Reciprocity condition in synchronously time-periodic bianisotropic materials

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In this paper, a sufficient reciprocity condition for general time-periodic modulated bianisotropic media is extracted from first principles. Reciprocity of various cases of significant importance, including stationary bianisotropic media, time-varying (TV) isotropic media, TV anisotropic media, and TV bianisotropic media, are investigated using this condition. We prove that synchronous time modulation of stationary bianisotropic yet reciprocal media (chiral, pseudochiral, and achiral) does not lead to nonreciprocity, unless the modulation function breaks time reversal symmetry. This is in contrast to recently published research. The theoretical results are validated using in-house finite difference time domain simulations.

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I. INTRODUCTION

Reciprocity is an important principle in electromagnetics which states that electromagnetic wave transmission between source and observer is invariant if the location of the source and observer are swapped [1,2]. Breaking reciprocity is a crucial necessity in many microwave and optical systems to realize nonreciprocal devices such as isolators, circulators, and directional amplifiers applicable in full-duplex communications. Utilizing magneto-optical materials (e.g., ferrites) to achieve nonreciprocity was the major viable option for decades, where biasing the material with dc magnetic field allows the nonreciprocity [3–7]. Though wide-spread for microwave communications, ferrite-based devices tend to be in general bulky, lossy, and costly, and they are incompatible with integrated CMOS (complementary metal-oxide semiconductor) compatible systems [8]. Recently, magnetless approaches have been developed which yield nonreciprocity. Nonlinear nonreciprocal systems exploit the difference of field strength for opposite directions [9–13]. This approach has some limitations, such as the need for signals with sufficiently high intensity, the tradeoff between transmissions in the forward direction, and the level of input intensity for which large isolation can be achieved [14,15]. Nonreciprocal acoustic devices use angular momentum biasing to produce nonreciprocity [16,17]. But extending these mechanical motions to optical frequencies and for integrated devices can be quite challenging. Nonreciprocity based on optomechanical effects appears to be a promising approach to overcome these challenges [18,19]. Although optomechanical nonreciprocal devices provide large isolation and low loss and noise, their bandwidth is limited [19,20].

Breaking reciprocity by introducing time variations to material parameters of electromagnetic structures and circuits [21–23] has gained much attention in recent years due to its compatibility with integrated photonics [24–28]. Nonreciprocity by using linear time modulation is usually achieved when either the time modulation is traveling and thus spatiotemporal or is piecewise uniform with different phases at space. In the traveling-wave modulation approach, the permittivity of the structure, such as a waveguide or a ring resonator, is modulated with the functionality $\cos(\Omega t - \kappa \cdot \mathbf{r})$ [24,25,29–33], meaning every point of the medium exhibits different time-variation profiles with respect to each other. In piecewise uniform time-varying (TV) nonreciprocal devices, the permittivities of different parts of the system are modulated with the functionality $\cos(\Omega t + \phi_i)$, in which the phase ϕ_i is different between these parts [34–39]. Tandem phase modulators [34] and isolators based on a sequence of two direct photonic transitions separated by a waveguide [35] are examples of this category.

The common feature in these structures is that all points in space do not have the same temporal functionality for permittivity. In other words the permittivity can not be split into a product of a function of a single variable t (time) and another function of a single variable $\mathbf{r}$ (position). We shall refer to this type of modulation as asynchronous modulation. In synchronous modulation, however, any time-modulated part or point of the domain has exactly the same temporal variation function, and thus the temporal variation can be factored out as a common separable function. Synchronous modulation of isotropic media is incapable of yielding nonreciprocity except for cases in which time variation is irreversible [28].

Anisotropic and bianisotropic media with time modulation are less investigated thus far. A curious question that arises is whether synchronous time modulation of reciprocal bianisotropic materials, in particular their magnetoelectric coupling terms, can lead to nonreciprocity. To great surprise it was recently proposed that, starting with a reciprocal stationary bianisotropic medium, one can reach nonreciprocity with a common temporal variation function (synchronous) [40]. In [40] a bulk bianisotropic material with an antisymmetric magnetoelectric coupling tensor (which is reciprocal in the absence of temporal modulation) was analyzed, and it was shown that the Green's function of this material is not

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symmetric, leading to the conclusion that it must be nonreciprocal. This peculiar concept is in conflict with the common observation in typical time-modulated isotropic media which require both time and space to achieve nonreciprocity.

In order to investigate the preceding question, we look into reciprocity for the most general bianisotropic time-periodic modulated media and extract a criterion using the Lorentz reciprocity theorem for investigating the reciprocity of time-periodic modulated media in Sec. II. The Lorentz reciprocity provides a sufficient condition for reciprocity, which helps us to investigate the reciprocity of some important media, including anisotropic, chiral, and bianisotropic modulated media. We look into these cases by starting from reciprocal media and adding synchronous time modulation, in Sec. III. We investigate the reciprocity/nonreciprocity of some examples of sinusoidal modulation and show that, despite what [40] proposed, such media still remain reciprocal under any synchronous sinusoidal time modulation. Also we show that in synchronously time-periodic modulated bianisotropic media, if the modulation function has generalized time reversal symmetry, the media remain reciprocal, even in those cases where the time modulation is applied to the magnetoelectric coupling terms. We find the most general synchronous modulation for bianisotropic materials which does not result in nonreciprocal response. Finally, we validate our analytical results for some examples of sinusoidal time-modulated slabs by using in-house finite difference time domain (FDTD) simulations in Sec. IV.

II. RECIPROCITY THEOREM IN TIME-PERIODIC BIANISOTROPIC MEDIA

We wish to investigate the reciprocity/nonreciprocity of the most general linear, spatially local, dispersive, time-periodic bianisotropic media. The constitutive relations for a general linear spatially local dispersive time-varying medium can be written in time and space domains as [41–43]

$$\begin{aligned}\mathscr{D}(\mathbf{r},t) &= \int_{-\infty}^{t} \bar{\bar{\varepsilon}}(\mathbf{r},t;t-t')\cdot\mathscr{E}(\mathbf{r},t')dt' \\ &\quad + \int_{-\infty}^{t} \bar{\bar{\zeta}}(\mathbf{r},t;t-t')\cdot\mathscr{H}(\mathbf{r},t')dt', \\ \mathscr{B}(\mathbf{r},t) &= \int_{-\infty}^{t} \bar{\bar{\mu}}(\mathbf{r},t;t-t')\cdot\mathscr{H}(\mathbf{r},t')dt' \\ &\quad + \int_{-\infty}^{t} \bar{\bar{\xi}}(\mathbf{r},t;t-t')\cdot\mathscr{E}(\mathbf{r},t')dt',\end{aligned} \tag{1}$$

where $\bar{\bar{\varepsilon}}$ and $\bar{\bar{\mu}}$ are permittivity and permeability tensors, and $\bar{\bar{\zeta}}$ and $\bar{\bar{\xi}}$ are magneto-electric coupling tensors. For simplicity's sake, we assume time-periodic media such that $\bar{\bar{\varepsilon}}(\mathbf{r},t;\tau)=\bar{\bar{\varepsilon}}(\mathbf{r},t+2\pi/\Omega;\tau)$, $\bar{\bar{\mu}}(\mathbf{r},t;\tau)=\bar{\bar{\mu}}(\mathbf{r},t+2\pi/\Omega;\tau)$, $\bar{\bar{\xi}}(\mathbf{r},t;\tau)=\bar{\bar{\xi}}(\mathbf{r},t+2\pi/\Omega;\tau)$, $\bar{\bar{\zeta}}(\mathbf{r},t;\tau)=\bar{\bar{\zeta}}(\mathbf{r},t+2\pi/\Omega;\tau)$, where Ω is an arbitrary modulation frequency, and $\tau=t-t'$. For convenience of writing, the functionality of constitutive parameters is shown by a function $\bar{\bar{\nu}}(\mathbf{r},t;\tau)=\bar{\bar{\nu}}(\mathbf{r},t+2\pi/\Omega;\tau)$, which can be each of the parameters $\bar{\bar{\varepsilon}}, \bar{\bar{\mu}}, \bar{\bar{\xi}}, \bar{\bar{\zeta}}$. Because of time periodicity these

