Superconductivity out of a non-Fermi liquid: Free energy analysis

Shang-Shun Zhang [1], Yi-Ming Wu [2], Artem Abanov [3] and Andrey V. Chubukov [1]

[1] School of Physics and Astronomy and William I. Fine Theoretical Physics Institute, University of Minnesota, Minneapolis, Minnesota 55455, USA

[2] Institute for Advanced Study, Tsinghua University, Beijing 100084, China

[3] Department of Physics, Texas A&M University, College Station, Texas 77843, USA

(Received 29 August 2022; revised 10 October 2022; accepted 12 October 2022; published 31 October 2022)

In this paper, we present an in-depth analysis of the condensation energy E_c for a superconductor in a situation when superconductivity emerges out of a non-Fermi liquid due to pairing mediated by a massless boson. This is the case for electronic-mediated pairing near a quantum-critical point in metal, for pairing in SYK-type models, and for phonon-mediated pairing in the properly defined limit, when the dressed Debye frequency vanishes. We consider a subset of these quantum-critical models, in which the pairing in a channel with a proper spatial symmetry is described by an effective $0+1$ dimensional model with the effective dynamical interaction $V(\Omega_m) = \bar{g}^\gamma/|\Omega_m|^\gamma$, where γ is model-specific (the γ model). In previous papers, we argued that the pairing in the γ model is qualitatively different from that in a Fermi liquid, and the gap equation at $T = 0$ has an infinite number of topologically distinct solutions, $\Delta_n(\omega_m)$, where an integer n, running between 0 and infinity, is the number of zeros of $\Delta_n(\omega_m)$ on the positive Matsubara axis. This gives rise to the set of extrema of E_c at $E_{c,n}$, of which $E_{c,0}$ is the global minimum. The spectrum $E_{c,n}$ is discrete for a generic $\gamma < 2$ but becomes continuous at $\gamma = 2$–0. Here, we discuss in more detail the profile of the condensation energy near each $E_{c,n}$ and the transformation from a discrete to a continuous spectrum at $\gamma \to 2$. We also discuss the free energy and the specific heat of the γ model in the normal state.

DOI: 10.1103/PhysRevB.106.144513

I. INTRODUCTION

This work extends our previous analysis [1–6] of the interplay between non-Fermi liquid (NFL) physics and superconductivity for quantum-critical (QC) itinerant fermionic systems, whose low-energy dynamics can be described by an effective model of dispersion-full electrons, interacting by exchanging fluctuations of the critical order parameter. Viewed in the particle-hole channel, this interaction gives rise to singular self-energy $\Sigma(k, \omega)$ and NFL physics. Viewed in the particle-particle channel, it gives rise to a strong attraction in at least one pairing channel and to a spontaneous appearance of an anomalous pairing vertex $\Phi(k, \omega)$ at an elevated T_c. The two phenomena compete with each other: a NFL self-energy destroys Cooper logarithm in the particle-particle channel, while the opening of a pairing gap transfers the spectral weight out of low energies, rendering a coherent Fermi liquid behavior of low-energy fermions. Because both NFL and pairing come from the same source, the characteristic scales for the two phenomena are comparable. This makes the analysis of the competition quite challenging.

We analyze the interplay between pairing and NFL for a subset of quantum-critical systems, in which critical order parameter fluctuations are slow modes compared to fermions for one reason or another. Examples include systems at the verge of spin-density-wave order [7–15], charge-density wave order [16–26], ferromagnetic and Ising-nematic order [27–31] in 3D and 2D systems, fermions at a half-filled Landau level [32–36], electron-phonon systems [37–41] in the properly defined limit when the dressed Debye frequency vanishes.[1] In all these cases, fermionic self-energy, excluding the thermal piece, can be treated as momentum-independent $\Sigma(\omega)$ (barring potential logarithms [15,43]), while the pairing vertex $\Phi(k, \omega)$, is either k-independent or can be factorized as $\Phi(k, \omega) = f_k \Phi(\omega)$, where f_k is the spatial gap structure for the most strongly divergent channel (s-wave, d-wave, etc.). The theory then becomes effectively $0+1$ dimensional and reduces to the set of two coupled equations for $\tilde{\Sigma}(\omega) = \omega + \Sigma(\omega)$ and $\Phi(\omega)$ with an effective dynamical four-fermion interaction $V(\Omega)$. These can be obtained diagrammatically by restricting to rainbow approximation for the self-energy and ladder approximation for the pairing vertex. On the Matsubara axis, the two equations are

$$\Phi(\omega_m) = \pi T \sum_{m'} \frac{\Phi(\omega_{m'})}{\sqrt{\tilde{\Sigma}^2(\omega_{m'}) + \Phi^2(\omega_{m'})}} V(\omega_m - \omega_{m'}),$$
$$\tilde{\Sigma}(\omega_m) = \omega_m + \pi T \sum_{m'} \frac{\tilde{\Sigma}(\omega_{m'})}{\sqrt{\tilde{\Sigma}^2(\omega_{m'}) + \Phi^2(\omega_{m'})}} V(\omega_m - \omega_{m'}). \quad (1)$$

[1] At the small dressed Debye frequency, the physics that we study can be preempted by the development of a bipolaronic superconductivity [42]. We assume without proof that this does not happen.

At $T = 0$, $\pi T \sum_{m'} = (1/2) \int d\omega'_m$. The interaction $V(\Omega_m)$ is real and positive (attractive in our sign convention) and at a small but finite bosonic mass ω_D has the form

$$V(\Omega_m) = \frac{\bar{g}^\gamma}{\left(\Omega_m^2 + \omega_D^2\right)^{\gamma/2}}. \quad (2)$$

The $0 + 1$ dimensional model with $V(\Omega_m)$ as in Eq. (2) has been nicknamed the γ model. Different γ correspond to different physical realizations (see Ref. [1] for details). At a critical point, where $\omega_D = 0$, $V(\Omega_m)$ becomes a singular function of frequency, $V(\Omega_m) = (\bar{g}/|\Omega_m|)^\gamma$. For electronic pairing, the full set contains an additional equation describing the feedback from the pairing on the bosonic propagator, but we will neglect it as the feedback preserves Eqs. (1) and only changes γ into γ_{eff}, which depends on T. The physics of the feedback can then be analyzed by moving along the line $\gamma = \gamma_{\text{eff}}(T)$ in the (γ, T) plane.

A rather similar, although not identical set of equations holds for dispersion-less fermions randomly coupled to each other [44,45], or to phonons [46–49]. The equations for $\Sigma(\omega_m)$ and $\Phi(\omega_m)$ can be partly decoupled by introducing $\Delta(\omega_m) = \Phi(\omega_m)/Z(\omega_m)$, often called the gap function, and $Z(\omega_m) = 1 + \Sigma(\omega_m)/\omega_m$, which is the inverse quasiparticle residue. The equations for $\Delta(\omega_m)$ and $Z(\omega_m)$ are straightforwardly obtained from (1):

$$\Delta(\omega_m) = \pi T \sum_{m'} \frac{\Delta(\omega_{m'}) - \Delta(\omega_m)\frac{\omega_{m'}}{\omega_m}}{\sqrt{(\omega_{m'})^2 + \Delta^2(\omega_{m'})}} V(\omega_m - \omega_{m'}), \quad (3)$$

$$Z(\omega_m) = 1 + \frac{\pi T}{\omega_m} \sum_{m'} \frac{\omega_{m'}}{\sqrt{(\omega_{m'})^2 + \Delta^2(\omega_{m'})}} V(\omega_m - \omega_{m'}). \quad (4)$$

Equations (3) and (4) have the same form as Eliashberg equations for electron-phonon problem with a finite $V(0)$, and it is customary to keep calling them Eliashberg equations even when $V(0)$ diverges.

Observe that Eq. (3) is an integral equation that does not contain $Z(\omega_m)$, and Eq. (4) expresses $Z(\omega_m)$ via $\Delta(\omega_{m'})$. Note also that the potentially dangerous self-action term with $m = m'$ can be safely eliminated in (3) because the numerator there vanishes at $m = m'$. It should, however, be kept in (4).

In writing Eqs. (3) and (4), we assumed that $Z(\omega_m)$ is positive for all ω_m. The authors of Ref. [50] analyzed potential solutions with sign-changing $Z(\omega_m)$. We do not consider such solutions here as they only exist at a finite T, while our primary interest is to understand system behavior at zero temperature. We comment on the solutions with sign-changing $Z(\omega_m)$ in Appendix F. We argue that (i) at $T = 0$ such solutions do not exist and (ii) a finite T solutions exist, but do not obey the causality principle.

For large enough bosonic mass ω_D, the normal state is a Fermi liquid. Solving the gap equation, one obtains a pairing instability at some $T_{p,0}$. The corresponding gap function $\Delta_0(\omega_m)$ is a sign-preserving function of frequency. As ω_D decreases, new pairing instabilities emerge at $T_{p,n} < T_{p,0}$ (Ref. [2]). The number of solutions increases with decreasing ω_D and becomes infinite at $\omega_D = 0$, when a pairing boson becomes massless. In this limit, there exists an infinite, discrete set of solutions of the nonlinear gap equation at $T = 0$.

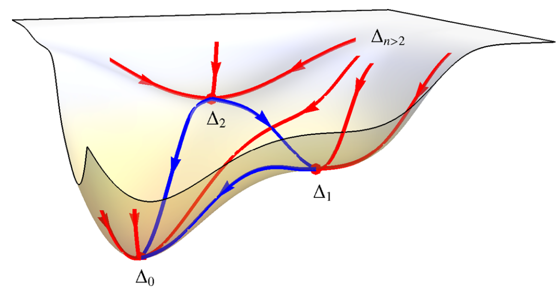

FIG. 1. A schematic plot of the free energy configuration in the functional space and the saddle points corresponding to the gap function $\Delta_n(\omega_m)$. The $n > 0$ solution is a saddle point of the free energy, which has n unstable principle axes connected with the $n' < n$ solutions, as illustrated by the flows in the plot. The $n = 0$ solution is a stable minimum.

One end of the set is the sign-preserving solution $\Delta_0(\omega_m)$. The other end is $\Delta_\infty(\omega_m)$ with infinitesimally small amplitude. For the latter, we found the exact analytical expression by solving exactly the linearized gap equation at $T = 0$ (Refs. [1,4]). All gap functions from the set are analytic in the upper half-plane of complex frequency $z = \omega' + i\omega''$, i.e., the corresponding Green's functions are causal.

In this communication, we address two issues. One is the profile of the condensation energy around $\Delta_n(\omega_m)$. The condensation energy $E_c = F_{\text{sc}} - F_n$ is the difference between the free energy of a superconductor, F_{sc}, and that of a would be normal state at the same temperature. The condensation energy is a functional of $\Delta(\omega_m)$ and $Z(\omega_m)$, and the Eliashberg equations are stationary conditions $\delta E_c/\delta\Delta = 0$ and $\delta E_c/\delta Z = 0$. We expand E_c to second order in deviations from $E_{c,n}$ for stationary gap function $\Delta_n(\omega_m)$ with different n and analyze the eigenvalues. For $n = 0$, all eigenvalues are positive, i.e., $\Delta_0(\omega_m)$ is a minimum of E_c. All other $\Delta_n(\omega_m)$ are saddle points, but rather specific ones (see Fig. 1). Namely, for $n = 1$, we show that there is a single negative eigenvalue, and the corresponding eigenfunction shifts $\Delta_1(\omega_m)$ towards $\Delta_0(\omega_m)$. For $n = 2$, there are two negative eigenvalues, which shift $\Delta_2(\omega_m)$ towards $\Delta_0(\omega_m)$ and towards $\Delta_1(\omega_m)$. For a generic $\Delta_n(\omega_m)$, there are n negative eigenvalues. One can find a "direction" in the space of $\Delta(\omega_m)$, along which all $E_{c,n}$ are connected by a single curve. Along this direction, each $E_{c,n}$ with $n > 0$ is an inflection point (see Fig. 7).

The other issue is the evolution of $\Delta_n(\omega_m)$ and $E_{c,n}$ near $\gamma = 2$. We argued in Refs. [4,5] that at $T = 0$ and $\omega_D = 0$, the set of $\Delta_n(\omega_m)$ becomes continuous at $\gamma \to 2$ in the particular double limit, which we discuss below. Specifically, we argued that all $\Delta_n(\omega_m)$ with finite n become equivalent to $\Delta_0(\omega_m)$ down to $\omega_m \to 0$, while $\Delta_n(\omega_m)$ with $n \to \infty$ form a one-parameter continuous set $\Delta(\epsilon, \omega_m)$, in which ϵ is a function of (n^*/n), where $n^* \sim |\ln(2 - \gamma)|/(2 - \gamma)$. By construction, ϵ runs between $0+$, for which $\Delta(0+, \omega_m)$ is infinitesimally small and $\epsilon_{\max}$, for which $\Delta(\epsilon_{\max}, \omega_m) = \Delta_0(\omega_m)$. We further argued that the condensation energy $E_{c,n}$ also becomes continuous $E_c(\epsilon)$ at $\gamma \to 2$. We emphasize that this happens only at $T = \omega_D = 0$. At any finite ω_D or T, $\Delta_n(\omega_m)$ and $E_{c,n}$ remain discrete at $\gamma = 2$.

Here we discuss the transformation from a discrete spectrum to a continuous one for both $\Delta_n(\omega_m)$ and $E_{c,n}$ in more detail. We show that the limit $\gamma \to 2$ has to be taken with special care because $E_{c,n}$ formally diverges as $1/(2-\gamma)$ for all values of n. Specifically, we show that for large but finite n,

$$E_{c,n} = -\bar{g}^2 N_F \left(\frac{1}{2-\gamma} - cn + \cdots \right), \tag{5}$$

where $c = O(1)$ and $N_F \propto N/E_F$ is the total density of states at the Fermi surface (N is the number of electrons and E_F is the Fermi energy). The prefactor c in (5) tends to a finite value at $\gamma \to 2$, yet, because the first term diverges, the condensation energy can be reexpressed as

$$E_{c,n} = -\frac{\bar{g}^2 N_F}{2-\gamma}(1 - cn(2-\gamma) + \cdots). \tag{6}$$

At $\gamma \to 2$, the term proportional to b vanishes for all finite n. We computed this term at $n \to \infty$ and found

$$E_c(\epsilon) = -\frac{\bar{g}^2 N_F}{2-\gamma} g(\epsilon), \tag{7}$$

where ϵ is the same as for the set of the gap functions, and $g(0) = 0$ and $g(\epsilon_{\max}) = 1$.

The divergence in the overall factor in $E_c(\epsilon)$ can be avoided if we treat $\gamma \to 2$ as the double limit, in which N_F scales as $2-\gamma$, and the product $N_F/(2-\gamma)$ remains finite. In this double limit one obtains a nondivergent, continuous spectrum of $E_c(\epsilon)$ at $\gamma \to 2$.

The emergence of a dispersion-full $E_c(\epsilon)$ may sound surprising because all gap functions $\Delta(\epsilon, \omega_m)$ satisfy $\delta E_c[\Delta(\epsilon, \omega_m)]/\delta\Delta(\epsilon, \omega_m) = 0$, hence $dE_c(\epsilon)/d\epsilon = 0$ and $E_c(\epsilon)$ is apparently flat. We show here that while $dE_c(\epsilon)/d\epsilon$ vanishes, $dE_c^2(\epsilon)/d\epsilon^2$ and all higher-order derivatives diverge at $\gamma \to 2$. In this case, Taylor expansion around any ϵ has to be taken to an infinite order, and the summation of Taylor series yields the dispersion-full $E_c(\epsilon)$.

As a part of our analysis, we discuss two other issues. One is the relation between Luttinger-Ward (LW) variational free energy and the actual one, obtained using Hubbard-Stratonovich transformation. We argue that the variational free energy yields the correct Eliashberg equations as stationary conditions, but in general should not be used to expand around stationary points as it does not correctly describe fluctuations of inverse quasiparticle residue $Z(\omega)$. However, when stationary $Z(\omega)$ is infinite, which at $\omega_D = 0$ holds for $\gamma > 1$ at $T = 0$ and for all γ at a finite T, both stationary solutions and fluctuations around it are properly described by the variational (spin chain) free energy. For $\gamma = 2$, this result has been obtained in Ref. [50].

The second issue is the specific heat in the normal state. It was argued in Ref. [51] that $C(T)$ is negative for $\gamma \geqslant 2$ in some T range above the onset of pairing, i.e., the γ model is unstable. We argue that this comes about because the authors of Ref. [51] subtracted a particular temperature-dependent piece from the free energy. We analyze the free energy without the subtraction and find that $C(T)$ is positive at all T.

The structure of the paper is the following. In the next section, we present the expressions for the variational LW free energy F_{sc}^{var} and the actual, more complex free energy F_{sc}, obtained by introducing normal and anomalous self-energies $\Sigma(\omega_m)$ and $\Phi(\omega_m)$ as auxiliary fields and integrating out fermions using the Hubbard-Stratonovich transformation [50,52,53]. We show that the condition that the free energy is stationary with respect to variations of Σ and Φ yields the correct Eliashberg equations for both F_{sc}^{var} and F_{sc}. We argue that to account for fluctuation corrections one generally has to use F_{sc}, but when stationary $Z(\omega_m) = 1 + \Sigma(\omega_m)/\omega_m$ is infinite, one can use more simple F_{sc}^{var}. We discuss the free energy in the normal state and the specific heat in Sec. II B. In Sec. III, we analyze the free energy expansion around the discrete set of the gap functions $\Delta_n(\omega_m)$ for $\gamma < 2$. We discuss the discrete set in Sec. III A, obtain the free energy expansion around $\Delta_n(\omega_m)$ with $n = 0,$ 1, and 2 in Sec. III B, and obtain the condensation energy for different n in Sec. III C. In Sec. IV, we analyze the transformation from a discrete spectrum of $\Delta_n(\omega_m)$ to a continuous one-parameter set $\Delta(\epsilon, \omega_m)$ at $\gamma \to 2$ and discuss the corresponding transformation for the condensation energy and the need to take the double limit $\gamma \to 2$, $N_F \to 0$, $N_F/(2-\gamma) \to$ const. We present our conclusions in Sec. V.

II. ELIASHBERG EQUATIONS AS STATIONARY POINTS OF THE FREE ENERGY

As we said, the Eliashberg equations, Eqs. (1), or, equivalently, Eqs. (3) and (4), can be obtained diagrammatically in one-loop/ladder approximation. By general arguments, these equations also emerge as conditions on stationary points of the variational free energy. This has been demonstrated by Eliashberg, who extended to a superconductor the variational approach, pioneered by Luttinger and Ward (Ref. [54]). The LW functional is expressed via variational functions Σ_k and Φ_k, where $k = (\mathbf{k}, \omega_m)$ and $\omega_m = (2m+1)\pi T$ is fermionic Matsubara frequency. It has the form

$$F_{sc}^{var} = -VT \sum_k \left[\ln\left(-\det \hat{G}_k^{-1}\right) - i\mathrm{Tr}(\hat{\tilde{\Sigma}}_k \hat{G}_k) \right] + VT^2 \sum_{k,k'} [G_k V(k-k') G_{k'} - F_k^* V(k-k') F_{k'}] + \cdots, \tag{8}$$

where V is the system's volume, $\sum_k = \int d^d\mathbf{k}/(2\pi)^d \sum_m$ (d is the spatial dimensionality), $\tilde{\Sigma}_k = \omega_m + \Sigma_k$, $\epsilon_{\mathbf{k}}$ is the fermionic dispersion, and $V(k-k')$ is a four-fermion interaction, which in general depends on both frequency and momentum transfer.

The matrices $\hat{G}_k$ and $i\hat{\tilde{\Sigma}}_k$ are the single-particle Green's function and the self-energy in Nambu representation of electron operators, $\Psi_{\boldsymbol{k}} = (\psi_{\boldsymbol{k},\uparrow}, \psi^\dagger_{-\boldsymbol{k},\downarrow})^T$. The dots in (8) stand for higher-order terms in G_k and F_k.

In the explicit form,

$$\hat{G}_k = \begin{bmatrix} G_k & F_k \\ F_k^* & -G_{-k} \end{bmatrix}, \quad i\hat{\tilde{\Sigma}}_k = \begin{bmatrix} i\tilde{\Sigma}_k & i\Phi_k \\ -i\Phi_k^* & -i\tilde{\Sigma}_{-k} \end{bmatrix}, \tag{9}$$

where the diagonal (off-diagonal) elements are the normal (anomalous) components. The elements of $\hat{G}_k$ and $\hat{\Sigma}_k$ are related by the Dyson equation:

$$G_k = -\frac{\epsilon_{-\mathbf{k}} - i\tilde{\Sigma}_{-k}}{(\epsilon_{\mathbf{k}} - i\tilde{\Sigma}_k)(\epsilon_{-\mathbf{k}} - i\tilde{\Sigma}_{-k}) + |\Phi_k|^2},$$
$$F_k = i\frac{\Phi_k}{(\epsilon_{\mathbf{k}} - i\tilde{\Sigma}_k)(\epsilon_{-\mathbf{k}} - i\tilde{\Sigma}_{-k}) + |\Phi_k|^2}. \tag{10}$$

The stationary solutions for Σ_k and Φ_k are obtained from $\delta F_{\rm sc}^{\rm var}/\delta\Sigma_k = 0$ and $\delta F_{\rm sc}^{\rm var}/\delta\Phi_k = 0$. This leads to

$$\begin{bmatrix} 2\frac{\delta F_k}{\delta\Sigma_k} & -\frac{\delta G_k}{\delta\Sigma_k} & -\frac{\delta G_{-k}}{\delta\Sigma_k} \\ 2\frac{\delta F_k}{\delta\Sigma_{-k}} & -\frac{\delta G_k}{\delta\Sigma_{-k}} & -\frac{\delta G_{-k}}{\delta\Sigma_{-k}} \\ 2\frac{\delta F_k}{\delta\Phi_k} & -\frac{\delta G_k}{\delta\Phi_k} & -\frac{\delta G_{-k}}{\delta\Phi_k} \end{bmatrix}\begin{bmatrix} I_{1,k} \\ I_{2,k} \\ I_{2,-k}\end{bmatrix} = 0, \tag{11}$$

where

$$I_{1,k} = \Phi_k^* - iT\sum_{k'} V(k-k')F_{k'}^*,$$
$$I_{2,k} = \Sigma_k - iT\sum_{k'} V(k-k')G_{k'}. \tag{12}$$

The determinant of the 3×3 matrix in Eq. (11) equals to $2i(\epsilon_{\boldsymbol{k}} - i\tilde{\Sigma}_k)(\epsilon_{-\boldsymbol{k}} - i\tilde{\Sigma}_{-k})$ and is generally nonzero. The three equations are then independent, which implies that $I_{1,k} = I_{2,k} = 0$, i.e.,

$$\Phi_k^* = iT\sum_{k'} V(k-k')F_{k'}^*,$$
$$\Sigma_k = iT\sum_{k'} V(k-k')G_{k'}. \tag{13}$$

Assuming further that a soft boson is a slow excitation compared to a fermion, one can factorize the integration over $\mathbf{k}$ along and transverse to the Fermi surface. The integration transverse to the Fermi surface is over fermionic dispersion. When the Fermi energy E_F is much larger that fermion-boson coupling $\bar{g}$ in Eq. (2), one can convert the momentum integration into that over $\epsilon_{\mathbf{k}}$ in infinite limits, keeping the density of states at its value at the Fermi surface. The integration along the Fermi surface involves $V(k-k')$ with both $\mathbf{k}$ and $\mathbf{k}'$ on the Fermi surface. Restricting to hot regions on the Fermi surface, when the interaction is peaked at a finite momentum transfer or projecting into the most attractive pairing component, when the interaction is peaked at zero momentum transfer, we obtain the two Eliashberg equations (1). We refer a reader to Ref. [1] for more detailed discussion of how the γ-model description emerges from particular lattice models.

