Density matrix of electrons coupled to Einstein phonons and the electron-phonon entanglement content of excited states

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(Received 4 June 2022; revised 17 October 2022; accepted 18 October 2022; published 4 November 2022)

We derive the exact reduced density matrix for the electrons in an analytically solvable electron-phonon model. Here, the electrons are described as a Luttinger liquid that is coupled to Einstein phonons. We further derive analytical expressions for the electron-phonon entanglement, its spectrum, and mutual information at finite and zero temperature as well as for excited states. The entanglement entropy is additive in momentum for the quasiparticle excited states but not in the electron-phonon coupled eigenmodes of the system.

DOI: 10.1103/PhysRevB.106.195404

I. INTRODUCTION

The electron-phonon coupling plays an essential role for many phenomena in solid state physics, ranging all the way from Bardeen-Cooper-Schrieffer(BCS)-type superconductivity [1–6] to the Peierls instability [7–9] and charge-density waves [10–12]. Multidimensional electron-phonon systems are in general not integrable, instead one needs to rely on, for example, diagrammatic perturbation theory [13,14], Monte Carlo simulations [15–17], or the tensor-network approach [18]. In one dimension, the coupled system consisting of a Luttinger liquid and phonons is however integrable [9,19–21]; a series of exact results [9,21–24] exist and, in addition, an accurate variational ansatz [25–28].

In the present work, we derive the reduced density matrix of the electron subsystem in a Luttinger model coupled to Einstein phonons. Previous related work on the density matrix or the entanglement entropy includes: the entanglement between electrons at opposite momenta of the Luttinger liquid [24], the entanglement entropy between a single spin and a bosonic bath [29], and that between the electron and the protons in an H_2^+ ion [30]. For the Holstein model, the entanglement entropy between the electrons and the phonons has been investigated using numerical exact diagonalization [31], and the thus calculated entanglement spectrum has been used to characterize nonanalyticities of the ground state [32]. Also for the Su-Schrieffer-Heeger model [31,32] nonanalyticities in the entanglement spectrum have been found. In contrast, using the variational method a continuous crossover between small and large entanglement in the case of large and small polarons has been reported [33]. Further, bosonization has been used to calculate the entanglement entropy, mutual information and entanglement negativity between electrons and acoustic phonons in Ref. [22] and the entanglement entropy of polaronic systems in Refs. [34,35].

The entanglement entropy of excited states has become an active area of research in the last years. Among others, in critical spin chains the universal properties of the entanglement entropy has been derived using conformal field theory [36] and the entanglement content of the Heisenberg chain has been investigated using the Bethe ansatz [37]. In Ref. [38], quasiparticle excitations in one-dimensional fermionic system were studied using exact diagonalization and tensor networks, finding that the entanglement is proportional to the quasiparticle number. The entanglement of free fermions in excited states and its relationship to the Fermi surface has been investigated in Ref. [39] and that of two-particle excited states in Ref. [40]. The latter shows that the entanglement entropy is the sum of two terms corresponding to the two particles, if the momentum of the two quasi particles is not too close. Finally, in Ref. [41], the entanglement entropy of the excited states of the XY and the Heisenberg XXZ spin chains has been investigated between spatial blocks.

The above listed works demonstrates, for various special examples with spatial bi-sectioning, that the entanglement increase by quasiparticle excitations is additive if the momentum of the quasiparticles is not too close. A unifying derivation in the case of free, homogeneous integrable models of this observation has been performed in a series of works [42–44]. Here, the entanglement S between two spatial regions in a state which contains a finite number of quasiparticle excitations with different momenta $q_1, \ldots, q_n$ generally is conjectured to be of the form

$$S = S_{\mathrm{GS}} + \sum_{i=1}^{n} s(q_n), \tag{1}$$

where $s(q_n)$ is a model dependent function which describes the entanglement content of the quasiparticle excitation at momentum q_n, and S_{GS} is the ground state entanglement entropy in the same setting. Here, we instead investigate a nonspatial bipartition, i.e., the bipartition between the electron and the phonon subsystem and find a similar additivity in momentum space.

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The present work is organized as follows. In Sec. II, we introduce the model and determine its eigenmodes in Sec. III. From these we derive the reduced density matrix in Sec. IV and present results on the mutual information and the entanglement entropy at zero and finite temperature in Sec. V and on excited states in Sec. VI. Finally, in Sec. VII, we make some closing remarks.

II. LUTTINGER LIQUID DESCRIPTION

In a preceding paper [22], we investigated the entanglement between a Luttinger liquid and acoustic phonons that are coupled to the Luttinger liquid. We considered several physically interesting versions of the problem, however not the model corresponding to the Luttinger liquid coupled to Einstein phonons which is at the focus of the present paper. To motivate our model, let us start with the Holstein Hamiltonian in momentum space which reads

$$H = \sum_q \epsilon(q) c_q^\dagger c_q + \omega_0 \sum_q a_q^\dagger a_q + \frac{g}{\sqrt{2\omega_0 M}} \frac{1}{\sqrt{L}} \sum_q c_k^\dagger c_{k+q} (a_q^\dagger + a_{-q}), \tag{2}$$

where L is the linear size of the system, $c_k^\dagger$ and c_k are creation and annihilation operators for electrons, $\epsilon(k)$ is the one-particle electron energy, $a_q^\dagger$ and a_q are the creation and annihilation operators of the phonons, ω_0 is the frequency of the Einstein phonons, M is the oscillator mass, and g is the strength of the electron-phonon coupling. After linearization around the Fermi energy and introducing left (L_k, $L_k^\dagger$) and right movers (R_k, $R_k^\dagger$), the kinetic energy term becomes

$$\sum_k c_k^\dagger c_k \epsilon(k) \to \sum_k v_f k (R_k^\dagger R_k - L_k^\dagger L_k). \tag{3}$$

Further after bosonization [45] with boson operators b_q, $b_q^\dagger$ this can be rewritten as

$$\sum_q v_f |q| b_q^\dagger b_q. \tag{4}$$

Neglecting backward scattering one gets—now also including the phonons:

$$H_{LL} = \sum_{q=-\infty}^{\infty} v_f |q| b_q^\dagger b_q + \omega_0 \sum_{q=-\infty}^{\infty} a_q^\dagger a_q + \frac{g}{\sqrt{4\pi\omega_0 M}} \sum_{q=-\text{cutoff}}^{q_{\text{cutoff}}} \sqrt{|q|} (b_{-q}^\dagger + b_q)(a_q^\dagger + a_{-q}). \tag{5}$$

Here, there is a new parameter q_{cutoff} since the Luttinger description is only valid in the low energy sector, so the high-momentum states are secluded from the electron-phonon interaction [23]. In the following, we will investigate the Hamiltonian defined in Eq. (5), the Holstein model (2) merely serves as a motivation. The main differences are the (infinite) linear electron dispersion of Hamiltonian (5), and the lack of the backscattering term. In the following, we set $\omega_0 = M = v_F = 1$.

III. CANONICAL TRANSFORMATION

In this section, we diagonalize the model (5) using a Bogoliubov transformation, calculate the pair correlation functions and from this in turn reconstruct the density matrix of the electrons in terms of the bosonized operators. In the literature of entanglement measures of free bosonic systems, canonical impulse and coordinate operators are used more often than the creation and annihilation operators $b^\dagger$, b of the bosons, mainly due to historical reasons. To fit to this major part of the literature, we define cosine and sine (or even and odd) canonical modes for the lattice. Specifically, the cosine modes for the lattice read

$$\hat{Q}_{C,q} = \frac{1}{2\sqrt{\omega_0}} (a_q^\dagger + a_q + a_{-q}^\dagger + a_{-q}), \tag{6}$$

$$\hat{P}_{C_q} = \frac{i\sqrt{\omega_0}}{2} (a_q^\dagger - a_q + a_{-q}^\dagger - a_{-q}), \tag{7}$$

and the sine modes

$$\hat{Q}_{S,q} = \frac{1}{2\sqrt{\omega_0}} (a_q^\dagger + a_q - a_{-q}^\dagger - a_{-q}), \tag{8}$$

$$\hat{P}_{S_q} = \frac{i\sqrt{\omega_0}}{2} (a_q^\dagger - a_q - a_{-q}^\dagger + a_{-q}). \tag{9}$$

These represent cosine and sine real space movements, respectively.