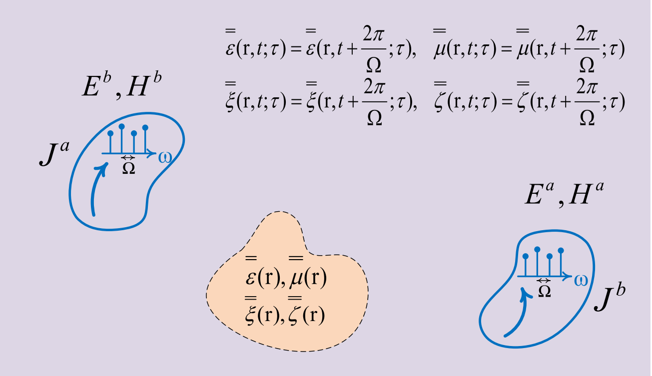

FIG. 1. Lorentz reciprocity theorem in inhomogeneous time-periodic modulated bianisotropic media. The figure depicts generalization of the Lorentz reciprocity theorem for time-periodic media such that the constitutive parameters of parts of the medium (purple region) are time modulated with a periodic function while some parts may not be modulated (orange region).

parameters can be written in the form of Fourier components $\bar{\bar{\nu}}_n(\mathbf{r};\tau)$.

In order to determine whether or not this medium is reciprocal, we utilize the Lorentz reciprocity theorem by considering two problems with current sources $\mathscr{J}^a$ and $\mathscr{J}^b$ as shown in Fig. 1. In such Ω time-periodic media, any source with frequency ω_0 will generate field harmonics at $\omega_m=\omega_0+m\Omega$. Therefore we assume the sources exciting the problem are also Ω periodic and thus can be written as a Fourier series, and represented with a vector of harmonics where the amplitude of side harmonic $\omega_m=\omega_0+m\Omega$ is shown by $\mathbf{J}_m^{a,b}$.

Under such circumstances, the Lorentz reciprocity condition for such time-periodic media in the frequency domain is [2]

$$\int_{V^a}\sum_m \frac{\mathbf{E}_m^b\cdot\mathbf{J}_m^a}{\omega_m}dV^a=\int_{V^b}\sum_m \frac{\mathbf{E}_m^a\cdot\mathbf{J}_m^b}{\omega_m}dV^b, \tag{2}$$

in which $\mathbf{E}_m^{a,b}$ shows the field harmonic at frequency ω_m. This relation states that in reciprocal media the interaction of the field harmonics $\mathbf{E}_m^a$ on the normalized source harmonics $\mathbf{J}_m^b/\omega_m$ is equal to the interaction of the field harmonics $\mathbf{E}_m^b$ on the normalized source harmonics $\mathbf{J}_m^a/\omega_m$ [2]. We will show that this expression is still not general enough as there exist reciprocal structures in which Eq. (2) is not satisfied unless a proper choice of time shift is applied.

The reciprocity condition for an arbitrary time-periodic modulation should be investigated by calculating the volume integrals in Eq. (2) and seeing if the equation holds, by solving the fields in the problem domain. This of course is a cumbersome if not impossible task for an arbitrary scenario. It would therefore be much more enticing to state the reciprocity condition based on the material parameters and without the need for the fields of the particular problem. In order to do so, we utilize the constitutive relations, which can be now written for a fully time periodic set of fields as

$$\mathbf{D}_m(\mathbf{r}) = \sum_n \left(\bar{\bar{\varepsilon}}^n_{m-n}(\mathbf{r}) \cdot \mathbf{E}_n(\mathbf{r}) + \bar{\bar{\zeta}}^n_{m-n}(\mathbf{r}) \cdot \mathbf{H}_n(\mathbf{r})\right),$$
$$\mathbf{B}_m(\mathbf{r}) = \sum_n \left(\bar{\bar{\mu}}^n_{m-n}(\mathbf{r}) \cdot \mathbf{H}_n(\mathbf{r}) + \bar{\bar{\xi}}^n_{m-n}(\mathbf{r}) \cdot \mathbf{E}_n(\mathbf{r})\right), \tag{3}$$

in which $\bar{\bar{\nu}}^n_{m-n}$ is the Fourier transform of $(m-n)$th series component at frequency ω_n, where $\bar{\bar{\nu}}$ shows each of the parameters $\bar{\bar{\varepsilon}}, \bar{\bar{\mu}}, \bar{\bar{\xi}}, \bar{\bar{\zeta}}$. The details of calculation are presented in Appendix A.

By considering two Ω periodic current sources a and b with current densities $\mathscr{J}^{a,b} = \sum_m \mathbf{J}^{a,b}_m e^{j(\omega_0+m\Omega)t}$ and substituting Eqs. (3) into Maxwell's equations, we obtain two sets of Maxwell's equations,

$$\nabla \times \mathbf{E}^{a,b}_m(\mathbf{r}) = -j\omega_m \sum_n \left(\bar{\bar{\mu}}^n_{m-n}(\mathbf{r}) \cdot \mathbf{H}^{a,b}_n(\mathbf{r}) + \bar{\bar{\xi}}^n_{m-n}(\mathbf{r}) \cdot \mathbf{E}^{a,b}_n(\mathbf{r})\right),$$
$$\nabla \times \mathbf{H}^{a,b}_m(\mathbf{r}) = \mathbf{J}^{a,b}_m(\mathbf{r}) + j\omega_m \sum \left(\bar{\bar{\varepsilon}}^n_{m-n}(\mathbf{r}) \cdot \mathbf{E}^{a,b}_n(\mathbf{r}) + \bar{\bar{\zeta}}^n_{m-n}(\mathbf{r}) \cdot \mathbf{H}^{a,b}_n(\mathbf{r})\right). \tag{4}$$

Using the identity $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$ and Maxwell's equations (4), we arrive at

$$\int_{V^a} \sum_m \frac{\mathbf{E}^b_m \cdot \mathbf{J}^a_m}{j\omega_m} dV^a - \int_{V^b} \sum_m \frac{\mathbf{E}^a_m \cdot \mathbf{J}^b_m}{j\omega_m} dV^b$$
$$= \int_V \Bigg[\sum_{m,n} \mathbf{H}^b_m \cdot \left(\bar{\bar{\mu}}^n_{m-n}(\mathbf{r}) - \bar{\bar{\mu}}^{m\,T}_{n-m}(\mathbf{r})\right) \cdot \mathbf{H}^a_n + \mathbf{E}^a_m \cdot \left(\bar{\bar{\varepsilon}}^n_{m-n}(\mathbf{r}) - \bar{\bar{\varepsilon}}^{m\,T}_{n-m}(\mathbf{r})\right) \cdot \mathbf{E}^b_n + \mathbf{H}^b_m \cdot \left(\bar{\bar{\xi}}^n_{m-n}(\mathbf{r}) + \bar{\bar{\zeta}}^{m\,T}_{n-m}(\mathbf{r})\right) \cdot \mathbf{E}^a_n - \mathbf{E}^b_m \cdot \left(\bar{\bar{\zeta}}^n_{m-n}(\mathbf{r}) + \bar{\bar{\xi}}^{m\,T}_{n-m}(\mathbf{r})\right) \cdot \mathbf{H}^a_n\Bigg] dV. \tag{5}$$

