



Supercurrent-induced resonant optical response

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The optical conductivity encodes the current response to a time dependent electric field. We develop a theory of the optical conductivity $\sigma(\omega)$ in the presence of a dc supercurrent. Current induced optical response is prohibited when current is conserved, an illustrative example of which is Galilean invariant systems. However, we show that lattice effects give rise to a pronounced current dependent peak in $\sigma(\omega)$ at the gap edge $\omega = 2\Delta$, which diverges in the clean limit. We demonstrate this in a model of a multiband superconductor. Our theory predicts the current induced peak in $\sigma(\omega)$ to scale quadratically with the super-current density, as was recently reported in experiments on NbN by Nakamura *et al.* [*Phys. Rev. Lett.* **122**, 257001 (2019)]. This provides the potential for a new mechanism for direct activation of the Higgs mode with light.

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I. INTRODUCTION

The nonlinear electromagnetic response of quantum materials at terahertz (THz) frequencies and below allows direct probing of electronic structure [1], and provides new inroads into the “THz gap”—a frequency range that has been barely exploited for practical applications in signal generation or detection [2]. In particular, in metals, the nonlinear Hall effect [3–5] and the linear/circular photogalvanic effects [6–10] at low frequency provide direct measurements of Fermi surface properties without generating high-energy excitations, and have established new connections between nonlinear conductivity and the quantum geometry of Bloch electrons [11–13].

In the context of superconductors, the nonlinear response has long been a focus of interest [14–20], where second-order effects in an ac electric field may enhance the superconducting gap and critical current [21–23].

In contrast, there has been less focus on nonlinear effects due to a combination of externally driven electric fields and currents. Notably, experiments on *s*-wave superconductor NbN observed a dc supercurrent to produce giant second-harmonic generation (SHG) in the THz range [24]. The supercurrent breaks inversion symmetry allowing for SHG. Separately, also in NbN, a dc supercurrent was seen to enhance the optical response at the edge of the superconducting gap, where peaks in the dissipative and reactive parts of the optical conductivity were observed [25].

In this work we analyze current enabled optical response in a superconductor. We study the effect of a uniform, dc supercurrent density \mathbf{j}_{sc} on the ac optical conductivity $\sigma(\omega)$ in a time-reversal symmetric superconductor at temperature $T = 0$. This dc supercurrent may be due to an external source, or a screening current induced by a magnetic field [26]. In clean systems, where the gap exceeds the scattering rate $2\Delta > \tau^{-1}$, we show the supercurrent to enable a large optical absorption peak of height $\propto j_{sc}^2$ at the gap edge $\omega = 2\Delta$. This peak is

due to single photon processes which excite quasiparticles across the gap. Naively, one might expect this peak to persist even for $\mathbf{j}_{sc} = 0$. However, when the Hamiltonian is inversion symmetric, this response is absent due to selection rules between quasiparticle modes at the gap edge [Fig. 1(a)], which cause the optical absorption to go continuously to zero at this frequency [27–32]. In order to obtain a large gap edge optical response, inversion symmetry must be broken. As superconductors with intrinsically broken inversion symmetry are rare, we consider extrinsic breaking due to a finite supercurrent [Fig. 1(b)].

A second important ingredient is current relaxation. The large gap edge response is thus absent in common minimal models of superconductors, where there is a single parabolic band, as the current is conserved (as follows from Galilean invariance). We provide a minimal model in the form of a multiband superconductor, as has been proposed for NbN [33,34] and experimentally reported in NbSe₂ [35–38], MgB₂ [39–43], and FeSe [44,45], or as may be realized via the proximity effect at a superconductor-metal interface.

We focus on a response that is linear in the electric field, and thus characterized by the optical conductivity $\sigma(\omega)$

$$\langle \mathbf{j}(\omega) \rangle = \sigma(\omega) \mathbf{E}(\omega). \quad (1)$$

We calculate the tensor $\sigma(\omega)$ in the presence of a dc supercurrent, for finite frequencies, via the Kubo formula

$$i\omega\sigma_{ab}(\omega) = -D_{ab} + iV \int_0^\infty dt e^{i\omega t} \langle [j_a(t), j_b(0)] \rangle, \quad (2)$$

where \mathbf{j} is the current density operator, V is the system volume, a, b are spatial indices, D_{ab} is the Drude weight in the clean noninteracting limit (for a parabolic band $D_{ab} = ne^2\delta_{ab}/m$ where n, m and e are electron density and mass, and charge), and $\langle \cdot \rangle$ denotes thermal expectation values calculated for zero-field $\mathbf{E}(\omega) = 0$. As the Kubo conductivity $\sigma_{ab}(\omega)$ depends only on the current \mathbf{j} and Hamiltonian H defined at zero field, we henceforth discuss these operators only as de-

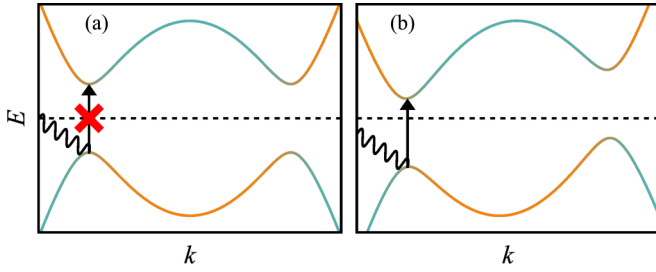


FIG. 1. Optical absorption in inversion symmetric superconductors: The quasiparticle band structure with electron (hole) orbitals colored orange (blue). (a) Inversion symmetry leads to selection rules which prevents optical excitation of quasiparticles across the gap. (b) A current breaks this symmetry, allowing excitation across the direct gap, resulting in an absorption peak at $\omega = 2\Delta$.

fined in this limit, neglecting the minimal coupling correction $\mathbf{k} \rightarrow \mathbf{k} + e\mathbf{A}$.

II. CURRENT CONSERVING SYSTEM

An illustrative example of a current conserving system is the canonical model of an *s*-wave superconductor: a single *parabolic* band with a contact interaction

$$H = \sum_{k\sigma} \xi_k c_{k\sigma}^\dagger c_{k\sigma} + g \sum_{\mathbf{q}} P_{\mathbf{q}}^\dagger P_{\mathbf{q}}. \quad (3)$$

Here the interaction is attractive $g < 0$, $\xi_k = k^2/2m - \mu$ is the dispersion, and $P_{\mathbf{q}}^\dagger = \sum_{\mathbf{k}} c_{\mathbf{k}+\mathbf{q}}^\dagger c_{-\mathbf{k}\downarrow}^\dagger$ creates a pair of electrons with total momentum \mathbf{q} .

The optical response of the system (3) is trivial and independent of the presence of the supercurrent or otherwise. This follows from the Galilean invariance of H , which entails that the current density operator is a conserved integral of motion

$$\mathbf{j} = \frac{e}{V} \sum_{k\sigma} \mathbf{v}_k c_{k\sigma}^\dagger c_{k\sigma} \Rightarrow [H, \mathbf{j}] = 0, \quad (4)$$

where $\mathbf{v}_k = \partial_k \xi_k = \mathbf{k}/m$ and V is the system volume. The conservation of current at zero-field $[H, \mathbf{j}] = 0$ entails that there are no energy absorbing processes which couple to the electric field. This current conservation thus forbids the electromagnetic energy dissipation, and results in a trivial optical conductivity. Specifically, the real part of the optical conductivity is zero at all finite frequencies, while the imaginary part, given by $\sigma(\omega) = iD/\omega$, encodes the oscillatory motion of the electron center of mass in response to an oscillating external electric field.

Let us now examine how the trivial optical conductivity of a current conserving superconductor may be correctly obtained in mean-field theory when the system is carrying a supercurrent \mathbf{j}_{sc} . For $\mathbf{j}_{\text{sc}} \neq 0$, Cooper pairing occurs at a finite momentum \mathbf{q} , this is related to the supercurrent density \mathbf{j}_{sc} and condensate velocity \mathbf{v}_c by

$$\mathbf{j}_{\text{sc}} = en\mathbf{v}_c, \quad \mathbf{v}_c = \mathbf{q}/2m. \quad (5)$$

The corresponding mean-field Hamiltonian, which depends on the Cooper pair momentum \mathbf{q} , is obtained in the usual way

$$\tilde{H} = \sum_{k\sigma} \xi_{\sigma k} c_{k\sigma}^\dagger c_{k\sigma} + \Delta(e^{i\theta} P_{\mathbf{q}} + e^{-i\theta} P_{\mathbf{q}}^\dagger), \quad (6)$$

where Δ and θ are the magnitude and phase of the superconducting order parameter, respectively. In the standard Bardeen-Cooper-Schrieffer (BCS) theory, the phase factor $e^{i\theta}$ is treated as a *c* number. Although this leads to an artefactual violation of electron number conservation $[\tilde{H}, N] \neq 0$ ($N = \sum_{k\sigma} c_{k\sigma}^\dagger c_{k\sigma}$), this treatment still allows for accurate calculation of the optical conductivity in clean and dirty systems at $\mathbf{j}_{\text{sc}} = 0$. The standard approach works for $\mathbf{q} = 0$ as the pairing and kinetic terms in \tilde{H} still conserve current as desired. However, for a finite supercurrent $\mathbf{j}_{\text{sc}} \neq 0$, Cooper pairing occurs at finite momentum \mathbf{q} , then the pair operators $P_{\mathbf{q}}$ in the mean-field Hamiltonian alter the current $[P_{\mathbf{q}}, \mathbf{j}] \neq 0$. In this case treating the condensate phase $e^{i\theta}$ as a *c* number leads to an additional artefactual violation of the current conservation: $[\tilde{H}, \mathbf{j}] \neq 0$ and consequently artefactual contributions to the optical conductivity.

