Hohenberg-Mermin-Wagner-type theorems and dipole symmetry

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We study the possibility of spontaneous symmetry breaking in systems with both charge and dipole symmetries. For d-dimensional systems at a positive temperature, we show that charge symmetry cannot be spontaneously broken for $d \leqslant 4$, while dipole symmetry cannot be spontaneously broken for $d \leqslant 2$. For $T = 0$, we show that charge symmetry cannot be spontaneously broken for $d \leqslant 2$ if the compressibility is finite. We also show that continuum systems with a dipole symmetry have infinite inertial mass density.

I. INTRODUCTION

The concepts of spontaneous symmetry breaking and long-range order play a central role in the theory of phase transitions. Peierls [1,2] and Landau [3] were the first to realize that in sufficiently low dimensions thermal fluctuations of the order parameter can preclude genuine long-range order. The impossibility of long-range order in one and two dimensions at positive temperatures has been proved by Hohenberg [4] in the case of superfluids and superconductors and by Mermin and Wagner [5] for (anti)ferromagnets. The relevant symmetries in these cases are either $U(1)$ or $SU(2)$. Analogous theorems for one-dimensional systems at zero temperature were proved by Pitaevskii and Stringari [6]. At $T = 0$ it is quantum fluctuations that destroy the order.

The impossibility of symmetry breaking in a particular system does not guarantee the absence of continuous phase transitions. A textbook example is the Berezinskii-Kosterlitz-Thouless phase transition [7–9]. In this case fluctuations of the order parameter are qualitatively different on the two sides of the transition, even though symmetry is not broken.

The main purpose of this paper is to generalize the above results to $U(1)$-invariant systems which conserve not only the charge but also the dipole moment. The additional symmetry possessed by such models is called the dipole symmetry. A simple noninteracting field theory with a dipole symmetry is the Lifshitz model which has a single real scalar field $\phi(t, \mathbf{x})$ and an action

$$S = \frac{1}{2} \int dt \, d^d x (\dot{\phi}^2 - C(\partial^2 \phi)^2). \qquad (1)$$

Here C is a positive constant and $\partial^2 = \partial_j \partial_j$. This model has both a charge symmetry $\phi \mapsto \phi + a$ and a dipole symmetry $\phi \mapsto \phi + \mathbf{b} \cdot \mathbf{x}$, where $\mathbf{b}$ is a constant vector. An example of

a lattice system with a dipole symmetry dubbed the dipolar Bose-Hubbard model was recently studied by Lake *et al.* [10]. Its Hamiltonian is given by

$$H = \sum_i \left(-t \sum_a \left[b_i^{\dagger 2} b_{i+a} b_{i-a} + b_{i-a}^\dagger b_{i+a}^\dagger b_i^2 \right] \right.$$
$$- t' \sum_{a,b,b \neq a} \left[b_i^\dagger b_{i+a+b}^\dagger b_{i+a} b_{i+b} + b_{i+b}^\dagger b_{i+a}^\dagger b_{i+a+b} b_i \right]$$
$$\left. + \frac{U}{2} n_i(n_i - 1) - \mu n_i \right), \qquad (2)$$

where b_i are boson creation operators and $n_i = b_i^\dagger b_i$, i goes over square lattice sites and a, b over basis vectors. The dipole symmetries are generated by $\sum_j j_a n_j$ where j_a is the a coordinate of the site j. The usual hopping term is forbidden by the dipole symmetry and particles can only hop in pairs. Reference [10] shows that within the mean-field approximation the model has several different phases characterized by spontaneous breaking of dipole and/or charge symmetries.

Our interest in such models stems from a connection between dipole symmetry and systems where the kinetic energy of charged quasiparticles is suppressed relative to the potential interaction energy, leading to strongly correlated physics. Whenever a system is effectively described by weakly interacting quasiparticles, suppression of the kinetic energy is tantamount to the flatness of the relevant band. For systems with strongly interacting quasiparticles, the concept of a flat band is problematic. A possible way out is to notice that an idealized exactly flat band has an additional symmetry of shifts by a constant momentum, or equivalently a Galilean boost symmetry. Invariance of the Hamiltonian under boost can be used as a definition of a "flat band" which does not rely on the weakness of the interaction. Although in the experimentally realized systems the band is never exactly flat (i.e., the commutator of the boost generator and the Hamiltonian

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is never exactly zero), the study of such idealized models is a good starting point to tackle the theoretical challenges of strong interactions in these materials.

To explain the connection to the dipole symmetry, consider a system of particles with mass m and density-density interactions. The generator of a shift in momentum space is given by

$$\mathbf{K} = \int d^d x \left(t \frac{\mathbf{p}(\mathbf{x})}{m} - \mathbf{x}\rho(\mathbf{x}) \right), \qquad (3)$$

where $\mathbf{p}(\mathbf{x})$, $\rho(\mathbf{x})$ are the momentum and particle densities and we divided the boost generator by m since we want it to shift momentum instead of velocity. In the limit $m \to \infty$ the kinetic energy becomes quenched and $\mathbf{K}$ starts to commute with the Hamiltonian, i.e., becomes a true symmetry instead of being part of the spectrum-generating algebra. In this limit, the generator (3) becomes the total dipole moment of the particles.

Dipole moment conservation on top of charge conservation leads to restricted mobility of charged particles but does not obstruct the propagation of neutral bound states. Similar phenomena appear in theories of fractons [11–13] and their connection to dipole symmetry is known [14–16]. Therefore models with dipole symmetries and fractonic behavior can be used as a playground for studying systems with vanishing kinetic energy. One can hope that their exotic features can also manifest themselves in more realistic "flat-band" systems.