We also introduce cosine and sine modes in the electronic subsystem, however we would like to mention that there is no simple connection between these operators, and the electron movements (only to a modulation of the electron density). For the electrons, the cosine modes read

$$\hat{q}_{C,q} = \frac{1}{2\sqrt{v_F|q|}} (b_q^\dagger + b_q + b_{-q}^\dagger + b_{-q}), \tag{10}$$

$$\hat{p}_{C_q} = \frac{i\sqrt{v_F|q|}}{2} (b_q^\dagger - b_q + b_{-q}^\dagger - b_{-q}), \tag{11}$$

and the sine modes

$$\hat{q}_{S,q} = \frac{1}{2\sqrt{v_F|q|}} (b_q^\dagger + b_q - b_{-q}^\dagger - b_{-q}), \tag{12}$$

$$\hat{p}_{S_q} = \frac{i\sqrt{v_F|q|}}{2} (b_q^\dagger - b_q - b_{-q}^\dagger + b_{-q}). \tag{13}$$

The Hamiltonian (5) then takes the following form:

$$H = \frac{1}{2} \sum_{q>0} [\hat{p}_{C,q}, \hat{P}_{C,q}] \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{p}_{C,q} \\ \hat{P}_{C,q} \end{bmatrix} + [\hat{q}_{C,q}, \hat{Q}_{C,q}] \begin{bmatrix} (v_F|q|)^2 & gq\sqrt{\frac{2v_F}{\pi M}} \\ gq\sqrt{\frac{2v_F}{\pi M}} & \omega_0^2 \end{bmatrix} \begin{bmatrix} \hat{q}_{C,q} \\ \hat{Q}_{C,q} \end{bmatrix} + [\hat{p}_{S,q}, \hat{P}_{S,q}] \begin{bmatrix} 1 & \frac{g}{\omega_0}\sqrt{\frac{2}{\pi M v_F}} \\ \frac{g}{\omega_0}\sqrt{\frac{2}{\pi M v_F}} & 1 \end{bmatrix} \begin{bmatrix} \hat{p}_{S,q} \\ \hat{P}_{S,q} \end{bmatrix} + [\hat{q}_{S,q}, \hat{Q}_{S,q}] \begin{bmatrix} (v_F|q|)^2 & 0 \\ 0 & \omega_0^2 \end{bmatrix} \begin{bmatrix} \hat{q}_{S,q} \\ \hat{Q}_{S,q} \end{bmatrix}. \tag{14}$$

The sine (cosine) mode of the lattice couples only to the sine (cosine) mode of the electrons. Next, we introduce new canon-

ical variables $\hat{p}_{1,q}$, $\hat{p}_{2,q}$, $\hat{p}_{3,q}$, $\hat{p}_{4,q}$, $\hat{q}_{1,q}$, $\hat{q}_{2,q}$, $\hat{q}_{3,q}$, and $\hat{q}_{4,q}$ to diagonalize the Hamiltonian

$$\begin{bmatrix}\hat{p}_{1,q}\\ \hat{p}_{2,q}\end{bmatrix} = \begin{bmatrix}A_q & -B_q\\ B_q & A_q\end{bmatrix}\begin{bmatrix}\hat{p}_{c,q}\\ \hat{P}_{c,q}\end{bmatrix}, \tag{15}$$

$$\begin{bmatrix}\hat{q}_{1,q}\\ \hat{q}_{2,q}\end{bmatrix} = \begin{bmatrix}A_q & -B_q\\ B_q & A_q\end{bmatrix}\begin{bmatrix}\hat{q}_{c,q}\\ \hat{Q}_{c,q}\end{bmatrix}. \tag{16}$$

Here, the A_q and B_q numbers [defined in Eqs. (23) and (22) below] are the components of the eigenvectors of the momentum matrix of the Hamiltonian. The sine modes can be diagonalized by a slightly modified unitary transformation:

$$\begin{bmatrix}\hat{p}_{3,k}\\ \hat{p}_{4,k}\end{bmatrix} = \begin{bmatrix}A_k\frac{\omega_{3,k}}{v_F k} & -B_k\frac{\omega_{3,k}}{\omega_0}\\ B_q\frac{\omega_{4,k}}{v_F k} & A_q\frac{\omega_{4,k}}{\omega_0}\end{bmatrix}\begin{bmatrix}\hat{p}_{s,q}\\ \hat{P}_{s,q}\end{bmatrix}, \tag{17}$$

$$\begin{bmatrix}\hat{q}_{3,q}\\ \hat{q}_{4,q}\end{bmatrix} = \begin{bmatrix}A_q\frac{v_F q}{\omega_{3,k}} & -B_q\frac{\omega_0}{\omega_{3,q}}\\ B_q\frac{v_F q}{\omega_{4,q}} & A_q\frac{\omega_0}{\omega_{4,q}}\end{bmatrix}\begin{bmatrix}\hat{q}_{s,q}\\ \hat{Q}_{s,q}\end{bmatrix}. \tag{18}$$

The thus diagonalized Hamiltonian takes the following simple form of four uncoupled harmonic oscillators:

$$H = \sum_{q>0}\sum_{i=1}^{4}\left(\frac{1}{2}\hat{p}_{i,q}^2 + \frac{1}{2}\omega_{i,q}^2\hat{q}_{i,q}^2\right). \tag{19}$$

Here, A_q, B_q, and $\omega_{1,q},\ldots,\omega_{4,q}$ are

$$\omega_{1,q}^2 = \omega_{3,q}^2 = \frac{(v_F q)^2+\omega_0^2}{2} + \sqrt{\frac{\left((v_F q)^2-\omega_0^2\right)^2}{4}+\frac{v_F g^2 q^2}{\pi M}}, \tag{20}$$

$$\omega_{2,q}^2 = \omega_{4,q}^2 = \frac{(v_F q)^2+\omega_0^2}{2} - \sqrt{\frac{\left((v_F q)^2-\omega_0^2\right)^2}{4}+\frac{v_F g^2 q^2}{\pi M}}\ ; \tag{21}$$

$$B_q = -\frac{gq}{\sqrt{N_q}}\sqrt{\frac{v_F}{\pi M}}, \tag{22}$$

$$A_q = \frac{1}{\sqrt{N_q}}\left((v_F|q|)^2-\omega_1^2\right), \tag{23}$$

$$N_q = q^2\frac{g^2q^2v_F}{\pi M} + \left((v_F|q|)^2-\omega_1^2\right)^2. \tag{24}$$

We can further rewrite the diagonalized Hamiltonian (19) again in terms of creation and annihilation operators

$$b_{i,q} = \sqrt{\frac{\omega_{i,q}}{2}}\left(\hat{q}_{i,q}+\frac{i}{\omega_{i,q}}\hat{p}_{i,q}\right), \tag{25}$$

$$b_{i,q}^\dagger = \sqrt{\frac{\omega_{i,q}}{2}}\left(\hat{q}_{i,q}-\frac{i}{\omega_{i,q}}\hat{p}_{i,q}\right). \tag{26}$$

With these operators the Hamiltonian becomes

$$H = \sum_{i,q>0}\omega_{i,q}(b_{i,q}^\dagger b_{i,q}+1/2)\ ; \tag{27}$$

and the excited states can be simply labeled by the occupation numbers $n_{i,q}$ for the bosonic degrees of freedom:

$$|\{n_{i,q}\}\rangle = \prod_{\substack{q=0,\ldots,\pi\\ i=1,\ldots,4}}(b_{i,q}^\dagger)^{n_{i,q}}|0\rangle. \tag{28}$$

The stability criterion of the system in our notations is $\pi\omega_0^2 v_F M > g^2$ [19,20,23].

IV. REDUCED DENSITY MATRIX

Next, we calculate the reduced density matrix S of the electrons from the pair correlation functions. Our quadratic Hamiltonian has been diagonalized by a canonical transformation. As a consequence, the Wick theorem holds in any subsystem, and the expectation value of any operator string can be calculated using the pair correlation functions. On the other hand, this means that if we find a Gaussian operator (exponential of a quadratic operator), which reproduces the pair correlation functions when used as a density matrix, this Gaussian operator must give the correct results for all operators, so it is the real density matrix of the subsystem. More details can be found in Ref. [46]. Hence the strategy is to calculate all pair correlation functions (there are a lot of zeros) and to find an appropriate Gaussian operator.