In order to obtain the optical conductivity correctly, we treat θ as an operator associated with the phase of the condensate, the canonically conjugate operator $-i\partial_\theta$ then counts the number of Cooper pairs making up the condensate. The physical picture [46,47] is that $e^{i\theta}$ increases the number of Cooper pairs by one, while $P_{\mathbf{q}}$ decreases the number of fermions by two. In this modified mean-field approach, the total charge consists of both charge-*e* fermions and charge-2*e* Cooper pairs. Thus, the total charge number operator takes the form

$$\tilde{N} = -2i\partial_\theta + \sum_{k\sigma} c_{k\sigma}^\dagger c_{k\sigma}. \quad (7)$$

Importantly this restores the conservation of the electron number to the mean-field theory, so that $[\tilde{H}, \tilde{N}] = 0$, respecting the charge conservation present in the original theory (3)

This modified mean-field approach also restores the current conservation of Galilean invariant superconductors. The total current consists of the currents carried by unpaired electrons and the condensate

$$\begin{aligned} \tilde{\mathbf{j}} &= \frac{e}{V} \left(-2i\mathbf{v}_c \partial_\theta + \sum_{k\sigma} \mathbf{v}_k c_{k\sigma}^\dagger c_{k\sigma} \right) \\ &= \frac{e}{V} \left(\mathbf{v}_c \tilde{N} + \sum_{k\sigma} (\mathbf{v}_k - \mathbf{v}_c) c_{k\sigma}^\dagger c_{k\sigma} \right), \end{aligned} \quad (8)$$

where in the second line, we have substituted Eq. (7) to eliminate ∂_θ . From Eq. (8) one obtains $[\tilde{\mathbf{j}}, \tilde{H}] = 0$ as required.

We have presented physical arguments for the form of the number and current density operators appropriate for a superconducting mean-field theory with a finite supercurrent. Importantly, these operators provide the correct bookkeeping for the charge and current carried by the condensate. In the Galilean invariant system considered, this restores conservation of current to the mean-field theory, and hence recovers the desired trivial optical conductivity. This approach is formalized in Appendix A where we show the mean-field current and number operators (7) and (8) may be obtained by extending the Hilbert space, and enforcing charge conservation as a

gauge constraint which must be respected by the mean-field theory. In Appendix B we further show that in addition to conservation of charge, current, and momentum, this approach restores the correct Galilean transformation properties of \tilde{H} .

III. LATTICE EFFECTS

Current conservation in a Galilean invariant system follows from the fact that the interaction conserves current. However, in solids, the presence of an underlying lattice means that current is generically not conserved: nonparabolic bands, Umklapp processes, or multiband effects all break Galilean invariance, leading to current relaxation, and allowing for a nontrivial optical response. To illustrate this, we now show how coupling between bands with different effective masses allows for a nontrivial optical response in a current carrying superconductor.

We consider a minimal model which, after the kinetic term has been diagonalized, consists of two parabolic bands $\alpha = 1, 2$, and interactions that scatter electrons within (g_{11} , g_{22}) and between ($g_{12} = g_{21}^*$) bands

$$H = \sum_{\alpha k \sigma} \xi_{\alpha k} c_{\alpha k \sigma}^\dagger c_{\alpha k \sigma} + \sum_{k \alpha \beta} g_{\alpha \beta} P_{\alpha q}^\dagger P_{\beta q}, \quad (9)$$

where $\xi_{\alpha k} = k^2/2m_\alpha - \mu_\alpha$, and $P_{\alpha q}^\dagger = \sum_k c_{\alpha k+q}^\dagger c_{\alpha-k}^\dagger$. For simplicity, we assume the primary pairing interaction occurs in the first band, and induces a pairing potential in the second band via the interband coupling, providing the dominant source of pairing in both bands

$$|g_{11}\langle P_{1q} \rangle| \gg |g_{12}\langle P_{2q} \rangle|, \quad |g_{12}\langle P_{1q} \rangle| \gg |g_{22}\langle P_{2q} \rangle|. \quad (10)$$

This may occur either due to a hierarchy of scales in the coupling $g_{11} \gg g_{12} \gg g_{22}$, or because the density of states at the Fermi surface is much greater in the first band. In this regime the mean-field Hamiltonian takes the form

$$\tilde{H} = \sum_{\alpha k \sigma} \xi_{\alpha k} c_{\alpha k \sigma}^\dagger c_{\alpha k \sigma} + \sum_{\alpha} \Delta_{\alpha} (e^{i\theta} P_{\alpha q} + \text{H.c.}), \quad (11)$$

where the primary gap is $\Delta_1 e^{i\theta} = g_{11}\langle P_{1q} \rangle$, and a smaller gap is induced in the second band $\Delta_2 = \Delta_1 g_{12}/g_{11}$. In Bogoliubov–de Gennes (BdG) form we obtain

$$\tilde{H} = \sum_{\alpha k} \psi_{\alpha k}^\dagger \tilde{H}_{\alpha k} \psi_{\alpha k}, \quad (12a)$$

where $\psi_{\alpha k}^\dagger = (c_{\alpha q/2+k}^\dagger, c_{\alpha q/2-k})$ is the Nambu spinor and we have defined the BdG matrices

$$\tilde{H}_{\alpha k} = \bar{\xi}_{\alpha k} \tau_3 + \delta \xi_{\alpha k} \tau_0 + \Delta_{\alpha} (e^{i\theta} \tau_+ + \text{H.c.}), \quad (12b)$$

where $\bar{\xi}_{\alpha k} = \frac{1}{2}(\xi_{\alpha, q/2+k} + \xi_{\alpha, q/2-k})$, $\delta \xi_{\alpha k} = \frac{1}{2}(\xi_{\alpha, q/2+k} - \xi_{\alpha, q/2-k})$, and τ_{α} are the usual Pauli matrices.

In this model Galilean invariance is broken, thus the current density is not conserved, but relaxes due to pair scattering between the bands. This nonconservation is quantitatively captured by the commutator of the current operator (4) and Hamiltonian (9) which details the rate of change of current due to each microscopic process

$$[\tilde{j}, H] = \delta \tilde{j} (g_{12} P_{1q}^\dagger P_{2q} - \text{H.c.}), \quad \delta \tilde{j} = \frac{e v_c \delta m}{m_2 V}, \quad (13)$$

where $\delta m = m_1 - m_2$, and $v_c = q/2m_1$ is the condensate velocity. Though for the parabolic bands considered here current is conserved upon *intra*band scattering, the current changes upon *inter*band scattering: pairs of electron from the second band are scattered with rate g_{12} into the first band, each causing a change to the current density of $\delta \tilde{j}$, and *vice versa*. The mean-field theory current operator must preserve this commutator in order to preserve the microscopic dynamics of current, correctly capture the current relaxation, and allow for accurate calculation of $\sigma(\omega)$.

In direct generalization of Eqs. (7) and (8) we use the mean-field current and number operators

$$\tilde{j} = \frac{e}{V} \left(v_c \tilde{N} + \sum_{\alpha k \sigma} (v_{\alpha k} - v_c) c_{\alpha k \sigma}^\dagger c_{\alpha k \sigma} \right),$$

$$\tilde{N} = -2i\partial_\theta + \sum_{\alpha k \sigma} c_{\alpha k \sigma}^\dagger c_{\alpha k \sigma}. \quad (14)$$

The mean-field operators have the commutator

$$[\tilde{j}, \tilde{H}] = \frac{e v_c \delta m}{m_2 V} (\Delta_2 e^{i\theta} P_{2q} - \text{H.c.}). \quad (15)$$

This commutation relation corresponds directly to the mean-field equivalent of Eq. (13). Precisely, one obtains Eq. (15) by making the mean-field replacement to both sides of Eq. (13). Note that Eq. (13) is symmetric upon exchanging the band labels, whereas Eq. (15) is not, this is a consequence of our assumption that both the primary gap Δ_1 and induced gap Δ_2 arise due to pairing in the first band (10). Note also that, as required, the mean-field current \tilde{j} is conserved in the limits of $m_1 = m_2$ and $g_{12} = 0$ in which Galilean invariance, and hence current conservation is restored. In BdG form we have

$$\tilde{j} = e v_c \tilde{N}/V + \sum_{\alpha k} \psi_{\alpha k}^\dagger \tilde{j}_{\alpha k} \psi_{\alpha k},$$

$$\tilde{j}_{1k} = e \bar{v}_{1k} \tau_0/V, \quad \tilde{j}_{2k} = e [\bar{v}_{2k} \tau_0 + v_c (\delta m/m_2) \tau_3]/V, \quad (16)$$

where $\bar{v}_{\alpha k} = \partial_k \bar{\xi}_{\alpha k} = k/m_\alpha$. We note the appearance of the τ_3 term in \tilde{j}_{2k} with a current dependent coefficient. This term measures the density of electrons in the second band. This term will be important to obtaining a large gap edge response, and is only possible in a multi-band system where electrons may be exchanged between bands. Within a single band system the conservation of the total number of electrons prevents the coupling of electron density to an external field.

