. It can be shown that $\exp(-W_{WZW})$ is independent of the values of $\tilde{Q}^{\mathbb{C}}$ for $u < 1$.

To show $\exp(-W_{\rm WZW})$ only depends on the value of $\tilde{Q}^{\mathbb{C}}(\tau, x, y, u = 1)$ we consider two different $\tilde{Q}^{\mathbb{C}}$, namely, $\tilde{Q}^{\mathbb{C}}_1(\tau, x, y, u)$ and $\tilde{Q}^{\mathbb{C}}_2(\tau, x, y, u)$, with

$$\tilde{Q}^{\mathbb{C}}_1(\tau, x, y, u = 1) = \tilde{Q}^{\mathbb{C}}_2(\tau, x, y, u = 1).$$

Since the WZW term is purely imaginary, the relative phase factor $\exp(-W_{\rm WZW})$ associated with $\tilde{Q}^{\mathbb{C}}_{1,2}$ is

$$\exp\left(-W_{\rm WZW}[\tilde{Q}^{\mathbb{C}}_1] + W_{\rm WZW}[\tilde{Q}^{\mathbb{C}}_2]\right). \quad (5)$$

In addition, because the WZW term involves ∂_u [see Eq. (4)], negating the sign of the WZW term can be accomplished by reversing the integration limit in u. Consequently we can regard $W_{\rm WZW}[\tilde{Q}^{\mathbb{C}}_1] - W_{\rm WZW}[\tilde{Q}^{\mathbb{C}}_2]$ as the integral from $u = 0$ to $u = 1$ back to $u = 0$ (recall that $\tilde{Q}^{\mathbb{C}}_{1,2}$ agree at $u = 1$). This is the WZW term defined on the closed manifold S^4. The condition that the $\exp(-W_{\rm WZW})$ is well defined requires Eq. (5) to be equal to 1, which means $W_{\rm WZW}[\tilde{Q}^{\mathbb{C}}_1] - W_{\rm WZW}[\tilde{Q}^{\mathbb{C}}_2]$ is an integer multiple of $2\pi i$. The reason the preceding statement holds is because the value of $W_{\rm WZW}/2\pi i$ on S^4 is precisely the topological invariant, namely, the wrapping number,

$$\mathcal{Q} = \frac{1}{256\pi^2}\int_{S^4} \mathrm{tr}[\tilde{Q}^{\mathbb{C}}\,(d\tilde{Q}^{\mathbb{C}})^4], \quad (6)$$

of the $S^4 \to$ OPM map. For the OPM given in Table IV of the SM [9] $\pi_4(\mathrm{OPM}) = \mathbb{Z}$, i.e., $\mathcal{Q} =$ integer.

V. BOSONIZED EQ. (1)

In Table V of the SM [9] we summarize the non-Abelian bosonized theory of Eq. (1) under closed boundary condition (S^D), where $\mathcal{Q}$ is precisely the topological invariant in Eq. (6).

[2]For such an extension to exist it requires the homotopy group of the space-time to the OPM map to be trivial, namely $\pi_D(\mathrm{OPM}) = 0$.

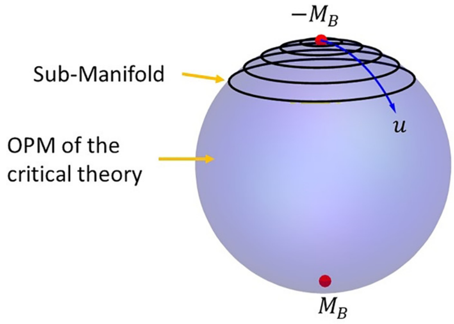

FIG. 1. Light blue sphere is a schematic representation of the OPM of the $m_0 = 0$ theory in Eq. (1). The black circles represent the one-parameter family of submanifolds in OPM used to derive the θ term in Table V of the SM [9] from the WZW term in Table VI. The red points represent $\mp M_B$ and the blue arrowed arc indicates the direction of the increasing u.

In Sec. IV we have seen that the WZW term in Table IV of the SM [9] becomes the topological term (the "θ term") in Table V of the SM [9] when the space-time manifold is closed in the u direction. If we interpret u as a spatial coordinate, since the NLσ model in Table IV of the SM [9] is the bosonized action of Eq. (2), it suggests the the NLσ model in Table V of the SM [9] is the bosonized action of Eq. (1). Conceptually this is the quantum wire construction [15–18], where u plays the role of the interwire coordinate. Depending on how the u loops are closed one may end with a topological trivial or nontrivial SPT.

In this section we consider closed boundary conditions. We start from the gapless critical point of the SPT phase transition by setting $m_0 = 0$ in Eq. (1) and apply the method of Ref. [8] to bosonize such a gapless theory. The results are given in Table VI of the SM [9]. To derive the boson theory for the gapped SPT phases we turn on mass terms which (1) gap the critical point and (2) reduce the symmetry of the critical point to that of the SPT phases.