The issue of spontaneous breaking of dipole symmetry is related to the question of whether systems that are invariant under boost symmetry in the above sense can nevertheless have quasiparticles with a nonzero dispersion. To see that this might be possible, consider the famous Landau relation between the bare mass m and the quasiparticle mass m^* in a Fermi liquid [17]:

$$\frac{1}{m^*} = \frac{1}{m} - \frac{2 p_f}{(2\pi)^3} \int d\Omega \, f(\theta) \cos\theta. \qquad (4)$$

This relation follows from Galilean invariance. One can see that in the limit $m \to \infty$ (and assuming that the system stays in the Fermi-liquid phase) the effective mass of quasiparticles can be generated by the interaction and remain finite. Since the Fermi surface is not invariant under translations in momentum space, for this mechanism to work the dipole symmetry of the $m = \infty$ model must be spontaneously broken.

Within the Lifshitz model, the effects of quantum and thermal fluctuations on the existence of phases with broken charge or dipole symmetries have been discussed in [18,19] and are reviewed in Sec. II. This type of analysis assumes that the only low-energy degrees of freedom are Goldstone bosons. However, spontaneous breaking of dipole symmetry (or more generally any symmetry that has a nontrivial commutator with translations) does not necessarily lead to a Goldstone mode [20,21]. For example, some Fermi liquids spontaneously break rotational and/or Galilean symmetries but their low-energy excitation spectrum consists of the particle-hole continuum without any bosonic collective modes. A further complication is that the choice of the fields describing the Goldstone modes is not unique. For example, in the traditional approach to crystalline solids, one does not introduce a separate field for the rotational symmetry, even though both

rotational and translational symmetries are broken into a discrete subgroup. But the same low-energy physics can also be described by a field theory with additional fields which play the role of the Goldstone bosons for rotations [22,23]. Thus it is desirable to prove theorems of Hohenberg-Mermin-Wagner (HMW) type without assuming anything about the low-energy degrees of freedom.

In Sec. III, we consider a general d-dimensional lattice system with a finite range Hamiltonian that commutes with both charge and dipole moment. We show that for $T > 0$ the dipole symmetry cannot be broken for $d \leqslant 2$, while the charge symmetry cannot be broken for $d \leqslant 4$. Thus the mere presence of dipole symmetry allows one to strengthen the conclusions of the HMW theorem. At $T = 0$ we show that the charge symmetry cannot be broken for $d \leqslant 2$ if the charge compressibility does not diverge. The theorem does not put any restriction on the dipole symmetry at zero temperature because the dielectric constant diverges for a general system with a dipole symmetry. Our results put on firm ground the analysis of [18,19] based on the Lifshitz model.

In Sec. IV we prove HMW-type theorems for continuum models with a dipole symmetry. Unlike lattice systems, such models have a conserved momentum $\mathbf{P}$, and thus one may also consider a generalized Gibbs ensemble with a nonzero velocity $\mathbf{v}$. We show that the inertial mass density in continuum models with a dipole symmetry is always infinite. This is a kind of converse to the statement that dipole symmetry emerges as the infinite-mass limit of the Galilean symmetry. Section V contains concluding remarks.

II. THE LIFSHITZ MODEL

The Lifshitz model (1) in d spatial dimensions is the simplest field theory which conserves both charge and its dipole moment. The charge symmetry is generated by a Hermitian operator Q satisfying $[Q, \phi] = -i$. The dipole symmetry is generated by a vector-valued Hermitian operator $\mathbf{D}$ satisfying $[\mathbf{D}, \phi(t, \mathbf{x})] = -i\mathbf{x}$. Naively, the Lifshitz model describes a phase where both charge and dipole symmetries are completely broken, with ϕ being the order parameter. Indeed, if $\langle \cdot \rangle$ denotes averaging over a ground state or a thermal state, then we have $\langle [Q, \phi(t, \mathbf{x})] \rangle = -i \neq 0$ and $\langle [\mathbf{D}, \phi(t, \mathbf{x})] \rangle = -i\mathbf{x} \neq 0$. By definition, this means that symmetries generated by both Q and $\mathbf{D}$ are spontaneously broken. But it is well known that quantum and thermal fluctuations may invalidate this conclusion by making ϕ ill-defined. Let us examine this issue in more detail.

For $T = 0$ and $d > 2$ the connected two-point function of ϕ is

$$\langle \phi(0, \mathbf{x})\phi(0, \mathbf{y}) \rangle_c \sim \frac{1}{|\mathbf{x} - \mathbf{y}|^{d-2}}. \qquad (5)$$

This is a well-defined two-point function that satisfies the cluster decomposition. Thus ϕ is a well-defined local field that may serve as an order parameter for both charge and dipole symmetry breaking. In contrast, for $d = 2$ the connected two-point function of ϕ is not well defined, because its spatial Fourier transform

$$\langle \phi(0)_{\mathbf{k}}\phi(0)_{\mathbf{k}} \rangle_c \sim \frac{1}{k^2} \qquad (6)$$

has a nonintegrable singularity at $\mathbf{k} = 0$.[1] Less formally, the average of $\phi(0, \mathbf{x})$ over a large region of diameter L has a standard deviation that grows like $(\log L)^{1/2}$, indicating that it cannot serve as an order parameter.

On the other hand, the two-point function of the vector-valued operator $\mathbf{u} = \nabla\phi$ has an integrable singularity in momentum space, and the two-point function in coordinate space is well defined and satisfies the cluster decomposition:

$$\langle u_j(0, \mathbf{x})u_l(0, \mathbf{y})\rangle_c \sim \frac{(x_j - y_j)(x_l - y_l)}{|\mathbf{x} - \mathbf{y}|^4}. \qquad (7)$$

Since $\mathbf{u}$ commutes with Q, it cannot serve as an order parameter for charge symmetry breaking. Neither are there any other local fields whose commutator with Q has a nonzero expectation value. Thus charge symmetry is unbroken in the $d = 2$ Lifshitz model at zero temperature. But since $\langle[D_j, u_l]\rangle = -i\delta_{jl}$, the dipole symmetry is spontaneously broken.