We may now evaluate the optical conductivity using Eqs. (12) and (16). As \tilde{N} , commutes with both the mean-field Hamiltonian \tilde{H} and the current operator \tilde{j} , we work in its diagonal basis, replacing \tilde{N} with its eigenvalue. By further writing \tilde{H} and \tilde{j} in terms of the chargeless fermions $f_{k\sigma} = e^{i\theta/2} c_{k\sigma}$ (which satisfy the anticommutation relations $\{f_{k\sigma}, f_{q\tau}^\dagger\} = \delta_{kq} \delta_{\sigma\tau}$, $\{f_{k\sigma}, f_{q\sigma}\} = 0$ as required) in this basis the Hamiltonian is independent of θ

$$\tilde{H} = \sum_{\alpha k \sigma} \bar{\xi}_{\alpha k} f_{\alpha k \sigma}^\dagger f_{\alpha k \sigma} + \sum_{\alpha} \Delta_{\alpha} \left(\sum_k f_{\alpha k+q}^\dagger f_{\alpha-k}^\dagger + \text{H.c.} \right), \quad (17)$$

and similarly for \tilde{j} . Consequently the current-current correlator to be evaluated using the usual fermionic algebra. In the clean system, which we consider first, only the latter ($\propto \tau_3$)

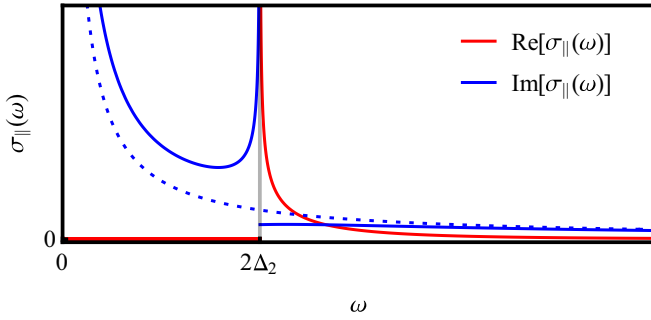


FIG. 2. Current induced optical conductivity of a clean superconductor: The red (blue) line show the dissipative (reactive) response (18). The dashed line shows the response for zero current $\mathbf{q} = 0$. The clean reactive response is shown (dashed) for comparison. Shown for a parabolic dispersion with parameters $(v_c \delta m / v_{2F} m_2)^2 = 1/2$.

term of Eq. (16) gives rise to a non-trivial response. To obtain this response, let us set the z direction parallel to \mathbf{q} , due to axial symmetry the optical conductivity then has two independent components $\sigma_{\parallel}(\omega) = \sigma_{zz}(\omega)$, and $\sigma_{\perp}(\omega) = \sigma_{xx}(\omega) = \sigma_{yy}(\omega)$. In the presence of a supercurrent the perpendicular component remains trivial $\text{Re}[\sigma_{\perp}(\omega)] = 0$, while the parallel component takes the form (plotted in Fig. 2, derivation in Appendix C)

$$\text{Re}[\sigma_{\parallel}(\omega)] = \frac{\rho_{2F}}{\omega} \left(\frac{ev_c \delta m}{m_2} \right)^2 f\left(\frac{\omega}{2\Delta_2}\right), \quad f(x) = \frac{\pi \theta(x^2 - 1)}{x \sqrt{x^2 - 1}}, \quad (18)$$

where we have set temperature $T = 0$, $\theta(x)$ is the usual step function, and ρ_{2F} is the normal phase Fermi surface density of states of the second band. The appearance of mass in this result is due to the dependency on the curvature of the bands at the Fermi surface. As is necessary, this current mediated contribution to the optical conductivity (18) disappears if the condensate velocity is zero $v_c = 0$, or in either of the current conserving cases of $m_1 = m_2$, or $g_{12} = 0$. The current mediated contribution is largest for in the case when the second band is much faster, $m_2 \ll m_1$, i.e. in the same regime as our assumption (10).

The current mediated optical response (18) is most significant at the gap edge where it diverges. This divergence is mirrored in the reactive (imaginary) part of the optical conductivity (obtained via the Kramers-Kronig relation) which has an equivalent divergence as the gap is approached from below

$$\text{Re}[\sigma_{\parallel}(2\Delta_2 + \delta\omega)] \sim \text{Im}[\sigma_{\parallel}(2\Delta_2 - \delta\omega)] = O(\delta\omega^{-1/2}), \quad (19)$$

where \sim indicates asymptotic equality as $\delta\omega \rightarrow 0$. For fields oscillating at frequencies just below the gap, where the reactive part is large, a large ac current is induced, carried by electrons in the second band. As the peak is found only in σ_{\parallel} , with σ_{\perp} remaining trivial, the induced ac current runs parallel to the supercurrent.

The large response is peaked at a frequency generically above the superconducting gap, and approaches the superconducting gap edge only in the limit of weak current. Specifically, the gap edge, where the onset of the real part optical conductivity occurs, is set by the indirect gap $\omega_{ig-} =$

$2\Delta_2 - qk_{2F}/2m_2 + O(q^2/m_2)$, this moves to lower frequencies as the supercurrent density increases due to the Doppler shift (the tilting of the dispersion). In contrast, the large response is always set by the current independent direct gap $2\Delta_2$. Moreover, as both the direct gap, and the change to the current (13) induced by scattering $\delta\mathbf{j}$ are independent of position on the Fermi surface, the current induced response (18) is obtained by integrating contributions from the entire Fermi surface.

IV. IMPURITY SCATTERING

Impurities couple different momentum sectors, altering the optical response. We consider the effect of quenched nonmagnetic disorder

$$\tilde{H} \rightarrow \tilde{H} + \sum_{kq\sigma} V_{kq} c_{k\sigma}^{\dagger} c_{q\sigma}, \quad (20)$$

and focus on its effect on the dissipative response, from which the reactive response follows via the Kramers-Kronig relation.

Let us recall some useful properties from the usual BCS theory of response functions for the case $\mathbf{j}_{sc} = 0$, $T = 0$. In the presence of time-reversal symmetry, response functions can be decomposed into *case I* (*case II*) terms which are generated by operators which are even (odd) under time-reversal symmetry [27–29]. In the usual Nambu basis $\psi_{\alpha k}^{\dagger} = (c_{k\uparrow}^{\dagger}, c_{-k\downarrow})$ these correspond to BdG matrices τ_3 (τ_0), respectively. At zero temperature, case I and case II responses exhibit markedly different behavior at the gap edge. To illustrate this consider an example case I (case II) operator O^+ (O^-) with BdG matrices $o_{kp}^+ = \tau_0$, ($o_{kp}^- = \tau_3$) coupling the \mathbf{k} and \mathbf{p} momentum sectors, we obtain matrix elements

$$|\langle \epsilon_k | o_{kp}^{\pm} | \epsilon_p \rangle|^2 = \frac{\epsilon_k \epsilon_p \mp \Delta_2^2 - \sqrt{(\epsilon_k^2 - \Delta_2^2)(\epsilon_p^2 - \Delta_2^2)}}{2\epsilon_k \epsilon_p}, \quad (21)$$

where $|\epsilon_k\rangle$ is the eigenvector of $\tilde{H}_{2,k}$ for zero current $\delta\xi_{2,k} = 0$. In the limit of states on either side of the gap edge $(\epsilon_k, \epsilon_p) \rightarrow (\Delta_2, -\Delta_2)$ we find that in case II $|\langle \epsilon_k | o_{kp}^- | \epsilon_p \rangle|^2 \rightarrow 0$, whereas in case I the matrix element $|\langle \epsilon_k | o_{kp}^+ | \epsilon_p \rangle|^2$ approaches double its normal phase ($\Delta_2 = 0$) value. Thus we see that, due to selection rules (or equivalently coherence factors), the case I (case II) response is enhanced (suppressed to zero) as $\omega \rightarrow 2\Delta_2^+$. Whereas inside the gap $|\omega| < 2\Delta_2$ both the case I and II responses are zero. Thus the (enhanced) case I response drops discontinuously to zero at $\omega = 2\Delta_2$. The distinct gap edge behavior of case I and case II operators relies only on the time-reversal symmetry of H [48], and thus holds in the presence of nonmagnetic disorder. Despite the possibility of enhanced linear response at the gap edge, no such large gap edge response is exhibited in the optical conductivity, as the current operator is odd under time-reversal symmetry, and thus purely case II [27–31, 49–54].