To obtain S, let us thus calculate every correlation function between the bosonized operators. The sine and cosine modes of the electron system are coupled to the corresponding phonon modes, but not to each other. Hence, the nonzero correlation functions of the electron subsystem are only $\langle q_{C,q}^2\rangle$, $\langle p_{C,q}^2\rangle$, $\langle p_{C,q}q_{C,q}\rangle$ and $\langle q_{S,q}^2\rangle$, $\langle p_{S,q}^2\rangle$, $\langle p_{S,q}q_{S,q}\rangle$. The coordinate-momentum correlation functions are constant $\langle p_{C,q}q_{C,q}\rangle = \langle p_{S,q}q_{S,q}\rangle = i$, which is a consequence of the commutator relations.

These correlation functions relate to the correlation functions of the eigenmodes $i = 1,\ldots,4$ as follows (any correlation function between different eigenmodes is zero):

$$\langle\hat{q}_{s,q}^2\rangle = \frac{\omega_{1,q}^2}{(v_F q)^2}A_q^2\langle\hat{q}_{3,q}^2\rangle + \frac{\omega_{2,q}^2}{(v_F q)^2}B_q^2\langle\hat{q}_{4,q}^2\rangle, \tag{29}$$

$$\langle\hat{p}_{s,q}^2\rangle = \frac{(v_F q)^2}{\omega_{1,q}^2}A_q^2\langle\hat{p}_{3,k}^2\rangle + \frac{(v_F q)^2}{\omega_{2,q}^2}B_q^2\langle\hat{p}_{4,q}^2\rangle, \tag{30}$$

$$\langle\hat{q}_{c,q}^2\rangle = A_q^2\langle\hat{q}_{1,q}^2\rangle + B_q^2\langle\hat{q}_{2,k}^2\rangle, \tag{31}$$

$$\langle\hat{p}_{c,q}^2\rangle = A_q^2\langle\hat{p}_{1,q}^2\rangle + B_q^2\langle\hat{p}_{2,q}^2\rangle. \tag{32}$$

These are in turn directly related to the expectation values $\langle b_{i,q}^\dagger b_{i,q}\rangle$, the bosonic occupation numbers, via

$$\langle\hat{q}_{i,q}^2\rangle = (2\langle b_{i,k}^\dagger b_{i,q}\rangle + 1)/(2\omega_{i,k}), \tag{33}$$

$$\langle\hat{p}_{i,q}^2\rangle = \omega_{i,k}(2\langle b_{i,q}^\dagger b_{i,q}\rangle + 1)/2. \tag{34}$$

For the ground state, excited states and thermal ensemble these expectation values are

$$\langle b_{i,q}^\dagger b_{i,q}\rangle = \begin{cases}0 & \text{in the ground state}\\ n_{i,q} & \text{in excited state } |\{n_{i,q}\}\rangle\\ \frac{1}{\exp(\beta\omega_{i,q})-1} & \text{at temperature } T=1/\beta\end{cases}. \tag{35}$$

This way we have listed all pair correlation functions. Since there is no (nontrivial) correlation between the sine and the cosine modes of the lattice, the density matrix has to be a product of a sine-mode and cosine-mode part as well as a product between different momenta $q > 0$, i.e.,

$$\begin{aligned}\rho = {}& \Pi_{q>0}^{q_{\text{cutoff}}}(1-e^{\beta_{c,q}})e^{-\sum_q \beta_{c,q}B_{c,q}^\dagger B_{c,q}}\\ & \times \Pi_{q>0}^{q_{\text{cutoff}}}(1-e^{\beta_{s,q}})e^{-\sum_q \beta_{s,q}B_{s,q}^\dagger B_{s,q}}\\ & \times \rho_{q>q_{\text{cutoff}}}.\end{aligned} \tag{36}$$

Here $B^{\dagger}_{s,q}$, $B_{s,q}$, $B^{\dagger}_{c,q}$, $B_{c,q}$ are bosonic creation and annihilation operators, which can be defined in the following way:

$$B^{\dagger}_{s,q} = \sqrt{\frac{\alpha_{s,q}}{2}}\left(\hat{q}_{s,q} + \frac{i}{\alpha_{s,q}}\hat{p}_{s,q}\right), \tag{37}$$

$$B_{s,q} = \sqrt{\frac{\alpha_{s,q}}{2}}\left(q_{s,q} - \frac{i}{\alpha_{s,q}}\hat{p}_{s,q}\right), \tag{38}$$

$$B^{\dagger}_{c,q} = \sqrt{\frac{\alpha_{c,q}}{2}}\left(\hat{q}_{c,q} + \frac{i}{\alpha_{c,q}}\hat{p}_{c,q}\right), \tag{39}$$

$$B_{c,q} = \sqrt{\frac{\alpha_{c,q}}{2}}\left(\hat{q}_{c,q} - \frac{i}{\alpha_{c,q}}\hat{p}_{c,q}\right), \tag{40}$$

where $\alpha_{C,q}$, $\alpha_{S,q}$, $\beta_{C,q}$, and $\beta_{S,q}$ are unknown parameters. These coefficients are the only remaining "free" parameters of the density matrix, because of the high symmetries (lots of zero correlations). One has to choose these parameters in such a way that the correct pair correlation functions as calculated above are restored. Now we calculated all pair correlation functions using the form Eq. (40) as the function of the unknown parameters (not shown here explicitly).

We first find that the root of $\langle\hat{p}^2_{S/C,q}\rangle/\langle\hat{q}^2_{S/C,q}\rangle$ is $\alpha_{C/S,q}$. Then we realize, that the product of these $(\langle\hat{p}^2_{S/C,q}\rangle\langle\hat{q}^2_{C/S,q}\rangle)$ is related to the occupation numbers. With these findings all parameters can be determined as follows:

$$\alpha_{C/S,k} = \sqrt{\frac{\langle\hat{p}^2_{S/C,k}\rangle}{\langle\hat{q}^2_{S/C,k}\rangle}}, \tag{41}$$

$$\beta_{C/S,k} = \ln\left(1 + \frac{1}{\sqrt{\langle\hat{p}^2_{S/C,k}\rangle\langle\hat{q}^2_{S/C,k}\rangle} - 1/2}\right). \tag{42}$$

These equations together with similar ones for the reduced density matrix of the phonons presented in Appendix B form the main result of this paper.

Generally, we have $\alpha_{C,k} \neq \alpha_{S,k}$, and $\beta_{S,k} \neq \beta_{C,k}$, but at zero temperature $\beta_{S,k} = \beta_{C,k}$. The bosonic $B_{s,q}$ and $B_{c,q}$ operators are related to the original bosonized b_q operators and the ladder operators of the harmonic oscillator through a Bogoliubov transformation.

From Eq. (40) it further follows that $\beta_{s,q}$, $\beta_{c,q}$ are the one-particle eigenvalues of the entanglement Hamiltonian $H_{\text{ent}} = -\ln\rho$, which is a bosonic free particle Hamiltonian. This one-particle spectrum of the entanglement Hamiltonian for various coupling strengths is shown in Fig. 1(a). Please note that the full entanglement spectrum is not simply the set of the one-particle eigenvalues of the entanglement Hamiltonian. Even in our case of a free bosonic entanglement Hamiltonian, the spectrum of H_{ent} still includes all multiples of the one-particle eigenvalues (and all combination of different multiples).

Although the one-particle eigenvalues (as a function of the coupling strength) do not cross each other, there can hence still be level crossings in the full entanglement spectrum. An illustrative example is shown in Fig. 1(b). If one arranges the entanglement eigenvalues by magnitude, the crossing simply implies a break in the derivative of the nth entanglement eigenvalue [but the eigenvalue itself remains a continuous function of the coupling, see Fig. 1(b).]