The situation for $d = 1$ is similar. The two-point function of the field ϕ in momentum space again has a nonintegrable singularity (6). The standard deviation of the spatial average of ϕ grows like $\sqrt{L}$ for large L. Since one lacks any order parameter for the charge symmetry breaking, it remains unbroken. But the two-point function of the field $u = \partial_x\phi$ is constant in momentum space and thus we have

$$\langle u(0, x)u(0, y)\rangle_c \sim \delta(x - y). \qquad (8)$$

This is a well-defined (although somewhat unusual) answer, and u can serve as an order parameter for the dipole symmetry breaking. Thus the dipole symmetry in the $d = 1$ Lifshitz model is broken at $T = 0$, as noted in [19].[2]

For $T > 0$ the static two-point function of the Lifshitz field ϕ in momentum space is

$$\langle \phi(0)_{\mathbf{k}}\phi(0)_{\mathbf{k}}\rangle_c \sim \frac{1}{k^4}. \qquad (9)$$

For $d > 4$ it is integrable, in which case the connected thermal two-point function of ϕ in coordinate space is well defined. Since ϕ is well defined and shifts under charge symmetry, the charge symmetry (and therefore also the dipole symmetry) are spontaneously broken for $d > 4$. For $d \leqslant 4$, the field ϕ is not well defined, and there are no other order parameters for charge symmetry breaking. Thus the charge symmetry is not broken for $d \leqslant 4$. As for dipole symmetry, for $d > 2$ it is spontaneously broken, while for $d \leqslant 2$ it is also unbroken. Indeed, for $d > 2$ we can take $u_j = \partial_j\phi$ as an order parameter, since its connected thermal two-point function has an integrable singularity $\sim 1/k^2$ in momentum space and decays as $1/x^{d-2}$ in coordinate space.

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[1]If one cuts off the integral at $k = \Lambda_{\text{IR}}$, one finds a logarithmically growing two-point function.

[2]Spontaneous breaking of a continuous symmetry in a one-dimensional (1D) system might seem surprising, but it does not contradict any known general theorems. Another example is a system of nonrelativistic fermions in 1D which spontaneously breaks boost invariance by forming a Luttinger liquid.

III. LATTICE SYSTEMS

A. Definitions

We will consider a lattice system with discrete translation symmetry and finite-range interactions. More formally, let Λ be a lattice in $\mathbb{R}^d$ with an action of translation group $\mathbb{Z}^d$. The Hilbert space is the product of an infinite number of copies of a finite-dimensional Hilbert space of a single site. Local operators are operators which act nontrivially only on a finite number of sites. We can formally combine all sites in a single unit cell into a single site and reduce the lattice Λ to $\mathbb{Z}^d$ without a loss of generality. If the translational symmetry is broken to a sublattice symmetry (as in antiferromagnets), we can formally combine the resulting unit cell to one cell and again assume that the lattice is $\mathbb{Z}^d$.

The Hamiltonian density operator $H_{\mathbf{n}}$ is assumed to be translationally invariant and finite range. The integer-valued index $\mathbf{n}$ enumerates the lattice $\mathbb{Z}^d$ sites. Moreover, we assume that the system has a charge symmetry and an associated dipole symmetry. Namely, there exist charge density operators $Q_{\mathbf{n}}$ which are finite range, translationally invariant, and satisfy

$$[Q_{\mathbf{n}}, Q_{\mathbf{m}}] = 0, \qquad (10)$$

$$\sum_{\mathbf{n} \in \mathbb{Z}^d} [Q_{\mathbf{n}}, H_{\mathbf{m}}] = 0, \qquad (11)$$

$$\sum_{\mathbf{n} \in \mathbb{Z}^d} n_i[Q_{\mathbf{n}}, H_{\mathbf{m}}] = 0, \quad \text{for all } i = 1, \dots, d, \qquad (12)$$

where n_i is the ith component of the $\mathbf{n}$ in $\mathbb{Z}^d$. The latter condition means that the dipole moment is conserved. We will define the dipole charge density as $D_{\mathbf{n}}^i = n_i Q_{\mathbf{n}}$ and the dipole charge as $D^i = \sum_{\mathbf{n} \in \mathbb{Z}^d} D_{\mathbf{n}}^i$. The operators $Q_{\mathbf{n}}, H_{\mathbf{n}}$ are assumed to be local operators supported on balls of diameter d_{int} centered at $\mathbf{n}$.

We also assume that the state of the system has a clustering property. More precisely, following [24,25], we assume that for any two local operators A, B one has

$$|\langle AT_{\mathbf{n}}(B)\rangle - \langle A\rangle\langle B\rangle| \leqslant \frac{C_{AB}}{|\mathbf{n}|^{\delta_{AB}}} \quad \text{as} \quad |\mathbf{n}| \to \infty, \qquad (13)$$

where $T_{\mathbf{n}}$ is the translation operator and δ_{AB}, C_{AB} are positive numbers. In reality, we only use the clustering property when $A = B$ and A is a local operator which is a candidate for the order parameter. Some sort of clustering is also implicit in [4,5], since it is assumed there that the Fourier transform of a two-point correlator is a well-defined integrable function rather than a distribution.

B. HMW theorem for charge symmetry at a nonzero temperature

We repeat the argument of [24,25] making only small adjustments. Suppose the charge symmetry is spontaneously broken, i.e., there exists a local operator A_0 such that

$$\langle[Q, A_0]\rangle \neq 0, \qquad (14)$$

where brackets denote the thermal average. Without a loss of generality, we may assume that A_0 is Hermitian, has zero average $\langle A_0\rangle = 0$, and is localized around the point $\mathbf{n} = 0$.