In the presence of the current this picture is altered, the pairing is at finite momentum, and thus the BdG matrices τ_3 and τ_0 are not odd/even under time reversal. However, for weak scattering, the distinction in behaviors of the matrix elements close to the gap remains, and so we retain

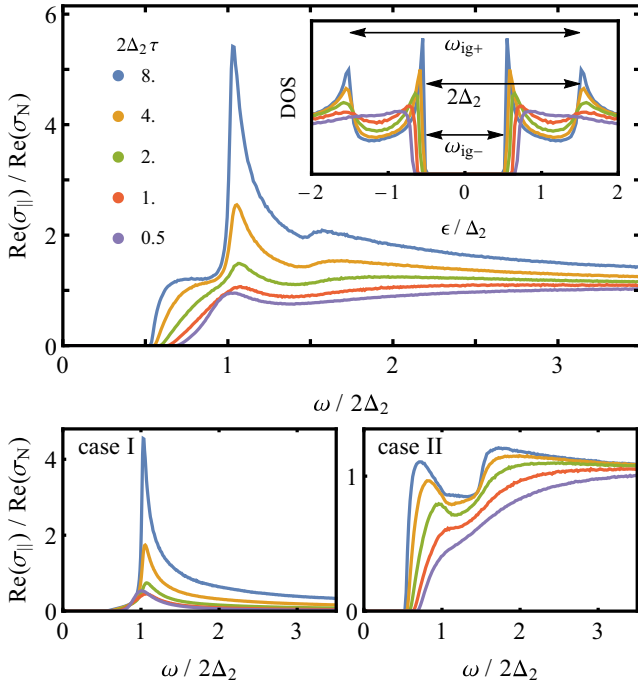


FIG. 3. *Current mediated response in a disordered superconducting wire:* The dissipative optical conductivity is shown in units of the normal phase dissipative response $\text{Re}\sigma_N$ for different scattering times (legend inset). The dissipative response onsets at the indirect gap $\omega_{\text{ig}-}$, is peaked at the direct gap 2Δ , and converges $\text{Re}(\sigma_{\parallel})/\text{Re}(\sigma_N) \sim 1$ far from the gap. The peak height becomes large $O(\tau^{3/2})$ in the clean limit ($2\Delta_2\tau \gtrsim 1$). The line shape is a sum of case I and case II contributions are plotted separately in the lower panels. The gaps $\omega_{\text{ig}-}$ and $2\Delta_2$ are labeled on the quasiparticle particle density of states (inset, upper panel). Parameters: $(v_c \delta m/m_2 v_{2F}) = 0.3$, $\delta\xi_{2F} = qk_{2F}/2m_2 = \Delta_2/2$, $2\Delta_2\tau$ given in legend.

the nomenclature of case I and case II for τ_3 and τ_0 , respectively. Using this convention, we find the case I term in the current (16) leads directly to a case I term in the optical conductivity (2), in addition to the usual case II term

$$\sigma(\omega) = \sigma_{\text{I}}(\omega) + \sigma_{\text{II}}(\omega). \quad (22)$$

In Eq. (22) the case I/II cross terms are omitted as they disorder average to zero (see Appendix D 2). Moreover, the optical conductivity is further altered by the current induced Doppler shift, which causes the indirect (momentum nonconserving) gap $\omega_{\text{ig}\pm} = 2\Delta_2 \pm qk_{2F}/2m_2 + O(q^2/m_2)$ to become smaller than the direct (momentum conserving) gap $2\Delta_2$ (see Fig. 1 and Fig. 3 inset).

We now discuss the case I and case II contributions to the optical conductivity in the presence of a finite current. These are obtained via exact diagonalization in Fig. 3, where the scattering term (3) is used for Gaussian distributed elements $V_{\mathbf{k},\mathbf{q}} = V_{-\mathbf{k},-\mathbf{q}}^*$ (numerical details in Appendix D). The current mediated case I contribution consists of a single peak at 2Δ . Unlike in the clean case, where the response is zero at frequencies below the peak $\omega < 2\Delta_2$, in the presence of scattering a tail extends down to the indirect gap at $\omega_{\text{ig}-} < 2\Delta_2$. Most significantly, the presence of a current means the sharp peak

at the direct gap $\omega = 2\Delta_2$ is not protected by time-reversal symmetry, and becomes rounded. This effect is quantitatively captured by repeating the calculation of Eq. (18) with a finite quasiparticle lifetime τ (see Appendix C) yielding

$$\sigma_{\parallel}(\omega) = \frac{\rho_{2F}}{\omega} \left(\frac{ev_c \delta m}{m_2} \right)^2 f\left(\frac{\omega\tau - i}{2\Delta_2\tau}\right), \quad f(z) = \frac{2i \arcsin(z)}{z\sqrt{1-z^2}}. \quad (23)$$

We find Eq. (23) the clean divergences in the dissipative and reactive parts both become truncated at a maximal height $\sigma(\omega) = O(\tau^{1/2})$. This may be compared with the normal phase response at the same frequency $\text{Re}(\sigma_N) = D\tau/(1 + \omega^2\tau^2)$ [55]. In the clean limit $2\Delta_2\tau \gtrsim 1$, at the gap edge $\omega = 2\Delta_2$, we thus find a peak contrast of

$$\begin{aligned} \frac{\text{Re}(\sigma_{\parallel})}{\text{Re}(\sigma_N)} &\sim \frac{\pi \rho_{2F}}{2D} \left(\frac{ev_c \delta m}{m_2} \right)^2 (2\Delta_2\tau)^{\frac{3}{2}} \\ &= \frac{\pi d}{2} \left(\frac{v_c \delta m}{v_{2F} m_2} \right)^2 (2\Delta_2\tau)^{\frac{3}{2}}, \end{aligned} \quad (24)$$

where \sim indicates asymptotic equality at large τ , and the second equality in Eq. (24) applies for a parabolic band in dimension d with Fermi velocity v_{2F} . We find the $O(\tau^{3/2})$ scaling is in good agreement with numerics (see Fig. 3 and Appendix D). Lastly we note that, as before in Eq. (19), the real and imaginary peaks are asymptotically equal in this limit $\text{Re}(\sigma_{\parallel}) \sim \text{Im}(\sigma_{\parallel})$.

The case II contribution (Fig. 3) exhibits no dramatic changes from the $j_{\text{sc}} = 0$ case, altered only due to the change in the gap structure: σ_{II} converges to the normal phase Drude response far above the gap $\text{Re}(\sigma_{\text{II}})/\text{Re}(\sigma_N) \sim 1$, and remains of comparable scale at all frequencies above the indirect gap. In the absence of a current, the case II response goes continuously to zero at the gap edge $\omega = 2\Delta_2$. In the presence of the supercurrent this divides into three onsets, one at each of the gaps—in one spatial dimension the indirect gaps are linear onsets, whereas the direct gap is softer due to additional suppression of the matrix elements. In larger spatial dimension the onsets at the indirect gaps become softer, as only parts of the Fermi surface where \mathbf{k} is parallel to \mathbf{q} contribute to the density of states at the indirect gap. The case I and case II contributions are shown separately in the lower panels of Fig 3.

V. DISCUSSION

In this manuscript we have described a supercurrent enabled optical response present in BCS superconductors. Such a supercurrent may be due to an external current source, or a screening current induced by a magnetic field. An important ingredient was the breaking of current conservation, which itself requires the breaking of Galilean symmetry, i.e., the presence of some form of lattice physics. As an example we considered pair scattering between bands with different Fermi velocities. However, the type of lattice physics employed is unimportant, as the current mediated response should be expected in any system where electron scattering does not conserve the total current. Other examples of physics which provide such nonconservation of current include Umklapp scattering, and strongly nonparabolic dispersion at the

Fermi surface. These examples, though generically providing a weaker response, may be appealing avenues for future study as they require only a single band.

A current enabled optical response was recently reported in the experimental analysis of Ref. [25]. In this experiment, a current enabled change to the optical conductivity was observed, which manifested in an absorption (reactance) peak beginning at $\omega = 2\Delta$ and extending just above (below) the gap. These peaks were observed to reach a height quadratic in the current, in agreement with (23), and were of typical sizes $\text{Re}(\sigma_{\parallel})/\text{Re}(\sigma_{\text{N}}) \sim 0.025$ [c.f. Eq. (24)].

In this manuscript, we have provided a proof of principle analysis showing that the presence of multiple gaps is sufficient to lead to such a current enabled response such as that observed in Ref. [25]. For simplicity, we assume a specific hierarchy of gap scales (10) for which the optical conductivity may be analytically calculated. However, we do not expect this assumption to be an important ingredient in obtaining the gap edge response, which should be present whenever there are low-energy scattering processes which allow the current to relax.

Our theory differs from the alternative theoretical proposal of Ref. [56], which considered a model of a dirty single band superconductor in the diffusive limit, and showed here too, one finds a current induced peak in $\sigma(\omega)$, consistent with the observations of Ref. [25]. The difference in origin of the gap edge peak uncovered in the present manuscript and in Ref. [56] manifests in different limiting behavior with disorder strength: the gap edge peak predicted by Ref. [56] becomes small in the limits of weak disorder (in which the clean limit of $\sigma_{ab}(\omega) = iD_{ab}/\omega$ is approached), and in the limit of strong disorder (where the absence of time-reversal symmetry means that the gap edge features are strongly smeared by the disorder), providing only an intermediate crossover regime where the effect is present. In contrast, in the present analysis the peak in the optical conductivity follows from the gap structure, rather than scattering, produces a response which grows asymptotically as $\tau^{3/2}$ with the scattering time τ as the clean limit $\tau \rightarrow \infty$ is approached, providing a large response in clean samples. One direction which will be interesting to pursue in future work is to quantitatively determine the relative sizes of the different contributions to the gap edge peak for different materials, in particular for the NbN system studied in Ref. [25].