Derivation of Eliashberg equations from variational $F_{\rm sc}^{\rm var}$ has been reported several times in the literature [55–57]. There is one caveat here, which will be relevant to our consideration below. Namely, under the assumptions used to derive Eq. (1), $\Sigma_k = \Sigma(\omega_m)$ and $\Phi_k = \Phi(\omega_m)$. Then one can integrate over momentum right in Eq. (8). The integration is straightforward and yields [1,55]

$$F_{\rm sc}^{\rm var} = -2\pi TN_F\sum_m \frac{\omega_m\tilde{\Sigma}_m}{\sqrt{\tilde{\Sigma}_m^2 + |\Phi_m|^2}} - \pi^2T^2N_F\sum_{m,m'} V(m-m')\frac{\tilde{\Sigma}_m\tilde{\Sigma}_{m'} + \frac{1}{2}(\Phi_m\Phi_{m'}^* + \Phi_m^*\Phi_{m'})}{\sqrt{\tilde{\Sigma}_m^2 + |\Phi_m|^2}\sqrt{\tilde{\Sigma}_{m'}^2 + |\Phi_{m'}|^2}}, \tag{14}$$

where $N_F \sim N/E_F$ is the total density of states (N is the number of particles in the system), and we introduced the shorthand notations $\Phi_m = \Phi(\omega_m)$, $\tilde{\Sigma}_m = \tilde{\Sigma}(\omega_m)$, and $V(m-m') = V(\omega_m - \omega_{m'})$. From $\delta F_{\rm sc}^{\rm var}/\delta\Sigma_m = 0$ and $\delta F_{\rm sc}^{\rm var}/\delta\Phi_m = 0$, we then obtain, instead of Eq. (11),

$$\Phi_m\tilde{\Sigma}_m I_1 - |\Phi_m|^2 I_2 = 0, \quad \tilde{\Sigma}_m^2 I_1 - \Phi_m^*\tilde{\Sigma}_m I_2 = 0. \tag{15}$$

One can immediately verify that the determinant of the 2×2 matrix vanishes, hence there is only one independent equation. Introducing Z_m and Δ_m as $Z_m = 1 + \Sigma_m/\omega_m$, $\Delta_m = \Phi_m/Z_m$, we find that this is Eq. (3) for Δ_m. There is no Eq. (4) on Z_m. This can be seen already from Eq. (14). Indeed, expressing Φ_m and Σ_m in terms of Δ_m and Z_m, we immediately find that $F_{\rm sc}^{\rm var}$ depends only on Δ_m:

$$F_{\rm sc}^{\rm var} = -2\pi TN_F\sum_m \frac{\omega_m^2}{\sqrt{\omega_m^2 + |\Delta_m|^2}} - \pi^2T^2N_F\sum_{m,m'} V(m-m')\frac{\omega_m\omega_{m'} + \frac{1}{2}(\Delta_m\Delta_{m'}^* + \Delta_m^*\Delta_{m'})}{\sqrt{\omega_m^2 + |\Delta_m|^2}\sqrt{\omega_{m'}^2 + |\Delta_{m'}|^2}}. \tag{16}$$

The independence of $F_{\rm sc}^{\rm var}$ on Z_m seem to imply that fluctuations of Z_m decouple from fluctuations of Δ_m for any ω_D. However, variational $F_{\rm sc}^{\rm var}$ is *not* the free energy, which appears in the partition function $\mathcal{Z} = \int \mathcal{D}[\Delta, Z]e^{-F_{\rm sc}(\Delta,Z)/T}$. To obtain $F_{\rm sc}$, one has to follow a standard procedure: introduce auxiliary fields Σ_m and Φ_m and integrate out fermions using the Hubbard-Stratonovich transformation. This has been implemented in Refs. [50,52,53], and the result is

$$F_{\rm sc} = -2\pi TN_F\sum_m Z_m\sqrt{\omega_m^2 + |\Delta_m|^2} + T^2N_F\sum_{m,m'} V^{-1}(m-m')\left[(Z_m-1)(Z_{m'}-1)\omega_m\omega_{m'} + \frac{1}{2}Z_mZ_{m'}(\Delta_m\Delta_{m'}^* + \Delta_m^*\Delta_{m'})\right], \tag{17}$$

where the inverse matrix $V^{-1}(m-m')$ satisfies $T^2\sum_{m''}V(m-m'')V^{-1}(m''-m')=\delta_{m,m'}$ (see Appendix A for detail). The stationary conditions $\delta F_{\rm sc}/\delta\Delta_m=0$, $\delta F_{\rm sc}/\delta Z_m=0$ yield Eqs. (3) and (4), as one can straightforwardly verify.

The free energy (17) depends on both Δ_m and Z_m, and the quadratic form in deviations from a stationary solution generally contains terms with variations of Δ_m and Z_m. The authors of Ref. [50] argued that for $\gamma=2$ (the case they considered) fluctuations of Z_m get steeper when the bosonic mass gets smaller, and the free energy $F_{\rm sc}(\Delta_m, Z_m)$ with arbitrary Δ_m and stationary Z_m, given by Eq. (4), coincides with $F^{\rm var}_{\rm sc}(\Delta_m)$ up to regular corrections in powers of $\omega_D/\bar g$. We extended the analysis of Ref. [50] to $\gamma<2$ and found that this holds when the stationary Z_m diverges at $\omega_D\to 0$. This is the case for $\gamma>1$ at $T=0$ and for all $\gamma>0$ at a finite T. We show the details in Appendix A.

For the rest of the paper, we consider the limit of vanishing ω_D and use $F^{\rm var}_{\rm sc}(\Delta_m)$ for the free energy both at and away from stationary points. At $T=0$, we will be chiefly interested in the system behavior for γ close to 2, when the equivalence between $F^{\rm var}_{\rm sc}$ and $F_{\rm sc}$ holds. We will also present some numerical results for smaller γ. These results are obtained at a small, but finite T, when the equivalence between $F^{\rm var}_{\rm sc}$ and $F_{\rm sc}$ again holds. To simplify the notation, we ignore the superscript "var" for the variational free energy in the rest of the main text.

A. Condensation energy

Condensation energy E_c is the difference between the free energy of a superconductor with a stationary Δ_m and that of a would be normal state at the same T. For an s-wave BCS superconductor $E_c=-N_F\Delta^2/2$ at zero temperature. In our case, stationary $F_{\rm sc}$ is given by (16), and the free energy in the normal state is

$$F_{\rm norm}=-2\pi TN_F\sum_m|\omega_m|-\pi^2T^2N_F\times\sum_{m,m'}V(\omega_m-\omega_{m'})\,{\rm sgn}(\omega_m\omega_{m'}). \tag{18}$$

The condensation energy

$$E_c=2\pi TN_F\sum_m\left(|\omega_m|-\frac{\omega_m^2}{\sqrt{\omega_m^2+\Delta_m^2}}\right)-\pi^2T^2N_F\sum_{m,m'}V(\omega_m-\omega'_m)\times\left(\frac{\omega_m\omega'_m+\Delta_m\Delta_{m'}}{\sqrt{\omega_m^2+\Delta_m^2}\sqrt{(\omega'_m)^2+\Delta_{m'}^2}}-{\rm sgn}(\omega_m\omega_{m'})\right). \tag{19}$$

In writing (19), we assumed for simplicity that Δ_m is real, i.e., we set its U(1) phase to be zero.

We note that both $F_{\rm sc}$ and $F_{\rm norm}$ contain thermal contribution from $m=m'$. It is proportional to $V(0)=(\bar g/\omega_D)^\gamma$ and diverges at $\omega_D=0$ for any $\gamma>0$.[2] This term gives rise to singular temperature-independent entropy at $T>\omega_D$, which we discuss in Sec. II B below. At the same time, the thermal pieces in (16) and (18) are identical and cancel out in the expression for the condensation energy, Eq. (19), which therefore remains nonsingular at $\omega_D=0$.

Using the gap equation (3), one can reexpress E_c as

$$E_c=-\pi TN_F\sum_m|\omega_m|\frac{(\sqrt{1+D^2(\omega_m)}-1)^2}{\sqrt{1+D^2(\omega_m)}}-\frac{1}{2}\pi^2T^2N_F\sum_{m,m'}V(\omega_m-\omega'_m)\times\frac{{\rm sgn}(\omega_m\omega_{m'})}{\sqrt{1+D^2(\omega_m)}\sqrt{1+D^2(\omega_{m'})}}\times(\sqrt{1+D^2(\omega_m)}-\sqrt{1+D^2(\omega_{m'})})^2, \tag{20}$$

where we remind that $D(\omega_m)=\Delta(\omega_m)/\omega_m$. The advantage of using Eq. (20) is in that it explicitly shows that $E_c<0$, i.e., that there is an energy gain from pairing. To see this explicitly one should convert the sum into the one over $m,m'\geqslant 0$ and use $V(\omega_m-\omega_{m'})>V(\omega_m+\omega_{m'})$.

At $T=0$, $2\pi T\sum_{m\geqslant 0}\to\int_0^\infty d\omega_m$. At $\gamma<2$, the two integrals in (20) are convergent both in the infrared and ultraviolet limits even when $\omega_D=0$. Convergence in the ultraviolet limit follows from the fact that at large ω_m, $D(\omega_m)$ falls off as $1/|\omega_m|^{\gamma+1}$. In the infrared limit, $\Delta(\omega_m)$ approaches a finite value Δ, and $D(\omega_m)$ becomes large. Keeping only the leading terms in (20), we obtain

$$E_c\propto\int_0^\Delta d\omega_m\int_0^\Delta d\omega_{m'}\frac{(\omega_m-\omega_{m'})^2}{\omega_m\omega_{m'}}\times\left(\frac{1}{|\omega_m-\omega_{m'}|^\gamma}-\frac{1}{|\omega_m+\omega_{m'}|^\gamma}\right). \tag{21}$$

This integral is convergent for all $\gamma<2$.

B. Specific heat

The specific heat is $C(T)=T\,dS(T)/dT$, where the entropy $S(T)=-dF/dT$ and $F=F_{\rm norm}+E_c$.

Computation of the temperature dependence of the free energy for any $\gamma>0$ requires care for two reasons, both related to $F_{\rm norm}$. First, $F_{\rm norm}$ contains a singular thermal contribution from $m=m'$. Second, the frequency summation in both terms in (18) does not converge, raising a possibility that there is a contribution to the specific heat that depends on the upper energy cutoff of the model. The frequency sums in Eq. (20) for E_c are ultraviolet-convergent, so the jump of $C(T)$ at $T=T_c-0$ is cutoff-independent.

Because it is $F_{\rm norm}$ that is potentially problematic for the entropy and the specific heat, we consider the normal state.

[2] Similarly, at $T=0$, both $F^{\rm var}_{\rm sc}$ and $F^{\rm var}_{\rm norm}$ contain $\int V(\Omega)d\Omega$. This integral is regular in the infrared for $\gamma<1$, but becomes singular at $\gamma>1$ and scales as $(\bar g/\omega_D)^{\gamma-1}$.

The first term in (18) is the free energy of the free Fermi gas,

$$F^{\rm free}_{\rm norm} = -2\pi T N_F \sum_m |\omega_m|. \tag{22}$$

The summation over Matsubara frequencies can be performed using the Euler-Maclaurin formula in the form presented in Ref. [58]:

$$F^{\rm free}_{\rm norm} = -8\pi^2 T^2 N_F \sum_{m=0}^{\infty}(m+1/2) \approx -2N_F \int_0^\infty x dx - \frac{\pi^2 T^2 N_F}{3}. \tag{23}$$

The upper limit of the integral is actually the upper energy cutoff in the theory, Λ, then $F^{\rm free}_{\rm norm} = -N_F(\Lambda^2 + \pi^2 T^2/3)$. The Λ^2 term is T independent and therefore is irrelevant for $S(T)$ and $C(T)$. Keeping the second term, we reproduce the known result for a Fermi gas: $S(T) = C(T) = (2/3)\pi^2 T N_F$.

Alternatively, we can evaluate $T\sum_m |\omega_m|$ directly, by restricting to M Matsubara numbers for positive and for negative ω_m, i.e., keeping the particle-hole symmetry. An elementary calculation then sets the relation between M and Λ:

$$M = \frac{\Lambda}{2\pi T} + \frac{\pi T}{12\Lambda} + O\left(\frac{1}{\Lambda^2}\right). \tag{24}$$

It is essential that there is no Λ-independent term in this relation. We verified the absence of the Λ-independent term in the complimentary calculation, in which we integrated over fermionic dispersion ϵ_k over a finite range between $-\Lambda$ and Λ, where $\Lambda \sim E_F \gg \bar{g}$, and then summed over all Matsubara frequencies, as in this calculation the Matsubara sum converges at $|\omega_m| > \Lambda$.

We now use Eq. (24) for the calculation of the second (interacting) term in Eq. (18),

$$F^{\rm int}_{\rm norm} = -\pi^2 T^2 N_F \bar{g}^\gamma \sum_{m,m'} \frac{{\rm sgn}(\omega_m \omega_{m'})}{(|\omega_m - \omega_{m'}|^2 + \omega_D^2)^{\gamma/2}}. \tag{25}$$

Summing over M positive and M negative Matsubara frequencies, we obtain after some straightforward algebra that for $T \gg \omega_D$ the second term in the free energy of Eq. (18) is

$$\begin{aligned} F^{\rm int}_{\rm norm} = &-N_F \pi T \Lambda \left(\frac{\bar{g}}{\omega_D}\right)^\gamma \\ &+ N_F \bar{g}^\gamma \Lambda^{2-\gamma} \frac{2(2^{-\gamma}-1)}{(1-\gamma)(2-\gamma)} \\ &- N_F \bar{g}^\gamma (2\pi T)^{1-\gamma} \Lambda \zeta(\gamma) \\ &+ \frac{3}{2} N_F \bar{g}^\gamma (2\pi T)^{2-\gamma} \zeta(\gamma-1) + O(1/\Lambda). \end{aligned} \tag{26}$$

This result is valid for $\gamma \neq 1$ (we discuss $\gamma = 1$ below). In obtaining this expression, we used the fact that $\sum_1^m 1/k^\gamma$ is a Harmonic number $H_\gamma(m)$, used the summation formula [59]

$$\sum_{m=1}^{n} H_\gamma(m) = (n+1)H_\gamma(n) - H_{\gamma-1}(n) \tag{27}$$

and the asymptotic expansion of $H_\gamma(m)$ at large m

$$H_\gamma(m) = \frac{m^{1-\gamma}}{1-\gamma} + \zeta(\gamma) + \frac{1}{2m^\gamma} + O\left(\frac{1}{m^{\gamma+1}}\right). \tag{28}$$

The first term in $F^{\rm int}_{\rm norm}$ comes from thermal fluctuations. It gives a singular contribution to the entropy, which we discuss below but does not contribute to the specific heat. The second term is the interaction part of the ground state energy. It does depend on the upper cutoff as $\Lambda^{2-\gamma}$, but is just a constant. The third term is a nonuniversal, cutoff-dependent temperature contribution to the free energy. It scales as $\Lambda T^{1-\gamma}$ and comes from fermions with $|\omega_m - \omega_{m'}| = O(T)$, while $\omega_m, \omega_{m'} = O(\Lambda)$. Finally, the last term is a universal, cutoff-independent term that scales as $T^{2-\gamma}$.[3]

Differentiating $F^{\rm int}_{\rm norm}$ twice over temperature, we obtain for the specific heat

$$C(T) = C_\Lambda(T) + C_{\rm univ}(T), \tag{29}$$

where

$$C_\Lambda(T) \approx 2\pi \Lambda N_F \left(\frac{\bar{g}}{2\pi T}\right)^\gamma \gamma(\gamma-1)\zeta(\gamma) \tag{30}$$

is a cutoff-dependent, nonuniversal contribution, and

$$C_{\rm univ}(T) = \frac{2}{3}\pi^2 T N_F (1 + 9(2-\gamma) \times (\gamma-1)\zeta(\gamma-1))\left(\frac{\bar{g}}{2\pi T}\right)^\gamma \tag{31}$$

is a much smaller universal piece. In (30), the product $\gamma(\gamma-1)\zeta(\gamma)$ is positive for all $\gamma > 0$ and evolves smoothly through $\gamma = 1$. At $\gamma = 1$, the corresponding term in $F^{\rm int}_{\rm norm}$ scales as $\Lambda \ln T$. In (31), $(2-\gamma)(\gamma-1)\zeta(\gamma-1)$ is positive for $\gamma < 1$ and negative for larger γ.

On a more careful look, we found that $C_\Lambda(T)$ originates from the nonanalytic $T^{1-\gamma}$ term in the fermionic self-energy at large ω_m. Indeed, the self-energy is

$$\Sigma(\omega_m) = \bar{g}^\gamma \left\{\frac{\pi T}{\omega_D^\gamma} + (2\pi T)^{1-\gamma}\left[H_\gamma(m) + 0.5\left(H_\gamma\left(\frac{\Lambda}{2\pi T} - 1 + m\right) - H_\gamma\left(\frac{\Lambda}{2\pi T} + m\right)\right)\right]\right\} \tag{32}$$

(we set $\omega_m > 0$ for definiteness). At $1 \ll m \ll \Lambda/(2\pi T)$, $H_\gamma(m)(2\pi T)^{1-\gamma} \approx (\omega_m)^{1-\gamma}/(1-\gamma) + \zeta(\gamma)(2\pi T)^{1-\gamma}$, hence[4]

$$\Sigma(\omega_m) \approx \bar{g}^\gamma \left(\frac{\pi T}{\omega_D^\gamma} + \frac{\omega_m^{1-\gamma}}{1-\gamma} + \zeta(\gamma)(2\pi T)^{1-\gamma}\right). \tag{33}$$

The first two terms in the self-energy do not contribute to the specific heat, while the last $T^{1-\gamma}$ term gives rise to Λ/T^γ scaling of $C_\Lambda(T)$, as one can straightforwardly verify by substituting this asymptotic form into (32) and then into $F^{\rm int}_{\rm norm} = -2\pi T N_F \sum_{m \geqslant 0} \Sigma(\omega_m)$.

Taken as a face value, Eq. (30) implies that $C(T) \approx C_\Lambda(T) \propto \Lambda/T^\gamma$ is positive. The upper energy cutoff Λ is

[3]Alternatively, these results can be obtained by subtracting and adding back the divergent terms from the summand [60].

[4]The prefactor for the $T^{1-\gamma}$ is negative for $\gamma < 1$, hence for these γ, $\Sigma(\omega_m)$ has a negative temperature-dependent offset. This is consistent with Ref. [61], where a negative offset has been found for $\gamma = 1/3$ and $1/2$.

generally a fraction of the Fermi energy E_F. The density of states $N_F \sim N/E_F$, where N is the number of electrons, hence $C_\Lambda(T)$ per particle is of order $(\bar{g}/T)^\gamma$.

The result $C(T) \propto \Lambda/T^\gamma$ holds for the γ model, whose free energy in the normal state is defined by Eq. (18), but generally does not hold for the underlying microscopic models, for which the γ model has been argued to be the proper low-energy model for the analysis of superconductivity. The reason is that the free energy in (18) does not include the contribution from the critical boson

$$F_{\text{bos}} = \frac{1}{2}T\sum_q[\ln D^{-1}(q) + \Pi(q)D(q)], \tag{34}$$

where $q = (\vec{q}, \Omega_m)$, $D(q)$ is the bosonic propagator, and $\Pi(q)$ is the bosonic polarization [$D^{-1}(q) = D_0^{-1}(q) - \Pi(q)$, where D_0 is a bare bosonic propagator]. The argument for neglecting F_{bos} is that the bosonic piece either does not change between the normal and the superconducting state and hence cancels out in the condensation energy, or its change is not relevant from physics perspective and can be effectively described by making the exponent γ T-dependent.

Such an argument, however, does not apply to the normal state. There, F_{bos} has to be kept because, bosonic Π is created by the coupling to fermions. We analyzed the full free energy for several underlying fermion-boson models [62] and in all cases found that the contribution to the specific heat from F_{bos} cancels out $C_\Lambda(T)$ from Eq. (30). It also gives rise to another universal contribution to $C(T)$. For Ising-nematic case ($\gamma = 1/3$), this universal contribution has the same form as $C_{\text{univ}}(T)$ in (31), while for the phonon case ($\gamma = 2$) it contains a much larger positive T-independent term (see Ref. [62] for details). For critical models towards an order with a finite Q, e.g., antiferromagnetic $Q = (\pi, \pi)$, the universal term in $C(T)$ differs from C_{univ} in (31) by yet another reason – the self-energy depends on the position on the Fermi surface and has different forms in hot regions, where the γ-model description is valid, and in the rest of the Fermi surface. Pairing involves predominantly hot fermions and is adequately described by the γ model. However, the free energy in the normal state comes from the full Fermi surface and its singular part at criticality has an extra T-dependent factor proportional to the width of the hot region.

The specific heat of the γ model has been recently analyzed in Ref. [51]. The authors of this work regularized the free energy (18) by subtracting from it the term

$$-\pi^2T^2N_F\sum_{m,m'}V(\omega_m - \omega_{m'}) \tag{35}$$

(the same subtraction has been used in their other works in Refs. [50,63]). This substraction changes $F^{\text{int}}_{\text{norm}}$ to

$$F^{\text{int}}_{\text{norm, reg}} = -\pi^2T^2N_F\sum_{m,m'}V(\omega_m - \omega_{m'})(\text{sgn}(\omega_m\omega_{m'}) - 1). \tag{36}$$

Because the numerator is nonzero only when ω_m and $\omega_{m'}$ have opposite sign, thermal contribution from $m = m'$ vanishes, hence (36) can be evaluated right at $\omega_D = 0$. The full outcome of the subtraction is that the first and the third term in (26) disappear. The analysis in Ref. [51] has been restricted to $\gamma \geqslant 2$, in which case the second term in (26) also vanishes. The last, universal term, proportional to $T^{2-\gamma}$, remains almost intact: the only effect of subtraction is the change of the prefactor. Specifically,

$$F^{\text{int}}_{\text{norm, reg}} = 4\pi^2N_F\zeta(\gamma - 1)(\bar{g}/2\pi)^\gamma T^{2-\gamma}. \tag{37}$$

Differentiating twice over T, the authors of Ref. [51] found the regularized specific heat is negative for $\gamma \geqslant 2$ in some T range above T_c and conjectured that this implies that the normal state is unstable. Like we said, we found in the analysis of the microscopic models at various γ (including models with continuously varying γ) that the total specific heat of a coupled electron-boson model is positive. Whether a negative sign of $C(T)$ in the regularized γ model, which is a portion of the full $C(T) > 0$, implies an instability of the normal state, remains to be seen.[5]

Returning to Eq. (29), the specific heat for the underlying fermion-boson model for $\gamma = 1/3$ has been recently analyzed using quantum Monte Carlo technique [64]. In the regime of vanishing bosonic mass, the specific heat increases with decreasing T at strong enough coupling. At a face value, this is consistent with $C(T) \propto 1/T^{1/3}$. It is however, possible that the observed increase is due to superconducting fluctuations above T_c, as the authors of Ref. [64] suggested.

C. Applicability of the γ model in the normal state at finite T and $\omega_D \to 0$

We now argue that there is another issue with the analysis of the specific heat within the γ model in the normal state at a finite T—the T range, where one can neglect thermal vertex corrections, shrinks when the bosonic mass $\omega_D \to 0$ and remains valid above T_c only if one rescales the Fermi energy in line with ω_D.

Below we consider the free energy given by (26), with no subtraction.

The argument is threefold. First, as we already stated, the analysis leading to Eq. (26) is valid only at $T > \omega_D$. At smaller T, the system displays a Fermi liquid behavior and the entropy and the specific heat per particle scale as T, even without subtraction. We show the numerical result for the entropy in Fig. 2.

Second, the Eliashberg equation for the self-energy is valid as long as vertex corrections are small. A straightforward computation of vertex corrections at a finite $T \gg \omega_D$ when thermal fluctuations are the strongest, shows that they are small as long as

$$\left(\frac{\bar{g}}{\omega_D}\right)^\gamma \ll \frac{\Lambda}{T} \tag{38}$$

or, equivalently, as long as $T \ll T_{\max}$, where $T_{\max} \sim \Lambda(\omega_D/\bar{g})^\gamma$. The temperature window, where Eq. (26) and the

[5]We note in passing that the regularization, aimed to eliminate the singular thermal piece, is not unique. In particular, one can subtract $-\pi^2T^2N_F\bar{g}^\gamma\sum_{m,m'}(1 + b - b\,\text{sgn}(\omega_m\omega_{m'}))/(|\omega_m - \omega_{m'}|^\gamma + \omega_D^\gamma)$ with arbitrary b. One can then choose b such that even after regularization $C(T)$ remains positive down to any chosen T.

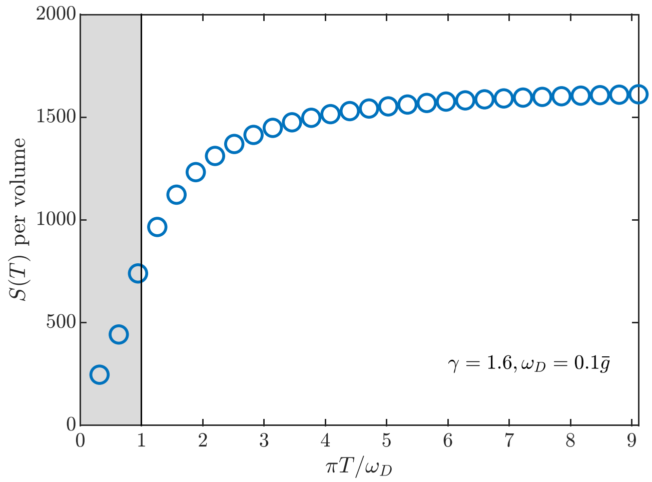

FIG. 2. Entropy per volume $S(T)$ as a function of $\pi T/\omega_D$, where we take the parameters $\omega_D = 0.1\bar{g}$ and $\gamma = 1.6$. A cutoff $\Lambda \sim 15\bar{g}$ is used to regularize the ultraviolet divergence. The grey region highlights the Fermi liquid regime where $\pi T \ll \omega_D$.

corresponding expressions for the entropy and the specific heat are valid, is then

$$\omega_D \ll T \ll T_{\max}. \tag{39}$$

This window collapses when $\omega_D \to 0$. The only way to keep it finite is to consider the double limit $\omega_D \to 0$, $\Lambda \to \infty$ (i.e., $E_F \to \infty$) such that $T_{\max}$ remains finite.

Third, the normal state description of the γ model is valid only above the pairing instability temperature T_p. The latter is of order $\bar{g}$ for a generic γ. The normal state analysis is then applicable when $T_{\max} > \bar{g}$. This strengthen the requirement on the double limit: it should be $\Lambda\omega_D^\gamma > \bar{g}^{\gamma+1}$. At the lower boundary $T = T_p \sim \bar{g}$, we then have

$$S(T_p) \sim \frac{\Lambda^2 N_F}{\bar{g}}, \quad C(T_p) \sim (\Lambda N_F) \leqslant N. \tag{40}$$

This implies that the specific heat per particle at $T_p < T < T_{\max}$ is at most O(1). The analysis of the free energy at $T > T_{\max}$ requires one to include series of vertex corrections and is beyond the scope of this paper.

We emphasize in this regard that both thermal self-energy and the singular part of the $T = 0$ self-energy at $\gamma > 1$ cancel out in the Eliashberg equation for the superconducting gap. To avoid vertex corrections at $\omega_D \to 0$, one still needs to employ the double limit $\omega_D \to 0$, $E_F \to \infty$,[6] however, the Eliashberg gap equation does not contain E_F, so the need for the double limit does not affect the pairing problem.