The necessary and sufficient condition for satisfying the definition in Eq. (2), requires left-hand side of Eq. (5) to be zero, and thus the right side of Eq. (5) equals zero. One obvious way to ensure the satisfaction of the preceding integral equation is to make its integrand zero, which is further guaranteed when each summand of the summation over harmonics m and n in the integrand is zero. Therefore, it can be easily seen that if the following conditions are met, reciprocity is guaranteed:

$$\bar{\bar{\mu}}^n_{m-n}(\mathbf{r}) = \bar{\bar{\mu}}^{m\,T}_{n-m}(\mathbf{r}), \quad \bar{\bar{\varepsilon}}^n_{m-n}(\mathbf{r}) = \bar{\bar{\varepsilon}}^{m\,T}_{n-m}(\mathbf{r}),$$
$$\bar{\bar{\xi}}^n_{m-n}(\mathbf{r}) = -\bar{\bar{\zeta}}^{m\,T}_{n-m}(\mathbf{r}). \tag{6}$$

Examination of this condition does not require electromagnetic field calculation and obviates the need for numerical simulations of time-periodic structures under study. However, the condition (6) is merely a sufficient condition for reciprocity and there are conceivable reciprocal scenarios in which condition (6) is not held. A simple isotropic medium which is modulated sinusoidally with a global nonzero constant phase is an example which, despite being reciprocal, does not satisfy the reciprocity condition of (6). In these cases the summands of the summation in Eq. (5) are not zero, but the summation on m and n is zero.

As the reciprocity condition in Eqs. (6) is based on the time Fourier components, it is affected by the choice of the time origin. Yet, obviously, the physical properties of a time-varying medium must not be affected by this choice. Therefore, the violation of Eqs. (6) and (2) does not necessarily imply nonreciprocity simply because these expressions do not take account of the arbitrariness of the time origin. The latter is not reflected in the mathematical expression of the reciprocity theorem in stationary media but leads to a subtle nuance in analysis of TV media for which the mathematical expression of the reciprocity theorem in the frequency domain depends on the choice of the time origin. Given that the choice of the time origin is arbitrary for temporally periodic media, we can apply a time translation $t \to t - \Delta T$ to the previous analysis and define the new constitutive parameters $\bar{\bar{\nu}}'_n$ as $\bar{\bar{\nu}}'_n = \bar{\bar{\nu}}_n(\mathbf{r};\tau)e^{-jn\Omega\Delta T}$. By applying this time translation, the integrands of the integral in Eq. (5) change. It can be shown that if there exists a time shift that satisfies the condition

$$\bar{\bar{\mu}}^{n\,\prime}_{m-n}(\mathbf{r}) = \bar{\bar{\mu}}^{m\,\prime T}_{n-m}(\mathbf{r}), \quad \bar{\bar{\varepsilon}}^{n\,\prime}_{m-n}(\mathbf{r}) = \bar{\bar{\varepsilon}}^{m\,\prime T}_{n-m}(\mathbf{r}),$$
$$\bar{\bar{\xi}}^{n\,\prime}_{m-n}(\mathbf{r}) = -\bar{\bar{\zeta}}^{m\,\prime T}_{n-m}(\mathbf{r}), \tag{7}$$

the TV media is reciprocal (see Appendix B). Consequently, the statement of the reciprocity theorem, Eq. (2), will change and the generalized reciprocity theorem can be expressed as

$$\int_{V^a} \sum_m \frac{\mathbf{E}^b_m \cdot \mathbf{J}^a_m}{\omega_m} e^{-2j\omega_m\Delta T} dV^a = \int_{V^b} \sum_m \frac{\mathbf{E}^a_m \cdot \mathbf{J}^b_m}{\omega_m} e^{-2j\omega_m\Delta T} dV^b, \tag{8}$$

which states that if there exists a time shift ΔT in which the fields satisfy Eq. (8), the media is reciprocal. This form of reciprocity means that if the frequency of two sources $\mathbf{J}^a$ and $\mathbf{J}^b$ is the same, the reciprocity condition, Eq. (2), is satisfied for any time shift, as in static media. However, if the two sources have different frequency components, the generalized reciprocity theorem of Eq. (8) must be used.

Accordingly, the condition in Eqs. (7) is a rather generalized sufficient condition for investigating the reciprocity in dispersive time-periodic modulated bianisotropic media. It is worth noting that there are many reciprocal examples in which Eqs. (6) do not hold but Eqs. (7) do. Still, the condition in Eqs. (7) remains a sufficient (and not necessary) condition. In other words, the reciprocity/nonreciprocity question is inconclusive whenever Eqs. (7) are violated. However, we do not have any reciprocal example in which the condition (7) is not satisfied.

For time-periodic modulated media, investigating the reciprocity condition, Eqs. (6) and (7), requires knowing the response of the media, that is, knowing $\bar{\bar{\varepsilon}}(\mathbf{r},t;\tau)$, $\bar{\bar{\mu}}(\mathbf{r},t;\tau)$, $\bar{\bar{\xi}}(\mathbf{r},t;\tau)$, and $\bar{\bar{\zeta}}(\mathbf{r},t;\tau)$. We usually do not have this information. However, a practical simplifying assumption is that the modulation frequency, Ω, is much lower than the source frequency, ω_0, and the dispersion can be ignored in the frequency range, which is logical for problems with nearly monochromatic electromagnetic source. This assumption is henceforward applied throughout the paper. In this case, the constitutive parameters can be written as $\bar{\bar{\nu}}(\mathbf{r}, t; \omega_0)$ (see Appendix C). By applying this assumption, Eqs. (6) can also be revised as

$$\begin{aligned} \bar{\bar{\mu}}_{m-n}(\mathbf{r}, \omega_0) &= \bar{\bar{\mu}}_{n-m}{}^{T}(\mathbf{r}, \omega_0),\\ \bar{\bar{\varepsilon}}_{m-n}(\mathbf{r}, \omega_0) &= \bar{\bar{\varepsilon}}_{n-m}{}^{T}(\mathbf{r}, \omega_0),\\ \bar{\bar{\xi}}_{m-n}(\mathbf{r}, \omega_0) &= -\bar{\bar{\zeta}}_{n-m}{}^{T}(\mathbf{r}, \omega_0), \end{aligned} \tag{9}$$

and the generalized reciprocity condition will be revised as

$$\begin{aligned} \bar{\bar{\mu}}_{m-n}{}'(\mathbf{r}, \omega_0) &= \bar{\bar{\mu}}_{n-m}{}'^{T}(\mathbf{r}, \omega_0),\\ \bar{\bar{\varepsilon}}_{m-n}{}'(\mathbf{r}, \omega_0) &= \bar{\bar{\varepsilon}}_{n-m}{}'^{T}(\mathbf{r}, \omega_0),\\ \bar{\bar{\xi}}_{m-n}{}'(\mathbf{r}, \omega_0) &= -\bar{\bar{\zeta}}_{n-m}{}'^{T}(\mathbf{r}, \omega_0). \end{aligned} \tag{10}$$

These two conditions (9) and (10) are special cases of conditions (6) and (7). Every isotropic time-periodic medium which satisfies Eqs. (10) has generalized time reversal symmetry, that is, $\tilde{\nu}(\mathbf{r}, t-\Delta T) = \tilde{\nu}(\mathbf{r}, -t-\Delta T)$ [28] [or satisfying Eqs. (10) in isotropic media is equal to having generalized time reversal symmetry]. This can easily be demonstrated by writing the generalized time reversal symmetry relation according to its harmonic components, that is, $\nu_n(\mathbf{r})e^{-jn\Omega\Delta T} = \nu_{-n}(\mathbf{r})e^{jn\Omega\Delta T}$, which satisfies the reciprocity condition (10).