More broadly, our theoretical treatment extends the standard BCS theory of superconductivity to situations where the condensate is flowing. In this setting, it is necessary to have accurate calculation of the current response as it necessitates correct accounting for the exchange of momentum between the condensate and excitations upon creation/destruction of a Cooper pair. In the absence of a supercurrent, this theory reduces to the standard BCS theory. Consequently, if one follows the Mattis-Bardeen treatment of scattering effects, one will obtain the usual form of the optical conductivity in the limit of zero supercurrent [27–31].

We comment on further interesting avenues for future studies. The large reactive response which is generated in the superconducting gap provides a mechanism for the coherent coupling of THz radiation and superconducting circuits [57]. Additionally, we note that the current enabled optical response

allows for excitation of the Higgs mode. The coherent generation of excitations above the superconducting gap results in a suppression of the pairing potential, and this perturbation decays amidst long-lived oscillations of the collective Higgs mode $\delta\Delta(t) \sim \cos(2\Delta t)/\sqrt{2\Delta t}$ [58–62]. Here we present a single photon process, second order in the current, which permits such a coupling. This provides an alternative mechanism to previous efforts which have focused on achieving such a coupling using multiphoton processes [62]. Lastly we note that our theory relies only on the scattering between bands, which may be taken as parabolic in the simplest example. We leave to future investigations the exploration of how effects due to the quantum geometry of electrons [32,63,64] may also contribute to the superconducting optical response.

Note added in proof. Recently a related work appeared [65].

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APPENDIX A: GAUGE CONSTRAINT DERIVATION

In this Appendix we formalize the arguments presented in the main text, showing how the current and number operators obtained in the main text (8) may be obtained from a BCS theory in an extended Hilbert space in which charge conservation is enforced using a gauge constraint.

We begin with a many fermion Fock space \mathcal{F} with associated creation (annihilation) operators $f_{\alpha k \sigma}^{\dagger}$ ($f_{\alpha k \sigma}$). We then extend this Hilbert space to include a degree of freedom which counts total number of particles in the system

$$\mathcal{H}_{\text{E}} = \mathcal{F} \otimes \mathcal{H}_{\phi}, \quad (\text{A1})$$

where $\mathcal{H}_{\phi} = L_2(S_1)$ is spanned by $|n\rangle$, $n \in \mathbb{Z}$, or equivalently in its dual basis $|\phi\rangle$, $\phi \in [0, 2\pi]$ in the canonically conjugate basis. Operators acting on \mathcal{H}_{ϕ} include the number operator $N_{\phi} = -i\partial_{\phi}$ and the raising (lowering) operator $e^{i\phi}$

$$N_{\phi}|n\rangle = n|n\rangle, \quad e^{i\phi}|n\rangle = |n+1\rangle. \quad (\text{A2})$$

We impose the gauge constraint that N_{ϕ} is equal to the total number of fermions $N = \sum_{k\sigma} f_{\alpha k \sigma}^{\dagger} f_{\alpha k \sigma}$

$$\mathcal{F}' = \text{span}(|\psi\rangle : |\psi\rangle \in \mathcal{H}_{\text{E}}, (N - N_{\phi})|\psi\rangle = 0), \quad (\text{A3})$$

or in more concise terms, we fix to a gauge

$$N = N_{\phi}. \quad (\text{A4})$$

One may verify that \mathcal{F}' is itself a fermionic Fock space with associated fermionic creation (annihilation) operators

$$c_{\alpha k \sigma}^{\dagger} = f_{\alpha k \sigma}^{\dagger} e^{i\phi}, \quad c_{\alpha k \sigma} = f_{\alpha k \sigma} e^{-i\phi}. \quad (\text{A5})$$

More specifically, \mathcal{F} and \mathcal{F}' are isomorphic, related by the unitary map

$$c_{\alpha k \sigma}^{\dagger} = U f_{\alpha k \sigma}^{\dagger} U^{\dagger}, \quad U = e^{i\phi N}. \quad (\text{A6})$$

A useful picture for understanding this construction is to view Eq. (A5) as a decomposition of the electron $c_{\alpha k \sigma}^\dagger$ into a chargeless fermionic quasiparticle $f_{\alpha k \sigma}^\dagger$ and a charge $e^{i\phi}$. In accordance with this intuition, the electrons have the global U(1) transformation

$$e^{i\alpha Q} c_{\alpha k \sigma}^\dagger e^{-i\alpha Q} = c_{\alpha k \sigma}^\dagger e^{ie\alpha}, \quad Q = eN_\phi. \quad (\text{A7})$$

Due to our extension of the Hilbert space, there is a gauge freedom in defining operator on \mathcal{F}

$$O \rightarrow O' = O + \lambda(N - N_\phi). \quad (\text{A8})$$

In the full theory this gauge choice is inconsequential. However, in the mean-field theory, this gauge choice leads to ambiguities which must be resolved.

We now turn to consider the full Hamiltonian

$$H = \sum_{\alpha k \sigma} \xi_{\alpha k} c_{\alpha k \sigma}^\dagger c_{\alpha k \sigma} + \sum_{\alpha q} g_{\alpha \beta} P_{\alpha q}^\dagger P_{\beta q}, \quad (\text{A9})$$

where $P_{\alpha q}^\dagger = \sum_k c_{\alpha k+q}^\dagger c_{\alpha-k}^\dagger$ and $\xi_{\alpha k} = k^2/2m_\alpha - \mu_\alpha$ creates a pair of electrons with total momentum q . The corresponding number, momentum, and current density operators are given by

$$N = \sum_{\alpha k \sigma} c_{\alpha k \sigma}^\dagger c_{\alpha k \sigma}, \quad (\text{A10a})$$

$$K = \sum_{\alpha k \sigma} k c_{\alpha k \sigma}^\dagger c_{\alpha k \sigma}, \quad (\text{A10b})$$

$$\mathbf{j} = \frac{e}{V} \sum_{\alpha k \sigma} \mathbf{v}_{\alpha k} c_{\alpha k \sigma}^\dagger c_{\alpha k \sigma}, \quad (\text{A10c})$$

where $\mathbf{v}_{\alpha k} = \partial_k \xi_{\alpha k}$. In a situation where electrons cannot flow in an out of the system from an external environment, the electronic charge of the system remains fixed to its initial value N_0 , thus we have the additional constraint

$$N_\phi = N_0. \quad (\text{A11})$$

The BCS mean-field analysis breaks conservation of a number of fermions. Here we seek to apply this analysis in a manner that preserves the conservation of charge, specifically we relax the constraint (A4) (the conservation of chargeless fermionic quasiparticles) while maintaining the constraint (A11) (the conservation of charge). This is achieved by applying the usual mean-field arguments in the basis of the $f_{\alpha k \sigma}^\dagger$, so following the example used in the main text, we consider pairing dominated by the α band

$$\langle F_{1k} \rangle = \delta_{kq} \Delta_2 / g_{21} = \delta_{kq} \Delta_1 / g_{11}, \quad \langle F_{2k} \rangle = 0 \quad (\text{A12})$$

where

$$F_{\alpha q}^\dagger = \sum_k f_{\alpha k+q}^\dagger f_{\alpha-k}^\dagger. \quad (\text{A13})$$

This leads to the usual mean-field replacement in the contact interaction

$$H_{\text{int}} = \sum_{\alpha q} g_{\alpha \beta} F_{\alpha q}^\dagger F_{\beta q} \rightarrow \tilde{H}_{\text{int}} = \sum_{\alpha} (\Delta_{\alpha} F_{\alpha q} + \text{H.c.}) - \Delta_1^2 / g_{11}. \quad (\text{A14})$$

Neglecting the overall constant this yields the mean-field Hamiltonian

$$\tilde{H} = \sum_{\alpha k \sigma} \xi_{\alpha k} f_{\alpha k \sigma}^\dagger f_{\alpha k \sigma} + \sum_{\alpha} (\Delta_{\alpha} F_{\alpha q} + \text{H.c.}), \quad (\text{A15})$$

which we may subsequently write in terms of the electron algebra

$$\tilde{H} = \sum_{k \sigma} \xi_{\alpha k} c_{\alpha k \sigma}^\dagger c_{\alpha k \sigma} + \sum_{\alpha} \Delta_{\alpha} (e^{2i\phi} P_{\alpha q} + \text{H.c.}). \quad (\text{A16})$$

In relaxing the constraint on the number of fermionic quasiparticles, we introduce a physical distinction between previously gauge equivalent operators (A8). It is thus necessary to fix a gauge consistent with the mean-field theory. In cases where there are conservation laws, this gauge fixing is easily achieved by requiring that the mean-field theory also respect the appropriate conservation laws. For example, in the case of momentum, we have $[H, \mathbf{K}] = 0$, and we seek to find $\tilde{\mathbf{K}} = \mathbf{K} + \lambda(N_\phi - N)$ such that $[\tilde{H}, \tilde{\mathbf{K}}] = 0$. One finds the solution

$$\tilde{\mathbf{K}} = \mathbf{K} + (q/2)(N_\phi - N) = qN_\phi/2 + \sum_{k \sigma} (\mathbf{k} - q/2) c_{\alpha k \sigma}^\dagger c_{\alpha k \sigma}. \quad (\text{A17})$$