There is an additional issue at a finite T, related to the very derivation of the Eliashberg equations at a given γ from the corresponding microscopic model with momentum and frequency dependent $V(k - k')$. The derivation assumes that bosons are slow modes compared to fermions, in which case the momentum integration can be factorized such that the integrations transverse and along the Fermi surfaces are performed separately in fermionic and bosonic propagators, respectively. This holds at $T = 0$, but may not hold at a finite T, when thermal fluctuations are present [61]. As a result, the thermal self-energy in the γ model and in the underlying microscopic model may differ even without vertex corrections.

[6] Vertex corrections at $T = 0$ become singular at $\gamma > 1$, and to keep them small, one needs to take the double limit $\omega_D \to 0$, $E_F \to \infty$ such that $E_F\omega_D^{\gamma-1} \ll \bar{g}^\gamma$.

III. A GENERIC $\gamma < 2$. A DISCRETE SET OF SOLUTIONS OF THE GAP EQUATION AT $T = \omega_D = 0$, AND FREE ENERGY EXPANSION AROUND THEM

In this section, we analyze the profile of the free energy in the superconducting state for a generic $\gamma < 2$.

To set the stage for our analysis, we first briefly review the results for $\Delta(\omega_m)$ at stationary points at $T = 0$. We consider even-frequency gap functions, for which $\Delta(-\omega_m) = \Delta(\omega_m)$. For the analysis of odd-frequency gap functions, see Refs. [65,66].

A. Discrete set of solutions, $\Delta_n(\omega_m)$

For large enough $\omega_D \geqslant \bar{g}$, the normal state is a Fermi liquid with $\Sigma(\omega_m) \approx \lambda\omega_m$, where $\lambda = (\bar{g}/\omega_D)^\gamma(\gamma/2)$. The gap equation has a single sign-preserving solution $\Delta_0(\omega_m)$. It has a finite value at $\omega_m = 0$ and decreases as $1/|\omega_m|^\gamma$ at large frequencies.

When $\omega_D/\bar{g}$ gets smaller, new solutions of the gap equation appear one-by-one. We label these solutions as $\Delta_n(\omega_m)$, where $n = 1, 2\ldots$ A function $\Delta_n(\omega_m)$ changes sign n times along the positive Matsubara axis. These solutions are topologically distinct as each zero of $\Delta_n(\omega_m)$ is the center of a dynamical vortex on the complex frequency plane $z = \omega' + i\omega''$.

At $\omega_D = 0$, the number of solutions become infinite, and they form a discrete set, in which n ranges from zero to infinity. The overall magnitude of $\Delta_n(\omega_m)$ decreases exponentially with n, and the "end point" of the set, $\Delta_\infty(\omega_m)$, is the solution of the linearized gap equation

$$\Delta_\infty(\omega_m) = \frac{\bar{g}^\gamma}{2}\int d\omega'_m \frac{\Delta_\infty(\omega'_m) - \Delta_\infty(\omega_m)\frac{\omega'_m}{\omega_m}}{|\omega'_m|}\frac{1}{|\omega'_m - \omega_m|^\gamma}. \tag{41}$$

As a proof that such set does exist, we obtained the exact analytical solution of this equation (Refs. [1,4,5]). At small $\omega_m \ll \bar{g}$, $\Delta_\infty(\omega_m)$ oscillates as a function of $\ln(\bar{g}/|\omega_m|)^\gamma$ as

$$\Delta_\infty(\omega_m) = 2\epsilon\bar{g}^{1-\gamma/2}|\omega_m|^{\gamma/2}\cos(\beta(\gamma)\ln(\bar{g}/|\omega_m|)^\gamma + \phi(\gamma)), \tag{42}$$

where $\beta(\gamma)$ and $\phi(\gamma)$ are γ-dependent parameters and ϵ is an arbitrarily small dimensionless overall factor. At larger ω_m, $\Delta_\infty(\omega_m) \propto \epsilon/|\omega_m|^\gamma$. In simple terms, at small ω_m, $\Delta_\infty(\omega_m)$ in the left-hand side (l.h.s.) of (41) can be neglected, which is equivalent to neglecting the bare ω_m in the fermionic propagator compared to the self-energy, as one can easily verify. The solution of (41) without the l.h.s. is the sum of the two power laws $\Delta_\infty(\omega_m) \propto |\omega_m|^a$ with complex exponents $a = \gamma/2 \pm i\beta_\gamma$. This is Eq. (42). A relative phase ϕ is a free parameter in this approximation. At large $\omega_m \gg \bar{g}$, the dependence $\Delta_\infty(\omega_m) \propto 1/|\omega_m|^\gamma$ follows from *a'posteriori* verified assumption that typical ω'_m in (41) are of order $\bar{g}$, in which case one can approximate $1/|\omega'_m - \omega_m|^\gamma$ in the right-hand side (r.h.s.) of (41) by $1/|\omega_m|^\gamma$. The exact solution shows that there exists a particular ϕ, for which $\Delta_\infty(\omega_m)$ gradually transforms between the limits of small and large $|\omega_m|/\bar{g}$.

Qualitative understanding of the appearance of a discrete set of solutions of Eq. (3) can be obtained by expanding the r.h.s. of the nonlinear gap equation (3) in powers of $D(\omega_m) = \Delta(\omega_m)/\omega_m$ and analyzing the structure of perturbation series in the gap amplitude ϵ, which we now treat as finite. The analysis is tedious but straightforward. At small $\omega_m \ll \bar{g}$, the expansion holds in powers of $\epsilon^* = \epsilon(\bar{g}/|\omega_m|)^{1-\gamma/2}$ and yields

$$\Delta(\omega_m) = 2\epsilon^*|\omega_m|[Q_1\cos(\psi) + (\epsilon^*)^2 Q_3\cos(3\psi+\phi_3) + (\epsilon^*)^4 Q_5\cos(5\psi+\phi_5) + \dots], \tag{43}$$

where

$$\psi = \beta(\gamma)\ln(\bar{g}/|\omega_m|)^\gamma + \phi_{\epsilon^*}, \tag{44}$$

$$\phi_{\epsilon^*} = \phi(\gamma) + s_1(\epsilon^*)^2 + s_2(\epsilon^*)^4 + \dots, \tag{45}$$

$$\phi_{2k+1} = t_{0,2k+1} + t_{1,2k+1}(\epsilon^*)^2 + t_{2,2k+1}(\epsilon^*)^4 + \dots, \tag{46}$$

$$Q_{2k+1} = r_{0,2k+1} + r_{1,2k+1}(\epsilon^*)^2 + r_{2,2k+1}(\epsilon^*)^4 + \dots \tag{47}$$

The coefficients s_i, $t_{l,2k+1}$ and $r_{l,2k+1}$ are γ-dependent functions, except for $r_{0,1} = 1$. We present the exact expressions in Appendix D. For a generic $\gamma < 2$, they all are of order one.

We see that the expansion generates higher-order harmonics with multiples of the argument ψ and phases ϕ_{2k+1}. These extra phase factors are most relevant to our reasoning. Without them, oscillations from $\cos\psi$, $\cos 3\psi$, $\cos 5\psi$...would all be in phase, and $\Delta(\omega_m)$ would oscillate as a function of $\ln(\omega_m/\bar{g})$ down to $\omega_m = 0$. Because of ϕ_{2k+1}, the positions of the nodal points in different harmonics are shifted by different amounts, and when the corrections from the nonlinear terms become of order one, oscillations get destroyed. At smaller frequencies, it is natural to assume that $\Delta(\omega_m)$ approaches a constant value $\Delta(0)$. For a generic $\gamma < 2$, this transformation occurs when ϵ^* becomes of order one, i.e., at a critical Matsubara frequency $|\omega_m| \sim \omega_c = \bar{g}\epsilon^{2/(2-\gamma)}$. To first approximation, the gap function then follows $\Delta_\infty(\omega_m)$ from (42) down to ω_c and saturates at $\Delta(\omega_m) \sim \omega_c$ at smaller frequencies. Such a gap function is smooth (to discontinuity of a derivative) if $\omega_m = \omega_c$ is one of the extrema of $\Delta_\infty(\omega_m)$. The largest $\omega_c = O(\bar{g})$ is above the highest frequency where $\Delta_\infty(\omega_m)$ changes sign. The corresponding solution of the nonlinear gap equation is sign-preserving $\Delta_0(\omega_m)$, and the corresponding $\epsilon_0 = O(1)$. However, this is not the only option – because $\Delta_\infty(\omega_m)$ oscillates, ω_c can be chosen as an extremum of $\Delta_\infty(\omega_m)$ after it changes sign n times. In this case, $\omega_c \sim \bar{g}e^{-n\pi/(\beta(\gamma)\gamma)}$. The corresponding solution is $\Delta_n(\omega_m)$ that changes sign n times at positive ω_m, the corresponding $\epsilon_n \sim e^{-\pi n(2-\gamma)/(2\beta(\gamma)\gamma)}$, and the corresponding $\Delta_n(0)$ is

$$\Delta_n(0) \sim \omega_c \sim \bar{g}e^{-\pi n/(\beta(\gamma)\gamma)}, \tag{48}$$

At large n, both ϵ_n and $\Delta_n(0)$ decrease exponentially with n. In Fig. 3, we show the trial functions $\Delta_n(\omega_m)$, constructed this way, for a representative $\gamma = 0.8$ and for $n = 0,\ 1,$ and 2. We see that this reasoning works quite well. Namely, upon substituting the trial functions into the r.h.s. of Eq. (3), one recovers the same function in the l.h.s. with reasonable accuracy (more on this below).

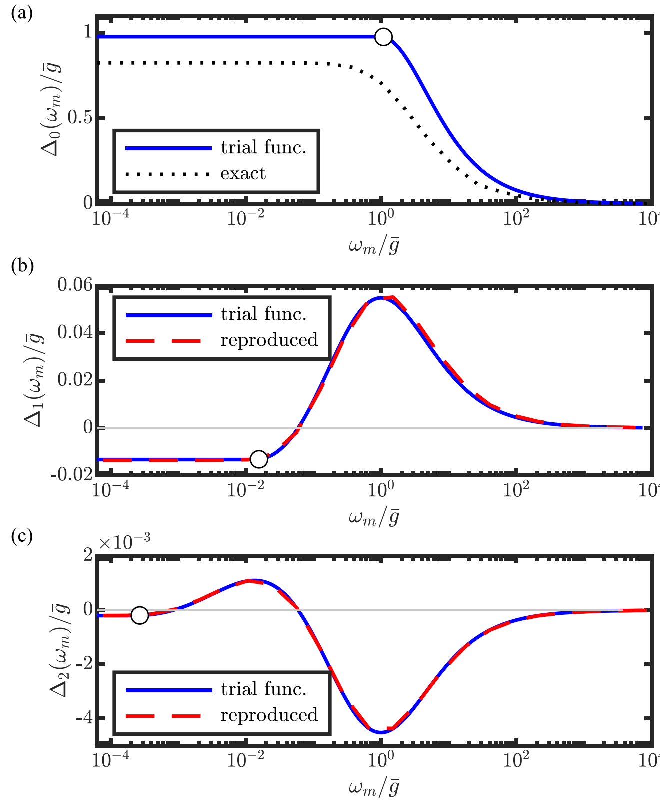

FIG. 3. Trial gap function (blue solid line) of (a) $n = 0$, (b) 1, and (c) 2 solutions, where $\gamma = 0.8$. Exact gap function of $n = 0$ solution is shown by the black dotted line. In (b) and (c), the reproduced gap function by substituting the trial gap function back to the gap equation is shown by red dashed lines. Circles indicate ω_c, below which the trial gap function $\Delta_n(\omega_m)$ becomes a constant.

These results present our most direct evidence of a discrete set of the gap functions $\Delta_n(\omega_m)$, which satisfy the nonlinear gap equation. The other evidences are (i) the exact analytical result for $\Delta_\infty(\omega_m)$, (ii) strong numerical evidence for the existence of a discrete set of the solutions of the nonlinear differential equation, which well approximates Eq. (3) at small γ (Ref. [1]), and (iii) a highly accurate numerical solution of the linearized gap equation at a finite T, yielding a discrete set of critical pairing temperatures $T_{p,n}$ for n up to 17 [2]. These critical $T_{p,n}$ decrease exponentially with n, consistent with $\Delta_n(0)$ in Eq. (48), and the corresponding eigenfunction changes sign n times along the positive Matsubara axis.

We emphasize that these results hold for γ, which are not particularly close to 2. For γ close to 2, the analysis of $\Delta_n(\omega_m)$ has to be modified, as we show in Sec. IV.

We also emphasize that all functions $\Delta_n(\omega_m)$ from the discrete set can be analytically continued into the upper half-plane of complex frequency. We verified this analytically for $n = \infty$ by replacing $i\omega_m$ by $z = \omega' + i\omega''$ in the analytical formula for $\Delta_\infty(\omega_m)$, and we checked this numerically for $n = 0,\ 1,$ and 2 by extending $\Delta_n(\omega_m)$ that we just obtained, to the upper frequency half-plane using Pade approximants.

B. Free energy expansion near $\Delta_n(\omega_m)$

We now analyze the form of the free energy F_{sc} near the stationary solutions $\Delta_n(\omega_m)$. Our goal here is to understand whether these solutions correspond to local minima or saddle points. To address this issue we expand F_{sc} to second order in $\delta\Delta(\omega_m) = \Delta(\omega_m) - \Delta_n(\omega_m)$ and analyze the quadratic form in $\delta\Delta(\omega_m)$.

We discuss the expansion of the free energy in detail in Appendix A and here just present the result. At $\omega_D = 0$, the free energy is $F_{sc} = F_{sc,n} + \delta F_{sc,n}$, where

$$F_{sc,n} = -N_F 2\pi T \sum_m \frac{\omega_m^2}{\sqrt{\omega_m^2 + \Delta_n^2(\omega_m)}} - \pi^2 T^2 N_F \sum_{m,m'} V(\omega_m - \omega'_m) \frac{\omega_m \omega'_m + \Delta_n(\omega_m)\Delta_n(\omega'_m)}{\sqrt{\omega_m^2 + \Delta_n^2(\omega_m)}\sqrt{(\omega'_m)^2 + \Delta_n^2(\omega'_m)}} \tag{49}$$

is the free energy for stationary $\Delta_n(\omega_m)$, and

$$\delta F_{sc,n} = N_F \pi^2 T^2 \sum_{m,m' \geqslant 0} \frac{1}{D_n(\omega_m) D_n(\omega_{m'})} \frac{1}{\left(1 + D_n^2(\omega_m)\right)^{3/2} \left(1 + D_n^2(\omega_{m'})\right)^{3/2}} Q_{m,m'} \tag{50}$$

with

$$Q_{m,m'} = \bar{g}^\gamma \Bigg[(\delta D(\omega_m) D_n(\omega_m) - \delta D(\omega_{m'}) D_n(\omega_{m'}))^2 \left(\frac{1}{|\omega_m - \omega_{m'}|^\gamma} + \frac{1}{|\omega_m + \omega_{m'}|^\gamma} \right) + \frac{\left(\delta D(\omega_m) D_n^2(\omega_m) - \delta D(\omega_{m'}) D_n^2(\omega_{m'})\right)^2}{|\omega_m - \omega_{m'}|^\gamma} + \frac{\left(\delta D(\omega_m) D_n^2(\omega_m) + \delta D(\omega_{m'}) D_n^2(\omega_{m'})\right)^2}{|\omega_m + \omega_{m'}|^\gamma} \Bigg] \tag{51}$$

is the variation of the free energy to the second order in $\delta\Delta(\omega_m)$. In (51) we introduced $D_n(\omega_m) = \Delta_n(\omega_m)/\omega_m$ and $\delta D(\omega_m) = \delta\Delta(\omega_m)/\omega_m$ and restricted to even-frequency variations $\delta\Delta(\omega_m) = \delta\Delta(-\omega_m)$.

Because $Q_{m,m'}$ is positive (the sum of full squares with positive prefactors), $\delta F_{sc,0}$ is definitely positive for the sign-preserving $D_0(\omega_m)$. For $n > 0$, this is not guaranteed, however, as $D_n(\omega_m)$ changes sign n times along the positive Matsubara axis, and the product $D_n(\omega_m) D_n(\omega_{m'})$ in the denominator of (51) can be of either sign.

It is convenient to reexpress $\delta F_{sc,n}$ as

$$\delta F_{sc,n} = (\pi T)^2 N_F \sum_{m,m'} \mathcal{K}_n(\omega_m, \omega_{m'}) \delta\Delta(\omega_m) \delta\Delta(\omega_{m'}), \tag{52}$$

where

$$\mathcal{K}_n(\omega_m, \omega_{m'}) = \frac{\delta_{mm'}}{2\pi T} \frac{\omega_m^2 Z_n(\omega_m)}{(\omega_m^2 + \Delta_n(\omega_m)^2)^{3/2}} - V(\omega_m - \omega_{m'}) \times \frac{\omega_m \omega_{m'} (\omega_m \omega_{m'} + \Delta_n(\omega_m) \Delta_n(\omega_{m'}))}{(\omega_m^2 + \Delta_n(\omega_m)^2)^{3/2} (\omega_{m'}^2 + \Delta_n(\omega_{m'})^2)^{3/2}}, \tag{53}$$

and $Z_n(\omega_m) = 1 + (\pi T/\omega_m) \sum_{m''} V(\omega_m - \omega_{m''}) \omega_{m''} / \sqrt{\omega_{m''}^2 + \Delta_n^2(\omega_{m''})}$ is the inverse quasiparticle residue at the stationary point.

The symmetric matrix $\mathcal{K}_n(\omega_m, \omega_{m'})$ can be rewritten in the diagonal form as

$$\mathcal{K}_n(\omega_m, \omega_{m'}) = \sum_p \lambda_n^{(p)} \Delta_n^{(p)}(\omega_m) \Delta_n^{(p)}(\omega_{m'}), \tag{54}$$

where $\lambda_n^{(p)}$ ($p = 1, 2 \ldots$) are the eigenvalues and $\Delta_n^{(p)}(\omega_m)$ are the eigenfunctions, which we will have to find. The eigenfunctions form a complete and orthogonal set and obey the normalization condition

$$T \sum_m \Delta_n^{(p)}(\omega_m) \Delta_n^{(p')}(\omega_m) = \delta_{p,p'}. \tag{55}$$

Expanding $\delta\Delta(\omega_m) = \sum_p C_n^p \Delta_n^{(p)}(\omega_m)$ and substituting into Eq. (52), we obtain $\delta F_{sc,n} = \sum_p \lambda_n^p (C_n^p)^2$. Obviously, if there is a negative λ_n^p for at least one value of p, the free energy gets reduced by deviations from $\Delta_n(\omega_m)$ along this direction. In this situation, $\Delta_n(\omega_m)$ is a saddle point of the free energy. If $\lambda_n^p > 0$ for all p, $\Delta_n(\omega_m)$ is a local minimum.

We obtained the eigenvalues and eigenfunctions of $\mathcal{K}_n(\omega_m, \omega_{m'})$ for $n = 0$, 1 and 2 numerically on a nonuniform mesh of 2000 × 2000 Matsubara frequencies. A nonuniform grid was chosen to reach extremely low temperatures $\sim 10^{-10}\bar{g}$ (see Ref. [2] for detail). For stationary $\Delta_n(\omega_m)$ we used the trial functions, constructed in the previous section (Fig. 3).

We used two computational procedures to obtain $\lambda_n^{(p)}$ and $\Delta_n^{(p)}(\omega_m)$. In the first, we set $\delta\Delta(\omega_m)$ to be real and even under $\omega_m \to -\omega_m$, but did not require that it is analytic in the upper half-plane of frequency. Like we just said, our $\Delta_n(z)$ are analytic functions of $z = \omega' + i\omega''$ at $\omega'' > 0$. However, fluctuating $\delta\Delta(\omega_m)$ do not have to be analytic. In the second procedure, we restricted $\delta\Delta(\omega_m)$ to analytic functions in the upper half-plane (see Appendix B for details). We found the same structure of eigenvalues and eigenfunctions in the two cases. This equivalence implies that for physically relevant fluctuations, $\delta\Delta(z)$ is an analytic function of z.

Below we present the results obtained using the first computational procedure. We show the eigenvalues $\lambda_n^{(p)}$ for a representative $\gamma = 0.8$ in Fig. 4. We see that for $n = 0$, all $\lambda_0^{(p)}$ are positive, as expected, i.e., $\Delta_0(\omega_m)$ is a minimum of the free energy functional (a global minimum as we will see momentarily). For $n = 1$, there is one *negative* eigenvalue $\lambda_1^{(1)}$. The corresponding eigenfunction $\Delta_1^{(1)}(\omega_m)$ sets the "direction", along which the free energy is reduced upon deviations from $\Delta_1(\omega_m)$. We show $\Delta_1^{(1)}(\omega_m)$ in Fig. 5(a). We see that it is sign-preserving and shifts the gap function towards the one with $n = 0$. This can also be seen by analyzing how our trial gap function $\Delta_1(\omega_m)$ evolves under iterations. As we showed in Fig. 3, this function satisfies the nonlinear gap equation quite well. Still, it is not exactly $\Delta_1(\omega_m)$, and likely

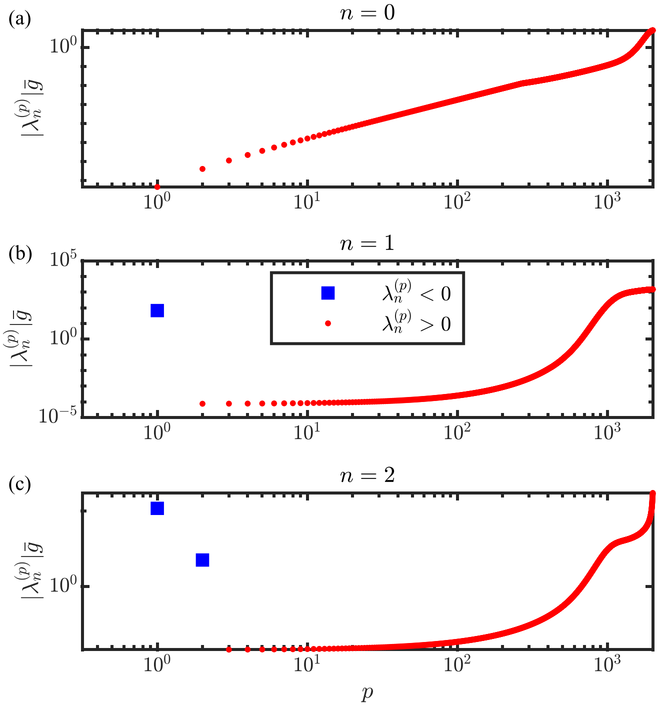

FIG. 4. Eigenvalues of the kernel matrix $\mathcal{K}_{\omega,\omega'}$ for (a) $n = 0$, (b) 1, and (c) 2 solutions, where $\gamma = 0.8$.

contains some amount of $\Delta_1^{(1)}(\omega_m)$. Then it is natural to expect that after a number of iterations the trial $\Delta_1(\omega_m)$ will start flowing towards $\Delta_0(\omega_m)$. Figure 6 shows that this is exactly the case: after the large number of iterations the trial $\Delta_1(\omega_m)$ approaches $\Delta_0(\omega_m)$.

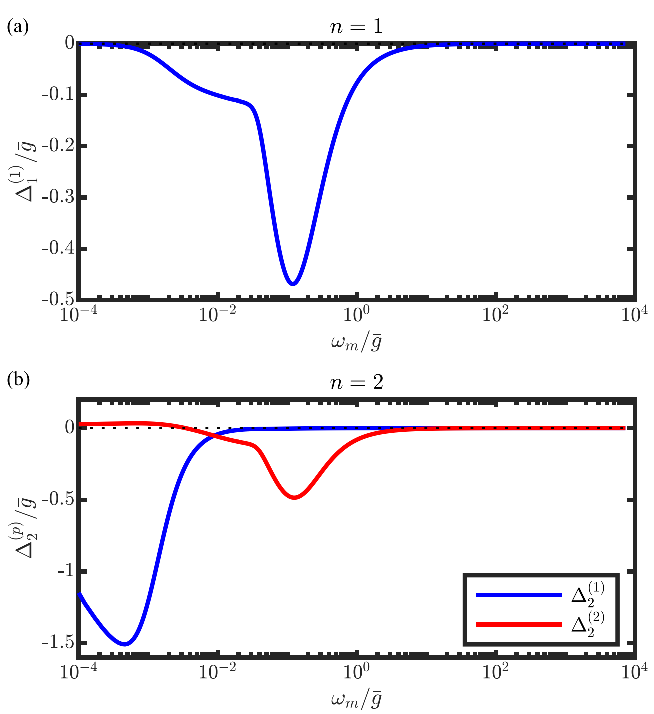

FIG. 5. Eigenfunctions corresponding to the negative eigenvalues of the kernel matrix $\mathcal{K}_{\omega,\omega'}$ for (a) $n = 1$ and (b) 2. We set $\gamma = 0.8$.

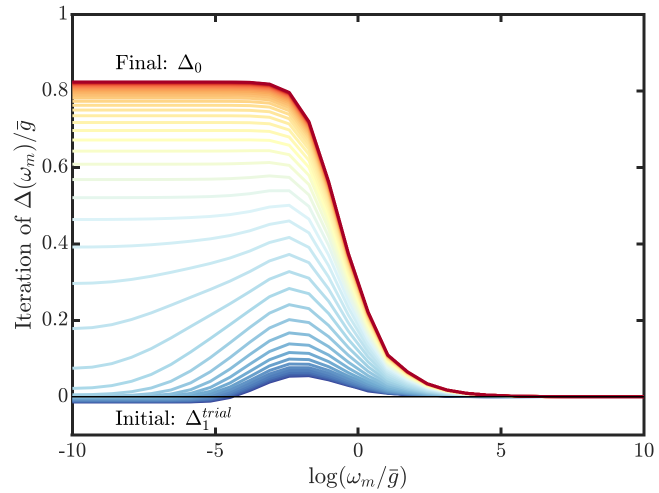

FIG. 6. Iteration process that solves the stationary points of the nonlinear gap equation, Eq. (3), which starts from the input $\Delta_1^{\text{trial}}(\omega_m)$ and saturates at $\Delta_0(\omega_m)$.