We will consider a spatially averaged version of A_0 defined as

$$A_R = \frac{1}{\text{Vol}(B_R^d)} \sum_{|\mathbf{n}| \leqslant R} T_\mathbf{n}(A_0), \qquad (15)$$

where $T_\mathbf{n}$ is the translation operator, R is some number, and $\text{Vol}(B_R^d)$ is the number of lattice sites in a d-dimensional ball of radius R. We assume that the full $\mathbb{Z}^d$ translational symmetry is preserved and thus

$$\langle [Q, A_0] \rangle = \langle [Q, A] \rangle. \qquad (16)$$

By increasing d_int if necessary, we may assume that A_0 is supported on a ball of diameter d_int with a center at 0. Then A_R is supported on a ball of diameter $2R + d_\text{int}$ centered at 0.

Without changing the commutator, we can also replace Q in Eq. (16) with $Q(f) = \sum_{\mathbf{n} \in \mathbb{Z}^d} f(\mathbf{n}) Q_\mathbf{n}$, where $f(\mathbf{n})$ is any compactly supported function equal to 1 whenever $\mathbf{n}$ belongs to a ball of radius $R + \frac{3}{2} d_\text{int}$ centered at 0.

We will use the Bogoliubov inequality valid for any two bounded operators A, B and a thermal (Kubo-Martin-Schwinger) state at a nonzero temperature T:

$$|\langle [A, B] \rangle|^2 \leqslant \frac{1}{2T} \langle [B, [H, B^\dagger]] \rangle \langle A^\dagger A + A A^\dagger \rangle. \qquad (17)$$

We will apply it to $A = A_R$ and $B = Q(f)$.

Estimates

We want to estimate $\langle [Q(f), [Q(f), H]] \rangle$. We can rewrite the commutators as

$$[Q(f), H] = \sum_{\mathbf{m}, \mathbf{k}} f(\mathbf{m})[Q_\mathbf{m}, H_\mathbf{k}]$$

$$= \sum_{\mathbf{m}, \mathbf{k}} (f(\mathbf{m}) - f(\mathbf{k}))[Q_\mathbf{m}, H_\mathbf{k}],$$

$$[Q(f), [Q(f), H]] = \sum_{\mathbf{n}, \mathbf{m}, \mathbf{k}} f(\mathbf{n})(f(\mathbf{m}) - f(\mathbf{k})) \qquad (18)$$

$$\times [Q_\mathbf{n}, [Q_\mathbf{m}, H_\mathbf{k}]]$$

$$= \sum_{\mathbf{n}, \mathbf{m}, \mathbf{k}} (f(\mathbf{n}) - f(\mathbf{k}))(f(\mathbf{m}) - f(\mathbf{k}))$$

$$\times [Q_\mathbf{n}, [Q_\mathbf{m}, H_\mathbf{k}]],$$

where the charge conservation (11) and the ultralocality of the charge (10) were used. The commutator $\langle [Q_\mathbf{n}, [Q_\mathbf{m}, H_\mathbf{k}]] \rangle$ is nonzero only if the supports of $H_\mathbf{k}$, $Q_\mathbf{n}$, and $Q_\mathbf{m}$ overlap. The expectation value can be estimated as

$$|\langle [Q_\mathbf{n}, [Q_\mathbf{m}, H_\mathbf{k}]] \rangle| \leqslant 4 \|Q_0\|^2 \|H_0\|, \qquad (19)$$

where the operator norm is defined as $\|A\| = \max_{v \in \mathcal{H}} \frac{|Av|}{|v|}$. The norm of operators is guaranteed to be finite only if the on-site Hilbert space is finite dimensional. For now, we will assume this and discuss the infinite-dimensional case in Sec. III E.

We can expand the function f in the Taylor series around $\mathbf{k}$ with the reminder in the Lagrange form

$$f(\mathbf{n}) = f(\mathbf{k}) + [(\mathbf{n} - \mathbf{k}) \cdot \nabla] f(\mathbf{k}) + \frac{1}{2}[(\mathbf{n} - \mathbf{k}) \cdot \nabla]^2 f(\xi), \qquad (20)$$

where $(\mathbf{n} - \mathbf{k}) \cdot \nabla$ is derivative in the direction of the vector $\mathbf{n} - \mathbf{k}$ and ξ is a point on the straight interval from $\mathbf{k}$ to $\mathbf{n}$. Due to dipole symmetry linear terms proportional to $[(\mathbf{n} - \mathbf{k}) \cdot \nabla] f(\mathbf{k})$ and $[(\mathbf{m} - \mathbf{k}) \cdot \nabla] f(\mathbf{k})$ in (18) will be zero and we can estimate

$$|\langle [Q(f), [Q(f), H]] \rangle| \leqslant C_d R_f^d d_\text{int}^4 \|Q_0\|^2 \|H_0\| \left(\max |\nabla^2 f| \right)^2, \qquad (21)$$

where R_f is half of the diameter of the support of the function f and C_d is a constant that depends only on d.

Choosing the function $f(\mathbf{r})$ to be 1 for $r \leqslant R + \frac{3}{2} d_\text{int}$, 0 for $r > 2R$ and smoothly interpolating between these two values with the second derivatives of order $O(\frac{1}{R^2})$, we find

$$\langle [Q(f), [Q(f), H]] \rangle = O(R^{d-4}). \qquad (22)$$

Using the clustering property and letting $\delta = \delta_{A_0 A_0}$, we can estimate

$$\langle A_R^\dagger A_R + A_R A_R^\dagger \rangle = \frac{2}{\text{Vol}(B_R^d)^2} \sum_{|\mathbf{n}| \leqslant R, |\mathbf{m}| \leqslant R} \langle T_\mathbf{n}(A_0) T_\mathbf{m}(A_0) \rangle$$

$$< \frac{2 C_{A_0 A_0}}{\text{Vol}(B_R^d)} \sum_{|\mathbf{n}| < 2R} \frac{1}{|\mathbf{n}|^\delta}$$

$$= \begin{cases} O\left(\frac{1}{R^\delta}\right) & \text{if } \delta < d, \\ O\left(\frac{\ln R}{R^d}\right) & \text{if } \delta = d, \\ O\left(\frac{1}{R^d}\right) & \text{if } \delta > d. \end{cases} \qquad (23)$$

Combining the two estimates and substituting them into the Bogoliubov inequality we find

$$|\langle [Q, A_0] \rangle|^2 = |\langle [Q(f), A_R] \rangle|^2 = O(R^{d-4-\delta}). \qquad (24)$$

Since R is arbitrary, this means that for $d \leqslant 4$ and any local A_0 we have $\langle [Q, A_0] \rangle = 0$. Thus the charge symmetry cannot be spontaneously broken for $d \leqslant 4$ if the temperature is nonzero.