Applying the same logic to the case of charge we obtain

$$\tilde{N} = N_\phi. \quad (\text{A18})$$

Current is generically not a conserved quantity, but we can nevertheless gauge fix in an analogous manner, requiring that the algebra of $[H, \mathbf{j}]$ is conserved. Specifically we want that as

$$[\tilde{\mathbf{j}}, P_{\alpha q}^\dagger] = e v_{\alpha q} P_{\alpha q}^\dagger / V, \quad (\text{A19})$$

and that its mean-field replacement $P_{\alpha q}^\dagger \rightarrow \langle P_{\alpha q}^\dagger \rangle = \delta_{\alpha 1} \Delta_1 e^{2i\phi} / g_{11}$ should act on the total current in an analogous way

$$[\tilde{\mathbf{j}}, e^{2i\phi}] = e v_{\alpha q} e^{2i\phi} / V \quad (\text{A20})$$

yielding the mean-field current operator

$$\tilde{\mathbf{j}} = \frac{e}{V} \left(\mathbf{v}_{1q} N_\phi + \sum_{k \sigma} (\mathbf{v}_k - \mathbf{v}_{1q}) c_{\alpha k \sigma}^\dagger c_{\alpha k \sigma} \right). \quad (\text{A21})$$

It is natural in the superconducting context to identify the ‘‘missing fermions’’ with the condensate. Specifically, the number of Cooper pairs in the condensate is taken to be

$$N_\theta = (N_\phi - N) / 2 \quad (\text{A22})$$

with a canonically conjugate coordinate

$$e^{i\theta} = e^{2i\phi}. \quad (\text{A23})$$

With these substitutions the results in the main text (6), Eqs. (7), (8), and (14) are readily obtained.

APPENDIX B: GALILEAN INVARIANCE, CONSERVATION OF MOMENTUM AND CHARGE CONSERVATION IN THE MEAN-FIELD THEORY

In this Appendix we show the argument presented in the main text and Appendix A restores the correct Galilean

transformation properties of the mean field Hamiltonian \tilde{H} . Corollaries of the restored behavior under Galilean transformations are the restored conservation of momentum, current, and particle number.

First we recap the technical content of Galilean invariance. A Galilean transformation to a moving reference frame $\mathbf{r} \rightarrow \mathbf{r}' = \mathbf{r} + \mathbf{u}t$ is implemented by the unitary

$$U_{\mathbf{u}} = \exp(i\mathbf{u} \cdot \mathbf{g}), \quad \mathbf{g} = M\mathbf{R} - \mathbf{K}t, \quad (\text{B1})$$

where \mathbf{K} is the total momentum and \mathbf{R} the center of mass

$$\mathbf{K} = \sum_{k\sigma} k c_{k\sigma}^\dagger c_{k\sigma}, \quad \mathbf{R} = M^{-1} \sum_{r\sigma} m r c_{r\sigma}^\dagger c_{r\sigma}, \quad (\text{B2})$$

and $M = mN$ is the total mass, with N the number of electrons and m their effective mass. Under this transformation position and momentum base creation operators transform as

$$\begin{aligned} U_{\mathbf{u}} c_{k,\sigma}^\dagger U_{\mathbf{u}}^\dagger &= c_{k+mu,\sigma}^\dagger \exp[-i\mathbf{k} \cdot \mathbf{u} - imu^2 t/2], \\ U_{\mathbf{u}} c_{r,\sigma}^\dagger U_{\mathbf{u}}^\dagger &= c_{r+ut,\sigma}^\dagger \exp[i\mathbf{m}\mathbf{r} \cdot \mathbf{u} + imu^2 t/2]. \end{aligned} \quad (\text{B3})$$

Galilean symmetry is the statement that the dynamics of $|\psi\rangle$ and $|\psi'\rangle = U_{\mathbf{u}}|\psi\rangle$ are generated by the same Hamiltonian

$$i\partial_t|\psi\rangle = H|\psi\rangle, \quad i\partial_t(U_{\mathbf{u}}|\psi\rangle) = H U_{\mathbf{u}}|\psi\rangle. \quad (\text{B4})$$

A system is thus Galilean symmetric if and only if the Hamiltonian satisfies

$$H = U_{\mathbf{u}} H U_{\mathbf{u}}^\dagger - i U_{\mathbf{u}} \partial_t U_{\mathbf{u}}^\dagger. \quad (\text{B5})$$

It is readily verified that the full Hamiltonian (3) satisfies the transformation rule (B5) provided the dispersion is parabolic.

However we cannot make the requirement of Galilean invariance on the mean-field Hamiltonian \tilde{H} . This is because the mean-field theory explicitly breaks Galilean symmetry by privileging a specific frame, the rest frame of the condensate. Instead the Galilean transformation relates a family of mean field Hamiltonians $\tilde{H}_{\mathbf{q}}$ parameterized by the Cooper pair momentum \mathbf{q}

$$\tilde{H}_{\mathbf{q}} = \sum_{k\sigma} \xi_k c_{k\sigma}^\dagger c_{k\sigma} + \Delta(e^{i\mathbf{q}} P_{\mathbf{q}} + \text{H.c.}) \quad (\text{B6})$$

Here, for clarity, we have explicitly labeled the momentum \mathbf{q} carried by the mean-field, writing $e^{i\mathbf{q}}$. As the cooper pair momentum transforms as $\mathbf{q} \rightarrow \mathbf{q}' = \mathbf{q} + 2m\mathbf{u}$ under Galilean transformations, the mean-field Hamiltonian must satisfy the transformation rule

$$\tilde{H}_{\mathbf{q}+2m\mathbf{u}} = \tilde{U}_{\mathbf{u}} \tilde{H}_{\mathbf{q}} \tilde{U}_{\mathbf{u}}^\dagger - i \tilde{U}_{\mathbf{u}} \partial_t \tilde{U}_{\mathbf{u}}^\dagger. \quad (\text{B7})$$

We now show that the mean-field Hamiltonian H satisfies the Galilean transformation rule (B7) if we make the appropriate requirements for the transformation properties of the superconducting phase $e^{i\mathbf{q}}$. In detail, if we define a mean Galilean transformation operator for the mean-field theory $\tilde{U}_{\mathbf{u}} = \exp(i\mathbf{u} \cdot \tilde{\mathbf{g}})$, where $\tilde{\mathbf{g}} = M\tilde{\mathbf{R}} - \tilde{\mathbf{K}}t$ where we require

$$\begin{aligned} [e^{i\mathbf{q}}, e^{-i\mathbf{q}}] &= 0, \quad [\tilde{\mathbf{N}}, e^{i\mathbf{q}}] = 2e^{i\mathbf{q}}, \\ [M\tilde{\mathbf{R}}, e^{i\mathbf{q}}] &= -2im\partial_{\mathbf{q}} e^{i\mathbf{q}}, \quad [\tilde{\mathbf{K}}, e^{i\mathbf{q}}] = \mathbf{q} e^{i\mathbf{q}}, \end{aligned} \quad (\text{B8})$$

then we recover the desired property, that $e^{i\mathbf{q}}$ transforms as a creation operator for a pair of electrons with total momentum

\mathbf{q}

$$U_{\mathbf{u}} e^{i\mathbf{q}} U_{\mathbf{u}}^\dagger = e^{i\mathbf{q}+2m\mathbf{u}} \exp[-i\mathbf{q} \cdot \mathbf{u} - imu^2 t] \quad (\text{B9})$$

and it is readily verified that the mean field Hamiltonian (B6) satisfies the transformation rule (B7).

It may be further verified that the mean-field Hamiltonian conserves momentum and particle number

$$[\tilde{H}_{\mathbf{q}}, \tilde{\mathbf{K}}] = 0, \quad [\tilde{H}_{\mathbf{q}}, \tilde{\mathbf{N}}] = 0, \quad [\tilde{H}_{\mathbf{q}}, \tilde{\mathbf{J}}] = 0, \quad (\text{B10})$$

where the current is given by $\tilde{\mathbf{J}} = -e\tilde{\mathbf{K}}/m$.

APPENDIX C: DERIVATION OF THE OPTICAL RESPONSE IN THE EFFECTIVE ONE-BAND MODEL

In this Appendix we derive the optical conductivity (18) as plotted in Fig. 1, and obtain the form (23) for the case I response in the presence of a finite scattering time.