Applying the reciprocity condition given in (9) or (10) for stationary bianisotropic unbounded media results in the conventional reciprocity condition $\bar{\bar{\varepsilon}} = \bar{\bar{\varepsilon}}^T$, $\bar{\bar{\mu}} = \bar{\bar{\mu}}^T$, and $\bar{\bar{\zeta}} = -\bar{\bar{\xi}}^T$. Therefore, a chiral medium with $\bar{\bar{\zeta}} = -\bar{\bar{\xi}} = j\kappa_0\bar{\bar{I}}$ is reciprocal and a Tellegen medium with $\bar{\bar{\zeta}} = \bar{\bar{\xi}} = \zeta_0\bar{\bar{I}}$ is nonreciprocal. In the next section, several examples of time-periodic bianisotropic media are presented and the reciprocity/nonreciprocity conditions of these examples are investigated using the reciprocity conditions (9) and (10).

III. SYNCHRONOUS MODULATION IN DIFFERENT MEDIA

In this section, we consider a general bianisotropic medium with synchronous modulation with the following constitutive parameters:

$$\bar{\bar{\nu}}(\mathbf{r}, t, \omega_0) = \bar{\bar{\nu}}_{st}(\mathbf{r}, \omega_0) + \bar{\bar{M}}_\nu(\mathbf{r}) f(t), \tag{11}$$

where $\bar{\bar{\nu}} = \bar{\bar{\varepsilon}}, \bar{\bar{\mu}}, \bar{\bar{\xi}}, \bar{\bar{\zeta}}$ and $\bar{\bar{\nu}}_{st}$ shows the static constitutive parameters and $f(t)$ is any periodic function. Synchronous modulation means that all points in a medium are modulated with the same function. This modulation function can be applied to all points of a medium with the same strength (constant $\bar{\bar{M}}$), which we call global synchronous modulation, or it can be applied to points with different amplitudes [$\bar{\bar{M}}(\mathbf{r})$], which we call local synchronous modulation. It is noteworthy that global modulation is a special case of synchronous modulation.

Now, we look into an unbounded isotropic material with synchronous sinusoidal modulation which has a time dependence of

$$\begin{aligned} \bar{\bar{\varepsilon}}(\mathbf{r}, t, \omega_0) &= \varepsilon_{st}(\mathbf{r}, \omega_0)\bar{\bar{I}} + M_\varepsilon \bar{\bar{I}} \cos(\Omega t + \phi),\\ \bar{\bar{\mu}} &= \mu_0 \bar{\bar{I}}, \quad \bar{\bar{\zeta}} = \bar{\bar{\xi}} = 0, \end{aligned} \tag{12}$$

where $\varepsilon_{st}(\mathbf{r}, \omega_0)$ is the static permittivity, M_ε is the modulation strength function, and Ω is the modulation frequency. The Fourier components of constitutive parameters in this media are

$$\varepsilon_0 = \varepsilon_{st}(\mathbf{r}, \omega_0), \quad \varepsilon_{-1} = \frac{M_\varepsilon}{2} e^{-j\phi}, \quad \varepsilon_{+1} = \frac{M_\varepsilon}{2} e^{j\phi}, \tag{13}$$

and the condition (9) is not met. However, if we change the time t to $t - \Delta T$ with $\Delta T = \phi/\Omega$, the condition (10) will be met and the media is in fact found to be reciprocal, which is an expected result [38]. It should be noted that this result is not attainable with the sufficient condition (9) but is deducible from Eq. (10). For this reason, the condition (9) is not a general sufficient condition as mentioned before. It should be noted that according to (9) a sinusoidally time-modulated isotropic slab having a modulation strength $M_\varepsilon(\mathbf{r})$ is also reciprocal. Therefore, synchronous sinusoidal modulation in isotropic media cannot result in nonreciprocity.

A. Synchronous sinusoidal time modulation in bianisotropic media

Now, we assume an unbounded bianisotropic reciprocal material in which $\bar{\bar{\xi}} = -\bar{\bar{\zeta}}^T$. An arbitrary tensor $\bar{\bar{\xi}}$ can be decomposed into a linear combination of three basic tensors as follows:

$$\bar{\bar{\xi}} = T\bar{\bar{I}} + \sum_{i=1}^{3} P_i \mathbf{a}_i \mathbf{a}_i + A(\mathbf{b} \times \bar{\bar{I}}), \tag{14}$$

where T, P_i, and A are complex amplitudes, $\bar{\bar{I}}$ is the unit dyadic, and a_i and b_i are unit vectors [44]. Each of these amplitudes defines a special behavior of the material. It represents reciprocal bianisotropic materials which can be divided into three main categories: Chiral, pseudochiral, and uniaxial omega, based on which of these amplitudes is nonzero. In the following Secs. III A 1 to III A 3, we will apply synchronous time modulation to these three categories which are reciprocal in the absence of time modulation, and investigate whether or not time modulation in anisotropy and bianisotropy results in nonreciprocity.

1. Chiral media

First, assume an example of unbounded chiral material which is modulated sinusoidally as follows:

$$\begin{aligned} \bar{\bar{\varepsilon}}(\mathbf{r}, t, \omega_0) &= \bar{\bar{\varepsilon}}_{st}(\mathbf{r}, \omega_0) + \bar{\bar{M}}_\varepsilon \cos(\Omega t),\\ \bar{\bar{\mu}}(\mathbf{r}, t, \omega_0) &= \bar{\bar{\mu}}_{st}(\mathbf{r}, \omega_0) + \bar{\bar{M}}_\mu \cos(\Omega t),\\ \bar{\bar{\zeta}}(\mathbf{r}, t, \omega_0) &= \bar{\bar{\zeta}}_{st}(\mathbf{r}, \omega_0) + M_c\bar{\bar{I}} \cos(\Omega t),\\ \bar{\bar{\xi}}(\mathbf{r}, t, \omega_0) &= \bar{\bar{\xi}}_{st}(\mathbf{r}, \omega_0) - M_c\bar{\bar{I}} \cos(\Omega t), \end{aligned} \tag{15}$$

where $\bar{\bar{I}}$ is the unit dyadic, and $\bar{\bar{\varepsilon}}_{st} = \bar{\bar{\varepsilon}}_{st}{}^T$, $\bar{\bar{\mu}}_{st} = \bar{\bar{\mu}}_{st}{}^T$, $\bar{\bar{\zeta}}_{st} = -\bar{\bar{\xi}}_{st}{}^T$, and $\bar{\bar{M}}_{\varepsilon,\mu} = \bar{\bar{M}}_{\varepsilon,\mu}{}^T$. Here, Fourier components of constitutive parameters are

$$\bar{\bar{\varepsilon}}_0 = \bar{\bar{\varepsilon}}_{st}(\mathbf{r}, \quad \omega_0), \bar{\bar{\varepsilon}}_{+1} = \bar{\bar{\varepsilon}}_{-1} = \frac{\bar{\bar{M}}_\varepsilon}{2},$$

$$\bar{\bar{\mu}}_0 = \bar{\bar{\mu}}_{st}(\mathbf{r}, \omega_0), \quad \bar{\bar{\mu}}_{+1} = \bar{\bar{\mu}}_{-1} = \frac{\bar{\bar{M}}_\mu}{2},$$
$$\bar{\bar{\zeta}}_0 = \bar{\bar{\zeta}}_{st}(\mathbf{r}, \omega_0), \quad \bar{\bar{\zeta}}_{+1} = \bar{\bar{\zeta}}_{-1} = \frac{M_c}{2}\bar{\bar{I}},$$
$$\bar{\bar{\xi}}_0 = \bar{\bar{\xi}}_{st}(\mathbf{r}, \omega_0), \quad \bar{\bar{\xi}}_{+1} = \bar{\bar{\xi}}_{-1} = -\frac{M_c}{2}\bar{\bar{I}}. \tag{16}$$

Therefore, $\bar{\bar{\varepsilon}}_{+1} = \bar{\bar{\varepsilon}}_{-1}^T$, $\bar{\bar{\mu}}_{+1} = \bar{\bar{\mu}}_{-1}^T$, $\bar{\bar{\zeta}}_{+1/-1} = \frac{M_c}{2}\bar{\bar{I}} = -\bar{\bar{\xi}}^T_{-1/+1} = \frac{M_c}{2}\bar{\bar{I}}^T$ and the condition (9) is satisfied. As a result, the chiral material with synchronous sinusoidal time modulation is reciprocal. Note that, as mentioned before, if the cosine function has an arbitrary global phase, the condition (10) is satisfied and the medium remains reciprocal.