Another way to see this is to construct a one-parameter trial gap function

$$\Delta_{01}(\omega_m; \alpha) = \cos\alpha\, \Delta_1(\omega_m) + \sin\alpha\, \Delta_0(\omega_m), \qquad (56)$$

which interpolates between $\Delta_1(\omega_m)$ at $\alpha = 0$ and $\Delta_0(\omega_m)$ at $\alpha = \pi/2$, and compute the free energy $F_{\rm sc}(\alpha)$ as a function of α. The result is shown in Fig. 7. We see that $F_{\rm sc}(\alpha)$ indeed gets smaller when α increases. We verified that $F_{\rm sc}(\alpha)$ is quadratic in α in the vicinity of Δ_1, however the numerical prefactor is very small.

For $n = 2$ the same calculation yields two *negative* eigenvalues, $\lambda_2^{(1)}$ and $\lambda_2^{(2)}$. We show the result in Fig. 4(c). The eigenfunction $\Delta_2^{(1)}$ corresponding to $\lambda_2^{(1)}$ is sign-preserving and the eigenfunction $\Delta_2^{(2)}$, corresponding to $\lambda_2^{(2)}$ has one nodal point [see Fig. 5(b)]. The free energy gets reduced upon deviations from $\Delta_2(\omega_m)$ along both directions. To show this more explicitly, we construct the trial functions

$$\Delta_{12}(\omega_m; \alpha) = \cos(\alpha)\Delta_2(\omega_m) + \sin(\alpha)\Delta_1(\omega_m), \qquad (57)$$

which interpolates between $\Delta_2(\omega_m)$ and $\Delta_1(\omega_m)$, and

$$\Delta_{02}(\omega_m; \alpha) = \cos(\alpha)\Delta_2(\omega_m) + \sin(\alpha)\Delta_0(\omega_m), \qquad (58)$$

which interpolates between $\Delta_2(\omega_m)$ and $\Delta_0(\omega_m)$. For both functions, we compute $F_{\rm sc}$ as a function of α. We show the results in Fig. 7. We see that $F_{\rm sc}$ monotonically decreases when the trial gap function moves from Δ_2 to Δ_1 or to Δ_0. This incidentally also shows that $F_{\rm sc}$ increases upon deviation from $\Delta_1(\omega_m)$ in the direction of $\Delta_2(\omega_m)$, consistent with the result that there is only one negative eigenvalue for $n = 1$.

One can also construct a two-parameter trial gap function that smoothly interpolates between $\Delta_0(\omega_m)$, $\Delta_1(\omega_m)$, and $\Delta_2(\omega_m)$, see Fig. 7(b). For this trial function, $F_{\rm sc}$ for $n = 0$ is a minimum, $F_{\rm sc}$ for $n = 2$ is a maximum, and $F_{\rm sc}$ for $n = 1$ is an inflection point.

C. A discrete set of condensation energies $E_{c,n}$

The condensation energy for the stationary $\Delta_n(\omega_m)$ is given by Eq. (20). For $\gamma < 2$, $\Delta_n(\omega_m)$ form a discrete set,

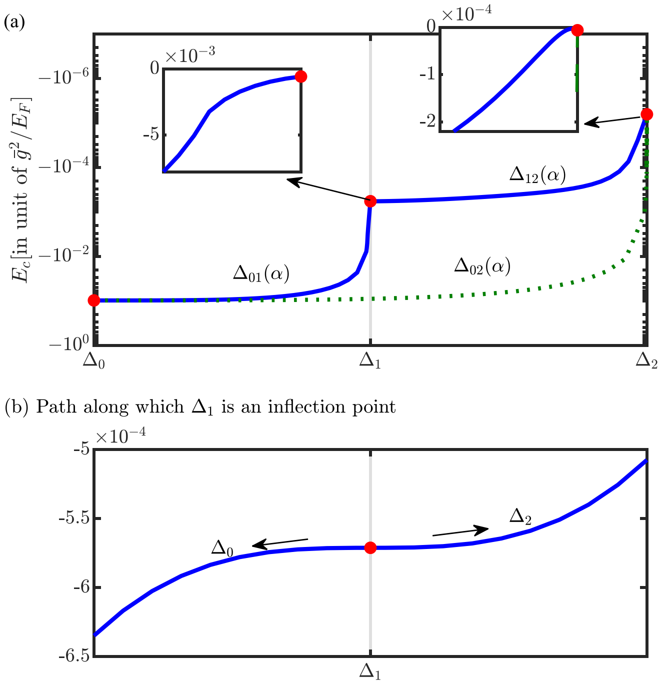

FIG. 7. (a) Free energy relative to the normal state value F_n [i.e., the condensation energy in Eq. (19)] along different paths that smoothly connect $\Delta_0(\omega_m)$, $\Delta_1(\omega_m)$, and $\Delta_2(\omega_m)$ (red dots). These paths correspond to the trial functions $\Delta_{01}(\omega_m;\alpha)$, $\Delta_{12}(\omega_m;\alpha)$, and $\Delta_{0z2}(\omega_m;\alpha)$. We set $\gamma = 0.8$. Insets show zoom-ins around $\Delta_1(\omega_m)$ and $\Delta_2(\omega_m)$. (b) The two-parameter trial gap function, smoothly interpolating between Δ_0, Δ_1, and Δ_2. For this function, Δ_1 is an inflection point.

and accordingly $E_{c,n}$ also form a discrete set. In Fig. 7(a), we present $E_{c,n}$ for representative $\gamma = 0.8$ for $n = 0,\ 1$, and 2, using the gap functions from Fig. 3. We see that $E_{c,0}$ is indeed the largest by magnitude. At large n, $E_{c,n}$ is exponentially small in n, as expected.

IV. LIMIT $\gamma \to 2$

In this section, we discuss how this set $\Delta_n(\omega_m)$ and the condensation energy $E_{c,n}$ evolve as $\gamma \to 2$. The analysis below is for $T = 0$.

A. $\Delta_n(\omega_m)$ at $\gamma = 2$–0

We argued in the previous section that at $\gamma < 2$ the nonlinear gap equation (3) has an infinite number of solutions $\Delta_n(\omega_m)$, whose amplitudes decrease exponentially with n. We obtained this result by using the exact solution of the linearized gap equation, $\Delta_\infty(\omega_m)$, as the starting point and analyzing the structure of the expansion in the gap amplitude ϵ. We found that ϵ is quantized into a discrete set of ϵ_n. The quantization follows from the fact that at $\omega_m \ll \bar{g}$, each term in the expansion oscillates as a function of $\ln|\omega_m|$ with its own phase. The explicit result for the expansion is given by Eq. (43), which we reproduce here for convenience of a reader:

$$\Delta(\omega_m) = 2\epsilon^*|\omega_m|[Q_1\cos(\psi) + (\epsilon^*)^2 Q_3 \cos(3\psi + \phi_3) + (\epsilon^*)^4 Q_5\cos(5\psi+\phi_5) + \dots], \tag{59}$$

where $\epsilon^* = \epsilon(\bar{g}/|\omega_m|)^{1-\gamma/2}$, $\psi = \beta(\gamma)\ln(\bar{g}/|\omega_m|)^\gamma + \phi$, and Q_{2k+1}, ϕ_{2k+1}, and ϕ are functions of $(\epsilon^*)^2$. The quantization sets ϵ to be $\epsilon_n \sim e^{-bn(2-\gamma)/2}$, where $b = O(1)$.

We now analyze Eq. (59) at $\gamma \to 2$. Here $\epsilon^* \approx \epsilon$ down to smallest ω_m. The series Q_{2k+1} in Eq. (59) remain nonsingular and evolve continuously at $\gamma \to 2$ towards

$$Q_1 = 1 + 1.3049\epsilon^2 + 2.99051\epsilon^4 + \cdots, \tag{60}$$

$$Q_3 = 0.222548 + 0.618278\epsilon^2 + \cdots, \tag{61}$$

$$Q_5 = 0.0426647 + \cdots \tag{62}$$

The series for ψ do become singular and transform $\beta(\gamma = 2) = 0.38187$ (Ref. [5]) into ϵ-dependent β_ϵ. Explicitly,

$$\psi = 2\beta(2)\ln\left(\frac{\bar{g}}{|\omega_m|}\right)(1 - 0.5\epsilon^2 - 0.889\epsilon^4 + \cdots) + \phi_\epsilon = 2\beta_\epsilon \ln\left(\frac{\bar{g}}{|\omega_m|}\right) + \phi_\epsilon, \tag{63}$$

where

$$\beta_\epsilon = \beta(2)(1 - 0.5\epsilon^2 - 0.889\epsilon^4 + \cdots). \tag{64}$$

The transformation $\beta(2) \to \beta_\epsilon$ shifts the frequencies, at which $\psi = \pi/2 + n\pi$, to $\omega_m \propto e^{-n\pi/2\beta_\epsilon}$. The corrections to phase $\phi(2)$, defined in Eq. (45), are also singular and transform it into $\phi_\epsilon = \phi(2) + (\beta_2\epsilon^2/(2-\gamma))(1 + 0.889\epsilon^2 + \dots)$. This transformation is, however, irrelevant as the phase is defined up to an integer number of 2π.

Finally, the series for ϕ_3, ϕ_5, etc., also remain regular, but each term contains $2-\gamma$ as a prefactor. In explicit form,

$$\phi_3 = (2-\gamma)[1/8 - 0.577535(\epsilon^*)^2 + \cdots], \tag{65}$$

$$\phi_5 = (2-\gamma)[0.452288 + \cdots]. \tag{66}$$

We keep ϵ^* here as it will be relevant for the understanding of the behavior at small but finite $2-\gamma$. We see that all ϕ_{2k+1} vanish at $\gamma = 2$. In this case, the scale, at which oscillations would be destroyed due to phase randomization, vanishes. This eliminates the argument for the discretization of ϵ.

B. Crossover from a discrete to a continuous set at $\gamma \to 2$

It is instructive to analyze in more detail how the continuous set of Δ_ϵ emerges. At $\gamma \leqslant 2$ we expect by continuity that the phases ϕ_{2k+1} become of order one at some ϵ^*. This happens when the rest of ϕ_{2k+1} compensates the overall $2-\gamma$. This happens either because prefactors become singular at some finite ϵ^*, as we suggested in Ref. [5], or because they diverge at $\epsilon^* \to \infty$ as, e.g., $(\epsilon^*)^{1/a}$. In the last case, which we analyze here, oscillations survive down to $\omega_m \sim \bar{g}(\epsilon(2-\gamma)^a)^{2/(2-\gamma)}$. Matching this scale with the position of one of the extrema of $\cos\psi$ at $\omega_m \propto \bar{g}e^{-n\pi/2\beta_\epsilon}$, we obtain self-consistent equation on discretized ϵ_n

$$\epsilon_n \sim \frac{\exp\left[-\frac{\pi}{4\beta_{\epsilon_n}}n(2-\gamma)\right]}{(2-\gamma)^a}. \tag{67}$$

We see that at $\gamma \to 2$, all ϵ_n with finite $n \ll n^* = |\ln(2-\gamma)|/(2-\gamma)$ become equal to $\epsilon_{n=0} = \epsilon_{\max} \sim \frac{1}{(2-\gamma)^a}$.

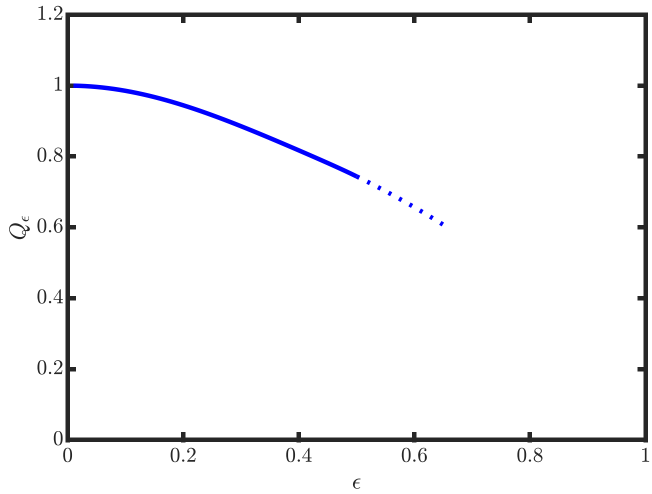

FIG. 8. The series for Q_ϵ given by Eq. (69). The series converge for $\epsilon \leqslant 0.5$. The dashed line is an extrapolation to somewhat larger ϵ.

As we anticipated, $\epsilon_{\max} = \infty$ at $\gamma = 2$.[7] Simultaneously, ϵ_n with $n \geqslant n^* \to \infty$ form a continuous spectrum, and at $\gamma = 2–0$, $\Delta_n(\omega_m)$ form a continuous one-parameter set

$$\Delta(\epsilon, \omega_m) = 2\epsilon|\omega_m|(Q_1(\epsilon)\cos(\psi(\epsilon)) + Q_3(\epsilon)\epsilon^2\cos(3\psi(\epsilon)) + \cdots), \quad (68)$$

As expected, all $\Delta(\epsilon, \omega_m)$ with finite ϵ change sign infinite number of times at $\omega_m \propto e^{-\pi k/(2\beta_\epsilon)}$, i.e., in our classification, they all are $n = \infty$ solutions. The lower point of the set is $\epsilon = 0+$, which corresponds to the solution of the linearized gap equation, $\Delta_\infty(\omega_m)$. It indeed changes sign infinite number of times.

The other end point is $\epsilon \to \infty$. We argue below that this limit is continuous, i.e., this end point is the sign-preserving $\Delta_{n=0}(\omega_m)$. This is the case if $\beta_\infty = 0$ as then ψ becomes independent on frequency and $\Delta(\infty, \omega_m)$ reduces to a constant [up to corrections in powers of $\omega_m/\bar{g}$, which we neglected in (68)]. The vanishing of β_∞ in turn implies that the frequencies $\omega_m \propto e^{-\pi k/(2\beta_{\epsilon_n})}$, at which $\Delta_n(\omega_m)$ with a finite n changes sign, all vanish at $\gamma \to 2$, as they should.

Below we present several arguments in favor of our assertion that $\beta_\infty = 0$. First, we look at series expansion in ϵ. Equation (64) already shows that β_ϵ decreases with ϵ. To obtain more precise result, we redo the expansion in ϵ self-consistently, by evaluating each term assuming that β_ϵ is vanishingly small. We obtain $\beta_\epsilon^2 \propto Q_\epsilon$, where

$$Q_\epsilon \equiv 1 - \frac{3}{2}\epsilon^2 + 3\epsilon^4 - \frac{103}{16}\epsilon^6 + \frac{915}{64}\epsilon^8 - \frac{4149}{128}\epsilon^{10} + \frac{19075}{256}\epsilon^{12} - \frac{354279}{2048}\epsilon^{14} + \cdots \quad (69)$$

Our assertion is valid if $Q_\infty = 0$. We plot Q_ϵ in Fig. 8. The series converge well up to $\epsilon \sim 0.5$. Within this range, Q_ϵ smoothly decreases with increasing ϵ. The behavior is at least consistent with the vanishing of Q_ϵ and hence β_ϵ at $\epsilon = \infty$.

Second, we reevaluate $\Delta_\epsilon(\omega_m)$ along the same lines as before, by redoing the expansion in ϵ keeping β_ϵ vanishingly small. The result is, up to order ϵ^{15},

$$\begin{aligned}\Delta(\epsilon, \omega_m) = \frac{\epsilon|\omega_m|}{1024}[&2048\epsilon\cos(\psi) + \epsilon^3(1024 - 768\epsilon^2 \\ &+ 768\epsilon^4 - 960\epsilon^6 + 1392\epsilon^8 - 2226\epsilon^{10} \\ &+ 3810\epsilon^{12} + \cdots)\cos(3\psi) \\ &- \epsilon^5(-768 + 1280\epsilon^2 - 1920\epsilon^4 + 3000\epsilon^6 \\ &- 4970\epsilon^8 + 8670\epsilon^{10} + \cdots)\cos(5\psi) \\ &+ \epsilon^7(640 - 1680\epsilon^2 + 3360\epsilon^4 - 6328\epsilon^6 \\ &+ 11886\epsilon^8 + \cdots)\cos(7\psi) - \epsilon^9(-560 + 2016\epsilon^2 \\ &- 5040\epsilon^4 + 11130\epsilon^6 + \dots)\cos(9\psi) \\ &+ \epsilon^{11}(504 - 2310\epsilon^2 + 6930\epsilon^4 + \cdots)\cos(11\psi) \\ &- \epsilon^{13}(-462 + 2574\epsilon^2 + \cdots)\cos(13\psi) \\ &+ \epsilon^{15}(429 + \cdots)\cos(15\psi) + \cdots]. \end{aligned} \quad (70)$$

We expect that at $\epsilon \to \infty$, $\Delta(\epsilon, \omega_m)$ becomes independent on ω_m at $\omega_m \to 0$. This holds when the expression in square brackets in (70) compensates the overall factor $|\omega_m|$ in the r.h.s. of (70). A simple experimentation shows that this can happen if $\phi_\infty = \pi/2$, in which case $\cos[(2k+1)\psi] = (-1)^{k+1}[(2k+1)2\beta_\epsilon \ln(\bar{g}/|\omega_m|) + \dots.]$, where dots stand for higher order of the logarithm. The expression in square brackets in (70) then becomes the series in $\ln(\bar{g}/|\omega_m|)$ with ϵ-dependent prefactors. If the series are exponential, they can cancel the overall $|\omega_m|$ in (70), however, for this the prefactors must tend to finite values at $\epsilon = \infty$

We found that the prefactor C_ϵ for $\ln(\bar{g}/|\omega_m|)$ is proportional to Q_ϵ from (69):

$$C_\epsilon = -4\epsilon Q_\epsilon \beta_\epsilon. \quad (71)$$

As $Q_\epsilon \propto \beta_\epsilon^2$, the condition that C_∞ is finite yields $\beta_\epsilon \sim (1/\epsilon)^{1/3}$ (hence $Q_\epsilon \propto 1/\epsilon^{2/3} \to 0$).

Third, as an independent check, we set ϵ to be large but finite and evaluated $\Delta_\epsilon(\omega_m)$ near each nodal point ω_p, for which $\psi = \pi/2 + \pi p$. Setting $\omega_m = \omega_p + \delta\omega$ and expanding in $\delta\omega$, we obtain

$$\Delta_\epsilon(\omega_p + \delta\omega) \approx 4(-1)^p\epsilon Q_\epsilon\beta_\epsilon\delta\omega. \quad (72)$$

If $\epsilon Q_\epsilon \beta_\epsilon$ tends to a constant at $\epsilon \to \infty$, as we anticipated, $\Delta_\epsilon(\omega_p + \delta\omega) \sim (-1)^p\delta\omega$. This is consistent with the behavior of $\Delta_n(\omega_m)$ with finite n at small but finite $2 - \gamma$ in the frequency range, where $\Delta_n(\omega_m)$ changes sign n times.

Substituting $\beta_\epsilon \propto 1/\epsilon^{1/3}$ into (67), we find the relation between ϵ and n at $\gamma = 2–0$:

$$\frac{n}{n^*} \sim \frac{4a}{\epsilon^{1/3}}. \quad (73)$$

This relation holds when $\epsilon \gg 1$, i.e., when $n \ll n^*$. A more general expression, valid also for $\epsilon \leqslant 1$, is $\beta_\epsilon = (\pi/4a)n/n^*$. A more sophisticated analysis is required to relate ϵ and n at smaller ϵ.

[7]If the phases ϕ_{2k+1} become $O(1)$ at a finite ϵ^*, the results are the same, only $a = 0$, hence $\epsilon_{\max}$ is finite.

C. Continuum spectrum of the condensation energy

The appearance of a continuum of solutions of the Eliashberg gap equation at $\gamma = 2$–0 and $\omega_D = 0$ poses the question about the spectrum of the condensation energy. At a first glance, the condensation energy should be flat as a function of ϵ because each $\Delta_\epsilon(\omega_m)$ satisfies the stationary condition $\delta F_{\rm sc}/\delta\Delta = \delta E_c/\delta\Delta = 0$, hence $dE_c/d\epsilon = \delta E_c/\delta\Delta_\epsilon \times d\Delta_\epsilon/d\epsilon = 0$, We argue that in our case the situation is more tricky because the condensation energy formally diverges at $\gamma = 2$–0 for all ϵ, which creates "zero times infinity" conundrum. We argue that after proper regularization, the condensation energy becomes a regular nonflat function of ϵ.

To understand how to reconcile a nonflat $E_c(\epsilon)$ with the fact that each Δ_ϵ is a stationary solution, consider γ slightly below 2. For any $\gamma < 2$, $\epsilon = \epsilon_n$ are discrete, and the condensation energy, given by Eq. (19), is also discrete $E_{c,n}$. Let's start with $n = 0$. At small frequencies the gap function $\Delta_0(\omega_m)$ tends to a finite value $\Delta_0 \sim \bar{g}$. The first term in Eq. (19) is regular and of order $\bar{g}^2 N_F$. The leading contribution to the second term in Eq. (19) comes from small frequencies and has the form

$$-N_F \bar{g}^\gamma \int_0^{\omega^*} d\omega_m \int_0^{\omega^*} d\omega_{m'} \frac{1}{|\omega_m + \omega_{m'}|^\gamma}, \tag{74}$$

where the upper limit $\omega^* \sim \bar{g}$. Evaluating the integral we find that the condensation energy diverges at $\gamma \to 2$ as

$$E_{c,0} \approx -\bar{g}^2 N_F \frac{1}{2-\gamma}. \tag{75}$$

Consider next the condensation energy for a finite n. Because all $\Delta_n(\omega)$ tend to finite $\Delta_n(0)$ at vanishing ω_m, the divergent $1/(2-\gamma)$ term is the same as in (75). The difference $\delta E_n = E_{c,n} - E_{c,0}$ comes from the range where $\Delta_n(\omega_m)$ changes sign n times between the largest $\omega_m \sim \bar{g}e^{-\pi/2\beta_{\epsilon_n}}$ and the smallest $\omega_m \sim \bar{g}e^{-\pi n/2\beta_{\epsilon_n}}$. We obtained (see Appendix E for details) $\delta E_c = c\bar{g}^2 N_F n$, where $c = O(1)$. Together with Eq. (75) this yields

$$\begin{aligned} E_{c,n} &= -\bar{g}^2 N_F \left(\frac{1}{2-\gamma} - cn + \cdots\right) \\ &= E_{c,0}(1 - cn(2-\gamma) + \cdots). \end{aligned} \tag{76}$$

Equation (76) is valid as long as $n \ll n^*$, more accurately, when $n \leqslant n^*/((2-\gamma)^{a/3}|\ln(2-\gamma)|$ $[\epsilon \approx \epsilon_{\max} \sim 1/(2-\gamma)^a]$. For larger n, the factor $n(2-\gamma)$ in the r.h.s. of (76) is replaced by

$$J_n = \frac{2\beta_\epsilon}{\pi}\left(1 - e^{-\frac{\pi n(2-\gamma)}{2\beta_\epsilon}}\right) = \frac{2\beta_\epsilon}{\pi}(1 - (\epsilon(2-\gamma)^a)^2) \approx \frac{2\beta_\epsilon}{\pi} \tag{77}$$

(see again Appendix E 2 for details). This result holds for $n \sim n^*$, i.e., $\epsilon \sim 1$. For numerically large ϵ (but $\epsilon \ll \epsilon_{\max} \sim 1/(2-\gamma)^a$), $\beta_\epsilon \sim 1/\epsilon^3$, hence

$$E_c(\epsilon) \approx E_{c,0}\left(1 - \frac{\bar{c}}{\epsilon^{1/3}}\right), \tag{78}$$

where $\bar{c} = O(1)$. In the opposite limit $n \gg n^*$, i.e., at $\epsilon \to 0$, the gap function $\Delta_\infty(\omega_m) \propto \epsilon$ does not saturate at a finite value at $\omega_m = 0$. Yet we found in explicit calculation in Appendix E 2 that $E_c(\epsilon) \propto \epsilon^2/(2-\gamma)$ still diverges as $1/(2-\gamma)$. For a generic ϵ, we expect

$$E_c = E_{c,0}\, g(\epsilon), \tag{79}$$

where $g(\epsilon) \approx 1 - \bar{c}/\epsilon^{1/3}$ for large ϵ and $g(\epsilon) \sim \epsilon^2$ for small ϵ.[8]

The singular $1/(2-\gamma)$ dependence in E_c is the reason why $E_c(\epsilon)$ disperses with ϵ despite that for each ϵ, $\Delta_\epsilon(\omega_m)$ is a stationary point of $F_{\rm sc}$. To see this explicitly, we substituted $\Delta_\epsilon(\omega_m)$ from (68) into the expression for E_c, Eq. (19), and expanded around some representative $\epsilon = \epsilon_0$. The linear term in the expansion of $E_c(\epsilon)$ in $\epsilon - \epsilon_0$ vanishes, as it should, but the prefactors for all higher derivatives diverge. In this special case, the Taylor expansion in $\epsilon - \epsilon_0$ must be kept to an infinite order to obtain the correct functional form of $E_c(\epsilon)$ near ϵ_0.

We see from (79) that if we measure $E_c(\epsilon)$ in units of $E_c(\infty)$, we obtain the dispersing $E_c/E_c(\infty) = g(\epsilon)$. This, however, makes sense if $E_c(\infty)$ remains finite at $\gamma = 2$–0, i.e., if $1/(2-\gamma)$ divergence is regularized. The way to do this is to take the double limit $\gamma \to 2$ and $N_F/N \to 0$ (i.e., $E_F \sim \Lambda \to \infty$), such that the product $N_F/(N(2-\gamma))$ remains finite at $\gamma = 2$–0. We illustrate this in Fig. 9. Once $E_c(\infty)$ is finite, the spectrum of the condensation energies becomes a regular gapless function of ϵ. In particular, for a finite n, the difference $(E_{c,n} - E_{c,0})/N \sim N_F n/N$ vanishes.