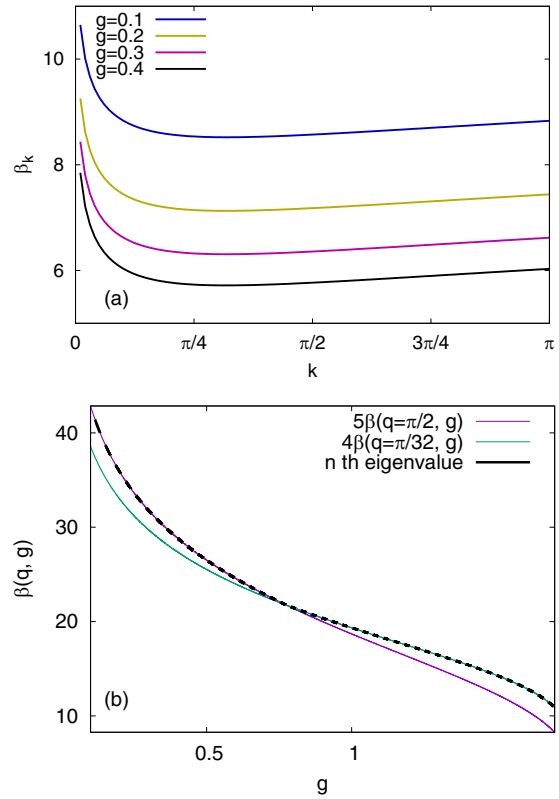

FIG. 1. (a) One-particle spectrum $\beta_{C,k} = \beta_{S,k} = \beta_k$ of the entanglement Hamiltonian for various coupling strengths g at zero temperature. (b) Crossing of different eigenvalues of the entanglement Hamiltonian. Here and in the following $\omega_0 = M = v_F = 1$.

In Ref. [32], it has been found by exact diagonalization of a Peierls-type Hamiltonian, that the entanglement spectrum is a nonanalytical function of the coupling strength, the nth entanglement eigenvalue is a continuous function of the coupling strength, but its derivative is not continuous. Our integrable model gives a simple explanation at least for some similar singularities. Due to the finite bandwidth of the Holstein Hamiltonian, singularities of a different origin may appear. Indeed not all singularities presented in Ref. [32] seems to be consistent with level crossing. On the other hand, the results of Ref. [32] have been obtained by exact diagonalization; and despite the usage of state-of-art methods, only a very small part of the entanglement spectrum can be explored with exact diagonalization. Hence, we do think that the crossing presented here exist also in finite bandwidth systems, but additional singularities may originate from the band edge.

V. ENTANGLEMENT MEASURES AT ZERO AND FINITE TEMPERATURE

In this section, we present results for the entanglement entropy

$$S \equiv S_e = -\text{Tr}\rho\ln\rho \tag{43}$$

for the ground and thermally excited state with the reduced density operator ρ for the electronic subsystem given by

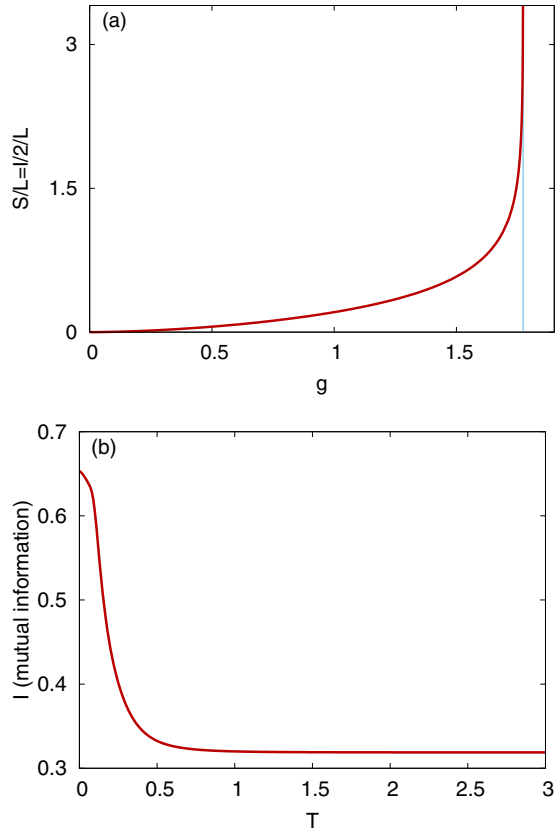

FIG. 2. Upper panel: Entanglement entropy S for the electronic subsystem (with a bipartition into electrons and phonons) as the function of the coupling strength g for the ground state. Lower panel: Mutual information I as the function of temperature at interaction $g = 0.1$.

Eq. (36), and for the mutual information

$$I = S_e + S_{\rm ph} - S_{\rm tot}, \tag{44}$$

where $S_{\rm ph}$ and $S_{\rm tot}$ are in analogy to Eq. (43) the entanglement entropies of the phonon (see Appendix B) and the total system, respectively. The results for the present model are shown in Fig. 2 (see Appendix A for calculational details). The entanglement entropy S grows with the electron-phonon coupling strength g and diverges at the Wentzel-Bardeen singularity. At zero temperature [panel (a)], the mutual information is $I = 2S$ since for our bipartition $S_e = S_{\rm ph}$ and $S_{\rm tot} = 0$. However, at finite temperatures $T > 0$ [panel (b)], neither of the latter two equations hold and there are deviations. Specifically, the mutual information decreases with increasing temperature, and after a local minimum around $T \approx 0.6$ reaches a plateau value for $T \to \infty$.

VI. ENTANGLEMENT CONTENT OF EXCITED STATES

In this section, we investigate the electron-phonon entanglement content of excited states $|\{n_{i,k}\}\rangle$. The entanglement entropy can be written as (see Appendix A)

$$S = \sum_{q>0} s\big(\langle \hat{p}^2_{C,q}\rangle\langle \hat{q}^2_{C,q}\rangle\big) + s\big(\langle \hat{p}^2_{S,q}\rangle\langle \hat{q}^2_{S,q}\rangle\big) \tag{45}$$

with $s(x) = (\sqrt{x}+1/2)\ln(\sqrt{x}+1/2) - (\sqrt{x}-1/2)\ln(\sqrt{x}-1/2)$. It is clear from Eq. (45), that the entanglement content of the different momenta is additive $S = \sum_q S_q$. This agrees with Ref. [42] and confirms Eq. (1), and further shows that a generalization to nonspatial bipartitions is possible. However, the entanglement content of the excitations $b^\dagger_{i,q}$ at the same momentum is not linearly additive for $i = 1, \ldots, 4$. If it were additive, the green and blue lines in Fig. 3(c) would coincide. The entanglement is also not proportional to the occupation number as observed in Ref. [38]: clearly the red line in Fig. 3(c) is not linear. The entanglement content of the first and third and the second and fourth kind of excitations are equal, see panels (a) and (b) of Fig. 3.

The entanglement negativity can be written as the following sum over the momentum

$$\mathcal{E} = -2\sum_{q=0,\pm}^{q_{\rm cutoff}} \ln\min(1, \sqrt{\Lambda_{C,q,\pm}}) + \ln\min(1, \sqrt{\Lambda_{S,q,\pm}}), \tag{46}$$

the derivation and the $\Lambda_{C/S,k,\pm}$ values are given in Appendix C. In Fig. 4, the terms corresponding to a given momentum k are shown for different occupation numbers n_1. Quite surprisingly, the contribution of a given mode to the entanglement negativity varies in a nonmonotonic way with the occupation number. For small occupations, the negativity become smaller then the ground state negativity, before starting to increase again for high occupation numbers. This is a rather different behavior than the entanglement entropy. The entanglement entropy always grows with excitations, one can define a positive entanglement content for each one-particle excitation. As we have seen, the negativity can decrease with one-particle excitations. If one defines a "negativity content of excitations," this content could be positive or negative.

VII. CONCLUSION AND OUTLOOK

We have derived an analytical formula for the entanglement entropy and the entanglement spectrum of a Luttinger liquid coupled to an Einstein phonon. The entanglement spectrum is $\sum_{q>0}\beta_{C,q}n_{C,q} + \sum_{q>0}\beta_{S,k}n_{S,q}$, where $n_{C,q} = 0, 1, 2, \ldots$ and $n_{S,q} = 0, 1, 2, \ldots$ are the occupation numbers and $\beta_{C/S,q}$ from Eq. (42) the eigenvalues of the bosonic modes of the entanglement Hamiltonian. The entanglement spectrum and thus the entanglement entropy is additive in momentum $q > 0$. For each q the Luttinger liquid coupled to Einstein phonons has four bosonic eigenmodes $\beta_{i,q}$, $i = 1, \ldots, 4$, because q and $-q$ couple into independent sine (S) and cosine (C) linear combinations [24]. For each of these in turn, electrons and phonons couple into two bosonic eigenmodes. In terms of the occupation of these four eigenmodes $\beta_{i,q}$ at fixed q, the entanglement entropy is not additive. In other words, while the momentum additivity conjectured in Eq. (1) holds in our model, there is no such additivity for the quasiparticle excitations at each momentum. As in exact diagonalization for a Peierls-type Hamiltonian [32], we find that the entanglement spectrum is a nonanalytical function of the coupling strength, caused by a level crossing of the eigenvalues.