C. HMW theorem for the dipole symmetry at a nonzero temperature

We can use the estimates from the previous section to constrain dipole symmetry breaking.

Suppose dipole symmetry is spontaneously broken, with a local order parameter A_0:

$$\langle [D^i, A_0] \rangle \neq 0. \qquad (25)$$

We will assume that the charge symmetry is not broken.[3] This will allow us to act with a translation operator on the left-hand side of (25) without affecting the right-hand side. We will consider a spatially averaged A_0 given by (15). We can replace D^i in this equation with $D^i(g) = \sum_{\mathbf{n} \in \mathbb{Z}^d} g(\mathbf{n}) D_\mathbf{n}^i$, where $g(\mathbf{n})$ is any compactly supported function equal to 1 whenever $\mathbf{n}$ belongs to a ball of radius $R + \frac{3}{2} d_\text{int}$ centered at 0.

We will use the Bogoliubov inequality (17) with $A = A_R$ and $B = D^i(g)$.

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[3]If the charge symmetry is broken, then the dipole symmetry is broken as well.

Estimates

We now want to estimate

$$\langle [D^i(g), [D^i(g), H]] \rangle = \sum_{\mathbf{n},\mathbf{m}} g(\mathbf{n}) g(\mathbf{m}) \langle [D^i_{\mathbf{n}}, [D^i_{\mathbf{m}}, H]] \rangle$$

$$= \sum_{\mathbf{n},\mathbf{m}} n^i g(\mathbf{n}) m^i g(\mathbf{m}) \langle [Q_{\mathbf{n}}, [Q_{\mathbf{m}}, H]] \rangle.$$

The considerations of the previous section did not use a specific form of the function f. Therefore, we can use the same estimate (21) with the function $f(\mathbf{n}) = n^i g(\mathbf{n})$:

$$|\langle [D^i(g), [D^i(g), H]] \rangle|$$
$$\leqslant C_d (R_f)^d d_{\text{int}}^4 \|Q_0\|^2 \|H_0\| (\max |\nabla^2 f|)^2. \quad (26)$$

Since $g(\mathbf{r})$ must be 1 for $r \leqslant R + \frac{3}{2} d_{\text{int}}$ the function $f(\mathbf{r})$ must be equal to x^i for $r \leqslant R + \frac{3}{2} d_{\text{int}}$. We can smoothly interpolate it to zero for $r > 2R$ while keeping the second derivatives bounded by $O(\frac{1}{R})$. Then we get

$$|\langle [D^i(f), [D^i(f), H]] \rangle| = O(R^{d-2}). \quad (27)$$

Using this estimate, the Bogoliubov inequality, and the estimate (23), we find

$$|\langle [D^i, A_0] \rangle|^2 = |\langle [D^i(f), A] \rangle|^2 = O(R^{d-2-\delta}). \quad (28)$$

Since R is arbitrary, the dipole symmetry cannot be spontaneously broken in $d \leqslant 2$ at nonzero temperatures.

D. HMW-type theorems at zero temperature

At zero temperatures instead of the Bogoliubov inequality we use the uncertainty relation [6]

$$|\langle [A, B] \rangle|^2 \leqslant \langle A^\dagger A + A A^\dagger \rangle \langle B^\dagger B + B B^\dagger \rangle, \quad (29)$$

where brackets denote the average over any state and we assumed that the operators A, B have zero averages. We will apply it to the case when $A = A_R - \langle A_R \rangle$, $B = Q(f) - \langle Q(f) \rangle$, and the state is the ground state of H. We can upper bound the right-hand side of this inequality in terms of more physical quantities as follows. Recall that for any two local operators B, B' one defines the structure factor $S_{BB'}(\omega)$ as

$$S_{BB'}(\omega) = \int_{-\infty}^{\infty} [\langle B(t) B' \rangle - \langle B \rangle \langle B' \rangle] e^{i\omega t} dt. \quad (30)$$

Since the average is over the ground state, $S_{BB'}(\omega) = 0$ for $\omega < 0$. If $B' = B^\dagger$, then $S_{BB^\dagger}(\omega)$ is positive and is known as the spectral density of B.

Another useful quantity is the static susceptibility $\chi_{BB'}$ defined as the change in the expectation value of B under an infinitesimal variation of the Hamiltonian by $-\epsilon B'$:

$$\langle B \rangle_{H-\epsilon B'} = \langle B \rangle_H + \epsilon \chi_{BB'} + O(\epsilon^2). \quad (31)$$

Here it is assumed that B, B' are Hermitian, but one can formally extend $\chi_{BB'}$ to general operators by linearity. In linear response theory, it is shown that $\chi_{BB'}$ can be expressed through $S_{BB'}(\omega)$. At $T = 0$ the formula looks as follows:

$$\chi_{BB'} = \int_0^\infty \frac{S_{BB'}(\omega) + S_{B'B}(\omega)}{2\omega} \frac{d\omega}{\pi}. \quad (32)$$