In Fig. 1 we schematically represent the OPM of the $m_0 = 0$ theory by the light blue sphere, where $\pm M_B$ corresponds to the two red points. If the order parameter fluctuates uniformly in the light blue sphere, the full symmetry of the $m_0 = 0$ fermion theory will be recovered. This symmetry is larger than the symmetry in Table I of the SM [9] for $m_0 \neq 0$. To reduce the symmetry, we select a family of submanifolds in the OPM of the $m_0 = 0$ theory, so that after the order parameter uniformly fluctuates in the submanifold, only the symmetries in Table I of the SM [9] are restored. These submanifolds are parametrized by u and have the properties that (i) at $u = 0$ and $u = 1$ the submanifold shrinks to $\mp M_B$, (ii) the fermions remain gapped in the submanifolds at any value of u, and (iii) at any value of u, after the order parameter uniformly fluctuates, only the symmetry in Table I of the SM [9] is restored. These submanifolds are shown schematically as the black circles in Fig. 1, while the direction of increasing u is shown as the blue arrowed arc.

Mathematically these submanifolds are given in the rows labeled as "Sub-OPM" in Table VII of the SM [9]. The fermion mass in the submanifolds is parametrized by $M(u)$ and the $Q^{\mathbb{R},\mathbb{C}}$ corresponding to $M(u)$ are given in the rows labeled by "$Q^{\mathbb{R},\mathbb{C}}$" in Table VII of the SM [9]. Note that the submanifolds formed by $Q^{\mathbb{R},\mathbb{C}}$ agree with the OPM in Table V of the SM [9]. Substituting the $Q^{\mathbb{R},\mathbb{C}}$ in Table VII of the SM [9] into the W_{WZW} in Table VI of the SM [9] computes the accumulated Berry's phase as u increases from 0 to 1. It is straightforward to show that the result is $i\theta\mathcal{Q}$, where $\mathcal{Q}$ is given in Table V of the SM [9], and the value of θ depends on the end value of u. Tuning the end value of u from 0 to 1 changes θ from 0 to 2π. The above derivation of the θ term from the WZW term is analogous to that carried out in Ref. [19], where the authors derive the O(3) NLσ with $\theta = \pi$ term from the O(4) NLσ model with WZW term in $D = 1 + 1$.

VI. CRITICAL SPIN LIQUIDS AS THE BOUNDARY OF SPIN SPTS

In Secs. 15 and 16 of Ref. [8] it is shown that the $d = 1$ $O(4)$ and $d = 2$ $O(5)$ NLσ models with level-1 WZW term can be derived from the "π-flux" [20] "spinon" mean-field theory after the "charge"-$SU(2)$ confinement. The $SU(2)$ confinement is to enforce the no double occupation constraint of Mott insulators [21], so that the low energy degrees of freedom are *spins* rather than spinons. This confinement requires the low energy spinon mass to be $SU(2)$ singlet, which selects the four masses in the $d = 1$ $O(4)$ NLσ model and the five masses in the $d = 2$ $O(5)$ NLσ model, respectively.

The gamma matrices and the symmetry generators of the π flux phase spinons are given in the column of Table III of the SM [9] labeled by "boundary of topological superconductors." [The reason that we consider the boundary of topological superconductors rather than the "boundary of topological insulators" is that the charge $SU(2)$ gauge fields break the spinon conservation [8].] The relevant flavor number is $n = 4$ for $D = 1 + 1$ and $n = 8$ for $D = 2 + 1$, respectively (note these values are greater than the respective n_c in Table II of the SM [9]).

The "critical spin liquid" in $D = 0 + 1$ corresponds to free spin 1/2's. This was not discussed in Ref. [8], where the flavor number is $n = 4$ (here $n_c = 4$ also). Hence there are four Majorana fermion zero modes (or two complex fermion zero modes—one for spin up and the other for spin down). Under the charge-$SU(2)$ confinement only one of the spin zero modes is occupied. This gives rise to a spin 1/2. The coherent state path integral for such spin 1/2 is the $O(3)$ NLσ model with the WZW term.

The criticality of the $d = 0, 1, 2$ spin liquids is protected by the $O(3)$, $O(4)$, and $O(5)$ symmetries, respectively. The OPM and the WZW term of the NLσ models associated with these spin liquids are given in the second column of Table VIII of the SM [9]. These critical spin liquids are realized at the boundary of $d = 1, 2, 3$ bosonic SPTs whose NLσ models and the associated θ terms are given in the third column of Table VIII of the SM [9].

VII. CONCLUSION

We have bosonized a class of $\mathbb{Z}$-classified fermion SPTs. The results are NLσ models with the topological θ term.

In particular, the trivial SPT corresponds to $\theta = 0$ and the nontrivial SPT corresponds to $\theta = 2\pi$. Tuning the θ value from 0 to 2π triggers the SPT phase transition. Finally, we present the bosonic SPTs whose boundaries are critical spin liquids. It is satisfying that after bosonization the field theories for the fermion and bosonic SPTs, and their boundaries, are unified.

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