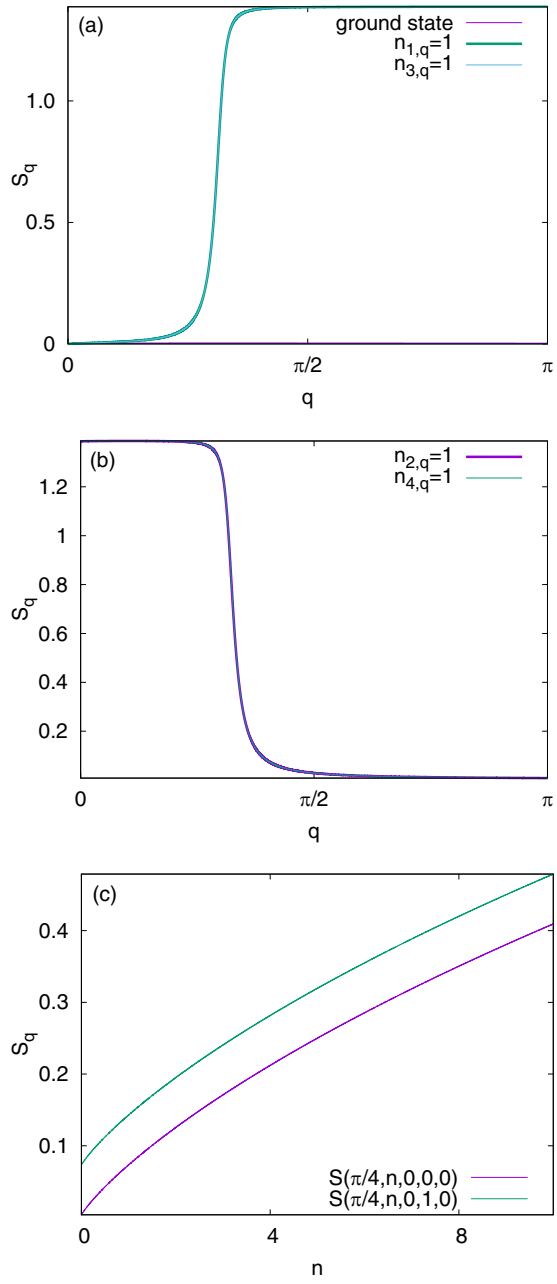

FIG. 3. (a) Entanglement content of the mode with momentum q in the ground state, and in the excited state with $n_{1,q} = 1$ ($n_{3,q} = 1$) and all other occupation numbers zero. Here and for all subpanels $g = 0.1$. (b) Entanglement content of the mode with momentum q in the excited states where $n_{2,q} = 1 (n_{4,q} = 1)$ and all other occupation numbers are zero. (c) Entanglement content of the excited state with $n_{1,\pi/4} \neq 0$ and all other occupation number zero as the function of the occupation number $n = n_{1,\pi/4}$ (purple line). The green line shows the entanglement content of the excited state where $n_{1,\pi/4} \neq 0$ and, in addition, $n_{3,\pi/4} = 1$. The sum of the entanglement contents of the two states with only one nonzero occupation number (blue line) differs from the state where both are excited simultaneously (green line).

In our calculations, we did not include the electron spin. If included, there will be spin modes and charge modes, and only the charge modes couple to the phonon system. Another simple way of generalization is to include the Coulomb

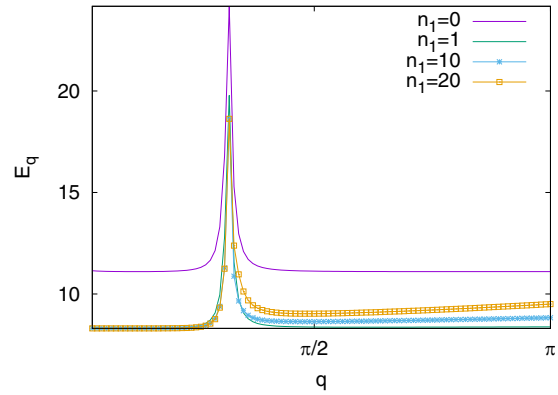

FIG. 4. Logarithmic entanglement negativity content of modes with a given wave number at coupling $g = 0.1$; $v_F = M = \omega_0 = 1$.

interaction between the electrons, and to consider acoustic phonons. In this case, the form of the electron density matrix remains unchanged, only its coefficients change, as discussed in Appendix D.

Our results are predominately of fundamental, theoretical interest. However, for prospective applications let us mention that similar one-dimensional electron-phonon models describe the low energy behavior of carbon nanotubes [47–50] and the surface states of thin topological insulator wires [51,52]. There are also three-dimensional systems such as $Li_{0.9}Mo_6O_{17}$ [53] where the electron system is quasi-one-dimensional, and can be modeled as several parallel chain, each described by the Luttinger theory. For these systems, our results provide a first, general point of understanding, which need to be further detailed to actually describe these materials. Another nontrivial generalization for higher dimension is the inclusion the phonons to the coupled-wire description of topological systems [54].

ACKNOWLEDGMENTS

G. R. thanks Carsten Timm, Zoltan Zimborás, and Ferenc Iglói for useful discussions. This work was supported in part by the Quantum Information National Laboratory of Hungary, and by the National Research, Development and Innovation Office NKFIH under Grant No. K128989. It profited from discussions within the Special Research Programme (SFB) F 86 Correlated Quantum Materials and Solid-State Quantum Systems of the Austrian Science Funds (FWF).

APPENDIX A: CALCULATION OF THE ENTANGLEMENT ENTROPY AND THE MUTUAL INFORMATION

The von Neumann entropy of the reduced density matrix is the entanglement entropy in the case of pure states (e.g., at zero temperature). It is also used in the definition of the mutual information in the case of general (for example, thermal) states. The calculation of this quantity has been published in Refs. [22,46,55,56], but to be self-contained we summarize it here in a nutshell.

The investigated subsystem contains N bosonic modes with $\hat{q}_1, \ldots, \hat{q}_N$ and $\hat{p}_1, \ldots, \hat{p}_N$ canonical conjugated position-momentum operators with $[q_i, p_j] = i\delta_{i,j}$. To calculate the entropy one first evaluates all pair-correlation

functions $Q_{ij} = \langle \hat{q}_i \hat{q}_j \rangle$ and $P_{i,j} = \langle \hat{p}_i \hat{p}_j \rangle$ and $\mathrm{Re}\langle \hat{q}_1 \hat{p}_l \rangle$. Next, one forms the following correlation matrix of the correlation functions:

$$M = \begin{pmatrix} \langle \hat{q}_1 \hat{q}_1 \rangle & \dots & \langle \hat{q}_1 \hat{q}_l \rangle & \mathrm{Re}\langle \hat{q}_1 \hat{p}_1 \rangle & \dots & \mathrm{Re}\langle \hat{q}_1 \hat{p}_l \rangle \\ \vdots & & \vdots & \vdots & & \vdots \\ \langle \hat{q}_l \hat{q}_1 \rangle & \dots & \langle \hat{q}_l \hat{q}_l \rangle & \mathrm{Re}\langle \hat{q}_l \hat{p}_1 \rangle & \dots & \mathrm{Re}\langle \hat{q}_l \hat{p}_l \rangle \\ \mathrm{Re}\langle \hat{p}_1 \hat{q}_1 \rangle & \dots & \mathrm{Re}\langle \hat{p}_1 \hat{q}_l \rangle & \langle \hat{p}_1 \hat{p}_1 \rangle & \dots & \langle \hat{p}_1 \hat{p}_l \rangle \\ \vdots & & \vdots & \vdots & & \vdots \\ \mathrm{Re}\langle \hat{p}_l \hat{q}_1 \rangle & \dots & \mathrm{Re}\langle \hat{p}_l \hat{q}_l \rangle & \langle \hat{p}_l \hat{p}_1 \rangle & \dots & \langle \hat{p}_l \hat{p}_l \rangle \end{pmatrix}. \tag{A1}$$