Starting the effective Hamiltonian (12) and current operator (16), given to first order in (t/ξ_{2k}) and zeroth order in $(\delta\xi_{1k} - \delta\xi_{2k})/\xi_{2k}$ by

$$\begin{aligned} \tilde{H}_{2k} &= \bar{\xi}_{2k}\tau_3 + \delta\xi_{2k}\tau_0 + \Delta_2(e^{i\theta}\tau_+ + \text{H.c.}), \\ \tilde{\mathbf{J}}_{2k} &= e[\bar{\mathbf{v}}_{2k}\tau_0 + \mathbf{v}_c(\delta m/m_2)\tau_3]/V. \end{aligned} \quad (\text{C1})$$

The current correlator is then easily evaluated

$$\langle [J_a(t), J_a(0)] \rangle = \sum_{k\alpha\beta} (f_{k\alpha} - f_{k\beta}) |\langle k\alpha | \tilde{\mathbf{J}}_{a,2k} | k\beta \rangle|^2 e^{i(E_{k\beta} - E_{k\alpha})t}, \quad (\text{C2})$$

where $\alpha = 1, 2$ enumerates the eigenvectors of the effective Hamiltonian $\tilde{H}_{2k} |k\alpha\rangle = E_{k\alpha} |k\alpha\rangle$, and $f_{k\alpha} = f(E_{k\alpha})$ is population of the α th level, given by the Fermi-Dirac distribution. Evaluating this at $T = 0$, including a finite quasiparticle lifetime $\tau > 0$ and we obtain

$$\begin{aligned} C(\omega) &= V \int_0^\infty dt e^{i\omega t} \langle [j_a(t), j_a(0)] \rangle \\ &= \int_{-\infty}^\infty d\xi \frac{\rho_{2F} j_{aq}^2 \Delta_2^2}{\Delta_2^2 + \xi^2} \sum_{s=\pm} \frac{is}{\omega + i/\tau - 2s\sqrt{\xi^2 + \Delta_2^2}} \\ &= \int_{2\Delta}^\infty \frac{4\rho_{2F} j_{aq}^2 \Delta_2^2 d\omega'}{\omega' \sqrt{\omega'^2 - 4\Delta_2^2}} \sum_{s=\pm} \frac{is}{\omega - s\omega' + i/\tau}, \end{aligned} \quad (\text{C3})$$

where ρ_{1F} is the normal phase density of states of the first band at the Fermi surface and we have set

$$j_{aq} = \frac{eq_a \delta m}{2m_1 m_2} = \frac{ev_{c,a} \delta m}{m_2}, \quad (\text{C4})$$

where $\mathbf{v}_c = \mathbf{q}/2m_1$ is the condensate velocity. To obtain $C(\omega)$ in the $\tau \rightarrow \infty$ limit we apply the Sokhotski-Plemelj theorem and evaluate the subsequent integrals to obtain

$$\sigma_{aa}(\omega) = \frac{iD_{aa}}{\omega} + \frac{e^2 \rho_{2F}}{\omega} \cdot \left(\frac{v_{c,a} \delta m}{m_2} \right)^2 \cdot f\left(\frac{\omega}{2\Delta_2}\right), \quad (\text{C5a})$$

where

$$f(x) = -f^*(-x) = \begin{cases} \frac{\pi - 2i \operatorname{arccosh}(x)}{x\sqrt{x^2 - 1}} & x > 1 \\ \frac{2i \operatorname{arcsin}(x)}{x\sqrt{1 - x^2}} & 0 < x < 1. \end{cases} \quad (\text{C5b})$$

Taking the real part we obtain Eq. (18) as desired, both the real and imaginary parts are shown in Fig. 1.

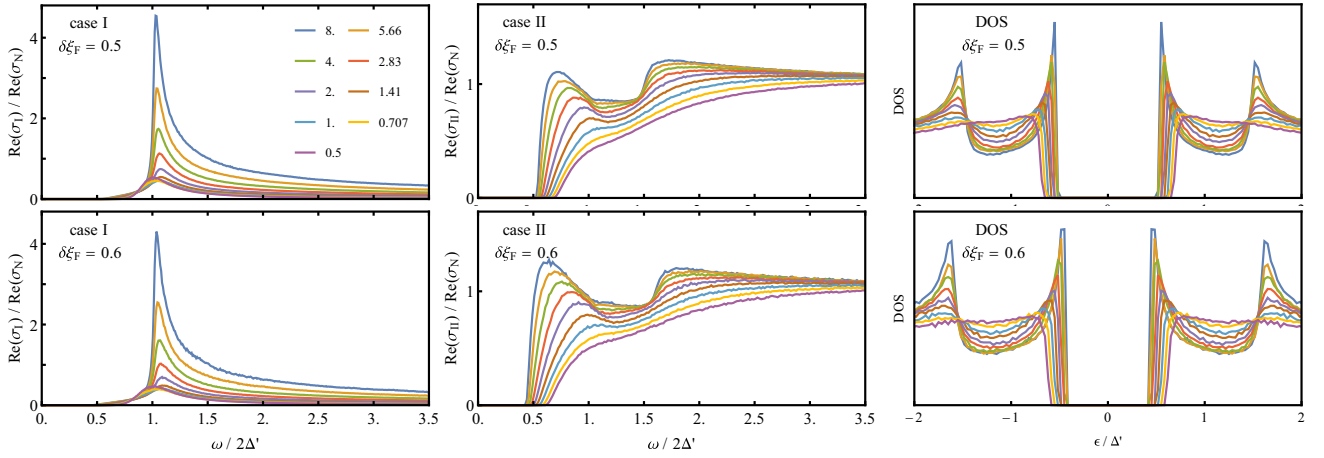


FIG. 4. *Additional data, as in Fig. 3:* Here we show data, as in Fig. 3, for additional parameters. Data is shown for two different values of the current, parameterized by $\delta\xi_{2F} = qk_{2F}/2m_2 = 0.5, 0.6$ (top and bottom row, respectively). The columns show the Case I part of the optical conductivity, the Case II part of the optical conductivity, and the density of states, respectively. The Case I part is not significantly effected by vary the current but far a small change to the overall scale factor. Increasing the current does however change the indirect gap, moving the onset of the Case II part to lower frequencies (second column), whereas in terms of the quasiparticle density of states (third column) ones sees the inner pair of edges (at $\epsilon = \pm|\Delta - \delta\xi_{2F}|$) move to lower frequencies, and the outer pair (at $\epsilon = \pm|\Delta + \delta\xi_{2F}|$) to higher frequencies.

For finite τ we may evaluate Eq. (C3) directly, obtaining

$$\sigma_{aa}(\omega) = \frac{ine^2}{\omega m} + \frac{\rho_{1F} j_{aq}^2}{\omega} f\left(\frac{\omega - i/\tau}{2\Delta_2}\right) \quad (C6)$$

with

$$f(z) = \frac{2i \arcsin(z)}{z\sqrt{1-z^2}} \quad (C7)$$

and it may be verified that in the limit of $\tau \rightarrow \infty$ (i.e., in the limit of $z \in \mathbb{R}$) this coincides with the previous result Eq. (C5).

In the limit of large τ , in the vicinity of the gap edge, this form simplifies

$$\sigma_{aa}(\omega) \sim \frac{ine^2}{2\Delta_2 m} + \frac{\pi \rho_{2F} j_{aq}^2}{2\Delta_2} \sqrt{\frac{\Delta_2}{\omega - 2\Delta_2 - i/\tau}}, \quad (C8)$$

where \sim indicates asymptotic equality when the limit of large τ is taken with $\tau(\omega - 2\Delta_2)$ held fixed. This reveals the maximum height of $\sigma_{aa}(2\Delta_2) = O(\sqrt{\tau})$ which holds for the complex conductivity, and for the real and imaginary parts individually.

We comment that this approach does not recover the Case I response, i.e., the term which reproduces Drude conductivity in the limit $\Delta_2 \rightarrow 0$. This term must be obtained by other means [27–31,49–53,55].

APPENDIX D: EXACT DIAGONALIZATION

In this section we provide details and additional data regarding the random matrix model used to numerically obtain forms for the dissipative part of the optical conductivity in Fig. 3 for a superconducting wire.

Data for the cases of $\delta\xi_{2F} = qk_{2F}/2m_2 = 0.5, 0.6$ is provided in Fig. 4, while Fig. 5 provides numerical confirmation of the scaling of the current induced absorption peak stated in Eq. (24).

1. Numerical details

We begin from the BdG Hamiltonian (12). For simplicity we assume a hierarchy of scales that allows us to consider effects due to the second band only. Specifically we work in the frequency window $\omega \in [-\omega_c, \omega_c]$ such that

$$2\Delta_2 + \frac{qk_{2F}}{2m_2} + \frac{1}{\tau} < \omega_c < 2\Delta_1 - \frac{qk_{1F}}{2m_1} - \frac{1}{\tau}. \quad (D1)$$

In words: we assume that ω_c is at least τ^{-1} above the larger indirect gap of the second band, and at least τ^{-1} below the smaller indirect gap of the first band. In this frequency range the current operator for the first band has no spectral weight, and we may calculate the optical conductivity considering consider only the current operator for the second band. At the end of this section we discuss violations of this assumption.

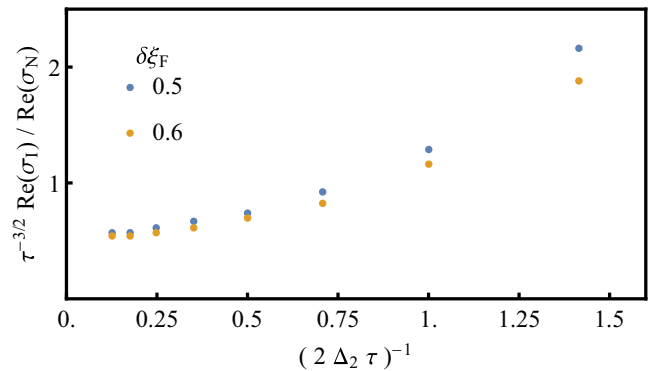


FIG. 5. *Scaling of the peak value of the Case I response:* Here we show the peak values of the Case I response (left column of Fig 4) rescaled by $\tau^{-3/2}$. The two series correspond to distinct values of the current (values of $\delta\xi_{2F}$ inset). The data displays apparent asymptotic convergence to a finite constant, confirming the relationship $\text{Re}(\sigma_I)/\text{Re}(\sigma_N) = O(\tau^{3/2})$ discussed in the main text, and derived in Appendix C.