2. *Pseudochiral media*

As the second case, we look into a pseudochiral material with the following constitutive parameters:

$$\bar{\bar{\bar{\varepsilon}}}(\mathbf{r}, t, \omega_0) = \bar{\bar{\varepsilon}}_{st}(\mathbf{r}, \omega_0) + \bar{\bar{M}}_\varepsilon \cos(\Omega t),$$
$$\bar{\bar{\bar{\mu}}}(\mathbf{r}, t, \omega_0) = \bar{\bar{\mu}}_{st}(\mathbf{r}, \omega_0) + \bar{\bar{M}}_\mu \cos(\Omega t),$$
$$\bar{\bar{\bar{\zeta}}}(\mathbf{r}, t, \omega_0) = \bar{\bar{\zeta}}_{st}(\mathbf{r}, \omega_0) + M_p\bar{\bar{U}} \cos(\Omega t),$$
$$\bar{\bar{\bar{\xi}}}(\mathbf{r}, t, \omega_0) = \bar{\bar{\xi}}_{st}(\mathbf{r}, \omega_0) - M_p\bar{\bar{U}} \cos(\Omega t), \tag{17}$$

where $\bar{\bar{U}} = \bar{\bar{U}}^T$ is a symmetric matrix, and $\bar{\bar{\varepsilon}}_{st} = \bar{\bar{\varepsilon}}_{st}{}^T$, $\bar{\bar{\mu}}_{st} = \bar{\bar{\mu}}_{st}{}^T$, $\bar{\bar{\zeta}}_{st} = -\bar{\bar{\xi}}_{st}{}^T$. Like the previous example, the reciprocity conditions for $\bar{\bar{\varepsilon}}$ and $\bar{\bar{\mu}}$ are satisfied. The Fourier components of magneto-electric coupling parameters are

$$\bar{\bar{\zeta}}_0 = \bar{\bar{\zeta}}_{st}(\mathbf{r}, \omega_0), \quad \bar{\bar{\zeta}}_{+1} = \bar{\bar{\zeta}}_{-1} = \frac{M_p}{2}\bar{\bar{U}},$$
$$\bar{\bar{\xi}}_0 = \bar{\bar{\xi}}_{st}(\mathbf{r}), \quad \bar{\bar{\xi}}_{+1} = \bar{\bar{\xi}}_{-1} = -\frac{M_p}{2}\bar{\bar{U}}. \tag{18}$$

Therefore, $\bar{\bar{\zeta}}_{+1/-1} = \frac{M_p}{2}\bar{\bar{U}} = -\bar{\bar{\xi}}^T_{-1/+1} = \frac{M_p}{2}\bar{\bar{U}}^T$, and synchronous sinusoidal time modulation of magneto-electric coupling tensors in pseudochiral material does not result in nonreciprocity.

3. *Uniaxial omega media*

Now, consider a unbounded uniaxial omega material with

$$\bar{\bar{\bar{\varepsilon}}}(\mathbf{r}, t, \omega_0) = \bar{\bar{\varepsilon}}_{st}(\mathbf{r}, \omega_0) + \bar{\bar{M}}_\varepsilon \cos(\Omega t),$$
$$\bar{\bar{\bar{\mu}}}(\mathbf{r}, t, \omega_0) = \bar{\bar{\mu}}_{st}(\mathbf{r}, \omega_0) + \bar{\bar{M}}_\mu \cos(\Omega t),$$
$$\bar{\bar{\bar{\zeta}}}(\mathbf{r}, t, \omega_0) = \bar{\bar{\zeta}}_{st}(\mathbf{r}, \omega_0) + M_\Omega\bar{\bar{J}} \cos(\Omega t),$$
$$\bar{\bar{\bar{\xi}}}(\mathbf{r}, t, \omega_0) = \bar{\bar{\xi}}_{st}(\mathbf{r}, \omega_0) + M_\Omega\bar{\bar{J}} \cos(\Omega t), \tag{19}$$

where $\bar{\bar{J}} = \hat{z} \times \bar{\bar{I}}$ is the transverse vector product dyadic, and $\bar{\bar{\varepsilon}}_{st} = \bar{\bar{\varepsilon}}_{st}{}^T$, $\bar{\bar{\mu}}_{st} = \bar{\bar{\mu}}_{st}{}^T$, $\bar{\bar{\zeta}}_{st} = -\bar{\bar{\xi}}_{st}{}^T$. Similarly to previous examples, Fourier components of constitutive parameters can be written as $\bar{\bar{\zeta}}_{+1/-1} = \frac{M_\Omega}{2}\bar{\bar{J}} = -\bar{\bar{\xi}}^T_{-1/+1} = -\frac{M_\Omega}{2}\bar{\bar{J}}^T$, and, in contrast to what [40] proposed, nonreciprocity can not be achieved in bianisotropic bulk material with synchronous sinusoidally time-modulated antisymmetric magnetoelectric coupling tensors. The reciprocity in [40] is investigated using the reciprocity condition on the Green's operator of the media. But due to a sign error in the derivation of the transpose of the Green's operator, an incorrect conclusion is obtained. In order to validate the analytical results, several examples are demonstrated in Sec. IV.

Therefore, it is impossible to achieve nonreciprocity in statically reciprocal bianisotropic media insofar as the time modulation is sinusoidal in accordance with Eqs. (10). However, introducing phase difference between modulation functionality of different parts of magneto-electric tensors (14) may result in nonreciprocal response.

A special case of bianisotropic materials is anisotropic media in which $\bar{\bar{\xi}} = \bar{\bar{\zeta}} = 0$. Therefore, statically reciprocal anisotropic media are also reciprocal under synchronous sinusoidal modulation. Another important note is that in an anisotropic medium, if the diagonal elements of the matrix $\bar{\bar{\varepsilon}}$ or $\bar{\bar{\mu}}$ are modulated with different phases, the material is reciprocal in the absence of modulation but the reciprocity condition (9) and also (10) are not met in the presence of modulation and we should investigate its reciprocity/nonreciprocity using another approach.

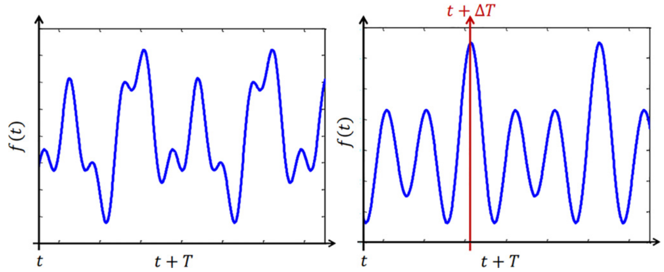

FIG. 2. Two different modulation functions. The right function satisfies the generalized time reversal symmetry but the left one does not.

B. Synchronous time-periodic modulation in bianisotropic media

According to the reciprocity condition (10), reciprocity is by no means limited to sinusoidal modulation and many other examples of $f(t)$ will result in reciprocal response. It can be easily proved that in synchronously time-periodic bianisotropic media illuminated by a nearly monochromatic source the medium is reciprocal if the modulation function satisfies the generalized time reversal symmetry. Two different functions are shown in Fig. 2. The left one does not satisfy the generalized time reversal symmetry but the right one is an even function with a time translation $t \to t - \Delta T$.