The emergence of a continuum spectrum of $E_c(\epsilon)$ at $\gamma = 2 + 0$ due to a collapse of all $E_{c,n}$ with finite n onto $E_{c,0}$ has a profound effect on the behavior of the superfluid stiffness ρ_s at $T = \Omega_D = 0$ as it opens up a gapless channel of "longitudinal" gap fluctuations. In Ref. [5], we argued semi-phenomenologically that these fluctuations give rise to singular downward renormalization of the stiffness and give rise to vanishing T_c at $\gamma = 2$–0. We leave the detailed analysis of ρ_s, based on the formula for $E_c(\epsilon)$, obtained in this paper, to a separate study.

Note that the $1/(2-\gamma)$ divergence can also be regularized by a finite T or a finite ω_D. In both cases, the would be divergent term in E_c becomes

$$-\bar{g}^2 N_F \ln\frac{\bar{g}}{T} \quad \text{or} \quad -\bar{g}^2 N_F \ln\frac{\bar{g}}{\omega_D}. \tag{80}$$

The spectrum of $E_{c,n}$ then remains discrete at $\gamma = 2$, but becomes continuous in the properly defined double limit $T, \omega_D \to 0$ and $N_F \to 0$. Incidentally, the same double limit is also required to keep vertex corrections to the Eliashberg equations small. Vertex corrections at $T = 0$ are controlled by $\lambda_E = (N_F/N)\bar{g}^2/\omega_D)$, and to keep them small one has to set N_F/N to zero along with $\omega_D \to 0$.

V. CONCLUSIONS

In this work, we extended our earlier analysis of pairing in a metal tuned to a $T = 0$ quantum critical point, at which it develops a spontaneous order in the particle-hole channel. At this point, bosonic fluctuations of a critical order parameter become massless. In earlier works [1–6], we analyzed the pairing mediated by a massless boson and argued that this

[8]The emergence of the overall factor $1/(2-\gamma)$ in E_c was not noted in Ref. [5]. Equation [40] in that paper is incorrect.

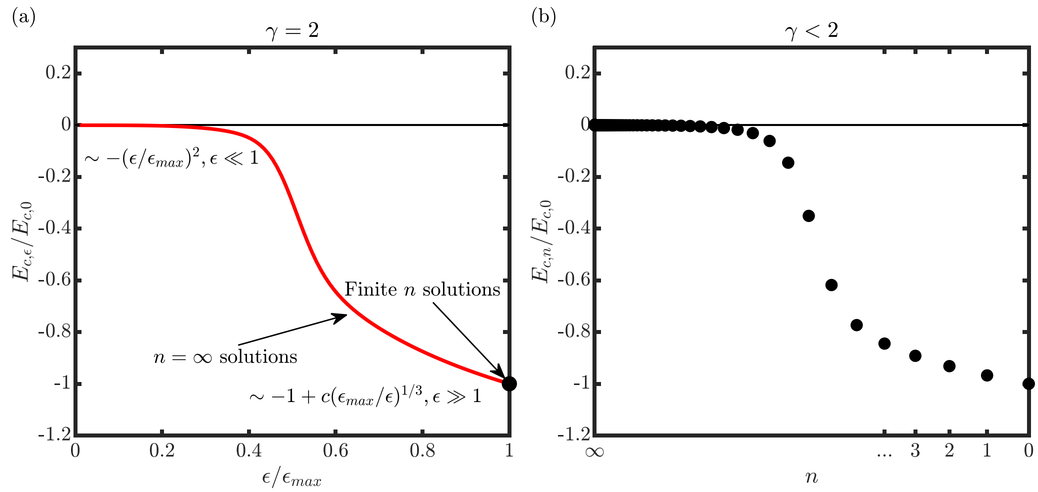

FIG. 9. (Schematic) condensation energy $E_{c,n}$ for (a) $\gamma = 2$ and (b) a generic $\gamma < 2$.

problem is qualitatively different from the BCS/Eliashberg pairing in a metal away from a critical point. Specifically, a massless boson gives rise not only to the pairing, but also to a NFL behavior in the normal state. The competition between NFL and pairing leads to new physics, not seen in noncritical metals. We argued that under certain conditions low-energy properties of a critical metal are described by a $0+1$ dimensional dynamical model with an effective four-fermion interaction $V(\Omega) \propto 1/|\Omega|^\gamma$. This model has been nicknamed the γ model. We argued that for $\gamma < 2$, the ground state is a superconductor with a regular, sign-preserving gap function $\Delta_0(\omega_m)$ on the Matsubara axis. Yet, the gap equation at $T = 0$ also allows an infinite, discrete set of topologically distinct stationary solutions $\Delta_n(\omega_m)$ with n nodal points at positive ω_m ($n = 1, 2, \ldots$). We argued that the set evolves with increasing γ and eventually becomes a continuous one at $\gamma = 2$–0. At this γ, the system undergoes a topological transition into a state with novel measurable features.

Here we performed an in-depth analysis of the free energy of the γ model for a generic $\gamma < 2$ and for γ infinitesimally close to 2. One goal of our study was to understand the profile of the free energy near each stationary $\Delta_n(\omega_m)$, another was to understand in more detail how a discrete set of $\Delta_n(\omega_m)$ becomes continuous at $\gamma = 2$–0 and how the spectrum of condensation energies evolves near this γ.

For the free energy analysis, we first compared the two forms of the free energy—the variational one, obtained by extending the Luttinger-Ward-Eliashberg variational approach to arbitrary γ, and the actual one, which appears in the partition function after Hubbard-Stratonovich transformation. The variational free energy has a simple form and yields the stationary gap equation, which agrees with the one obtained diagrammatically by summing up ladder diagrams with the fully dressed Green's functions. Yet, for a massive pairing boson it does not adequately describe fluctuations around a stationary point as it neglects variations of the quasiparticle residue $Z(\omega_m)$. We showed that for a massless boson fluctuations of Z around its stationary value become irrelevant, and the variational free energy can be used to obtain not only stationary solutions, but also fluctuations around them. For $\gamma = 2$, this has been earlier found in Ref. [50].

To obtain stationary $\Delta_n(\omega_m)$ at $T = 0$, we used as a point of departure the exact solution for infinitesimally small $\Delta_\infty(\omega_m)$, which we found in previous works, expanded in the gap amplitude ϵ, and explicitly demonstrated that ϵ is discretized into ϵ_n. Using this reasoning, we obtained $\Delta_n(\omega_m)$ with $n = 0$, 1, and 2, which satisfy the gap equation with high numerical accuracy.

We then expanded the free energy near stationary $\Delta_n(\omega_m)$ with these n to second order in deviations and analyzed the stability of each solution. We found that the $n = 0$ solution is a global minimum, while the expansion around $\Delta_1(\omega_m)$ yields one negative eigenvalue for deviations towards $\Delta_0(\omega_m)$, and the expansion around $\Delta_2(\omega_m)$ yields two negative eigenvalues for deviations towards $\Delta_0(\omega_m)$ and $\Delta_1(\omega_m)$. By continuity, we expect n negative eigenvalues for $\Delta_n(\omega_m)$. This result implies that Δ_0 is a true minimum, while Δ_n with $n > 0$ is a saddle point with n unstable directions.

As a biproduct of this analysis, we obtained the free energy and the specific heat $C(T)$ in the normal state and compared the results with recent studies by other groups.

For our second goal, we analyzed in more detail than before the transformation from a discrete set of $\Delta_n(\omega_m)$ into a continuous one-parameter set $\Delta(\epsilon, \omega_m)$ at $\gamma = 2$–0. We demonstrated that when the exponent γ increases towards 2, the set of discretized gap amplitudes ϵ_n progressively splits into two subsets: the ones with $n < n^* \sim |\ln(2-\gamma)|/(2-\gamma)$ all approach ϵ_0, while the ones with $n > n^*$ form a continuum set $\Delta(\epsilon, \omega_m)$, where ϵ is a function of the ratio n/n^* when both tend to infinity. At $\gamma = 2$–0, all functions $\Delta(\epsilon, \omega_m)$ oscillate infinite number of times down to $\omega_m = 0$, i.e., in the count of nodal points they all are $n = \infty$ solutions (the solution of the linearized gap equation is the one at infinitesimally small ϵ). This holds for all finite ϵ. However, the end point at $\epsilon = \infty$ is a sign-preserving $\Delta_0(\omega_m)$, which at $\gamma = 2$–0 incorporates all gap functions $\Delta_n(\omega_m)$ with finite n. We showed that the range, where $\Delta_n(\omega_m)$ changes sign n times, shrinks to progressively lower frequencies as $\gamma \to 2$.

We used these results to obtain the condensation energy $E_{c,n}$—the difference between free energy of a superconductor and of a would be normal state at the same T. We found $E_c(\epsilon) = E_c(\infty)g(\epsilon)$, where $E_c(\infty) = \bar{g}^2 N_F(1/(2-\gamma)$, and $g(\epsilon \gg 1) = 1 - O(1/\epsilon^{1/3})$ and $g(\epsilon \ll 1) \sim \epsilon^2$. We argued that this result makes sense if $E_c(\infty)$ remains finite at $\gamma = 2$–0, i.e., if $1/(2-\gamma)$ divergence is regularized. The way to do this is to take the double limit $\gamma \to 2$ and $N_F/N \to 0$ (i.e., $E_F \sim \Lambda \to \infty$), such that the product $N_F/(N(2-\gamma))$ remains finite at $\gamma = 2$–0. At the same time, we argued that $1/(2-\gamma)$ divergence in $E_c(\infty)$ is is the reason why $E_c(\epsilon)$ disperses with ϵ despite that all $\Delta_\epsilon(\omega_m)$ are stationary points of the free energy. We argued that if we expand $E_c(\epsilon)$ around some ϵ_0, the linear term in the expansion vanishes, as it should, but the prefactors for all higher derivatives diverge. In this singular case, Taylor expansion in $\epsilon - \epsilon_0$ must be kept to an infinite order to obtain the correct functional form of $E_c(\epsilon)$.

We argued in Ref. [6] that $\gamma = 2$ is a critical point of a topological phase transition. The argument is that the gap function $\Delta_0(\omega_m)$, for which the condensation energy is the largest by magnitude for $\gamma < 2$ and $\gamma > 2$, lives on different Riemann surfaces at $\gamma < 2$ and $\gamma > 2$. A related argument that $\gamma = 2$ is special is that in the upper frequency half-plane, $\Delta_0(z)$, where $z = \omega' + i\omega''$, possesses dynamical vortices, and their number becomes infinite when γ approaches 2 from either side. It is natural to expect a gapless branch of excitations at the critical point, and the emergence of a gapless continuum of $E_c(\epsilon)$ is in line with this reasoning.

The emergence of a continuum spectrum of $E_c(\epsilon)$ at $\gamma = 2 + 0$ due to a collapse of all $E_{c,n}$ with finite n onto $E_{c,0}$ has a profound effect on the behavior of the superfluid stiffness ρ_s at $T = \Omega_D = 0$ as it opens up a gapless channel of "longitudinal" gap fluctuations. In [5] we argued semi-phenomenologically that these fluctuations give rise to singular downward renormalization of the stiffness and give rise to vanishing T_c at $\gamma = 2$–0. We leave the detailed analysis of ρ_s, based on the formula for $E_c(\epsilon)$, obtained in this paper, to a separate study. This analysis has to be made at small, but finite ω_D. The limit $\omega_D \to 0$ also has to be taken together with $E_F \to \infty$ to keep vertex corrections to the Eliashberg gap equation small.

ACKNOWLEDGMENTS

We thank everyone with whom we discussed this and previous works. S.-S.Z. and A.V.C. were supported by the NSF DMR-1834856. Y.-M.W. is grateful to the support of Shuimu Fellow Foundation at Tsinghua.

APPENDIX A: EXPANSION OF FREE ENERGY AND EQUIVALENCE OF F_{sc} AND F_{sc}^{var} at $\omega_D \to 0$

In this Appendix, we show that for a finite ω_D, the expansion of the actual free energy of an Eliashberg superconductor, F_{sc}, to second order in variations around a stationary solution $\Delta_n(\omega_m)$ is non equivalent to the expansion of the variational free energy F_{sc}^{var}. However, at $\omega_D \to 0$, the expansions become equivalent for any value of γ at a finite T and for $\gamma > 1$ at $T = 0$. We verified that F_{sc}^{var} and F_{sc} remain equivalent also beyond the second order.

1. Computation of $V^{-1}(\Omega_m)$

The free energy F_{sc} is given by Eq. (17). It contains the "inverse" interaction $V^{-1}(\Omega_m)$, which is the Fourier transform of $1/V(\tau)$. We will need the explicit form of $V^{-1}(\Omega_m)$ at a finite T and $\omega_D \ll T$. We compute it in this subsection and use the result in the next subsection.

The interaction $V(\Omega_m)$ is given by Eq. (2):

$$V(\Omega_m) = \frac{\bar{g}^\gamma}{\left(\Omega_m^2 + \omega_D^2\right)^{\gamma/2}}. \tag{A1}$$

Its inverse Fourier transform is

$$V(\tau) = T\left(\frac{\bar{g}}{\omega_D}\right)^\gamma \left[1 + \left(\frac{\omega_D}{2\pi T}\right)^\gamma \times (Li_\gamma(e^{2\pi T\tau}) + Li_\gamma(e^{-2\pi T\tau}))\right], \tag{A2}$$

where $Li_\gamma(x)$ is a polylogarithmic function. For $\omega_D \ll T$, the second term is a small correction. Then

$$V^{-1}(\tau) = \frac{1}{T}\left(\frac{\omega_D}{\bar{g}}\right)^\gamma \left[1 - \left(\frac{\omega_D}{2\pi T}\right)^\gamma \times (Li_\gamma(e^{2\pi T\tau}) + Li_\gamma(e^{-2\pi T\tau}))\right]. \tag{A3}$$

The Fourier transform of (A3) is

$$V^{-1}(\Omega_m) = \int_0^{1/T} V^{-1}(\tau)\cos(\Omega_m\tau) = \frac{1}{T^2}\left(\frac{\omega_D}{\bar{g}}\right)^\gamma \left[\delta_m - \left(\frac{\omega_D}{2\pi T}\right)^\gamma I_m\right], \tag{A4}$$

where $\delta_{m,0}$ is a Kronecker symbol and

$$I_m = \frac{1}{2\pi}\int_0^{2\pi} dx\cos(mx)(Li_\gamma(e^x) + Li_\gamma(e^{-x})). \tag{A5}$$

We verified that $I_0 = 0$ and $I_{m>0} = 1/|m|^\gamma$. As a result,

$$V^{-1}(\Omega_m) = \frac{1}{T^2}\left(\frac{\omega_D}{\bar{g}}\right)^\gamma \left[\delta_{m,0} - (1-\delta_{m,0})\left(\frac{\omega_D}{|\Omega_m|}\right)^\gamma\right]. \tag{A6}$$

The original $V(\Omega_m)$ can be equally represented as

$$V(\Omega_m) = \left(\frac{\bar{g}}{\omega_D}\right)^\gamma \left[\delta_{m,0} + (1-\delta_{m,0})\left(\frac{\omega_D}{|\Omega_m|}\right)^\gamma\right]. \tag{A7}$$

We will use Eqs. (A6) and (A7) below.

We note in passing that at $T = 0$ the computation of $V^{-1}(\Omega_m)$ for a continuous Ω_m is more involved and requites one to regularize the integral over τ and then take the proper double limit. To be brief, we show how it works for $\gamma = 2$. We have

$$V(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\Omega_m e^{i\Omega_m\tau} V(\Omega_m) = \frac{\bar{g}^2}{2\omega_D} e^{-\omega_D|\tau|}. \tag{A8}$$

Then

$$V^{-1}(\tau) = \frac{2\omega_D}{\bar{g}^2} e^{\omega_D|\tau|}. \tag{A9}$$

The inverse interaction $V^{-1}(\Omega_m) = \int_{-\infty}^{\infty} d\Omega_m e^{-i\Omega_m\tau} V^{-1}(\tau)$. To make this integral convergent, we add to the integrand

$e^{-\delta\tau^2}$ and set $\delta \to 0$ at the end of calculations. Integrating over τ, we obtain

$$V^{-1}(\Omega_m) = \frac{2\omega_D}{\bar{g}}(S(\Omega_m) + S(-\Omega_m)), \tag{A10}$$

where at large $|\Omega_m| \gg (\delta)^{1/2}$,

$$S(\Omega_m) = -\frac{1}{\omega_D + i\Omega_m} + \sqrt{\frac{\pi}{\delta}} e^{\frac{(\omega_D + i\Omega_m)^2}{4\delta}}. \tag{A11}$$

Then

$$V^{-1}(\Omega_m) = \frac{4}{\bar{g}^2}\left[-\pi\omega_D\delta(\Omega_m) - \frac{\omega_D^2}{\Omega_m^2} + \frac{\sqrt{\pi}}{\sqrt{\delta}}\omega_D \cos\left(\frac{\omega_D\Omega_m}{2\delta}\right) e^{\frac{(\omega_D + i\Omega_m)^2}{4\delta}} \right]. \tag{A12}$$

A simple experimentation shows that in the double limit, in which we set $\omega_D \to 0$ first and set $\delta \to 0$ after that, the last term in (A12) reduces to $2\pi\omega_D\delta(\Omega_m)$ (To see this one can just integrate the last term in (A12) over Ω_m. The integration yields $2\pi\omega_D$.) The total $V^{-1}(\Omega_m)$ is then

$$V^{-1}(\Omega_m) = \frac{4\omega_D}{\bar{g}^2}\left[\pi\delta(\Omega_m) - \frac{\omega_D}{\Omega_m^2}\right]. \tag{A13}$$

In the same limit $\omega_D \to 0$, we can express $V(\Omega_m)$ as

$$V(\Omega_m) = \frac{\bar{g}^2}{\omega_D}\left[\pi\delta(\Omega_m) + \frac{\omega_D}{\Omega_m^2}\right]. \tag{A14}$$

This analysis can be easily extended to $\gamma < 2$.

2. Expansion of free energy $F_{\text{sc},n}$ in ω_D

The free energy, obtained in Hubbard-Stratonovich formalism, is given by Eq. (17). We select the stationary solution for the gap $\Delta_n(\omega_m)$ and the quasiparticle residue $Z_n(\omega_m) = Z(\Delta_n(\omega_m))$ with some n and expand around them to second order in $\delta\Delta(\omega_m) = \Delta(\omega_m) - \Delta_n(\omega_m)$ and $\delta Z(\omega_m) = Z(\omega_m) - Z_n(\omega_m)$. The zeroth-order term $F_{\text{sc},n}^{(0)}$ is

$$\begin{aligned} F_{\text{sc},n}^{(0)} &= -N_F 2\pi T \sum_m Z_n(\omega_m)\sqrt{\omega_m^2 + \Delta_n^2(\omega_m)} \\ &\quad + N_F T^2 \sum_{m,m'} V^{-1}(\omega_m - \omega_{m'}) \\ &\quad \times [Z_n(\omega_m)\Delta_n(\omega_m)Z_n(\omega_{m'})\Delta_n(\omega_{m'}) \\ &\quad + \omega_m\omega_{m'}(Z_n(\omega_m) - 1)(Z_n(\omega_{m'}) - 1)]. \end{aligned} \tag{A15}$$

Using the Eliashberg equations

$$Z_n(\omega_m)\Delta_n(\omega_m) = \pi T \sum_{m'} V(\omega_m - \omega_{m'}) \frac{\Delta_n(\omega_{m'})}{\sqrt{\omega_{m'}^2 + \Delta_n^2(\omega_{m'})}}, \tag{A16}$$

$$\omega_m(Z_n(\omega_m) - 1) = \pi T \sum_{m'} V(\omega_m - \omega_{m'}) \frac{\omega_{m'}}{\sqrt{\omega_{m'}^2 + \Delta_n^2(\omega_{m'})}}, \tag{A17}$$

we reexpress $F_{\text{sc},n}^{(0)}$ as

$$F_{\text{sc},n}^{(0)} = -N_F 2\pi T \sum_m Z_n(\omega_m)\sqrt{\omega_m^2 + \Delta_n^2(\omega_m)} + N_F \pi^2 T^2 \sum_{m,m'} \frac{\Delta_n(\omega_m)\Delta_n(\omega_{m'}) + \omega_m\omega_{m'}}{\sqrt{\omega_m^2 + \Delta_n^2(\omega_m)}\sqrt{\omega_{m'}^2 + \Delta_n^2(\omega_{m'})}} S(m, m'), \tag{A18}$$

where

$$S(m, m') = T^2 \sum_{m'', m'''} V^{-1}(\omega_m - \omega_{m''}) V(\omega_{m''} - \omega_{m'''}) V(\omega_{m'''} - \omega_{m'}). \tag{A19}$$

The sum is evaluated using the relation

$$T^2 \sum_{m''} V^{-1}(\omega_m - \omega_{m''}) V(\omega_{m''} - \omega_{m'}) = \delta_{m,m'}. \tag{A20}$$

Using (A20), we obtain $S(m, m') = V(\omega_m - \omega_{m'})$ and

$$F_{\text{sc},n}^{(0)} = -N_F 2\pi T \sum_m Z_n(\omega_m)\sqrt{\omega_m^2 + \Delta_n^2(\omega_m)} + N_F \pi^2 T^2 \sum_{m,m'} \frac{\Delta_n(\omega_m)\Delta_n(\omega_{m'}) + \omega_m\omega_{m'}}{\sqrt{\omega_m^2 + \Delta_n^2(\omega_m)}\sqrt{\omega_{m'}^2 + \Delta_n^2(\omega_{m'})}} V(\omega_m - \omega_{m'}). \tag{A21}$$

Using the Eliashberg equations once again, we re-express $F_{\text{sc},n}^{(0)}$ as

$$F_{\text{sc},n}^{(0)} = -N_F 2\pi T \sum_m \frac{\omega_m^2}{\sqrt{\omega_m^2 + \Delta_n^2(\omega_m)}} - \pi^2 T^2 N_F \sum_{m,m'} V(\omega_m - \omega_{m'}) \frac{\omega_m\omega_{m'} + \Delta_n(\omega_m)\Delta_n(\omega_{m'})}{\sqrt{\omega_m^2 + \Delta_n^2(\omega_m)}\sqrt{(\omega_{m'})^2 + \Delta_n^2(\omega_{m'})}}. \tag{A22}$$

This is the same expression as the variational free energy $F_{\text{sc}}^{\text{var}}$ for stationary $\Delta_n(\omega_m)$, Eq. (16).

First-order terms in the expansion of $F_{\text{sc},n}$ in $\delta\Delta$ and δZ vanish because F_{sc} is stationary with respect to variations around $\Delta_n(\omega_m)$ and $Z(\Delta_n(\omega_m))$. The expansion to second order in variations yields

$$\begin{aligned} F_{\text{sc},n}^{(2)} &= -N_F \pi T \sum_m \frac{(\delta\Delta_m)^2 Z_n(\omega_m)\omega_m^2}{\left(\omega_m^2 + \Delta_n^2(\omega_m)\right)^{3/2}} + N_F T^2 \sum_{m,m'} V^{-1}(\omega_m - \omega_{m'}) \\ &\quad \times [\delta\Delta_m\delta\Delta_{m'} Z_n(\omega_m)Z_n(\omega_{m'}) + \delta Z_m\delta Z_{m'}(\omega_m\omega_m' + \Delta_n(\omega_m)\Delta_n(\omega_m')) + 2\delta Z_m\delta\Delta_{m'} Z_n(\omega_{m'})\Delta_n(\omega_m)]. \end{aligned} \tag{A23}$$

For a generic ω_D, δZ_m and $\delta\Delta_m$ are two independent variations. For small ω_D, the largest contribution to $Z_n(\omega_m)$ comes from the term with $m = m'$ in the r.h.s. of (A16). If we keep only this term, we find

$$Z_n(\omega_m) = \left(\frac{\bar{g}}{\omega_D}\right)^{\gamma} \frac{\pi T}{\sqrt{\omega_m^2 + \Delta_n^2(\omega_m)}}. \tag{A24}$$

Based on this observation, we split δZ_m into two components, one of which satisfies Eq. (A24):

$$\delta Z_m = -\frac{\Delta_n(\omega_m) Z_n(\omega_m)\delta\Delta_m}{\omega_m^2 + \Delta_n^2(\omega_m)} + \bar{\delta} Z_m. \tag{A25}$$

The first term in (A25) would be the variation of $Z_n(\omega_m)$ if Eq. (A24) was valid for arbitrary $\Delta(\omega_m)$, not just a stationary $\Delta_n(\omega_m)$

Substituting δZ_m from (A25) into (A23), we find $F^{(2)}_{\text{sc},n}$ as the sum of three terms. The first one contains only $\delta\Delta$, the second one contains only $\bar{\delta}Z$, and the third is the mixed term. The term with $\bar{\delta}Z$ is

$$N_F T^2 \sum_{m,m'} V^{-1}(\omega_m - \omega_{m'})\bar{\delta} Z_m \bar{\delta} Z_{m'}(\omega_m \omega'_m + \Delta_n(\omega_m)\Delta_n(\omega'_m)). \tag{A26}$$

We now use Eq. (A6) for $V^{-1}(\Omega_m)$. Substituting into (A26), we find that this component of $F^{(2)}_{\text{sc},n}$ contains ω_D^{γ} as the overall factor and hence vanishes as $\omega_D \to 0$. The same happens with the mixed term. It is

$$-2N_F T^2 \sum_{m,m'} V^{-1}(\omega_m - \omega_{m'})\bar{\delta} Z_m \delta\Delta_{m'} \frac{Z_n(\omega_{m'})\omega_{m'}(\Delta_n(\omega_{m'})\omega_m - \Delta_n(\omega_m)\omega_{m'})}{\omega_{m'}^2 + \Delta_n^2(\omega_{m'})}. \tag{A27}$$

Substituting $V^{-1}(\omega_m - \omega_{m'})$ from (A6), we find that the leading term with $m = m'$ does not contribute because the summand in (A27) vanishes at $m = m'$. Combining the subleading term in V^{-1} with $Z_n(\omega_m) \propto (\bar{g}/\omega_D)^{\gamma}$, we find that the overall factor in (A27) again scales as ω_D^{γ} and vanishes at $\omega_D \to 0$. This implies that variations $\bar{\delta}Z$ do not affect the free energy, at least for quadratic deviations from a stationary point.