Now one has to diagonalize the matrix using symplectic transformations and in this way obtain the symplectic spectrum $\Lambda_1, \dots, \Lambda_N$. Every eigenvalue of the M matrix is twice degenerated, so N numbers gives the symplectic spectrum of this $2N \times 2N$ matrix. Having the eigenvalues of the matrix at hand, one gets the entanglement entropy of the reduced density matrix as

$$S = \sum_{j=1}^{N} (\Lambda_j + 1/2) \ln(\Lambda_j + 1/2) - (\Lambda_j - 1/2) \ln(\Lambda_j - 1/2). \tag{A2}$$

In our model, every coordinate-impulse correlation function is zero $\mathrm{Re}\langle \hat{q}_1 \hat{p}_l \rangle = 0$. In this case, it can be shown that the square of the symplectic eigenvalues are the eigenvalues of the PQ matrix, where P is matrix containing all momentum-momentum functions (all of $\langle \hat{p}_{S,k} \hat{p}_{S,q} \rangle$ and $\langle \hat{p}_{C,k} \hat{p}_{C,q} \rangle$), and Q is a matrix built up from the coordinate correlation functions in a similar manner.

When we consider the sine-cosine modes defined in Eqs. (11) and (13), the QP matrix further become diagonal, and in the diagonal there are the $\langle \hat{p}^2_{S,k} \rangle$ and the $\langle \hat{p}^2_{C,k} \rangle$ values. This leads to the simple integral formula of Eq. (45).

APPENDIX B: DENSITY MATRIX OF THE PHONONS

With a similar reasoning as we used in the main text to derive the density matrix of the electrons, one can obtain the reduced density matrix of the phonons:

$$\begin{aligned} \rho = & \prod_{q>0}^{q_{\mathrm{cutoff}}} \left(1 - e^{\beta^{\mathrm{Phonon}}_{c,q}}\right) e^{-\sum_q \beta^{\mathrm{Phonon}}_{c,q} D^\dagger_{c,q} D_{c,q}} \\ & \times \prod_{q>0}^{q_{\mathrm{cutoff}}} \left(1 - e^{\beta^{\mathrm{Phonon}}_{s,q}}\right) e^{-\sum_q \beta^{\mathrm{Phonon}}_{s,q} D^\dagger_{s,q} D_{s,q}} \\ & \times \rho^{\pi}_{q>q_{\mathrm{cutoff}}}. \end{aligned} \tag{B1}$$

Here $D^\dagger_{s,q}$, $D_{s,q}$, $D^\dagger_{c,q}$, $D_{c,q}$ are bosonic creation and annihilation operators, defined in the following way:

$$D^\dagger_{S,q} = \sqrt{\frac{\gamma_{S,q}}{2}} \left(\hat{Q}_{S,q} + \frac{i}{\gamma_{S,q}} \hat{P}_{S,q} \right), \tag{B2}$$

$$D_{S,q} = \sqrt{\frac{\gamma_{S,q}}{2}} \left(Q_{S,q} - \frac{i}{\gamma_{S,q}} \hat{P}_{S,q} \right), \tag{B3}$$

$$D^\dagger_{C,q} = \sqrt{\frac{\gamma_{C,q}}{2}} \left(\hat{Q}_{C,q} + \frac{i}{\gamma_{C,q}} \hat{P}_{c,q} \right), \tag{B4}$$

$$D_{C,q} = \sqrt{\frac{\gamma_{C,q}}{2}} \left(\hat{Q}_{C,q} - \frac{i}{\gamma_{C,q}} \hat{P}_{C,q} \right), \tag{B5}$$

where the parameters $\gamma_{C,q}$, $\gamma_{S,q}$, $\beta_{C,q}$, $\beta_{S,q}$ are obtained in a similar way as in the case of the density matrix of the electrons, yielding

$$\gamma_{C/S,q} = \sqrt{\frac{\langle \hat{P}^2_{S/C,q} \rangle}{\langle \hat{Q}^2_{S/C,q} \rangle}}, \tag{B6}$$

$$\beta^{\mathrm{Phonon}}_{C/S,q} = \ln \left(1 + \frac{1}{\sqrt{\langle \hat{P}^2_{S/C,q} \rangle \langle \hat{Q}^2_{S/C,q} \rangle} - 1/2} \right). \tag{B7}$$

Here the expectation values of the squares of the bosonic operators are the following:

$$\langle \hat{Q}^2_{C,q} \rangle = A^2_k \langle \hat{q}^2_{2,q} \rangle + B^2_k \langle \hat{q}^2_{1,q} \rangle, \tag{B8}$$

$$\langle \hat{P}^2_{C,q} \rangle = A^2_k \langle \hat{p}^2_{2,q} \rangle + B^2_k \langle \hat{p}^2_{1,q} \rangle, \tag{B9}$$

$$\langle \hat{Q}^2_{S,q} \rangle = A^2_k \frac{\omega^2_{4,q}}{\omega^2_0} \langle \hat{q}^2_{4,q} \rangle + B^2_k \frac{\omega^2_{3,q}}{\omega^2_0} \langle \hat{q}^2_{3,q} \rangle, \tag{B10}$$

$$\langle \hat{P}^2_{S,q} \rangle = A^2_k \frac{\omega^2_0}{\omega^2_{4,q}} \langle \hat{p}^2_{4,q} \rangle + B^2_k \frac{\omega^2_0}{\omega^2_{3,q}} \langle \hat{p}^2_{3,q} \rangle. \tag{B11}$$

APPENDIX C: CALCULATION OF THE ENTANGLEMENT NEGATIVITY

To calculate the entanglement negativity between the electrons and the phonons, one can consider the correlation matrix of the whole system, which is similar to M in Eq. (A1) but now contains all variables of the phonons and the bosonized fermions. Then one multiples all phonon momenta with -1. The thus transformed correlation matrix is the correlation matrix of the partial transpose of the density matrix.

One has to calculate the symplectic eigenvalues of the transformed matrix, let us denote these eigenvalues as λ. Then, the logarithmic negativity is calculated as

$$\mathcal{E} = -\sum_{\lambda} \ln \min(1, \lambda). \tag{C1}$$

In our problem the correlation matrix is block-diagonal, so one can find the symplectic eigenvalues block-by-block, and write the negativity as a sum over momentum, i.e.,

$$\mathcal{E} = -2 \sum_{q=0,\pm}^{q_{\mathrm{cutoff}}} \ln \min(1, \sqrt{\Lambda_{C,q,\pm}}) + \ln \min(1, \sqrt{\Lambda_{S,q,\pm}}), \tag{C2}$$

where

$$\Lambda_{C/S,q,\pm} = \tfrac{1}{2} \left[a_{C/S,q} \pm \sqrt{a^2_{C/S,q} + 4 b_{C/S,q} - 4 c_{C/S,q}} \right] \tag{C3}$$

with

$$a_{C/S,q} = \langle \hat{Q}_{C/S,q} \hat{Q}_{C/S,q} \rangle \langle \hat{P}_{C/S,q} \hat{P}_{C/S,q} \rangle + \langle \hat{q}_{C/S,q} \hat{q}_{C/S,q} \rangle \langle \hat{p}_{C/S,q} \hat{p}_{C/S,q} \rangle + 2 \langle \hat{P}_{C/S,q} \hat{p}_{C/S,q} \rangle \langle \hat{Q}_{C/S,q} \hat{q}_{C/S,q} \rangle, \tag{C4}$$

$$\begin{aligned} b_{C/S,q} = & \left(\langle \hat{Q}_{C/S,q} \hat{q}_{C/S,q} \rangle \langle \hat{P}_{C/S,q} \hat{P}_{C/S,q} \rangle - \langle \hat{q}_{C/S,q} \hat{q}_{C/S,q} \rangle \langle \hat{P}_{C/S,q} \hat{p}_{C/S,q} \rangle \right) \\ & \times \left(\langle \hat{Q}_{C/S,q} \hat{q}_{C/S,q} \rangle \langle \hat{p}_{C/S,q} \hat{p}_{C/S,q} \rangle - \langle \hat{P}_{C/S,q} \hat{p}_{C/S,q} \rangle \langle \hat{Q}_{C/S,q} \hat{Q}_{C/S,q} \rangle \right), \end{aligned} \tag{C5}$$

$$c_{C/S,q} = \big(\langle\hat{Q}_{C/S,q}\hat{Q}_{C/S,q}\rangle\langle\hat{P}_{C/S,q}\hat{P}_{C/S,q}\rangle - \langle\hat{P}_{C/S,q}\hat{p}_{C/S,q}\rangle\langle\hat{Q}_{C/S,q}\hat{q}_{C/S,q}\rangle\big) \times\big(\langle\hat{q}_{C/S,q}\hat{q}_{C/S,q}\rangle\langle\hat{p}_{C/S,q}\hat{p}_{C/S,q}\rangle - \langle\hat{P}_{C/S,q}\hat{p}_{C/S,q}\rangle\langle\hat{Q}_{C/S,q}\hat{q}_{C/S,q}\rangle\big). \tag{C6}$$