Therefore, the constitutive parameters for the most general synchronous time-periodic bianisotropic medium which is reciprocal in the absence of modulation and also reciprocal after adding time modulation can be written as follows:

$$\bar{\bar{\nu}}(\mathbf{r}, t, \omega_0) = \bar{\bar{\nu}}_{st}(\mathbf{r}, \omega_0) + \bar{\bar{M}}_\nu(\mathbf{r}) f(t),$$
$$f(t) = \sum_{n=0}^{\infty} a_n \cos(n\Omega t + n\phi), \tag{20}$$

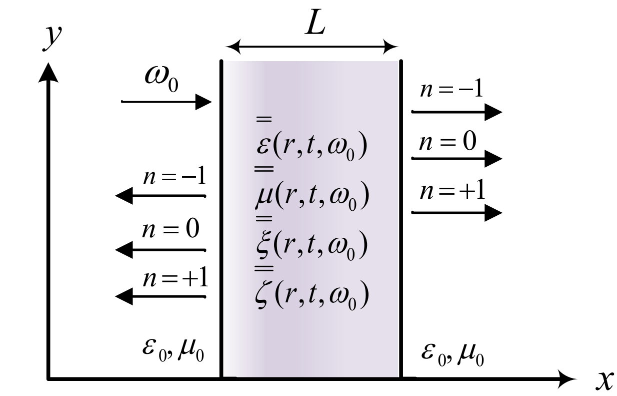

FIG. 3. Normal incidence on a time-periodic on-dimensional slab. The slab is infinite in y and z directions. A single harmonic field excites the slab from the left side and multiple harmonics at frequencies $\omega_n = \omega_0 + n\Omega$ are scattered.

which satisfies reciprocity condition (10), and also the generalized time reversal symmetry is satisfied for the function $f(t)$.

IV. NUMERICAL EXAMPLES

We have looked into media with globally synchronous time modulation in the examples so far. Here, we consider two examples with local synchronous sinusoidal modulation: Single time modulated slab and two cascaded slabs. To analyze these structures, we use in-house FDTD simulation, and the results are shown in the following subsections.

A. Single bianisotropic time-varying slab

As the first example, we consider a one-dimensional single bianisotropic slab whose constitutive parameters change periodically with time (Fig. 3). The time dependency of constitutive parameters of this media is as in Eq. (11), where the functionality of $M(\mathbf{r})$ is a step function. The theoretical condition (9) in this slab holds and now we want to check the FDTD simulation results. To investigate the nonreciprocity of this structure using simulation, as Fig. 3 shows, we excite it with an incident field at frequency f_s once from the right and again from the left, and the harmonics of transmitted and reflected electromagnetic fields have been calculated. According to the reciprocity condition (2), the difference between amplitudes of the harmonics at the incident frequency in two forward and backward problems is considered as the nonreciprocity criterion. We consider a bianisotropic slab of length 0.1 μm whose relative permittivity tensor is $\bar{\bar{\varepsilon}}_r = 3\bar{\bar{I}}$ and relative permeability and magnetoelectric coupling tensors are $\bar{\bar{\mu}}_r = \bar{\bar{I}}$, $\bar{\bar{\zeta}} = \bar{\bar{\xi}} = 0.5/c_0\bar{\bar{J}}$, which is reciprocal in the absence of time variation. We modulate these parameters using a sinusoidal temporal dependency as in Eqs. (19) whose parameters are $\bar{\bar{M}}_\varepsilon = 0.2\bar{\bar{I}}$, $\bar{\bar{M}}_\mu = 0$, $M_\Omega = 0.2/c_0$, $\omega_m = 0.2\omega_s$. According to Sec. III A 3, this media is reciprocal in the presence of time variation. We excite this slab with a plane wave with frequency $f_s = 3000$ THz normally impinging the slab. The left-to-right and right-to-left spectra of the transmitted and reflected fields versus frequencies can be seen in Fig. 4. The same results for these two problems validate the reciprocity of the synchronously sinusoidal time modulated bianisotropic slab.

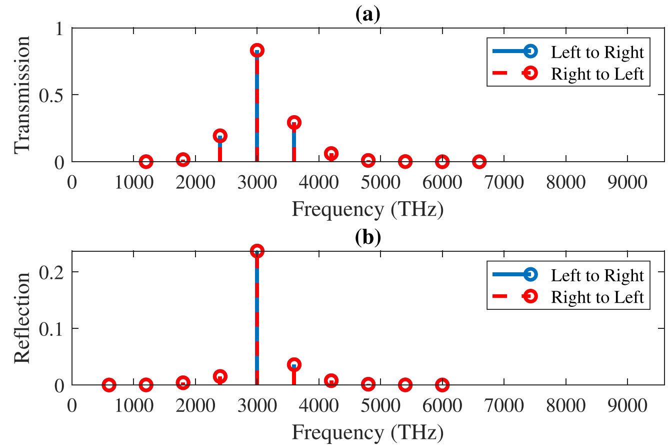

FIG. 4. FDTD numerical simulation of scattering from a sinusoidal globally time modulated bianisotropic slab in the forward and backward problems. (a) Comparison of transmitted electric field harmonics in forward and backward problems. (b) Comparison of reflected electric field harmonics in forward and backward problems.

Now, we assume a traveling-wave modulation profile, i.e., $\bar{\bar{\nu}} = \bar{\bar{\nu}}_{st} + \bar{\bar{M}}_\nu \cos(\Omega t - kx)$, where $\bar{\bar{\nu}} = \bar{\bar{\varepsilon}}, \bar{\bar{\mu}}, \bar{\bar{\zeta}}, \bar{\bar{\xi}}$ and $\bar{\bar{M}}_\varepsilon = 0.2\bar{\bar{I}}$, $\bar{\bar{M}}_\mu = 0$, $\bar{\bar{M}}_{\xi,\zeta} = 0.2/c_0\bar{\bar{J}}$, and the speed of modulation is $v_m = \Omega/k = c_0/\sqrt{3}$. Based on the relations (10), the reciprocity condition does not hold in this structure, but since the condition (10) is a sufficient condition for reciprocity, it cannot be said that this structure is nonreciprocal. The left-to-right and right-to-left spectra of the transmitted and reflected fields are demonstrated in Fig. 5. The difference between amplitudes at the main harmonic in the two forward and backward problems shows that the space-time modulated bianisotropic slab is nonreciprocal.

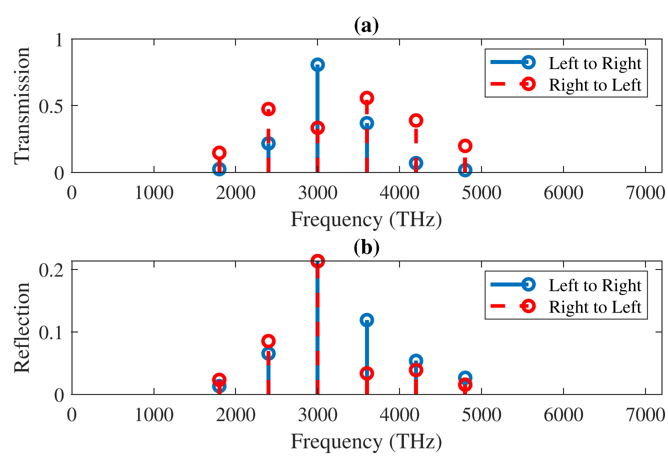

FIG. 5. FDTD numerical simulation of scattering from a sinusoidal space-time modulated bianisotropic slab in the forward and backward problems. (a) Comparison of transmitted electric field harmonics in forward and backward problems. (b) Comparison of reflected electric field harmonics in forward and backward problems.