The remaining term in $F^{(2)}_{\text{sc},n}$ is quadratic in $\delta\Delta$. Combing all contributions of this kind, we obtain

$$\begin{aligned} F^{(2)}_{\text{sc},n} = &-N_F \sum_m \frac{(\delta\Delta_m)^2 Z_n(\omega_m)\omega_m^2}{\left(\omega_m^2 + \Delta_n^2(\omega_m)\right)^{3/2}} + N_F T^2 \sum_{m,m'} V^{-1}(\omega_m - \omega_{m'})\delta\Delta_m \delta\Delta_{m'} \\ &\times \frac{Z_n(\omega_m) Z_n(\omega_{m'})\omega_m\omega'_m(\omega_m\omega'_m + \Delta_n(\omega_m)\Delta_n(\omega'_m))}{\left(\omega_m^2 + \Delta_n^2(\omega_m)\right)\left(\omega_{m'}^2 + \Delta_n^2(\omega_{m'})\right)}. \end{aligned} \tag{A28}$$

Substituting V^{-1} from (A6), we find after simple algebra that potentially divergent terms at $\omega_D \to 0$ cancel out, and the remaining piece is

$$\begin{aligned} F^{(2)}_{\text{sc},n} = &-N_F \pi T \sum_m \frac{(\delta\Delta_m)^2 Z_n(\omega_m)\omega_m^2}{\left(\omega_m^2 + \Delta_n^2(\omega_m)\right)^{3/2}} + N_F T^2 \sum_{m,m'} \delta\Delta_m \delta\Delta_{m'} V^{-1}(\omega_m - \omega_{m'}) \\ &\times \frac{Z_n(\omega_m) Z_n(\omega_{m'})\omega_m\omega'_m(\omega_m\omega'_m + \Delta_n(\omega_m)\Delta_n(\omega'_m))}{\left(\omega_m^2 + \Delta_n^2(\omega_m)\right)\left(\omega_{m'}^2 + \Delta_n^2(\omega_{m'})\right)}. \end{aligned} \tag{A29}$$

Using again Eq. (A6) for $V^{-1}(\omega_m - \omega_{m'})$ and reexpressing it via $V(\omega_m - \omega_{m'})$ from (A7), we obtain $F^{(2)}_{\text{sc},n}$ as the sum of the two terms: $F^{(2)}_{\text{sc},n} = F^{(2,a)}_{\text{sc},n} + F^{(2,b)}_{\text{sc},n}$, where

$$\begin{aligned} F^{(2,a)}_{\text{sc},n} = &N_F \pi T \sum_m \frac{(\delta\Delta_m)^2 \omega_m^2}{\left(\omega_m^2 + \Delta_n^2(\omega_m)\right)^{3/2}} + N_F \frac{\pi^2}{2} T^2 \sum_{m\neq m'} V\left(\omega_m - \omega_{m'}\right) \frac{\omega_m \omega'_m}{\left(\omega_m^2 + \Delta_n^2(\omega_m)\right)^{3/2}(\omega_{m'}^2 + \Delta_n^2(\omega_{m'}))^{3/2}} \\ &\times [(\delta\Delta_m\omega_{m'} - \delta\Delta_{m'}\omega_m)^2 + (\delta\Delta_m\Delta_n(\omega_{m'}) - \delta\Delta_{m'}\Delta_n(\omega_m))^2] \end{aligned} \tag{A30}$$

and

$$F^{(2,b)}_{\text{sc},n} = N_F \left(\frac{\omega_D}{\bar{g}}\right)^{\gamma} \pi T \sum_m \frac{(\delta\Delta_m)^2 \omega_m^2}{\left(\omega_m^2 + \Delta_n^2(\omega_m)\right)} \left(Z_n(\omega_m) - \frac{\pi T}{\sqrt{\omega_m^2 + \Delta_n^2(\omega_m)}}\left(\frac{\bar{g}}{\omega_D}\right)^{\gamma}\right)^2. \tag{A31}$$

Using Eq. (4), we find that the last bracket is the effective $Z_n(\omega_m)$, without the $m' = m$ term in the r.h.s. of (4). This expression is nonsingular at $\omega_D \to 0$. Then $F^{(2,b)}_{\text{sc},n}$ vanishes at vanishing ω_D because of the overall factor. Hence, in this limit, $F^{(2)}_{\text{sc},n} = F^{(2,a)}_{\text{sc},n}$.

The expression for $F^{(2)}_{\text{sc},n}$ can be further simplified by using the Eliashberg equation for $\Delta_n(\omega_m)$ and re-expressing the first term in (A30) as the double sum over m and m'. The result looks more compact when expressed in terms of $D_n(\omega_m) = \Delta_n(\omega_m)/\omega_m$

and $\delta D_m = \delta\Delta_m/\omega_m$. After simple algebra, we obtain

$$F^{(2)}_{\text{sc},n} = N_F \pi^2 T^2 \bar{g}^\gamma \sum_{m,m'\geqslant 0} \frac{1}{D_n(\omega_m) D_n(\omega_{m'})} \frac{1}{\left(1+D_n^2(\omega_m)\right)^{3/2}\left(1+D_n^2(\omega_{m'})\right)^{3/2}} Q_{m,m'}, \tag{A32}$$

where

$$\begin{aligned} Q_{m,m'} &= \bar{g}^\gamma(\delta D_m D_n(\omega_m) - \delta D_{m'} D_n(\omega_{m'})^2)\left(\frac{1}{|\omega_m-\omega_{m'}|^\gamma} + \frac{1}{|\omega_m+\omega_{m'}|^\gamma}\right) \\ &\quad + \frac{\left(\delta D_m D_n^2(\omega_m) - \delta D_{m'} D_n^2(\omega_{m'})\right)^2}{|\omega_m-\omega_{m'}|^\gamma} + \frac{\left(\delta D_m D_n^2(\omega_m) + \delta D_{m'} D_n^2(\omega_{m'})\right)^2}{|\omega_m+\omega_{m'}|^\gamma}. \end{aligned} \tag{A33}$$

Note that the term with $m = m'$ vanishes because of vanishing numerator (to see this clearly one should keep infinitesimally small ω_D in the denominator).

We now perform the same calculation staring from the variational free energy $F^{\text{var}}_{\text{sc}}$, Eq. (16). This free energy depends only on $\Delta(\omega_m)$. Expanding Eq. (16) to second order in $\delta\Delta_m = \Delta(\omega_m) - \Delta_n(\omega_m)$ and using the Eliashberg equation for stationary $\Delta_n(\omega_m)$, we obtain after long but straightforward algebra that $F^{\text{var}}_{\text{sc},n} = F^{\text{var},(0)}_{\text{sc},n} + F^{\text{var},(2)}_{\text{sc},n}$, where $F^{\text{var},(0)}_{\text{sc},n}$ and $F^{\text{var},(2)}_{\text{sc},n}$ are given by the same Eqs. (A22) and (A32) as $F^{(0)}_{\text{sc},n}$ and $F^{(2)}_{\text{sc},n}$, respectively. This implies that at vanishing ω_D, the expansion of the actual free energy around a stationary point coincides with the expansion of the variational free energy. For $\gamma = 2$, this has been established in Ref. [50].

More accurately, at a nonzero T, $F^{\text{var},(0)}_{\text{sc},n}$ coincides with $F^{(0)}_{\text{sc},n}$ for arbitrary ω_D, but $F^{\text{var},(2)}_{\text{sc},n} = F^{(2)}_{\text{sc},n}$ only up to corrections of order $(\omega_D/\bar{g})^\gamma$. This holds for all n, including $n = 0$. At $T = 0$, $F^{\text{var},(0)}_{\text{sc},n}$ and $F^{(0)}_{\text{sc},n}$ again coincide, and $F^{\text{var},(2)}_{\text{sc},n}$ and $F^{(2)}_{\text{sc},n}$ differ by terms of order $(\omega_D/\bar{g})^{\gamma-1}$. This difference is small for $\gamma > 1$, when $Z_n(\omega_m)$ diverges at vanishing ω_D. For these γ, $F^{(2)}_{\text{sc},n}$ is given by Eq. (A32) with $\pi^2T^2\sum_{m,m'\geqslant 0}$ replaced by $(1/4)\int_0^\infty\int_0^\infty d\omega_m d\omega_{m'}$. For $\gamma < 1$, $Z_n(\omega_m)$ remains finite at $\omega_D = 0$, and $F^{\text{var},(2)}_{\text{sc},n}$ and $F^{(2)}_{\text{sc},n}$ do not coincide, even at $\omega_D = 0$. Specifically, $F^{\text{var},(2)}_{\text{sc},n}$ is still given by Eq. (A32) with the sum replaced by the integral, but $F^{(2)}_{\text{sc},n}$ has an additional contribution from fluctuations of $\bar{\delta} Z_m$.

3. Beyond second order in $\delta\Delta_m$

We now go further and verify that at $T > 0$, $F^{\text{var}}_{\text{sc}}$ and F_{sc} remain equivalent to all orders in $\delta\Delta_m$, up to terms $O(\omega_D/\bar{g})^\gamma$. To see this, we use as inputs the facts that at vanishing ω_D and a finite T (i) $V^{-1}(\Omega_m)$ is represented by series of $(\omega_D/\bar{g})^\gamma$ as in Eq. (A6), (ii) Δ_m remains a regular function of ω_m, and (iii) Z_m contains a divergent thermal piece. We single out the divergent term and redefine Z_m as

$$Z_m = \frac{\pi T}{\sqrt{\omega_m^2+\Delta_m^2}}\left(\frac{\bar{g}}{\omega_D}\right)^\gamma + \bar{Z}_m. \tag{A34}$$

We then expand F_{sc} in series of $(\omega_D/\bar{g})^\gamma$. After straightforward algebra, done without using the Eliashberg gap equations, we obtain:

$$F_{\text{sc}} = F^{\text{var}}_{\text{sc}} + N_F\left(\frac{\omega_D}{\bar{g}}\right)^\gamma \sum_m \left[\bar{Z}_m^2(\omega_m^2+\Delta_m^2) - 2u_m\bar{Z}_m\omega_m^2 + v_m\omega_m^2\right] + O\left(\frac{\omega_D}{\bar{g}}\right)^{2\gamma}, \tag{A35}$$

where

$$u_m = 1 + 2\pi T\sum_{m\neq m'} V(m-m')\frac{\omega'_m/\omega_m + \Delta_{m'}\Delta_m/\omega_m^2}{\sqrt{\omega_{m'}^2+\Delta_{m'}^2}}, \quad v_m = 1 + 2\pi T\sum_{m\neq m'} V(m-m')\frac{\omega'_m/\omega_m}{\sqrt{\omega_{m'}^2+\Delta_{m'}^2}}. \tag{A36}$$

One can check that the prefactor for $(\omega_D/\bar{g})^\gamma$ is nonsingular. Eq (A35) then shows that F_{sc} and $F^{\text{var}}_{\text{sc}}$ differ by terms $O(\omega_D/\bar{g})^\gamma$. This difference vanishes in the limit $\omega_D \to 0$.

APPENDIX B: EXPANSION OF FREE ENERGY AROUND STATIONARY $F_{\text{sc},n}$ FOR THE SUBCLASS OF ANALYTIC GAP FUNCTIONS

In this Appendix, we discuss the details of our calculation of the matrix of quadratic deviations from $\Delta_n(\omega_m)$, assuming that the deviations $\delta\Delta(\omega_m)$ are analytic functions of complex $z = \omega' + i\omega''$, when extended into the upper half-plane of frequency.

An analytic $\delta\Delta(\omega_m)$ can by expressed via the Cauchy relation in terms of $\psi(\omega) = \text{Im}\,\delta\Delta(\omega)$ along the real axis. Substituting this relation into the expression for the free energy, we obtain $F^{(2)}_{\text{sc},n}$ in the form

$$F^{(2)}_{\text{sc},n} = N_F\int_0^\infty\int_0^\infty d\omega d\omega' \psi_n(\omega)\psi_n(\omega')\bar{\mathcal{K}}_{\omega,\omega'}, \tag{B1}$$

where

$$\bar{\mathcal{K}}_{\omega,\omega'} = \sum_p \lambda_n^p S_n^p(\omega) S_n^p(\omega') \tag{B2}$$

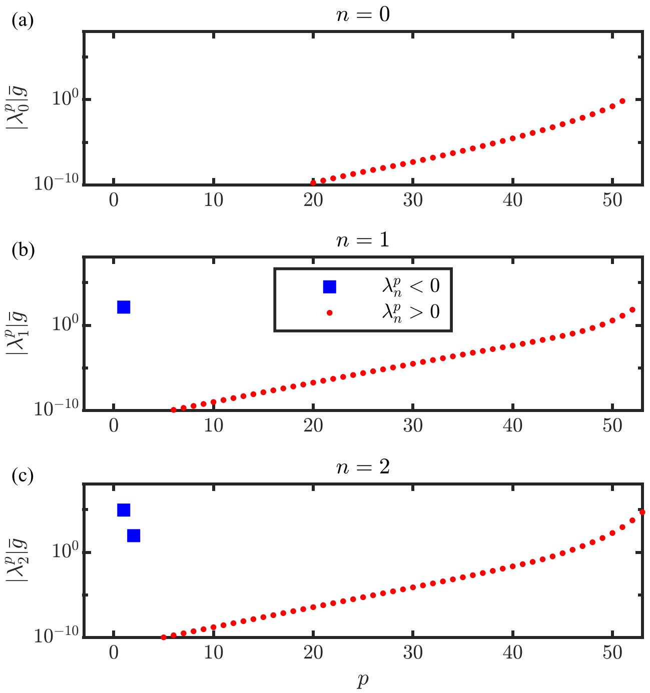

FIG. 10. Eigenvalues of the kernel matrix of $F^{(2)}_{\mathrm{sc},n}$ in a restricted subspace analytic in the upper half-plane of frequency, where (a) $n = 0$, (b) $n = 1$, and (c) $n = 2$ solutions. Here we take $\gamma = 0.8$.

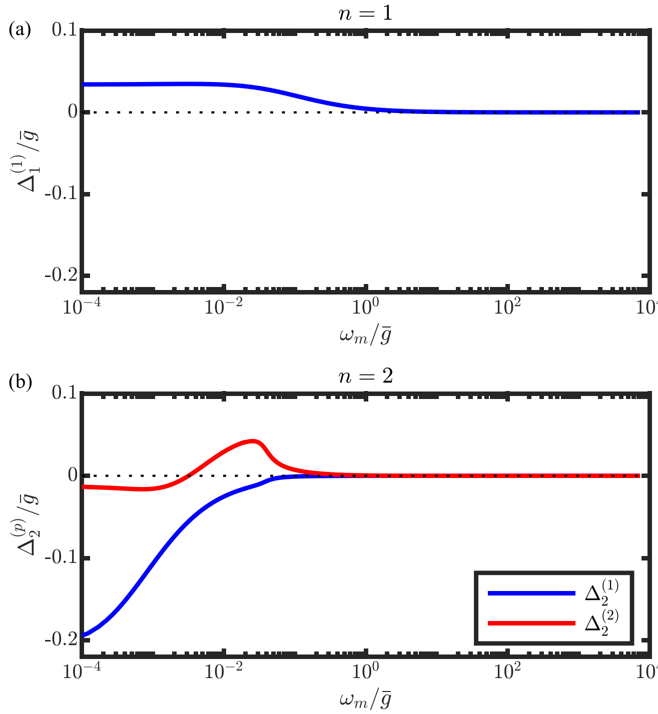

FIG. 11. Eigenfunctions corresponding to the negative eigenvalues shown in Fig. 10 (b) $n = 1$ and (c) $n = 2$, which are analytic in the upper half-plane of frequency. Here we take $\gamma = 0.8$.

and

$$S_n^p(\omega) = 2T \sum_m \frac{\Delta_n^p(\omega_m)\omega}{\omega^2 + \omega_m^2}. \tag{B3}$$

The functions $S_n^p(\omega)$ are not orthogonal and their set is overcomplete, as one can explicitly verify. We now introduce a complete set of orthogonal functions $\bar{S}_n^k(x)$, which satisfy

$$\int_0^\infty dx \bar{S}_n^k(x)\bar{S}_n^{k'}(x) = \delta_{k,k'}. \tag{B4}$$

Expressing $Q_{\omega,\omega'}$ and $S_n^p(x)$ in terms of these functions, we obtain

$$Q_{\omega,\omega'} = \sum_p \bar{\lambda}_n^p \bar{S}_n^p(\omega)\bar{S}_n^p(\omega'),$$
$$S_n^p(x) = \sum_k g_{p,k}\bar{S}_n^k(x). \tag{B5}$$

A simple manipulation then shows that

$$\bar{\lambda}_n^p = \sum_k \lambda_n^k g_{p,k}^2. \tag{B6}$$

In Fig. 10, we compare the eigenvalues $\bar{\lambda}_n^p$ ($n = 0, 1, 2$) with λ_n^p, which we obtained by diagonalizing $F^{(2)}_{\mathrm{sc},n}$ from Eq. (52) from the main text without imposing the condition of analyticity on $\delta\Delta(\omega_m)$. We see that the two sets are nonequivalent, yet in both cases there are n negative eigenvalues for the expansion around $\Delta_n(\omega_m)$. The eigenfunctions, which correspond to negative $\bar{\lambda}_n^p$, define the unstable directions in the restricted subspace towards $\Delta_{n'}(\omega_m)$ with $n' < n$, as shown in Fig. 11. This implies that all physically relevant perturbations are described by analytic $\delta\Delta(\omega_m)$. This result is also consistent with the analysis in the main text of the free energy for hybrid trial functions, made of combinations of stationary $\Delta_n(\omega_m)$. All stationary gap functions are analytic in the upper half-plane of frequency, as we explicitly verified, hence the hybrid trial functions are also analytic.

APPENDIX C: EVOLUTION OF TRIAL $\Delta_n^{\mathrm{trial}}(\omega_m)$ UNDER ITERATIONS

Our trial solutions "almost" satisfy the gap equation, but are not the exact $\Delta_n(\omega_m)$ and hence evolve under iterations. The analysis in the main text and in the previous Appendix shows that under iterations all trial $\Delta_n^{\mathrm{trial}}(\omega)$ should slowly move towards the actual $\Delta_0(\omega_m)$. To verify this, we re-express the gap equation as

$$\Delta(\omega_m) = \frac{\pi T \bar{g}^\gamma \sum_{m'\neq m} \frac{1}{|\omega_m - \omega'_m|^\gamma} \frac{\Delta(\omega'_m)}{\sqrt{(\omega'_m)^2 + \Delta^2(\omega'_m)}}}{1 + \pi T \frac{\bar{g}^\gamma}{2} \frac{\Delta(\omega_m)}{\omega_m} \sum_{m'\neq m} \frac{1}{|\omega_m - \omega'_m|^\gamma} \frac{\omega'_m}{\sqrt{(\omega'_m)^2 + \Delta^2(\omega'_m)}}} \tag{C1}$$

and initiate the iteration process by substituting the trial $\Delta_n^{\mathrm{trial}}(\omega_m)$ into the r.h.s.

For $n = 0$, the iteration procedure converges to the exact $\Delta_0(\omega_m)$, which we already used in the calculation of eigenfunctions for the free energy variations. For $n = 1$, the trial $\Delta_1^{\mathrm{trial}}(\omega_m)$ remains almost unchanged under first few iterations, but eventually gets modified and moves towards $\Delta_0(\omega_m)$ (see Fig. 6). The same happens with the trial $\Delta_2^{\mathrm{trial}}(\omega_m)$.

To quantify this analysis, we linearized the gap equation (C1) in deviations from the trial gap functions and found

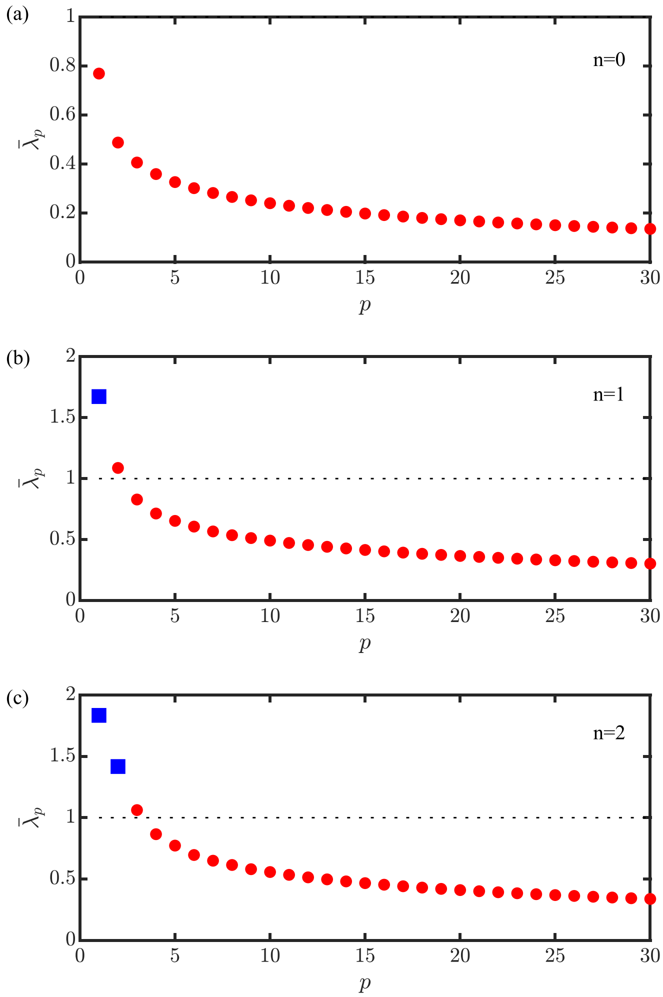

FIG. 12. Eigenvalues $\bar{\lambda}_p$ of the corresponding kernel of the iteration procedure following Eq. (C1) for the $n = 0$, $n = 1$, and $n = 2$ gap functions, respectively, where $\gamma = 0.8$. Solid symbols mark the eigenstates corresponding to unstable "directions" of the iteration matrix. The eigenvalue $\bar{\lambda}_p$ slightly above 1 for both $n = 1, 2$ solutions corresponds to stable "direction" of the iteration matrix, whose value becomes larger than one because the trivial gap function is close to not exactly located at the saddle point of the free-energy functional.

the eigenvalues of the corresponding kernel, $\bar{\lambda}_n^{(p)}$. One can easily verify that the trial function is stable (i.e., the iteration procedure converges) if $|\bar{\lambda}_n^{(p)}| < 1$ for all p and unstable if $|\bar{\lambda}_n^{(p)}| > 1$ for at least for one value of p. We show the result in Fig. 12. We see that for $n = 0$ all eigenvalues are below 1, while for $n = 1$ and $n = 2$, there are one and two eigenvalues larger than 1, respectively.

APPENDIX D: EXPANSION IN TERMS OF THE GAP AMPLITUDE

In this Appendix, we describe in some detail amplitude expansion for the function $D(\omega_m) = \Delta(\omega_m)/\omega_m$. Expanding the r.h.s. of Eq. (43) in powers of D, we obtain

$$\omega_m D(\omega_m) = \frac{\bar{g}^\gamma}{2}\int_{-\infty}^{\infty} d\omega'_m (D(\omega'_m) - D(\omega_m))\frac{\text{sgn}(\omega'_m)}{|\omega'_m - \omega_m|^\gamma} \times \left(1 - \frac{1}{2}D^2(\omega'_m) + \frac{3}{8}D^4(\omega'_m) + \cdots\right). \quad \text{(D1)}$$

We will be searching for the solution in the form

$$D(\omega_m) = \sum_j D^{(2j+1)}(\omega_m), \quad \text{(D2)}$$

where $j = 0, 1, 2, \ldots$ The first term $D^{(1)}(\omega_m)$ is the solution of the linearized gap equation [the one which we labeled as $D_\infty(\omega_m)$ in the main text]. We assign an overall factor ϵ to $D^{(1)}(\omega_m)$ and set $D^{(2j+1)}(\omega_m)$ to be proportional to $\epsilon^{(2j+1)}$. The expansion in (D2) then holds in powers of ϵ^2. At each order of the expansion in ϵ we obtain the equation on $D^{(2j+1)}(\omega_m)$ with the source term expressed in terms of $D^{(2j+1)}(\omega_m)$ with smaller j. Specifically,

$$\omega_m D^{(2j+1)}(\omega_m) - \frac{\bar{g}^\gamma}{2}\int_{-\infty}^{\infty} d\omega'_m \big(D^{(2j+1)}(\omega'_m) - D^{(2j+1)}(\omega_m)\big) \frac{\text{sgn}(\omega'_m)}{|\omega'_m - \omega_m|^\gamma} = K^{(2j+1)}(\omega_m), \quad \text{(D3)}$$

where

$$K^{(0)}(\omega_m) = 0, \quad \text{(D4)}$$

$$K^{(3)}(\omega_m) = -\frac{\bar{g}^\gamma}{4}\int_{-\infty}^{\infty} \frac{d\omega'_m \text{sgn}(\omega'_m)}{|\omega'_m - \omega_m|^\gamma} \times \big(D^{(1)}(\omega'_m) - D^{(1)}(\omega_m)\big)(D^{(1)})^2(\omega'_m), \quad \text{(D5)}$$

$$K^{(5)}(\omega_m) = \frac{\bar{g}^\gamma}{2}\int_{-\infty}^{\infty} \frac{d\omega'_m \text{sgn}(\omega'_m)}{|\omega'_m - \omega_m|^\gamma} \times \Bigg[-\frac{1}{2}\big(D^{(3)}(\omega'_m) - D^{(3)}(\omega_m)\big)(D^{(1)})^2(\omega'_m) + \big(D^{(1)}(\omega'_m) - D^{(1)}(\omega_m)\big) \times \left(\frac{3}{8}(D^{(1)})^4(\omega'_m) - D^{(1)}(\omega'_m)D^{(3)}(\omega'_m)\right)\Bigg], \quad \text{(D6)}$$

and so on.