The static susceptibility is not always well defined because of a possible nonintegrable singularity at $\omega = 0$. Assuming it exists for $B = B' = Q(f) - \langle Q(f) \rangle$ and using the Cauchy-Schwarz inequality, we get an estimate

$$\langle B^\dagger B + B B^\dagger \rangle = 2 \langle B(t) B \rangle \Big|_{t=0} = \int_0^\infty S_{BB}(\omega) \frac{d\omega}{\pi}$$

$$\leqslant \sqrt{\int_0^\infty \omega S_{BB}(\omega) \frac{d\omega}{\pi} \int_0^\infty \frac{S_{BB}(\omega)}{\omega} \frac{d\omega}{\pi}}$$

$$= \sqrt{|\langle [B, [H, B]] \rangle|} \sqrt{\chi_{BB}}. \quad (33)$$

The quantity χ_{BB} depends on the function f and thus also on R, but with the choice of f described at the end of Sec. III B 1, it can be upper bounded by a number of order R^d times an R-independent quantity with a simple thermodynamic interpretation:

$$\chi_{BB} = \sum_{\mathbf{n},\mathbf{m}} f(\mathbf{n}) f(\mathbf{m}) \chi_{Q_{\mathbf{n}} Q_{\mathbf{m}}} \leqslant C_d R^d \sum_{\mathbf{n}} \chi_{Q_0 Q_{\mathbf{n}}} + o(R^d)$$

$$= C_d R^d \frac{\partial \langle Q_0 \rangle}{\partial \mu} + o(R^d). \quad (34)$$

Here μ is the chemical potential. The non-negative quantity $\partial \langle Q_0 \rangle / \partial \mu$ is the derivative of the charge density with respect to the chemical potential, also known as charge compressibility. We assumed here that it is finite.

Combining (22), (23), (29), and (34) we find

$$|\langle [Q, A] \rangle|^2 \leqslant \sqrt{\frac{\partial \langle Q_0 \rangle}{\partial \mu}} \times O(R^{d-2-\delta}). \quad (35)$$

Therefore if charge compressibility is finite and the ground state is clustering, charge symmetry cannot be broken for $d \leqslant 2$. Note that in the absence of dipole symmetry one can only show that the charge symmetry cannot be broken for $d \leqslant 1$.

We cannot use the same method to exclude dipole symmetry breaking in $d = 1$. The main issue is that the analog of susceptibility for the dipole symmetry is

$$\chi_{D^i(f) D^i(f)} = \sum_{\mathbf{n},\mathbf{m}} n^i m^i f(\mathbf{n}) f(\mathbf{m}) \chi_{Q_{\mathbf{n}}, Q_{\mathbf{m}}}. \quad (36)$$

Even if we assume that $\chi_{Q_{\mathbf{n}}, Q_{\mathbf{m}}}$ is essentially nonzero only if $\mathbf{n}, \mathbf{m}$ are close to each other, the sum behaves as $O(R^{d+2})$. This is not enough to exclude symmetry breaking in $d = 1$. In fact, the Lifshitz model shows that dipole symmetry can be spontaneously broken in $d = 1$ for $T = 0$.

E. The dipolar Bose-Hubbard model

The above theorems apply, in particular, to lattice models of fermions with a dipole symmetry. They cannot be directly applied to the dipolar Bose-Hubbard model because the on-site Hilbert space is infinite dimensional, and the operators $Q_{\mathbf{n}}, H_{\mathbf{n}}$ are unbounded. However, after a slight modification, the proofs work for this model too, if local charge fluctuations in the relevant equilibrium state are not too wild. Indeed, we can write

$$H_{\mathbf{n}} = H_{\mathbf{n}}^{(1)} + H_{\mathbf{n}}^{(2)}, \quad (37)$$

where $H_{\mathbf{n}}^{(1)}$ is the dipole hopping term and $H_{\mathbf{n}}^{(2)}$ is a function of only $Q_{\mathbf{m}}$ on the neighboring sites. Then

$$[Q_{\mathbf{m}}, H_{\mathbf{k}}] = [Q_{\mathbf{m}}, H_{\mathbf{k}}^{(1)}] = \epsilon_{\mathbf{m}-\mathbf{k}} H_{\mathbf{k}}^{(1)}, \tag{38}$$

where $\epsilon_{\mathbf{m}-\mathbf{k}}$ is ± 2 if $\mathbf{m} = \mathbf{k}$, ± 1 if $\mathbf{m}, \mathbf{k}$ are nearest neighbors, and 0 otherwise. Therefore the estimate (19) can be replaced with

$$|\langle [Q_{\mathbf{n}}, [Q_{\mathbf{m}}, H_{\mathbf{k}}]]\rangle| \leqslant C_d |\langle H_0 \rangle|, \tag{39}$$

where C_d depends only on d. Further, it is easy to see that

$$|\langle H_0 \rangle| \leqslant C_d' |t| |\langle Q_{\mathbf{n}}^2 \rangle|, \tag{40}$$

where C_d' is another d-dependent constant. Thus if we assume that fluctuations of the charge density have a finite standard deviation, we can still bound the double commutator by a (state-dependent) constant. Note also that while the local operator A_0 need not be bounded (e.g., it could be b_0), the clustering property implies that it has a finite standard deviation. Then we still get the estimates (24), (28), and (35).

Similar arguments can be used for more general systems with unbounded $Q_{\mathbf{n}}$ and $H_{\mathbf{n}}$. As long as the fluctuations of the constituents are not too wild one can bound the expectation value on the left-hand side of (19).

IV. CONTINUUM THEORIES

A. HMW-type theorems for continuum theories

For continuum systems, the momentum density operator $p_j(\mathbf{x})$ is an obvious candidate for an order parameter for the dipole symmetry. Indeed, if we choose the dipole charge density to be $d_j(\mathbf{x}) = x_j \rho(\mathbf{x})$, where $\rho(\mathbf{x})$ is the charge density, we find

$$[D_j, p_k(\mathbf{x})] = i\delta_{jk}\rho(\mathbf{x}). \tag{41}$$

Thus, as long as the average charge density is not zero, the dipole symmetry appears to be broken. This statement does not depend on either temperature of dimensionality or space and suggests that the theorem of Sec. III C cannot be extended to continuum models with a dipole symmetry.