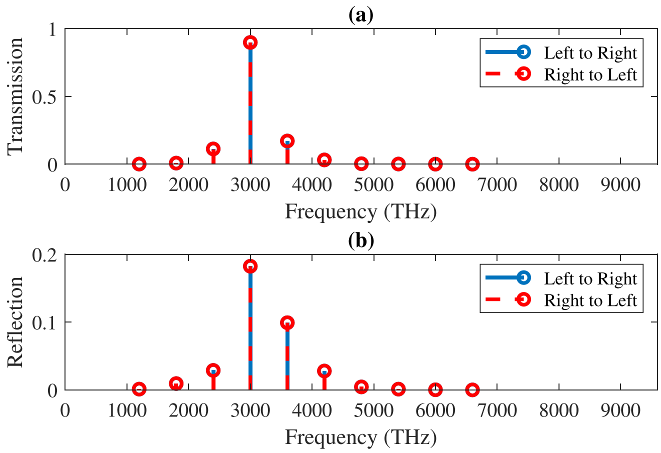

FIG. 6. FDTD numerical simulation of scattering from two cascaded sinusoidal time modulated bianisotropic slabs in the forward and backward problems. (a) Comparison of transmitted electric field harmonics in forward and backward problems. (b) Comparison of reflected electric field harmonics in forward and backward problems.

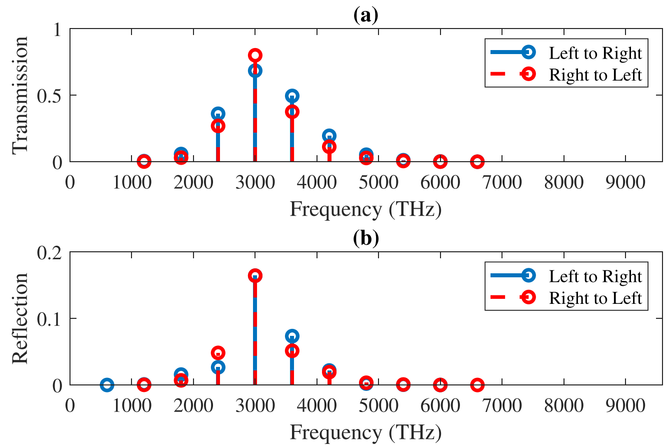

FIG. 7. FDTD numerical simulation of scattering from two cascaded sinusoidal time modulated bianisotropic slabs which have phase difference $\phi = 90^\circ$ in the forward and backward problems. (a) Comparison of transmitted electric field harmonics in forward and backward problems. (b) Comparison of reflected electric field harmonics in forward and backward problems.

B. Two cascaded bianisotropic time-varying slabs

Another example of locally synchronous time modulation is a structure that consists of two slab resonators. Here, we consider two cascaded time-varying slabs whose constitutive parameters are modulated. In the first case, these two slabs are modulated synchronously with each other, i.e., in such a way that there is not any phase difference between the modulation profiles of the two slabs. In the second case, there is a phase difference ϕ between the modulation profiles of the two slabs, or, in other words, there are at least two points in space which change with a phase difference with each other with time.

1. Locally synchronous time modulation

Here, we assume two cascaded bianisotropic slabs with the same thickness $L = 0.1\ \mu$m at a distance $d = 0.1\ \mu$m from each other. We consider the constitutive parameters of these two slabs to be the same as synchronous modulation in Sec. IV A. The left-to-right and right-to-left spectra of the transmitted and reflected fields versus frequency can be seen in Fig. 6. This figure shows that the amplitudes of different harmonics in forward and backward problems are the same and therefore this structure is reciprocal, which is compatible with the results shown in Sec. III A 3.

2. Locally asynchronous time modulation

If we choose a phase difference $\phi = 90^\circ$ between two slabs, we have the results shown in Fig. 7. This figure shows that the amplitudes of reflection/transmission field harmonics at incident frequency are different in the two forward and backward problems which shows that the structure is nonreciprocal. This nonreciprocal behavior also can be observed in two slabs of isotropic media which are modulated sinusoidally with a phase difference equal to 90°, as represented in [38].

V. CONCLUSION

In this paper, the reciprocity condition for general linear time-periodic bianisotropic media is extracted. It was demonstrated that achieving nonreciprocity from a linear statically reciprocal bianisotropic material whose permittivity, permeability, and magnetoelectric couplings are modulated synchronously in time using a sinusoidal time dependency is impossible. Also, we find the general time modulation function which does not result in nonreciprocity and we prove that every bianisotropic medium with synchronous time dependency which satisfies the generalized time reversal symmetry is reciprocal. Our analytical results were validated using in-house FDTD simulation for a synchronously sinusoidal time modulated slab of bianisotropic media and also two cascaded synchronous slabs. Similarly, multiple time-varying slabs whose constitutive parameters are modulated sinusoidally in time with the same phase have reciprocal response.

APPENDIX A: CALCULATION OF SIDE HARMONICS $\mathbf{D}_m$ AND $\mathbf{B}_m$

The constitutive parameters in a time-periodic bianisotropic medium with modulation frequency Ω can be written using Fourier series as follows:

$$\tilde{\bar{\bar{\nu}}}(\mathbf{r},t;\tau) = \sum_n \tilde{\bar{\bar{\nu}}}_n(\mathbf{r};\tau)e^{jn\Omega t}, \tag{A1}$$

in which $\tilde{\bar{\bar{\nu}}} = \tilde{\bar{\bar{\varepsilon}}}, \tilde{\bar{\bar{\mu}}}, \tilde{\bar{\bar{\xi}}}, \tilde{\bar{\bar{\zeta}}}$. Assuming a multitone current source in the form $\tilde{\mathbf{J}}^{a,b} = \sum_m \mathbf{J}_m^{a,b}e^{j(\omega_0+m\Omega)t}$, the electromagnetic fields are written as

$$\mathscr{E}(\boldsymbol{r},t) = \sum_m \mathbf{E}_m(\mathbf{r})e^{j\omega_m t},$$
$$\mathscr{H}(\boldsymbol{r},t) = \sum_m \mathbf{H}_m(\mathbf{r})e^{j\omega_m t}, \tag{A2}$$

where $\omega_m = \omega_0 + m\Omega$.

By substituting Eqs. (A1) and (A2) in Eq. (1), we will have

$$\begin{aligned}\mathscr{D}(\mathbf{r},t) &= \int_{-\infty}^{t}\sum_n e^{jn\Omega t}\bar{\bar{\varepsilon}}_n(\mathbf{r};\tau)\cdot\sum_m \mathbf{E}_m(\mathbf{r})e^{j\omega_m t'}dt' + \int_{-\infty}^{t}\sum_n e^{jn\Omega t}\bar{\bar{\zeta}}_n(\mathbf{r};\tau)\cdot\sum_m \mathbf{H}_m(\mathbf{r})e^{j\omega_m t'}dt' \\ &= \sum_{m,n}\int_0^{\infty}\bar{\bar{\varepsilon}}_n(\mathbf{r};\tau)e^{-j\omega_m\tau}d\tau\cdot\mathbf{E}_m(\mathbf{r})e^{j\omega_{m+n}t} + \sum_{m,n}\int_0^{\infty}\bar{\bar{\zeta}}_n(\mathbf{r};\tau)e^{-j\omega_m\tau}d\tau\cdot\mathbf{H}_m(\mathbf{r})e^{j\omega_{m+n}t}.\end{aligned} \tag{A3}$$

Then by using Fourier transform of parameters $\bar{\bar{\varepsilon}}_n(\boldsymbol{r};\tau)$, $\bar{\bar{\zeta}}_n(\boldsymbol{r};\tau)$, $\bar{\bar{\kappa}}_n(\boldsymbol{r};\tau)$, this equation can be revised as

$$\mathscr{D}(\mathbf{r},t) = \sum_{m,n}\bar{\bar{\varepsilon}}_n(\mathbf{r};\omega_m)\cdot\mathbf{E}_m(\mathbf{r})e^{j\omega_{m+n}t} + \sum_{m,n}\bar{\bar{\zeta}}_n(\mathbf{r};\omega_m)\cdot\mathbf{H}_m(\mathbf{r})e^{j\omega_{m+n}t} = \sum_{m,n}(\bar{\bar{\varepsilon}}_{m-n}(\mathbf{r};\omega_n)\cdot\mathbf{E}_n(\mathbf{r}) + \bar{\bar{\zeta}}_{m-n}(\mathbf{r};\omega_n)\cdot\mathbf{H}_n(\mathbf{r}))e^{j\omega_m t}. \tag{A4}$$