APPENDIX D: ACOUSTIC PHONONS

While the present work focuses on Einstein phonons, one can formulate the density matrix of the electrons coupled to acoustic phonons in a very similar way. From a technical side, changing the Einstein phonons to acoustic phonons is simple. However, physically, the acoustic phonon system with a local electron phonon coupling is unstable to lattice deformations. In the Luttinger description this is reflected by Hamiltonians with a spectrum unbounded from *below*.

However, if the coupling has a momentum dependence (i.e., is not exactly local) there are models with well-defined Hamiltonian and experimental relevance.

We consider here two types of acoustic phonon coupled to a plus Luttinger liquid. These two differ only in parameters, the first introduced by Bardeen [20] has linear electron-phonon coupling, the second has $\sqrt{k}$ electron phonon-coupling and is motivated by carbon nanotubes. The linear coupling has been introduced by an early analytical study [57], but it is only correct in a few materials [58], and it is even sensitive to magnetic field [59] and pressure [60].

The expressions for the correlation functions, Eq. (32), change for the acoustic phonons, but all later formulas for the reduced density matrix remain unchanged.

The original model investigated in Ref. [20] reads

$$H_{\text{LLB}} = \sum_{q=-\infty}^{\infty} v_f|q|b_q^\dagger b_q + \sum_{q=-\infty}^{\infty}\omega_q a_q^\dagger a_q + \frac{1}{\sqrt{4\pi v_S M}}\sum_{q=-\text{cutoff}}^{q_{\text{cutoff}}} gq(b_{-q}^\dagger + b_q)(a_q^\dagger + a_{-q}) \tag{D1}$$

$$\omega_q = v_S q. \tag{D2}$$

The second model considers reads

$$H_{\text{LL-NT}} = \sum_{q=-\infty}^{\infty} v_f|q|b_q^\dagger b_q + \sum_{q=-\infty}^{\infty}\omega_q a_q^\dagger a_q + \frac{1}{\sqrt{4\pi v_S M}}\sum_{q=-\text{cutoff}}^{q_{\text{cutoff}}} g\sqrt{q}(b_{-q}^\dagger + b_q)(a_q^\dagger + a_{-q}) \tag{D3}$$

$$+ \sum_q qh_q b_q^\dagger b_q + qf_q(b_q b_{-q} + b_q^\dagger b_{-q}^\dagger) \tag{D4}$$

In this later Hamiltonian, the bosonized interaction term corresponds to the following real-space interaction:

$$\frac{4}{L}\int_0^L\int_0^L dx\,dy\,(\hat{n}_L(x),\hat{n}_R(y)) \times\begin{pmatrix} h(x-y) & \frac{1}{2}f(x-y)\\ \frac{1}{2}f(x-y) & h(x-y)\end{pmatrix}\begin{pmatrix}\hat{n}_L(x)\\ \hat{n}_R(y)\end{pmatrix}, \tag{D5}$$

where $\hat{n}_R(x)$ and $\hat{n}_L(x)$ are the physical density of the right/left movers, the $h(x-y)$ ($f(x-y)$) functions describe nonlocal interactions between particles moving in the same (opposite) direction. The h_q and f_q coefficients in Eq. (D4) are the Fourier components of these functions.

The interaction term in the sine-cosine variables becomes

$$\frac{f_q}{2}\left(v_F q^2\hat{q}_{C,q}^2 + v_F q^2\hat{q}_{S,q}^2 + \frac{1}{v_F}\hat{p}_{C,q}^2 - \frac{1}{v_F}\hat{p}_{S,q}^2\right) + \frac{h_q}{2}\left(v_F q^2\hat{q}_{C,q}^2 + v_F q^2\hat{q}_{S,q}^2 + \frac{1}{v_F}\hat{p}_{C,q}^2 - \frac{1}{v_F}\hat{p}_{S,q}^2\right) \tag{D6}$$

so in the form Eq. (14) of the Hamiltonian only the diagonals of the matrices are changed, and the phonon frequency get a momentum dependence.

This Hamiltonian is diagonalized by a simple canonical transformation, and the expectation values needed to calculate the density matrix are

$$\langle q_{C,k}^2\rangle = \frac{1}{1+(f_q+h_q)/v_F}\big(C_q\langle q_{1,k}^2\rangle + D_q\big)\langle q_{2,k}^2\rangle, \tag{D7}$$

$$\langle p_{C,k}^2\rangle = (1+(f_q+h_q)/v_F)\big(C_q\langle p_{1,k}^2\rangle + D_q\big)\langle p_{2,k}^2\rangle, \tag{D8}$$

$$\langle q_{S,k}^2\rangle = \frac{1}{v_F q^2 + v_F q^2(f_q+h_q)/2}\big(E_q\langle q_{3,k}^2\rangle + F_q\langle q_{4,k}^2\rangle\big), \tag{D9}$$

$$\langle p_{S,k}^2\rangle = (v_F q^2 + v_F q^2(f_q+h_q)/2)\big(E_q\langle p_{3,k}^2\rangle + F_q\langle p_{4,k}^2\rangle\big). \tag{D10}$$

The frequencies are

$$\omega_{1,q}^2 = \frac{1}{2}\left(v_F q^2\frac{1+(f_q+h_q)/2}{1+(f_q+h_q)/v_F} + \omega_q^2 + \sqrt{\left(v_F q^2 v_F q^2(f_q+h_q)/2+\omega_q^2\right)^2 + 8\frac{g_k^2 q^2 v_F/\pi/M}{1+(f_q+h_q)/2}}\right), \tag{D11}$$

$$\omega_{2,q}^2 = \frac{1}{2}\left(v_F q^2\frac{1+(f_q+h_q)/2}{1+(f_q+h_q)/v_F} + \omega_q^2 + \sqrt{\left(v_F q^2 v_F q^2(f_q+h_q)/2+\omega_q^2\right)^2 + 8\frac{g_k^2 q^2 v_F/\pi/M}{1+(f_q+h_q)/2}}\right), \tag{D12}$$

$$\omega_{3,q} = \frac{1}{2}(v_F q^2(1+(f_q+h_q/2))(1+(f_q-h_q)/2) + \omega_q^2 \tag{D13}$$

$$+\sqrt{\left[v_F q^2(1+(f_q+h_q/2))(1+(f_q-h_q)/2)+\omega_q^2\right]^2+8\frac{g_k^2(v_F|q|+v_F q^2(f_q+h_q)/2)}{\pi M v_F}}\Bigg), \tag{D14}$$

$$\omega_{4,q}=\frac{1}{2}(v_F q^2(1+(f_q+h_q/2))(1+(f_q-h_q)/2)+\omega_q^2 \tag{D15}$$

$$-\sqrt{\left[v_F q^2(1+(f_q+h_q/2))(1+(f_q-h_q)/2)+\omega_q^2\right]^2+8\frac{g_k^2(v_F|q|+v_F q^2(f_q+h_q)/2)}{\pi M v_F}}\Bigg), \tag{D16}$$

$$C_q=\left(\omega_q^2-\omega_{1,q}\right)/\tilde{N}_q, \tag{D17}$$

$$D_q=-\frac{1}{\tilde{N}_q}\frac{g_k q\sqrt{\frac{2v_F}{\pi M}}}{\sqrt{1+(f_q+h_q)/v_f}}, \tag{D18}$$

$$\tilde{N}_q=\left(\omega_q^2-\omega_{1,q}\right)^2+\frac{g_k^2 q^2\frac{2v_F}{\pi M}}{1+(f_q+h_q)/v_f}, \tag{D19}$$

$$E_q=-\frac{g_k}{\omega_k}\frac{2}{\sqrt{\pi M v_F}}\sqrt{v_F|q|+v_F q^2(f_q+h_q)/2}/N', \tag{D20}$$

$$F_q=\left(\omega_q^2-\omega_{1,q}\right)/N'_q, \tag{D21}$$

$$N'_q=\left(\omega_q^2-\omega_{1,q}\right)^2+\frac{g_k^2}{\omega_k^2}\frac{4}{\pi M v_F}(v_F|q|+v_F q^2(f_q+h_q)/2). \tag{D22}$$

As already discussed, the further procedure is identical to that for Einstein phonons.