Below we focus on the solution for $D^{(2j+1)}(\omega_m)$ at small $\omega \ll \bar{g}$. At these frequencies,

$$D^{(1)}(\omega_m) = 2\epsilon^* \cos(\psi_0(\omega_m)), \quad \text{(D7)}$$

where $\epsilon^* = \epsilon|\omega_m|^{\gamma/2-1}$ and

$$\psi_0(\omega_m) = \beta \ln|\omega_m/\bar{g}|^\gamma + \phi, \quad \text{(D8)}$$

where $\beta = \beta_\gamma$ and $\phi = \phi(\gamma)$ are γ-dependent numbers (see the discussion around Eq. (42) in the main text and Refs. [1,4–6]). Substituting this $D^{(1)}(\omega_m)$ into (D5) we obtain the source term for $D^{(3)}(\omega_m)$ in the form

$$K^{(3)}(\omega) = (\epsilon^*)^3\bar{g}^\gamma|\omega_m|^{1-\gamma}(\epsilon^*)^3\big(e^{i\psi_\gamma(\omega)}I_1^{(3)} + e^{3i\psi_0(\omega)}I_3^{(3)}\big) + \text{c.c.}, \quad \text{(D9)}$$

where

$$I_1^{(3)} \equiv -\tfrac{1}{4}I(3\alpha + i\beta\gamma, 2\alpha + 2i\beta\gamma) - \tfrac{1}{2}I(3\alpha + i\beta\gamma, 2\alpha), \tag{D10}$$

$$I_3^{(3)} \equiv -\tfrac{1}{4}I(3\alpha + 3i\beta\gamma, 2\alpha + 2i\beta\gamma), \tag{D11}$$

$\alpha \equiv \gamma/2 - 1$, and

$$I(a, b) \equiv \int_{-\infty}^{\infty} \frac{dx}{|x-1|^\gamma}\big(|x|^a - \text{sgn}(x)|x|^b\big). \tag{D12}$$

This integral is infrared convergent for $\gamma < 3$. It contains the "high-energy" part, which is determined by internal frequencies of order $\bar{g}$, and the universal part, which comes from $x = O(1)$ and determines the behavior of $D^{(3)}(\omega_m)$ at $\omega_m \ll \bar{g}$. The universal part of $I(a, b)$ is

$$\begin{aligned} I(a, b) = {} & B(\gamma - 1 - a, 1 + a) + B(\gamma - 1 - b, 1 + b) + B(1 + a, 1 - \gamma) - B(1 + b, 1 - \gamma) \\ & + B(\gamma - 1 - a, 1 - \gamma) - B(\gamma - 1 - b, 1 - \gamma). \end{aligned} \tag{D13}$$

Substituting this into the expression for $K^{(3)}(\omega_m)$ and solving Eq. (D3) for $D^{(3)}(\omega_m)$, we obtain

$$D^{(3)}(\omega_m) = \text{sgn}(\omega_m)(\epsilon^*)^3\big(Q_1^{(3)}e^{i\psi_0(\omega_m)} + Q_3^{(3)}e^{3i\psi_0(\omega_m)}\big) + \text{c.c.}, \tag{D14}$$

where

$$Q_{2r+1}^{(3)} = -2\frac{I_{2r+1}^{(3)}}{I(3\alpha + i(2r+1)\beta\gamma, 0)}, \quad r = 0, 1. \tag{D15}$$

Adding $D^{(1)}(\omega_m)$ and $D^{(3)}(\omega_m)$, we see that $D^{(3)}(\omega_m)$ affects the prefactor for $\cos\psi_0(\omega_m)$ and also generates the higher harmonic $\cos 3\psi_0(\omega_m)$.

Further, we use the expressions for $D^{(1)}(\omega_m)$ and $D^{(3)}(\omega_m)$ and compute the source term for $D^{(5)}(\omega_m)$:

$$K^{(5)}(\omega_m) = \bar{g}^\gamma|\omega_m|^{\frac{3}{2}\gamma - 4}(\epsilon^*)^5\big(e^{i\psi_0(\omega_m)}I_1^{(5)} + e^{3i\psi_0(\omega_m)}I_3^{(5)} + e^{5i\psi_0(\omega_m)}I_5^{(5)}\big) + \text{c.c.}, \tag{D16}$$

where

$$\begin{aligned} I_1^{(5)} = {} & \tfrac{3}{16}[6I(5\alpha + i\beta\gamma, 4\alpha) + 4I(5\alpha + i\beta\gamma, 4\alpha + 2i\beta\gamma)] \\ & - \tfrac{1}{4}\big[2Q_1^{(3)}I(5\alpha + i\beta\gamma, 2\alpha) + Q_1^{(3)*}I(5\alpha + i\beta\gamma, 2\alpha + 2i\beta\gamma) + Q_3^{(3)}I(5\alpha + i\beta\gamma, 2\alpha - 2i\beta\gamma)\big] \\ & - \tfrac{1}{2}\big[Q_1^{(3)}I(5\alpha + i\beta\gamma, 4\alpha) + Q_1^{(3)*}I(5\alpha + i\beta\gamma, 4\alpha) + \big(Q_1^{(3)} + Q_3^{(3)*}\big)I(5\alpha + i\beta\gamma, 2\alpha + 2i\beta\gamma)\big], \end{aligned} \tag{D17}$$

$$\begin{aligned} I_3^{(5)} = {} & \tfrac{3}{16}[4I(5\alpha + 3i\beta\gamma, 4\alpha + 2i\beta\gamma) + I(5\alpha + 3i\beta\gamma, 4\alpha + 4i\beta\gamma)] \\ & - \tfrac{1}{4}\big[2Q_3^{(3)}I(5\alpha + 3i\beta\gamma, 2\alpha) + Q_1^{(3)}I(5\alpha + 3i\beta\gamma, 2\alpha + 2i\beta\gamma)\big] \\ & - \tfrac{1}{2}\big[\big(Q_1^{(3)} + Q_3^{(3)}\big)I(5\alpha + 3i\beta\gamma, 4\alpha + 2i\beta\gamma) + Q_3^{(3)*}I(5\alpha + 3i\beta\gamma, 4\alpha + 4i\beta\gamma)\big], \end{aligned} \tag{D18}$$

$$I_5^{(5)} = \tfrac{3}{16}I(5\alpha + 5i\beta\gamma, 4\alpha + 4i\beta\gamma) - \tfrac{1}{4}Q_3^{(3)}I(5\alpha + 5i\beta\gamma, 2\alpha + 2i\beta\gamma) - \tfrac{1}{2}Q_3^{(3)}I(5\alpha + 5i\beta\gamma, 4\alpha + 4i\beta\gamma). \tag{D19}$$

The induced solution is

$$D^{(5)}(\omega) = \text{sgn}(\omega_m)(\epsilon^*)^5\big(Q_1^{(5)}e^{i\psi_0(\omega_m)} + Q_3^{(5)}e^{3i\psi_0(\omega_m)} + Q_5^{(5)}e^{5i\psi_0(\omega_m)}\big) + \text{c.c.}, \tag{D20}$$

where

$$Q_{2r+1}^{(5)} = -2\frac{I_{2r+1}^{(5)}}{I(5\alpha + i(2r+1)\beta\gamma, 0)}. \tag{D21}$$

This expansion can be straightforwardly extended to higher orders. The generic form of the gap function is

$$\begin{aligned} D(\omega_m) = {} & \text{sgn}(\omega_m)\epsilon^*\big[\big(1 + (\epsilon^*)^2Q_1^{(3)} + (\epsilon^*)^4Q_1^{(5)} + \cdots\big)e^{i\psi_0(\omega_m)} + (\epsilon^*)^2\big(Q_3^{(3)} + (\epsilon^*)^2Q_3^{(5)} + \cdots\big)e^{3i\psi_0(\omega_m)} \\ & + (\epsilon^*)^4\big(Q_5^{(5)} + \ldots\big)e^{3i\psi_0(\omega_m)} + \cdots\big] + \text{c.c.} \end{aligned} \tag{D22}$$

The expression for $D(\omega_m)$ in the main text is obtained by further expressing $\sum_{j=k}^\infty Q_{2k+1}^{(2j+1)}(\epsilon^*)^{2j+1}$ in each line in (D22) as

$$\sum_{j=k}^\infty Q_{2k+1}^{(2j+1)}(\epsilon^*)^{2j+1} = (\epsilon^*)^{2k+1}Q_{2k+1}e^{i(\phi_{2k+1} + (2k+1)\psi)}, \tag{D23}$$

where Q_{2k+1}, ϕ_{2k+1}, and ψ are given by series in ϵ^*. These series are presented in Eqs. (44)–(47) in the main text.

APPENDIX E: CONDENSATION ENERGY NEAR $\gamma = 2$

The condensation energy for a stationary $\Delta_n(\omega_m)$ is given by Eq. (19). We reproduce this formula here for convenience of a reader. At $T = 0$, which we consider here, we have

$$E_{c,n} = N_F \int d\omega_m \left(|\omega_m| - \frac{\omega_m^2}{\sqrt{\omega_m^2 + \Delta_n^2(\omega_m)}} \right) - \frac{1}{4} N_F \iint d\omega_m d\omega'_m V(\omega_m - \omega'_m) \times \left(\frac{\omega_m \omega'_m + \Delta_n(\omega_m)\Delta_n(\omega'_m)}{\sqrt{\omega_m^2 + \Delta_n^2(\omega_m)}\sqrt{(\omega'_m)^2 + \Delta_n^2(\omega'_m)}} - \text{sgn}(\omega_m \omega'_m) \right). \tag{E1}$$

The first term in (E1) is infrared convergent and is determined by fermions with $\omega_m \sim \bar{g}$. This term is not singular at $\gamma = 2$ and below we just skip it. We argue in this Appendix that the second term in (E1) becomes singular in the limit $\gamma \to 2$ To see this we first consider $E_{c,0}$, which is the largest by magnitude.

1. Condensation energy for $\Delta_0(\omega_m)$

The gap function $\Delta_0(\omega_m)$ is sign-preserving and tends to $\Delta_0(0) \sim \bar{g}$ at $\omega_m \ll \bar{g}$. Using this, we express the condensation energy as

$$E_{c,0} \approx -\frac{1}{4} \bar{g}^\gamma N_F \left[\int_{-O(\bar{g})}^{O(\bar{g})} \frac{d\omega_m d\omega'_m}{|\omega_m - \omega'_m|^\gamma} (1 - \text{sgn}(\omega_m)\text{sgn}(\omega'_m)) + \text{regular terms} \right]. \tag{E2}$$

This integral can be easily evaluated and for $\gamma \lesssim 2$ yields

$$E_{c,0} \simeq -\bar{g}^2 N_F \left[\frac{1}{2-\gamma} + \text{regular terms} \right]. \tag{E3}$$

At a finite ω_D, the demominator in (E2) is replaced by $((\omega_m - \omega'_m)^2 + \omega_D^2)^{\gamma/2}$. The integral then remains finite even at $\gamma = 2$, and $E_{c,0}$ becomes

$$E_{c,0} \simeq -\bar{g}^2 N_F \left[\ln \frac{\bar{g}}{\omega_D} + \text{regular terms} \right]. \tag{E4}$$

The same happens when $\omega_D = 0$, but temperture T is finite. At $\gamma = 2$, we have

$$E_{c,0} \simeq -\bar{g}^2 N_F \left[\ln \frac{\bar{g}}{T} + \text{regular terms} \right], \tag{E5}$$

2. Condensation energy for $\Delta_n(\omega_m)$ with $n > 0$

For any finite n, the gap function $\Delta_n(\omega_m)$ still saturates at a finite value $\Delta_n(0)$ at $\omega_m = 0$. At low enough frequencies, $\Delta_n(0) \gg \omega_m$, and the integrand of the condensation energy $E_{c,n}$ has the same as for $n = 0$. Then $E_{c,n}$ contains the same divergent piece as in (E3).

We will also need the expression for the difference $E_{c,n} - E_{c,0}$. We show below that it primarily comes from the regions near the nodal points in $\Delta_n(\omega_m)$, where $|\Delta_n(\omega_m)| \leqslant |\omega_m|$. To demonstrate this, we divide the frequency range into two subranges – Ω_N around the nodal points and Ω_A away from nodal points. We use the convention

$$\Omega_N : \cup_p \{\omega^{(p)} - \delta\omega^{(p)} < |\omega_m| < \omega^{(p)} + \delta\omega^{(p)}\}, \tag{E6}$$

where $p = 1, \dots, n$ and $|\Delta_n(\omega^{(p)} \pm \delta\omega^{(p)})| = |\omega^{(p)} \pm \delta^{(p)}|$. Then $|\Delta_n(\omega_m)| < |\omega_m|$ for $\omega_m \in \Omega_N$ and $|\Delta_n(\omega_m)| > |\omega_m|$ for $\omega_m \in \Omega_A$. Because the positions of nodal points of $\Delta_n(\omega_m)$ depend on the logarithm of frequency, it is convenient to introduce the logarithmic variable $x = \ln(|\omega_m|/\bar{g})^\gamma$. The neighborhood of the nodal point ω_p transforms to $x_p - \delta_p < x < x_p + \delta_p$, where $x_p \simeq -\pi p/\beta_\epsilon$. We argued in the main text that for finite n and $\gamma \to 2$, $\epsilon_n \approx 1/(2-\gamma)^a$ $(a > 0)$, $\beta_\epsilon \sim \epsilon^{-1/3} \propto (2-\gamma)^{(a/3)}$, and $\Delta_n(\omega^{(p)} \pm \delta\omega_m)/|\omega_p| \simeq (-1)^p (x - x_p)$. The condition $|\Delta_n(\omega_m)| \sim |\omega_m|$ corresponds to $|x - x_p| \sim 1$.

Below we consider contributions to $\delta E_{c,n}$ from different combinations of ω_m, ω'_m, each within either Ω_N or Ω_A.

(1) $\omega_m, \omega'_m \in \Omega_N$.

In this frequency domain, $|\Delta(\omega_m)| < |\omega_m|$ and $|\Delta(\omega'_m)| < |\omega'_m|$. The the corresponding contribution to $\delta E_{c,n}$ is

$$\delta E_{c,n}^{(1)} \simeq -\frac{1}{4} \bar{g}^\gamma N_F \iint_{\omega_m, \omega'_m \in \Omega_N} \frac{d\omega_m d\omega'_m}{|\omega_m - \omega'_m|^\gamma} (\text{sgn}(\omega_m)\text{sgn}(\omega'_m) - 1), \tag{E7}$$

where the superscript (1) refers to the integration domain. Converting the integral in (E7) to positive ω_m and negative ω'_m or vice versa and changing the frequencies to $x = x_p + y$ and $x' = x_{p'} + y'$, we obtain to leading order in $2 - \gamma$

$$\delta E_{c,n}^{(1)} \simeq \frac{1}{4} \bar{g}^\gamma N_F \sum_{p,p'=1}^{n} \iint_0^{O(1)} dy dy' \left[2 \cosh \frac{x_p - x_{p'} + y - y'}{4} \right]^{-2} e^{\frac{2-\gamma}{4}(x_p + x_{p'})}. \tag{E8}$$

The integrand decays exponentially as a function of $|p-p'|$, hence the dominant contribution to the integral comes from $p=p'$. Keeping only this term, we obtain

$$\delta E_{c,n}^{(1)} \simeq \frac{1}{4}\bar{g}^{\gamma}N_F\iint_0^{O(1)} dydy'\left[2\cosh\frac{y-y'}{4}\right]^{-2}\sum_{p=1}^{n} e^{\frac{2-\gamma}{2}x_p} = \bar{g}^2 N_F\frac{c_1}{2-\gamma}J_n, \tag{E9}$$

where $J_n=\frac{2\beta_{\epsilon_n}}{\pi}(1-e^{-\frac{\pi n(2-\gamma)}{2\beta_{\epsilon_n}}})$ and $c_1=O(1)$. At $\gamma\to 2$ and finite n, the exponential factor in J_n can be expanded in n, yielding $\delta E_{c,n}^{(1)}=\bar{g}^2N_F c_1 n$.

(2) $\omega_m,\omega'_m\in\Omega_A$.

In this frequency range, $|\Delta(\omega_m)|>|\omega_m|$ and $|\Delta(\omega'_m)|>|\omega'_m|$. The contribution to the condensation energy is

$$\delta E_{c,n}^{(2)} = -\frac{1}{4}\bar{g}^{\gamma}N_F\iint_{\omega_m,\omega'_m\in\Omega_A}\frac{d\omega_m d\omega'_m}{|\omega_m-\omega'_m|^{\gamma}}(\text{sgn}(\Delta_n(\omega_m))\text{sgn}(\Delta_n(\omega'_m))-1). \tag{E10}$$

We further divide Ω_A into sub-intervals Ω_A^p between two adjacent nodal points ω_p and ω_{p+1}, where $p=1,2,\ldots,n$ and $\omega_{n+1}\equiv 0$. The sign structure of $\Delta_n(\omega_m)$ i such that ω_m and ω'_m must come from intervals Ω_A^p and $\Omega_A^{p'}$ with odd $p+p'$. Introducing again logarithmic variables $x=x_p-y$ and $x'=x_{p'+1}+y'$, we obtain after simple algebra

$$\delta E_{c,n}^{(2)} \simeq \frac{\bar{g}^2N_F}{4}\sum_{p+p'\in\text{odd}}\int_{O(1)}^{O(\pi/\beta_\epsilon)}dy\int_{O(1)}^{O(\pi/\beta_\epsilon)}dy'\left[\frac{1}{\left|2\sinh\frac{x_p-x_{p'+1}-y-y'}{2}\right|^2}+\frac{1}{\left(2\cosh\frac{x_p-x_{p'+1}-y+y'}{2}\right)^2}\right]e^{(x_p+x_{p'+1})(\frac{2-\gamma}{4})}. \tag{E11}$$

Given that $x_p-x_{p'+1}\propto(p'-p+1)/\beta_\epsilon$ and $\beta_\epsilon\ll 1$, this integral is exponentially small unless $p'+1=p$. Keeping only such term, we obtain

$$\delta E_{c,n}^{(2)} \simeq \frac{\bar{g}^{\gamma}N_F}{4}\int_{O(1)}^{O(\pi/\beta_\epsilon)}dy\int_{O(1)}^{O(\pi/\beta_\epsilon)}dy'\left[\frac{1}{\left|2\sinh\frac{y+y'}{2}\right|^2}+\frac{1}{\left(2\cosh\frac{y+y'}{2}\right)^2}\right]\sum_{p=1}^{n}e^{\frac{2-\gamma}{2}x_p} = \bar{g}^2N_F\frac{c_2}{2-\gamma}J_n, \tag{E12}$$

where $c_2=O(1)$. The integral over y,y' is confined to $y,y'\sim 1$, hence it is not surprising that $\delta E_{c,n}^{(2)}$ is of the sam order as $\delta E_{c,n}^{(1)}$.

(3) $\omega_m\in\Omega_N$, $\omega'_m\in\Omega_A$.

On general grounds we expect that the contribution from (3) is the same as from (1) and (2), but it is instructive to explicitly verify this. We compute the contribution from $|\Delta(\omega_m)|<|\omega_m|$ and $|\Delta(\omega'_m)|>|\omega'_m|$ and multiply the result by 2. The contribution to the condensation energy is

$$\delta E_{c,n}^{(3)} \simeq \frac{1}{2}\bar{g}^{\gamma}N_F\iint_{\omega_m\in\Omega_N,\omega'_m\in\Omega_A}\frac{d\omega_m d\omega'_m}{|\omega_m-\omega'_m|^{\gamma}}. \tag{E13}$$

In terms of logarithmic variables, it becomes

$$\delta E_{c,n}^{(3)} \simeq \frac{1}{4}\bar{g}^{\gamma}N_F\sum_{p=1}^{n}\int_{x_p-\delta_p}^{x_p+\delta_p}dx\sum_{p'=1}^{n}\int_{x_{p'+1}+\delta_{p'+1}}^{x_{p'}-\delta_{p'}}dx'\left(\frac{1}{\left|2\sinh\frac{x-x'}{2}\right|^2}+\frac{1}{\left(2\cosh\frac{x-x'}{2}\right)^2}\right)e^{(x+x')(\frac{2-\gamma}{4})}. \tag{E14}$$

The dominant contribution again comes from two neighboring patches. Keeping only this contribution, we obtain

$$\delta E_{c,n}^{(3)} \simeq \frac{1}{4}\bar{g}^{\gamma}N_F\int_{-O(1)}^{O(1)}dy\int_{O(1)}^{O(\pi/\beta_\epsilon)}dy'\left(\frac{1}{\left|2\sinh\frac{y-y'}{2}\right|^2}+\frac{1}{\left(2\cosh\frac{y-y'}{2}\right)^2}\right)\left(\sum_{p=2}^{n+1}+\sum_{p=1}^{n}\right)e^{\frac{2-\gamma}{2}x_p} = \bar{g}^2N_F\frac{c_3}{2-\gamma}J_n, \tag{E15}$$

The integral over y,y' is free from divergences as y and y belong to different ranges. It is again confined to $y'\geqslant 1$, $y\leqslant 1$.

In total, $\delta E_{c,n}=\bar{g}^2N_F\frac{c}{2-\gamma}J_n$, where $c=c_1+c_2+c_3$. This is what we used in the main text.

3. Condensation energy for infinitesimally small gap function

The reasoning in the main text for the dispersion of $E_c(\epsilon)$ at $\gamma=2$–0 implies that the overall factor $1/(2-\gamma)$ is present in $E_c(\epsilon)$ for all nonzero ϵ, including the smallest $\epsilon\to 0$. At vanishing ϵ, $\Delta(\omega_m)=\epsilon\Delta_\infty(\omega_m)$ is the solution of the linearized gap equation. This gap function does not saturate at $\omega_m=0$ and instead keep oscillating down to the smallest frequencies. In this situation, the argument about the overall factor $1/(2-\gamma)$, which we presented in the previous two sections, does not hold, and it is a'priori unclear whether $E_c(\epsilon)$ at $\epsilon\to 0$ does contain $1/(2-\gamma)$. We show that it does.

It is convenient to use the condensation energy in the form given by Eq. (20). The first term in (20) is not singular for any γ, so we focus on the second term. For small $D(\omega_m)$ we have

$$E_c \simeq -N_F\frac{\bar{g}^{\gamma}}{16}\iint d\omega_m d\omega'_m\left(\frac{1}{|\omega_m-\omega'_m|^{\gamma}}-\frac{1}{|\omega_m+\omega'_m|^{\gamma}}\right)(D^2(\omega_m)-D^2(\omega'_m))^2, \tag{E16}$$

where

$$D(\omega_m) = 2\epsilon^* \cos\left[(\beta(\gamma)\ln(\bar{g}/|\omega_m|)^\gamma + \phi(\gamma))\right], \tag{E17}$$

and $\epsilon^* = \epsilon(\bar{g}/|\omega_m|)^{1-\gamma/2}$ [see Eq. (43)]. For simplicity, we absorbed the numerical factor Q_1 into ϵ). For $\gamma \leqslant 2$, ϵ^* increases for decreasing ω_m, and becomes of order one at $\omega_m = \omega_* \sim \bar{g}\epsilon^{2/(2-\gamma)}$. At smaller ω_m, $|D(\omega_m)|$ becomes larger than one. Eq. (E16) is valid at ω_m and ω'_m larger than ω_*, which then sets the lower limit of integration in (E16). At infinitesimally small ϵ, ω_* is also infinitesimally small and further decreases at $\gamma \to 2$. Yet, for any finite ϵ and finite $2-\gamma$, ω_* remains finite.

We now evaluate the 2D integral in (E16). It is convenient to introduce logarithmic variables $x = \ln(\bar{g}/\omega_m)^\gamma$ and $x' = \ln(\bar{g}/\omega'_m)^\gamma$. The integration over x and over x' is between $O(1)$ and $x_* = \gamma\ln(\bar{g}/\omega_*) = 2\gamma/(2-\gamma)\ln(C/\epsilon)$, where $C = O(1)$. We further introduce $a = x + x'$ and $b = x - x'$. The condensation energy is re-expressed in terms of a and b as

$$E_c = -N_F\bar{g}^2\frac{\epsilon^4}{16\gamma^2 2^\gamma}\int_{O(1)}^{2x_*} da\, e^{\frac{2-\gamma}{2\gamma}a}\int_0^{2x_*-a} db\left(\frac{1}{(\sinh\frac{b}{2\gamma})^\gamma} - \frac{1}{(\cosh\frac{b}{2\gamma})}\right)\Bigg[\sin(\beta(\gamma)a + 2\phi(\gamma))\sin\beta(\gamma)b\cosh\left(\frac{\gamma-2}{2\gamma}b\right)$$
$$+ (1+\cos(\beta(\gamma)a + 2\phi(\gamma))\cos(\beta(\gamma)b))\sinh\left(\frac{\gamma-2}{2\gamma}b\right)\Bigg]^2. \tag{E18}$$

We see that the integral over a is confined to the upper limit $a \sim x_*$, but the one over b is confined to $b = O(1)$. We can then safely extend the upper limit of the integration over b to infinity. Taking the limit $\gamma \to 2$ in the last factor in (E18) and averaging over rapidly oscillating factor $\sin^2(\beta(\gamma)a + 2\phi(\gamma))$, we obtain

$$E_c - N_F\bar{g}^2\frac{\epsilon^4}{512}I\int_{O(1)}^{2x_*} da\, e^{\frac{2-\gamma}{4}a}, \tag{E19}$$

where

$$I = \int_0^\infty db\left(\frac{1}{\sinh^2\frac{b}{4}} - \frac{1}{\cosh^2\frac{b}{4}}\right)\sin^2(\beta(2)b)$$
$$\simeq 5.75683. \tag{E20}$$

Evaluating the remaining integral over a, we obtain

$$E_c = -N_F\bar{g}^2\frac{\epsilon^4}{512}I\frac{4}{2-\gamma}e^{(2-\gamma)x_*/2}. \tag{E21}$$

Substituting the expression for x_*, we obtain

$$E_c = -N_F\bar{g}^2\frac{C^2 I}{128}\frac{\epsilon^2}{2-\gamma}. \tag{E22}$$

The computation of the prefactor C requires more sophisticated analysis.