Therefore, $\mathscr{D}(\mathbf{r},t)$ can be written as $\mathscr{D}(\mathbf{r},t) = \sum_m \mathbf{D}_m(\mathbf{r})e^{j\omega_m t}$ and so on $\mathscr{B}(\mathbf{r},t)$, where

$$\begin{aligned}\mathbf{D}_m(\mathbf{r}) &= \sum_n \big(\bar{\bar{\varepsilon}}^n_{m-n}(\mathbf{r})\cdot\mathbf{E}_n(\mathbf{r}) + \bar{\bar{\zeta}}^n_{m-n}(\mathbf{r})\cdot\mathbf{H}_n(\mathbf{r})\big), \\ \mathbf{B}_m(\mathbf{r}) &= \sum_n \big(\bar{\bar{\mu}}^n_{m-n}(\mathbf{r})\cdot\mathbf{H}_n(\mathbf{r}) + \bar{\bar{\xi}}^n_{m-n}(\mathbf{r})\cdot\mathbf{E}_n(\mathbf{r})\big),\end{aligned} \tag{A5}$$

in which $\bar{\bar{\varepsilon}}^n_{m-n} = \bar{\bar{\varepsilon}}_{m-n}(\mathbf{r},\omega_n)$ is the Fourier component at frequency ω_n and so on, which is defined as

$$\bar{\bar{\varepsilon}}_n(\mathbf{r},\omega) = \int_0^{\infty}\bar{\bar{\varepsilon}}(\mathbf{r};\tau)e^{-j\omega\tau}d\tau. \tag{A6}$$

APPENDIX B: DERIVATION OF THE GENERALIZED RECIPROCITY CONDITION

If we apply a time translation $t \to t - \triangle T$ to Eq. (5), we will have

$$\begin{aligned}&\int_{V^a}\sum_m \frac{\mathbf{E}^b_m\cdot\mathbf{J}^a_m}{j\omega_m}e^{-2j\omega_m\triangle T}dV^a - \int_{V^b}\sum_m \frac{\mathbf{E}^a_m\cdot\mathbf{J}^b_m}{j\omega_m}e^{-2j\omega_m\triangle T}dV^b \\ &= \int_V\Bigg[\sum_{m,n}\mathbf{H}^b_m e^{-j\omega_m\triangle T}\cdot\big(\bar{\bar{\mu}}^{n}_{m-n}{}'(\mathbf{r}) - \bar{\bar{\mu}}^{m}_{n-m}{}'^T(\mathbf{r})\big)\cdot\mathbf{H}^a_n e^{-j\omega_n\triangle T} + \mathbf{E}^a_m e^{-j\omega_m\triangle T}\cdot\big(\bar{\bar{\varepsilon}}^{n}_{m-n}{}'(\mathbf{r}) - \bar{\bar{\varepsilon}}^{m}_{n-m}{}'^T(\mathbf{r})\big)\cdot\mathbf{E}^b_n e^{-j\omega_n\triangle T} \\ &\quad + \mathbf{H}^b_m e^{-j\omega_m\triangle T}\cdot\big(\bar{\bar{\xi}}^{n}_{m-n}{}'(\mathbf{r}) + \bar{\bar{\zeta}}^{m}_{n-m}{}'^T(\mathbf{r})\big)\cdot\mathbf{E}^a_n e^{-j\omega_n\triangle T} - \mathbf{E}^b_m e^{-j\omega_m\triangle T}\cdot\big(\bar{\bar{\zeta}}^{n}_{m-n}{}'(\mathbf{r}) + \bar{\bar{\xi}}^{m}_{n-m}{}'^T(\mathbf{r})\big)\cdot\mathbf{H}^a_n e^{-j\omega_n\triangle T}\Bigg]dV.\end{aligned} \tag{B1}$$

Thus, the generalized reciprocity condition, Eq. (7), is obtained.

APPENDIX C: CONSTITUTIVE PARAMETERS IN THE CASE OF NEARLY MONOCHROMATIC SOURCE

In the case of assuming monochromatic source, Eq. (1) can be written as

$$\begin{aligned}\mathscr{D}(\mathbf{r},t) &= \sum_m(\bar{\bar{\varepsilon}}_{\mathrm{r}}(\mathbf{r},t;\omega_m)\cdot\mathbf{E}_m(\mathbf{r}) + \bar{\bar{\zeta}}_{\mathrm{r}}(\mathbf{r},t;\omega_m)\cdot\mathbf{H}_m(\mathbf{r}))\cos(\omega_m t) \\ &\quad + \sum_m(\bar{\bar{\varepsilon}}_{\mathrm{im}}(\mathbf{r},t;\omega_m)\cdot\mathbf{E}_m(\mathbf{r}) + \bar{\bar{\zeta}}_{\mathrm{im}}(\mathbf{r},t;\omega_m)\cdot\mathbf{H}_m(\mathbf{r}))\sin(\omega_m t)\end{aligned} \tag{C1}$$

and

$$\begin{aligned}\mathscr{B}(\mathbf{r},t) &= \sum_m(\bar{\bar{\mu}}_{\mathrm{r}}(\mathbf{r},t;\omega_m)\cdot\mathbf{H}_m(\mathbf{r}) + \bar{\bar{\xi}}_{\mathrm{r}}(\mathbf{r},t;\omega_m)\cdot\mathbf{E}_m(\mathbf{r}))\cos(\omega_m t) \\ &\quad + \sum_m(\bar{\bar{\mu}}_{\mathrm{im}}(\mathbf{r},t;\omega_m)\cdot\mathbf{H}_m(\mathbf{r}) + \bar{\bar{\xi}}_{\mathrm{im}}(\mathbf{r},t;\omega_m)\cdot\mathbf{E}_m(\mathbf{r}))\sin(\omega_m t),\end{aligned} \tag{C2}$$

where the subscripts r and im show the real and imaginary parts of parameters. By applying the simplifying assumption $\Omega \ll \omega_0$, Eqs. (C1) and (C2) can be simplified as

$$\mathscr{D}(\mathbf{r},t) = \sum_m (\bar{\bar{\varepsilon}}_{\rm r}(\mathbf{r},t;\omega_0)\cdot\mathbf{E}_m(\mathbf{r}) + \bar{\bar{\zeta}}_{\rm r}(\mathbf{r},t;\omega_0)\cdot\mathbf{H}_m(\mathbf{r}))\cos(\omega_m t) + \sum_m (\bar{\bar{\varepsilon}}_{\rm im}(\mathbf{r},t;\omega_0)\cdot\mathbf{E}_m(\mathbf{r}) + \bar{\bar{\zeta}}_{\rm im}(\mathbf{r},t;\omega_0)\cdot\mathbf{H}_m(\mathbf{r}))\sin(\omega_m t) \tag{C3}$$

and

$$\mathscr{B}(\mathbf{r},t) = \sum_m (\bar{\bar{\mu}}_{\rm r}(\mathbf{r},t;\omega_0)\cdot\mathbf{H}_m(\mathbf{r}) + \bar{\bar{\xi}}_{\rm r}(\mathbf{r},t;\omega_0)\cdot\mathbf{E}_m(\mathbf{r}))\cos(\omega_m t) + \sum_m (\bar{\bar{\mu}}_{\rm im}(\mathbf{r},t;\omega_0)\cdot\mathbf{H}_m(\mathbf{r}) + \bar{\bar{\xi}}_{\rm im}(\mathbf{r},t;\omega_0)\cdot\mathbf{E}_m(\mathbf{r}))\sin(\omega_m t). \tag{C4}$$

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