In this section using a specific Hamiltonian as an example, we will show that the arguments of previous sections can be generalized without essential change to the continuum case. We discuss how the apparent contradiction is avoided.

Consider a bosonic field ψ with canonical commutation relations

$$[\psi(\mathbf{x}), \psi^\dagger(\mathbf{y})] = \delta(\mathbf{x} - \mathbf{y}), \tag{42}$$

$$[\psi(\mathbf{x}), \psi(\mathbf{y})] = [\psi^\dagger(\mathbf{x}), \psi^\dagger(\mathbf{y})] = 0, \tag{43}$$

and the Hamiltonian

$$H = H_{\text{kin}} + H_{\text{pot}}, \tag{44}$$

where the kinetic energy is

$$H_{\text{kin}} = \int d^d x \left[\frac{K_1}{2} |\partial_j \psi \partial_k \psi - \psi \partial_j \partial_k \psi|^2 + \frac{K_2}{2} (\partial_i \psi^\dagger \partial_j \psi^\dagger - \psi^\dagger \partial_j \partial_k \psi^\dagger)(\partial_j \psi \partial_k \psi - \psi \partial_j \partial_k \psi) \right],$$

and the potential energy depends only on the density $\rho = \psi^\dagger \psi$ but is arbitrary otherwise and can include long-range interactions. This Hamiltonian is invariant under charge and dipole transformations $\psi \to \exp(ia + ib_j x_j)\psi$. Similar Hamiltonians were studied in [18,26].

The charge conservation equation has the form

$$\dot{\rho} = \partial_j \partial_k j_{jk}, \tag{45}$$

where we defined

$$j_{jk} = i\frac{K_1}{2} \left(\partial_j \psi^\dagger \partial_k \psi^\dagger - \psi^\dagger \partial_j \partial_k \psi^\dagger \right)\psi^2 + i\frac{K_2}{2} \delta_{jk} \left(\partial_l \psi^\dagger \partial_l \psi^\dagger - \psi^\dagger \partial^2 \psi^\dagger \right)\psi^2 + \text{H.c.} \tag{46}$$

The commutator of the charge density with the current is

$$[\rho(\mathbf{x}), j_{jk}(\mathbf{y})] = -iK_1 \partial_j \partial_k \delta(\mathbf{x} - \mathbf{y})\psi^{\dagger 2}\psi^2(\mathbf{y}) - iK_2 \delta_{jk}\partial^2 \delta(\mathbf{x} - \mathbf{y})\psi^{\dagger 2}\psi^2(\mathbf{y}). \tag{47}$$

Using this commutation relation, we can do manipulations analogous to (18):

$$\int d^d x\, d^d y\, f(\mathbf{x})f(\mathbf{y})\langle [\rho(\mathbf{x}), [\rho(\mathbf{y}), H]]\rangle = i\int d^d x\, d^d y\, f(\mathbf{x})f(\mathbf{y})\langle [\rho(\mathbf{x}), \partial_j \partial_k j_{jk}(\mathbf{y})]\rangle$$

$$= \int d^d x\, d^d y\, (K_1 \partial_j \partial_k f(x)\partial_j \partial_k f(\mathbf{y}) + K_2 \partial^2 f(x)\partial^2 f(\mathbf{y}))\langle \psi^{\dagger 2}\psi^2(\mathbf{y})\rangle. \tag{48}$$

We can again apply the Bogoliubov inequality (17) to $A = A_R = \frac{1}{\text{Vol}B_R^d}\int_{B_R^d} d^d x[a(\mathbf{x}) - \langle a(\mathbf{x})\rangle]$ and $B = D_j(g) = \int d^d x\, x_j g(\mathbf{x})\rho(\mathbf{x})$, where $a(\mathbf{x})$ is order parameter, B_R^d is a d-dimensional ball of radius R centered at 0, and the function $g(\mathbf{x})$ is chosen to be 1 inside of this ball and interpolates to 0 outside of $|\mathbf{x}| > 2R$ so that second derivatives of

$f(\mathbf{x}) = x_j g(\mathbf{x})$ are bounded by $O(\frac{1}{R})$. We find

$$|\langle [D, a(0)]\rangle|^2 \leqslant \langle [\rho(f), [H, \rho(f)]]\rangle \langle A_R^\dagger A_R + A_R A_R^\dagger \rangle. \tag{49}$$

We can use (48) to show that

$$\langle [\rho(f), [H, \rho(f)]]\rangle = O(R^{d-2}). \tag{50}$$

Using the estimate (23), we find

$$|\langle [D, a(0)] \rangle|^2 \leqslant O(R^{d-2-\delta}). \tag{51}$$

Similarly, one can generalize the arguments for the charge symmetry at zero or nonzero temperature leading to the same results as for lattice systems.

There are several ways in which the HMW theorem for the dipole symmetry and the commutation relation (41) can be reconciled. Most obviously, the momentum density operator may fail to satisfy the clustering property. For example, take the Hamiltonian (44) and choose the potential term to have a minimum at $|\rho| = \rho_0$. Then we can ignore the fluctuation of the modulus of ψ and write approximately $\psi = \rho_0 \exp(i\phi)$. The effective action for ϕ is

$$\mathcal{S}_{\text{eff}} = \frac{k}{2} \int d^d x \, dt [\dot{\phi}^2 - C_1 (\partial^2 \phi)^2]. \tag{52}$$

This is the Lifshitz model discussed in Sec. II. The charge density operator in this approximation becomes $k\dot{\phi}$. Thus in a state with a constant charge density we have $\phi = \mu t + \eta$, where μ is the chemical potential and η is the fluctuating field with zero expectation value. The momentum density operator is $p_j = k\dot{\phi}\partial_j\phi$. To leading order in η we get $p_j = k\mu \partial_j \eta$. As discussed in Sec. II, the two-point function of $\partial_j \eta$ grows with distance if $T > 0$ and $d = 1, 2$. Therefore the same is true for p_j provided $\mu \neq 0$. Note that if the two-point function of a local operator grows with distance, fluctuations of its spatial average increase as the averaging region becomes larger. This is why it is natural to require an operator serving as an order parameter to satisfy clustering.