We see that E_c scales as $1/(2-\gamma)$, even for the smallest ϵ. We cited this result in the main text.

The $1/(2-\gamma)$ divergence at small ϵ can be regularized by a finite bosonic mass ω_D, like we found for larger ϵ. We found that at a finite ω_D the factor $1/(2-\gamma)$ is replaced by $\epsilon^2\ln\bar{g}/\omega_D$. As a result, the condensation energy scales as $E_c \propto \epsilon^4\ln\bar{g}/\omega_D$.

APPENDIX F: THE SOLUTIONS OF THE GAP EQUATION, FOUND IN Ref. [50] AND THEIR CAUSALITY

In the Ref. [50], the authors found the new class of solutions of Eliashberg equations for $\gamma = 2$ both in the normal and superconducting state. In the spin-chain language, introduced in [50], these solutions correspond to "spin-flip" configurations, in which spins on some lattice sites are antiparallel to the local field, generated by other spins. In terms of the Green's function, they correspond to sign-changing $G(k_F, \omega_m)$ along $\omega_m > 0$.

In this Appendix, we show that such solutions cannot be analytically continued to the upper half plane (UHP) of complex frequency without singularities, i.e., they violate causality principle. We also show explicitly that these solutions do not exist at $T = 0$. This last point was already made in Ref. [50].

The analysis of the analytic properties can be done most easily in the normal state. We focus on $k = k_F$ and define $G(\omega_m) = G(k_F, \omega_m)$. We recall that the solution on the Matsubara axis is casual if there exists a function $G(z)$ of complex $z = \omega' + i\omega''$, which is analytic and has no singularities in the UHP and for which $G(z = i\omega_m) = G(\omega_m)$. On the real axis, the function $G(\omega)$ is a conventional retarded Green's function; at large $|z|$ in the UHP, $G(z \to \infty) \approx 1/z$.

Like in Ref. [50], we assume particle-hole symmetry, in which case $G(z = i\omega'')$ must be real. Then if $G(\omega_m)$ changes sign once between Matsubara points ω_m and ω_{m+1}, then the real $G(z = i\omega'')$ must either have a pole or have a zero in the interval $z \in (i\omega_m, i\omega_{m+1})$. The former possibility immediately indicates that $G(z)$ has a pole in the UHP and thus violate the causality principle. Now we discuss the latter possibility and demonstrate that $G(z)$ still must have a pole somewhere in the UHP.

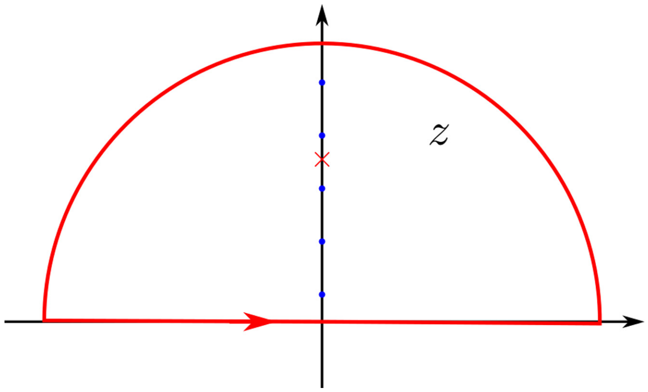

FIG. 13. The contour Γ used in the definition (F1). The radius of the arc is very large. The blue dots represent the Matsubara points. The red cross represents a zero of the function $G(z)$.

For the proof, consider the following integral

$$\frac{1}{2\pi}\mathrm{Im}\oint_\Gamma \partial_z \ln G(z)dz, \tag{F1}$$

where the closed contour Γ is shown in Fig. 13. The contour Γ consists of two parts: the large arc of radius $R \to \infty$ and the line along the real axis. The integral along the large arc can be evaluated using the asymptotic large z form $G(z) \approx 1/z$. Substituting this form into (F1) and using $z = Re^{i\phi}$, we find $\mathrm{Im}\int_{\rm arc} \partial_z \ln G(z)dz = -\pi$.

To evaluate the integral along the interval $(-R, R)$ on the real axis, we note that i) the Eliashberg self energy depends only on ω, and ii) the density of states, $\sim -\int dk \mathrm{Im}G(\omega, k)$, must be positive at all real ω. One can then make sure that ImG is negative and does not change sign on the real axis, while ReG is negative at $\omega \to -\infty$ and positive at $\omega \to +\infty$. It then immediately follows that $\mathrm{Im}\int_{-R}^{R} \partial_\omega \ln G(\omega)d\omega = \pi$. Hence

$$\mathrm{Im}\frac{1}{2\pi}\oint_\Gamma \partial_z \ln G(z)dz = 0 \tag{F2}$$

On the other hand, the function $\partial_z \ln G(z)$ is an analytic function in the UHP with a simple pole of residue +1 at the point where $G(z) = 0$. Using the residue theorem we then immediately find that

$$\mathrm{Im}\frac{1}{2\pi}\oint_\Gamma \partial_z \ln G(z)dz = 1 \tag{F3}$$

We see that (F3) is incomparable with (F2). This can only be avoided if $G(z)$ has a pole in the UHP as at this point $\partial_z \ln G(z)$ has a pole with residue −1, and the r.h.s of (F3) is $1 - 1 = 0$. In general, the r.h.s of (F3) is $n - p$, where n is the number of zeros and p is the number of poles, counted with their multiplicity. Then if $G(\omega_m)$ changes sign on the Matsubara axis more than once, there must be multiple poles of $G(z)$ in the UHP. The requirement that $G(z)$ must have a poles in the UHP contradicts the causality principle.

1. $T = 0$

Now we return to the Eliashberg equations and show that Z_ω cannot change sign on the positive Matsubara axis at $T = 0$ in the limit of vanishing bosonic mass. At $T = 0$, the system is in the superconducting state, and we analyze Eliashberg equations for a nonzero $\Delta(\omega_m)$.

First, we notice that at $T = 0$, Z_ω is a real function of ω_m. It also must be a continuous function, as the Green's function in the superconducting state must be analytic in the UHP.

As the authors of [50] noticed, the r.h.s. of the Eliashberg equation (4) $\mathrm{sgn}(Z_{\omega_{m'}})$ in the numerator under the sum. In the main text we set this term to 1 as we were only interested in the sign-preserving solutions for $Z(\omega_m)$. Now we keep this term. At $T = 0$, the equation for $Z(\omega_m)$ becomes

$$Z_{\omega_m} = 1 + \frac{g^\gamma}{2\omega_m}\int_{-\infty}^{\infty} \frac{d\omega'_m}{\sqrt{{\omega'_m}^2 + \Delta^2_{\omega'_m}}}\mathrm{sgn}(Z_{\omega'_m})\frac{1}{\left((\omega_m - \omega'_m)^2 + \omega_D^2\right)^{\gamma/2}}. \tag{F4}$$

We will be interested in the limit $\omega_D \to 0$.

Both functions Z_{ω_m} and Δ_{ω_m} are real and $Z_{\omega_m\to\infty} \to 1$, $\Delta_{\omega_m\to\infty} \to 0$.

We want to check, if it is possible that

(i) both Z_{ω_m} and Δ_{ω_m} are smooth functions;

(ii) Z_{ω_m} changes sign N times at frequencies ω_n, where $n = 0, \dots, N-1$.

We take the derivative of Z_{ω_m} over ω_m and evaluate the integral by parts

$$\begin{aligned}\partial_{\omega_m} Z_{\omega_m} &= -\frac{1}{\omega_m}(Z_{\omega_m} - 1) - \frac{g^\gamma}{2\omega_m}\int_{-\infty}^{\infty} \frac{d\omega'_m}{\sqrt{{\omega'_m}^2 + \Delta^2_{\omega'_m}}}\mathrm{sgn}(Z_{\omega'_m})\partial_{\omega'_m}\frac{1}{\left((\omega_m - \omega'_m)^2 + \omega_D^2\right)^{\gamma/2}} \\ &= -\frac{1}{\omega_m}(Z_{\omega_m} - 1) - \frac{g^\gamma}{2\omega_m}\left[\frac{\mathrm{sgn}(Z_{\omega'_m})}{\sqrt{{\omega'_m}^2 + \Delta^2_{\omega'_m}}}\frac{1}{\left((\omega_m - \omega'_m)^2 + \omega_D^2\right)^{\gamma/2}}\Bigg|_{-\infty}^{\infty} \right. \\ &\left. - \int_{-\infty}^{\infty} d\omega'_m \frac{1}{\left((\omega_m - \omega'_m)^2 + \omega_D^2\right)^{\gamma/2}}\partial_{\omega'_m}\frac{\mathrm{sgn}(Z_{\omega'_m})}{\sqrt{{\omega'_m}^2 + \Delta^2_{\omega'_m}}}\right]\end{aligned}$$

$$= -\frac{1}{\omega_m}(Z_\omega - 1) + \frac{g^\gamma}{2\omega_m}\left[2(-1)^{N+1}\sum_{n=0}^{N-1}\frac{(-1)^n}{\left((\omega_m-\omega_n)^2+\omega_D^2\right)^{\gamma/2}}\frac{1}{\sqrt{{\omega_n}^2+\Delta_{\omega_n}^2}}\right.$$
$$\left.+\int_{-\infty}^{\infty}d\omega'_m\frac{\text{sgn}(Z_{\omega'_m})}{\left((\omega_m-\omega'_m)^2+\omega_D^2\right)^{\gamma/2}}\partial_{\omega'_m}\frac{1}{\sqrt{{\omega'_m}^2+\Delta_{\omega'_m}^2}}\right], \tag{F5}$$

where we used $\partial_{\omega'_m}\text{sgn}(Z_{\omega'_m}) = 2(-1)^{N+1}\sum_{n=0}^{N-1}(-1)^n\delta(\omega_m-\omega_n)$. By assumption, Z_{ω_m} is a smooth function, so $Z_{\omega_n}=0$, *for all* n. Setting $\omega_m=\omega_k$ in (F5), we obtain

$$\partial_{\omega_m}Z_{\omega_m}|_{\omega_m=\omega_k} = \frac{1}{\omega_k} + \frac{g^\gamma}{2\omega_k}\int_{-\infty}^{\infty}d\omega'_m\frac{\text{sgn}(Z_{\omega'_m})}{\left((\omega_k-\omega'_m)^2+\omega_D^2\right)^{\gamma/2}}\partial_{\omega'_m}\frac{1}{\sqrt{{\omega'_m}^2+\Delta_{\omega'_m}^2}}$$
$$+\frac{g^\gamma}{\omega_k}(-1)^{N+1}\sum_{n=0}^{N-1}\frac{(-1)^n}{\left((\omega_k-\omega_n)^2+\omega_D^2\right)^{\gamma/2}}\frac{1}{\sqrt{{\omega_n}^2+\Delta_{\omega_n}^2}}. \tag{F6}$$

We see that in the limit $\omega_D\to 0$, the most singular term is the term in the sum with $n=k$. Keeping only this term, we get in this limit

$$\lim_{m\to 0}\partial_{\omega_m}Z_{\omega_m}|_{\omega_m=\omega_k} \sim \frac{g^\gamma}{\omega_k}\frac{(-1)^{k+N+1}}{\omega_D^{\gamma/2}}\frac{1}{\sqrt{{\omega_k}^2+\Delta_{\omega_k}^2}} \to \pm\infty. \tag{F7}$$

This shows that in the limit $\omega_D\to 0$, the sign-changing function Z_{ω_m} is not smooth function as its derivative over ω_m diverges at any finite ω_m.

[1] A. Abanov and A. V. Chubukov, Interplay between superconductivity and non-Fermi liquid at a quantum critical point in a metal. i. the γ model and its phase diagram at $T=0$: The case $0<\gamma<1$, Phys. Rev. B **102**, 024524 (2020).
[2] Y.-M. Wu, A. Abanov, Y. Wang, and A. V. Chubukov, Interplay between superconductivity and non-fermi liquid at a quantum critical point in a metal. ii. the γ model at a finite T for $0<\gamma<1$, Phys. Rev. B **102**, 024525 (2020).
[3] Y.-M. Wu, A. Abanov, and A. V. Chubukov, Interplay between superconductivity and non-fermi liquid behavior at a quantum critical point in a metal. iii. the γ model and its phase diagram across $\gamma=1$, Phys. Rev. B **102**, 094516 (2020).
[4] Y.-M. Wu, S.-S. Zhang, A. Abanov, and A. V. Chubukov, Interplay between superconductivity and non-fermi liquid at a quantum critical point in a metal. iv. the γ model and its phase diagram at $1<\gamma<2$, Phys. Rev. B **103**, 024522 (2021).
[5] Y.-M. Wu, S.-S. Zhang, A. Abanov, and A. V. Chubukov, Interplay between superconductivity and non-fermi liquid behavior at a quantum-critical point in a metal. v. the γ model and its phase diagram: The case $\gamma=2$, Phys. Rev. B **103**, 184508 (2021).
[6] S.-S. Zhang, Y.-M. Wu, A. Abanov, and A. V. Chubukov, Interplay between superconductivity and non-fermi liquid at a quantum critical point in a metal. vi. the γ model and its phase diagram at $2\gamma<3$, Phys. Rev. B **104**, 144509 (2021).
[7] A. J. Millis, Nearly antiferromagnetic fermi liquids: An analytic eliashberg approach, Phys. Rev. B **45**, 13047 (1992).
[8] B. L. Altshuler, L. B. Ioffe, A. I. Larkin, and A. J. Millis, Spin-density-wave transition in a two-dimensional spin liquid, Phys. Rev. B **52**, 4607 (1995).
[9] S. Sachdev, A. V. Chubukov, and A. Sokol, Crossover and scaling in a nearly antiferromagnetic fermi liquid in two dimensions, Phys. Rev. B **51**, 14874 (1995).
[10] A. Abanov, A. V. Chubukov, and A. M. Finkel'stein, Coherent vs. incoherent pairing in 2d systems near magnetic instability, Europhys. Lett. **54**, 488 (2001); A. Abanov, A. V. Chubukov, and J. Schmalian, Quantum-critical theory of the spin-fermion model and its application to cuprates: Normal state analysis, Adv. Phys. **52**, 119 (2003); A. Abanov and A. V. Chubukov, A Relation between the Resonance Neutron Peak and ARPES Data in Cuprates, Phys. Rev. Lett. **83**, 1652 (1999); A. Abanov, A. V. Chubukov, and J. Schmalian, Fingerprints of spin mediated pairing in cuprates, J. Electron Spectrosc. Relat. Phenom. **117-118**, 129 (2001); A. Abanov, A. V. Chubukov, and M. R. Norman, Gap anisotropy and universal pairing scale in a spin-fluctuation model of cuprate superconductors, Phys. Rev. B **78**, 220507(R) (2008).
[11] S. Sachdev, M. A. Metlitski, Y. Qi, and C. Xu, Fluctuating spin density waves in metals, Phys. Rev. B **80**, 155129 (2009).
[12] E. G. Moon and S. Sachdev, Competition between spin density wave order and superconductivity in the underdoped cuprates, Phys. Rev. B **80**, 035117 (2009).
[13] N. E. Bonesteel, I. A. McDonald, and C. Nayak, Gauge Fields and Pairing in Double-Layer Composite Fermion Metals, Phys. Rev. Lett. **77**, 3009 (1996).
[14] Y. Wang and A. V. Chubukov, Superconductivity at the Onset of Spin-Density-Wave Order in a Metal, Phys. Rev. Lett. **110**, 127001 (2013).
[15] M. A. Metlitski and S. Sachdev, Quantum phase transitions of metals in two spatial dimensions. ii. spin density wave order, Phys. Rev. B **82**, 075128 (2010).

[16] M. Vojta and S. Sachdev, Charge Order, Superconductivity, and a Global Phase Diagram of Doped Antiferromagnets, Phys. Rev. Lett. **83**, 3916 (1999).
[17] K. B. Efetov, H. Meier, and C. Pepin, Pseudogap state near a quantum critical point, Nat. Phys. **9**, 442 (2013).
[18] H. Meier, C. Pépin, M. Einenkel, and K. B. Efetov, Cascade of phase transitions in the vicinity of a quantum critical point, Phys. Rev. B **89**, 195115 (2014).
[19] K. B. Efetov, Quantum criticality in two dimensions and marginal fermi liquid, Phys. Rev. B **91**, 045110 (2015).
[20] A. M. Tsvelik, Ladder physics in the spin fermion model, Phys. Rev. B **95**, 201112(R) (2017).
[21] J. Bauer and S. Sachdev, Real-space eliashberg approach to charge order of electrons coupled to dynamic antiferromagnetic fluctuations, Phys. Rev. B **92**, 085134 (2015).
[22] C. Castellani, C. Di Castro, and M. Grilli, Singular Quasiparticle Scattering in the Proximity of Charge Instabilities, Phys. Rev. Lett. **75**, 4650 (1995).
[23] A. Perali, C. Castellani, C. Di Castro, and M. Grilli, d-wave superconductivity near charge instabilities, Phys. Rev. B **54**, 16216 (1996).
[24] S. Andergassen, S. Caprara, C. Di Castro, and M. Grilli, Anomalous Isotopic Effect Near the Charge-Ordering Quantum Criticality, Phys. Rev. Lett. **87**, 056401 (2001).
[25] Y. Wang and A. V. Chubukov, Enhancement of superconductivity at the onset of charge-density-wave order in a metal, Phys. Rev. B **92**, 125108 (2015).
[26] D. Chowdhury and S. Sachdev, Density-wave instabilities of fractionalized fermi liquids, Phys. Rev. B **90**, 245136 (2014).
[27] Z. Wang, W. Mao, and K. Bedell, Superconductivity near Itinerant Ferromagnetic Quantum Criticality, Phys. Rev. Lett. **87**, 257001 (2001).
[28] R. Roussev and A. J. Millis, Quantum critical effects on transition temperature of magnetically mediated p-wave superconductivity, Phys. Rev. B **63**, 140504(R) (2001).
[29] A. V. Chubukov, A. M. Finkel'stein, R. Haslinger, and D. K. Morr, First-Order Superconducting Transition near a Ferromagnetic Quantum Critical Point, Phys. Rev. Lett. **90**, 077002 (2003).
[30] D. Dalidovich and S.-S. Lee, Perturbative non-fermi liquids from dimensional regularization, Phys. Rev. B **88**, 245106 (2013).
[31] S. Lederer, Y. Schattner, E. Berg, and S. A. Kivelson, Enhancement of Superconductivity near a Nematic Quantum Critical Point, Phys. Rev. Lett. **114**, 097001 (2015).
[32] P. A. Lee, Gauge field, aharonov-bohm flux, and high-T_c superconductivity, Phys. Rev. Lett. **63**, 680 (1989).
[33] B. Blok and H. Monien, Gauge theories of high-t_c superconductors, Phys. Rev. B **47**, 3454 (1993).
[34] C. Nayak and F. Wilczek, Non-fermi liquid fixed point in $2 + 1$ dimensions, Nucl. Phys. B **417**, 359 (1994).
[35] B. L. Altshuler, L. B. Ioffe, and A. J. Millis, Low-energy properties of fermions with singular interactions, Phys. Rev. B **50**, 14048 (1994).
[36] Y. B. Kim, A. Furusaki, X.-G. Wen, and P. A. Lee, Gauge-invariant response functions of fermions coupled to a gauge field, Phys. Rev. B **50**, 17917 (1994).
[37] P. B. Allen and R. C. Dynes, Transition temperature of strong-coupled superconductors reanalyzed, Phys. Rev. B **12**, 905 (1975).
[38] A. Karakozov, E. Maksimov, and S. Mashkov, Effect of the frequency dependence of the electron-phonon interaction spectral function on the thermodynamic properties of superconductors, Zh. Eksp. Teor. Fiz. **68**, 1937 (1975).
[39] F. Marsiglio and J. P. Carbotte, Gap function and density of states in the strong-coupling limit for an electron-boson system, Phys. Rev. B **43**, 5355 (1991), for more recent results see F. Marsiglio and J.P. Carbotte, "Electron-Phonon Superconductivity", in "The Physics of Conventional and Unconventional Superconductors", Bennemann and Ketterson eds., Springer-Verlag, (2006) and references therein; F. Marsiglio, Eliashberg theory: A short review, Annals of Physics, **417**, 168102 (2020).
[40] R. Combescot, Strong-coupling limit of eliashberg theory, Phys. Rev. B **51**, 11625 (1995).
[41] A. V. Chubukov, A. Abanov, I. Esterlis, and S. A. Kivelson, Eliashberg theory of phonon-mediated superconductivity – when it is valid and how it breaks down, Ann. Phys. **417**, 168190 (2020).
[42] C. Zhang, J. Sous, D. R. Reichman, M. Berciu, A. J. Millis, N. V. Prokof'ev, and B. V. Svistunov, Bipolaronic high-temperature superconductivity, arXiv:2203.07380.
[43] M. A. Metlitski and S. Sachdev, Quantum phase transitions of metals in two spatial dimensions. i. ising-nematic order, Phys. Rev. B **82**, 075127 (2010).
[44] C. Li, S. Sachdev, and D. G. Joshi, Superconductivity of non-fermi liquids described by Sachdev-Ye-Kitaev models, arXiv:2208.05493.
[45] D. Chowdhury and E. Berg, Intrinsic superconducting instabilities of a solvable model for an incoherent metal, Phys. Rev. Res. **2**, 013301 (2020).
[46] I. Esterlis and J. Schmalian, Cooper pairing of incoherent electrons: An electron-phonon version of the Sachdev-Ye-Kitaev model, Phys. Rev. B **100**, 115132 (2019).
[47] Y. Wang, Solvable Strong-Coupling Quantum-Dot Model with a Non-Fermi-Liquid Pairing Transition, Phys. Rev. Lett. **124**, 017002 (2020).
[48] D. Hauck, M. J. Klug, I. Esterlis, and J. Schmalian, Eliashberg equations for an electronphonon version of the Sachdev–Ye–Kitaev model: Pair breaking in non-Fermi liquid superconductors, Ann. Phys. **417**, 168120 (2020).
[49] L. Classen and A. Chubukov, Superconductivity of incoherent electrons in the Yukawa Sachdev-Ye-Kitaev model, Phys. Rev. B **104**, 125120 (2021).
[50] E. A. Yuzbashyan and B. L. Altshuler, Migdal-eliashberg theory as a classical spin chain, Phys. Rev. B **106**, 014512 (2022).
[51] E. A. Yuzbashyan, M. K.-H. Kiessling, and B. L. Altshuler, Superconductivity near a quantum critical point in the extreme retardation regime, Phys. Rev. B **106**, 064502 (2022).
[52] E. Yuzbashyan, Free energy for eliashberg spins using functional integral approach (private communication).
[53] M. Protter, R. Boyack, and F. Marsiglio, Functional-integral approach to gaussian fluctuations in eliashberg theory, Phys. Rev. B **104**, 014513 (2021).
[54] J. M. Luttinger and J. C. Ward, Ground-state energy of a many-fermion system. ii, Phys. Rev. **118**, 1417 (1960).
[55] R. Haslinger and A. V. Chubukov, Condensation energy in strongly coupled superconductors, Phys. Rev. B **68**, 214508 (2003).

[56] A. Secchi, M. Polini, and M. I. Katsnelson, Phonon-mediated superconductivity in strongly correlated electron systems: A luttinger–ward functional approach, Ann. Phys. **417**, 168100 (2020).
[57] A. Benlagra, K. Kim, and C. Pépin, The luttinger–ward functional approach in the eliashberg framework: a systematic derivation of scaling for thermodynamics near the quantum critical point, J. Phys.: Condens. Matter **23**, 145601 (2011).
[58] L. D. Landau and E. M. Lifshitz, *Statistical Physics: Volume 5* (Elsevier, 2013).
[59] M. Kronenburg, Some generalized harmonic number identities, arXiv:1103.5430.
[60] S. S. Pufu, Anomalous dimensions of monopole operators in three-dimensional quantum electrodynamics, Phys. Rev. D **89**, 065016 (2014).
[61] A. Klein, A. V. Chubukov, Y. Schattner, and E. Berg, Normal State Properties of Quantum Critical Metals at Finite Temperature, Phys. Rev. X **10**, 031053 (2020).
[62] S.-S. Zhang, E. Berg, and A. V. Chubukov (unpublished).
[63] E. A. Yuzbashyan and B. L. Altshuler, Breakdown of the migdal-eliashberg theory and a theory of lattice-fermionic superfluidity, Phys. Rev. B **106**, 054518 (2022).
[64] O. Grossman, J. S. Hofmann, T. Holder, and E. Berg, Specific Heat of a Quantum Critical Metal, Phys. Rev. Lett. **127**, 017601 (2021).
[65] Y.-M. Wu, S.-S. Zhang, A. Abanov, and A. V. Chubukov, Odd frequency pairing in a quantum critical metal, Phys. Rev. B **106**, 094506 (2022).
[66] J. Linder and A. V. Balatsky, Odd-frequency superconductivity, Rev. Mod. Phys. **91**, 045005 (2019).