[1] E. Maxwell, Phys. Rev. **78**, 477 (1950).
[2] C. A. Reynolds, B. Serin, W. H. Wright, and L. B. Nesbitt, Phys. Rev. **78**, 487 (1950).
[3] W. D. Allen, R. H. Dawton, M. Bär, K. Mendelson, and J. L. Olsen, Nature (London) **166**, 1071 (1950).
[4] L. N. Cooper, Phys. Rev. **104**, 1189 (1956).
[5] J. Bardeen, L. N. Cooper, and J. R. Schrieffer, Phys. Rev. **106**, 162 (1957).
[6] J. Bardeen, L. N. Cooper, and J. R. Schrieffer, Phys. Rev. **108**, 1175 (1957).
[7] R. E. Peierls, *Quantum Theory of Solids* (Oxford University Press, Oxford, 1955), p. 108.
[8] P. A. Lee, T. M. Rice, and P. W. Anderson, Phys. Rev. Lett. **31**, 462 (1973).
[9] B. Braunecker, G. I. Japaridze, J. Klinovaja, and D. Loss, Phys. Rev. B **82**, 045127 (2010).
[10] G. Grüner, Rev. Mod. Phys. **60**, 1129 (1988).
[11] A. Virosztek, B. Dóra, and A. Virosztek, Phys. Rev. B **73**, 165127 (2006).
[12] J. H. Miller, C. Ordóñez, and E. Prodan, Phys. Rev. Lett. **84**, 1555 (2000).
[13] A. B. Midgal, Sov. Phys. JETP **7**, 996 (1958).
[14] G. M. Éliashberg, Sov. Phys. JETP **11**, 696 (1960).
[15] A. S. Mishchenko, N. Nagaosa, and N. Prokof'ev, Phys. Rev. Lett. **113**, 166402 (2014).
[16] A. S. Mishchenko, N. V. Prokof'ev, A. Sakamoto, and B. V. Svistunov, Phys. Rev. B **62**, 6317 (2000).
[17] N. V. Prokof'ev and B. V. Svistunov, Phys. Rev. Lett. **81**, 2514 (1998).
[18] J. del Pino, F. A. Y. N. Schröder, A. W. Chin, J. Feist, and F. J. Garcia-Vidal, Phys. Rev. Lett. **121**, 227401 (2018).
[19] G. Wentzel, Phys. Rev. **83**, 168 (1951).
[20] J. Bardeen, Rev. Mod. Phys. **23**, 261 (1951).
[21] D. Loss and T. Martin, Phys. Rev. B **50**, 12160 (1994).
[22] G. Roósz and C. Timm, Phys. Rev. B **104**, 035405 (2021).
[23] S. Engelsberg and B. B. Varga, Phys. Rev. **136**, A1582 (1964).
[24] B. Dóra, I. Lovas, and F. Pollmann, Phys. Rev. B **96**, 085109 (2017).
[25] A. Weiße, H. Fehske, G. Wellein, and A. R. Bishop, Phys. Rev. B **62**, R747 (2000).
[26] L.-C. Ku, S. A. Trugman, and J. Bonča, Phys. Rev. B **65**, 174306 (2002).
[27] Y. Toyozawa, Prog. Theor. Phys. **26**, 29 (1961).
[28] O. S. Barišić, Phys. Rev. B **65**, 144301 (2002).
[29] T. A. Costi and R. H. McKenzie, Phys. Rev. A **68**, 034301 (2003).
[30] J. L. Sanz-Vicario, J. F. Pérez-Torres, and G. Moreno-Polo, Phys. Rev. A **96**, 022503 (2017).
[31] V. M. Stojanovic and M. Vanevic, Phys. Rev. B **78**, 214301 (2008).
[32] V. M. Stojanovic, Phys. Rev. B **101**, 134301 (2020).
[33] Y. Zhao, P. Zanardi, and G. Chen, Phys. Rev. B **70**, 195113 (2004).
[34] C. A. Perroni, V. Marigliano Ramaglia, and V. Cataudella, Phys. Rev. B **82**, 104303 (2010).
[35] J. Sun and Y. Zhao, and W. Liang, Phys. Rev. B **79**, 155112 (2009).
[36] M. I. Berganza, F. C. Alcaraz, G. Sierra, J. Stat. Mech. (2012) P01016.
[37] J. Mölter, T. Barthe, U. Schollwöck, and V. Alba, J. Stat. Mech. (2014) P10029.
[38] I. Pizorn, arXiv:1202.3336.

[39] M. Storms, R. R. P. Singh, Phys. Rev. E **89**, 012125 (2014).
[40] R. Berkovits, Phys. Rev. B **87**, 075141 (2013).
[41] V. Alba, M. Fagotti, P. Calabrese, J. Stat. Mech. (2009) P10020.
[42] O. A. Castro-Alvaredo, C. De Fazio, B. Doyon, I. M. Szécsényi, Phys. Rev. Lett. **121**, 170602 (2018).
[43] O. A. Castro-Alvaredo, C. De Fazio, B. Doyon, I. M. Szécsényi, J. Math. Phys. **60**, 082301 (2019).
[44] O. A. Castro-Alvaredo, C. De Fazio, B. Doyon, and I. M. Szécsényi, J. High Energy Phys. **2019**, 1 (2019).
[45] J. von Delft and H. Schoeller, Ann. Phys. **510**, 225 (1998).
[46] I. Peschel, Braz. J. Phys. **42**, 267 (2012).
[47] H. Suzuura and T. Ando, Phys. Rev. B **65**, 235412 (2002).
[48] R. Rosati, F. Dolcini, and F. Rossi, Phys. Rev. B **92**, 235423 (2015).
[49] A. De Martino and R. Egger, Phys. Rev. B **67**, 235418 (2003).
[50] A. Benyamini, A. Hamo, S. Viola Kusminskiy, F. von Oppen, and S. Ilani, Nat. Phys. **10**, 151 (2014).
[51] R. Egger, A. Zazunov, and A. Levy Yeyati, Phys. Rev. Lett. **105**, 136403 (2010).
[52] K. Dorn, A. De Martino, and R. Egger, Phys. Rev. B **101**, 045402 (2020).
[53] P. Chudzinski, T. N. Jarlborg, and T. Giamarchi, Phys. Rev. B **86**, 075147 (2012).
[54] T. Meng, Eur. Phys. J.: Spec. Top. **229**, 527 (2020).
[55] K. Audenaert, J. Eisert, M. B. Plenio, and R. F. Werner, Phys. Rev. A **66**, 042327 (2002).
[56] V. Eisler and Z. Zimborás, New J. Phys. **16**, 123020 (2014).
[57] F. Bloch, Z. Phys. **52**, 555 (1929).
[58] F. Giustino, Rev. Mod. Phys. **89**, 015003 (2017).
[59] C. Faugeras, M. Amado, P. Kossacki, M. Orlita, M. Sprinkle, C. Berger, W. A. de Heer, and M. Potemski, Phys. Rev. Lett. **103**, 186803 (2009).
[60] P. Postorino, A. Congeduti, P. Dore, A. Sacchetti, F. Gorelli, L. Ulivi, and A. Kumar and D. D. Sarma, Phys. Rev. Lett. **91**, 175501 (2003).