Another possible loophole in the HMW theorem is that the spatially smeared momentum density operator may fail to be bounded, and thus the expression $\langle A_R^\dagger A_R + A_R A_R^\dagger \rangle$ may be infinite. For this to happen, $\langle p_j(\mathbf{x}) p_k(\mathbf{y}) \rangle$ must have a nonintegrable singularity at $\mathbf{x} = \mathbf{y}$. However, this is a short-distance issue that can be fixed by replacing the momentum density operator with a suitably regularized expression.

B. Inertial mass density

In Sec. I we motivated the dipole symmetry by arguing that it emerges in the limit of infinite particle mass. For a large class of continuous systems with dipole symmetry, one can be more precise and prove that they have infinite inertial mass density.

It is well known that the Bogoliubov inequality can be strengthened as follows:

$$|\langle [A, B] \rangle|^2 \leqslant \chi_{AA^\dagger} \langle [B, [H, B^\dagger]] \rangle. \tag{53}$$

This is simply the Cauchy-Schwarz inequality for the Hermitian inner product $(A, C) = \chi_{AC^\dagger}$ applied to $C = i[H, B]$. Let us apply it to Hermitian operators $A = \frac{1}{\text{vol } B_R^d} \int_{B_R^d} p_j(\mathbf{x}) d^d x$ and $B = D_j(g)$, as above. Assuming that translational symmetry is unbroken, we get

$$\langle \rho(0) \rangle^2 \leqslant \chi_{AA} \langle [\rho(f), [H, \rho(f)]] \rangle. \tag{54}$$

Here $f(\mathbf{x}) = x_j g(\mathbf{x})$ as before. For the model considered in Sec. IV A, or for any continuum theory with a dipole symmetry where the commutator of ρ and j_{jk} has the form similar to

(47), we have the estimate (50). Hence

$$\chi_{AA} \geqslant \langle \rho(0) \rangle^2 O(R^{2-d}). \tag{55}$$

To relate χ_{AA} to inertial mass density, recall that whenever momentum is conserved, one can define a generalized Gibbs ensemble by replacing the Hamiltonian H with $H - \mathbf{v} \cdot \mathbf{P}$, where $\mathbf{P}$ is the total momentum. The chemical potential for $\mathbf{P}$ is the velocity $\mathbf{v}$. It is natural to define the inertial mass density as the tensor

$$m_{jk} = \frac{\partial \langle p_j(0) \rangle}{\partial v_k} = \int \chi_{p_j(0) p_k(\mathbf{x})} d^d x. \tag{56}$$

If m_{jk} is finite, then $\chi_{AA} \leqslant m_{jj} O(R^{-d})$, which contradicts the inequality (55) for any nonzero $\langle \rho(0) \rangle$.

V. DISCUSSION

In this paper, we generalized the Hohenberg-Mermin-Wagner theorem to the dipole symmetry. We showed that for systems with finite-range interactions and clustering, dipole symmetry cannot be broken if $d = 1, 2$ and $T > 0$. Thus a system of fermions with a microscopic dipole symmetry cannot flow to a Fermi liquid or any phase with a pronounced Fermi surface. Any phase of such a system must be rather exotic.

We also showed that the mere presence of a dipole symmetry allows one to strengthen the conclusions of the HMW and Pitaevskii-Stringari theorems as applied to charge symmetry. Namely, with the same assumptions, the charge symmetry cannot be broken if $d \leqslant 4$, $T > 0$ and for $d \leqslant 2$, $T = 0$ and finite compressibility.

For our argument to work it was essential that at least part of the translational symmetry is preserved. It is an interesting problem to understand how the disorder affects the results. For the Ising model, symmetry-breaking disorder pushes the critical dimension even higher than is required by the HMW theorem [27]. It is natural to expect similar behavior for systems with dipole symmetry. The case of symmetry-preserving disorder is trickier. On large scales, one could expect the disorder to self-average and the system to effectively become translationally invariant. Any clarification of these questions is an interesting open problem.

Systems with a dipole symmetry provide a counterexample to the widespread belief that spontaneous breaking of charge symmetry implies superconductivity (i.e., infinite dc conductivity). Indeed, for any system with a dipole symmetry the current operator vanishes at zero wave vector and thus conductivity vanishes at all frequencies. On the other hand, the example of the $d = 3$ Lifshitz model at $T = 0$ shows that systems with a dipole symmetry can break charge symmetry.

Charge symmetry and continuous translational symmetry are alike in some respects. For example, HMW-type theorems for them are essentially the same. Superficially, dipole symmetry should be similar to a continuous rotational symmetry: The dipole moment is an integral of charge density times a linear function of coordinates, just like the angular momentum is the integral of momentum density times a linear function of coordinates. Therefore it might seem surprising that rotational symmetry can be spontaneously broken in two dimensions for

$T > 0$ [28], while dipole symmetry cannot. The difference arises from the different group structure of the full symmetry group and the corresponding constraints on the local conservation laws. Dipole symmetry requires the charge current to be a total derivative: $j_j = -\partial_k j_{jk}$. In contrast, rotational symmetry does not require the momentum current (i.e., the stress tensor) to be a total derivative, it only requires it to be symmetric. Consequently, the bounds resulting from the Bogoliubov inequality do not preclude the spontaneous breaking of rotational symmetry in two dimensions. This difference is also responsible for the different structure of the effective action for the Goldstone bosons.

Finally, it would be interesting to consider similar issues for systems of fermions in a magnetic field projected to the lowest Landau level. They have an infinite-dimensional Girvin, MacDonald, and Platzman symmetry [29] which contains a 1D dipole symmetry.

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