We thus consider only the Hamiltonian of the second band (12). In the present context, where momentum is not conserved, it is convenient to write this as a block matrix

$$\tilde{H} = \Psi^\dagger \tilde{h} \Psi, \quad (\text{D2})$$

where Ψ is the vector of all ψ_k . It is convenient to choose a basis in which \tilde{h} takes the block diagonal form

$$\tilde{h} = \begin{pmatrix} \tilde{h}_0 + \sigma_3 \delta \xi_{2F} & \tilde{\Delta} \\ \tilde{\Delta}^\dagger & -\tilde{h}_0^T - \sigma_3 \delta \xi_{2F} \end{pmatrix}, \quad (\text{D3})$$

where here and throughout σ_α are the usual Pauli matrices. Here \tilde{h} is denoted as block matrix with the top left (bottom right) blocks correspond to the particle (hole) sector, and the off-diagonal blocks correspond to the pair creation/breaking processes. Within the particle sector \tilde{h}_0 there are two further block—left movers and right movers (which have a relative offset due to the Doppler shift $\sigma_3 \delta \xi_{2F}$), and within each of these blocks there are d states labeled by their different crystal momenta k —thus \tilde{h} is a $4d \times 4d$ matrix.

The normal phase Hamiltonian is the sum of kinetic and potential energy terms

$$\tilde{h}_0 = T_0 + V_0. \quad (\text{D4})$$

In the one-dimensional setting considered here, the kinetic energy T_0 may in turn be expressed as a 2×2 block diagonal matrix, with the two blocks corresponding to left and right movers, respectively,

$$T_0 = t_0 \sigma_0, \quad t_0 = v_{2F} (\hat{k} - k_{2F}) \sigma_0, \quad (\text{D5})$$

where \hat{k} is a matrix which measures the *unsigned* momentum. For exact diagonalization we keep only a finite number of states in the vicinity of the Fermi surface, such that \hat{k} is a $d \times d$ diagonal matrix whose diagonal values (i.e., eigenvalues) k_v are evenly spaced over the interval specified by $E_v = v_{2F}(k_v - k_{2F}) \in [-E_c, E_c]$, where we use an energy cutoff $E_c = \omega_c + \Delta_2 + \delta \xi_{2F} + \lambda_V$ where $\lambda_V = v\sqrt{2d}$ is the disorder bandwidth. E_c is chosen to exceed ω_c by an additional buffer

The pairing potential matrix may be written in this basis

$$\tilde{\Delta} = \begin{pmatrix} 0 & \Delta_2 \sigma_1 \\ \Delta_2 \sigma_1 & 0 \end{pmatrix} = \Delta_2 \sigma_1 \tau_1, \quad (\text{D6})$$

where Δ is the scalar pairing potential.

The scattering potential V_0 is a $2d \times 2d$ matrix which mixes couples left and right moving electrons. As the scattering potential is assumed to be real in the position basis, the matrix V satisfies

$$\sigma_1 V_0 \sigma_1 = V_0^*, \quad (\text{D7})$$

where σ_1 is the usual Pauli matrix, in this context implementing coordinate inversion, sending left movers to right movers and vice versa. We use a random matrix model of the scattering potential, assuming it induces a statistically identical coupling between all low energy momentum modes. Specifically, we use

$$V = URU^\dagger, \quad U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}, \quad (\text{D8})$$

where $R = R^T$, one may verify that V satisfies (D7) as required. We take R to be a Gaussian orthogonal matrix, i.e., the

matrix elements R_{ij} are real, identically distributed Gaussian random numbers with correlations

$$[R_{ij} R_{nm}] = v^2 (\delta_{in} \delta_{jm} + \delta_{im} \delta_{jn}), \quad (\text{D9})$$

where v sets the typical scale of the matrix elements of V . The matrix element scale v is related to the scattering time τ by Fermi's Golden rule

$$1/\tau = 2\pi \rho v^2, \quad (\text{D10})$$

where $\rho = d/E_c$ is the density of states coupled by the disorder, i.e., of T_0 . We note that for the Fermi Golden rule result to be valid, we must be in the regime where it has a nonperturbative effect on T_0 , the kinetic term, $\rho v = dv/E_c \gg 1$. However, we also want to remain in the regime where the disorder does not alter the density of states of the minimal model—that is the typical eigenvalue of T_0 is of scale $\lambda_T = E_c/\sqrt{3}$, far exceeds the typical eigenvalue scale of V_0 , $\lambda_V = v\sqrt{2d}$, yielding the condition $E_c \gg v\sqrt{6d}$.

The current operator is given by Eq. (14). We are interested only in the current carried by the second band, for which, in the present notation, the BdG matrix may be decomposed into its case I and case II pieces as

$$j_2 = j_1 + j_{II}, \quad j_1 = \frac{e(\delta v_2 - v_c)}{V} \tau_3, \quad j_{II} = \frac{ev_{2F}}{V} \tau_3 \sigma_3. \quad (\text{D11})$$

Finally we calculate the optical conductivity. We divide the dissipative part of the optical conductivity into corresponding case I and case II terms

$$\text{Re}[\sigma(\omega)] = \text{Re}[\sigma_I(\omega)] + \text{Re}[\sigma_{II}(\omega)]. \quad (\text{D12})$$

Subsequently, we obtain Fig. 3 numerically by diagonalising \tilde{h} , to obtain the diagonal orbitals $\tilde{h}|\epsilon_v\rangle = \epsilon_v|\epsilon_v\rangle$ and use the relation

$$\text{Re}[\sigma_m(\omega)] = \pi V \sum_{\epsilon_\mu < 0 < \epsilon_\nu} |\langle \epsilon_\mu | j_m | \epsilon_\nu \rangle|^2 \delta(\omega - \omega_{\mu\nu}), \quad (\text{D13})$$

where $\omega_{\mu\nu} = \epsilon_\mu - \epsilon_\nu$, and $m \in \{I, II\}$, and the cross term disorder averages to zero (see Appendix D 2)

$$\overline{\text{Re}[\sigma_{\text{cross}}(\omega)]} = \pi V \overline{\sum_{\epsilon_\mu < 0 < \epsilon_\nu} \langle \epsilon_\mu | j_I | \epsilon_\nu \rangle \langle \epsilon_\nu | j_{II} | \epsilon_\mu \rangle \delta(\omega - \omega_{\mu\nu})} = 0. \quad (\text{D14})$$

Repeating this calculation for $\Delta = 0$, $j_{sc} = 0$ provides σ_N as required.

Finally we discuss parameters. As noted in previous paragraphs, the parameter regimes where this model is accurate are constrained. Specifically we require the following:

(1) The disorder strength is nonperturbative $\rho v = dv/E_c \gg 1$.

(2) The disorder bandwidth is small compared to the window of kinetic energies $\lambda_T/\lambda_V = E_c/(v\sqrt{6d}) \gg 1$.

(3) We need sufficient density of states to observed the expulsion of levels from the superconducting gap $2\Delta_2\rho = 2d\Delta_2/E_c \gg 1$.

(4) The energetic bias determining the current must also be nonperturbative, requiring $\sqrt{d}\delta\xi_{2F}/E_c \gg 1$. This is the condition that the matrix elements of $\delta\xi_{2F}\sigma_3\tau_3$ (which encode the energy bias setting the finite current) are of scale $\delta\xi_{2F}/\sqrt{d}$

far exceed the density of states in the electron/hole sectors $\rho = d/E_c$.

When these constraints are satisfied, the calculation of the optical conductivity is accurate for $\omega < \omega_c$. For the numerics presented in the main text, we use these constraints, determining x to satisfy $x \gg 1$ if $x > 2$. Specifically we set $\Delta = 1$, $E_c = 8.5, 8.6$, and $d = 4624, 3450$ for the data with $\delta\xi_{2F} = 0.5, 0.6$ respectively.

2. Cancellation of the cross term

Sufficient conditions for the case I/II cross term to average to zero are provided by

(1) The kinetic energy ξ_k may be linearized at the Fermi surface $\xi_k = v_{2F}(\mathbf{k} - \mathbf{k}_{2F})$.

(2) The scattering matrix elements are identically distributed, and symmetric about 0 (i.e., V and $-V$ occur with equal probability).

To see this, as before let $T_0 = t_0\sigma_0$, V denote the kinetic and disorder terms in \tilde{H} . Let also u denote the normal involutory ($uu^\dagger = u^\dagger u = u^2 = (u^\dagger)^2 = 1$) which changes the sign of the kinetic energy $ut_0u = -t_0$. One may show that two Hamiltonians \tilde{H} and \tilde{H}' which are identical but for their disorder realisations V and $V' = -uVu$ make exactly canceling contributions to the case I/II cross term

$$\text{Re}[\sigma_{\text{cross}}(\omega)] = \pi V \sum_{\epsilon_\mu < 0 < \epsilon_\nu} \langle \epsilon_\mu | j_I | \epsilon_\nu \rangle \langle \epsilon_\nu | j_{II} | \epsilon_\mu \rangle \delta(\omega - \omega_{\mu\nu}). \tag{D15}$$

Thus if V and $V' = -uVu$ occur in the disorder ensemble with equal probabilities the cross term averages to zero. The first condition above is sufficient for the involution u to exist, and the second is sufficient for V and $V' = -uVu$ to occur with equal measure in the disorder ensemble.

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