Impurity scattering in superconductors revisited: Diagrammatic formulation of the supercurrent-supercurrent correlation and Higgs-mode damping

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The diagrammatic formalism and transport equation are conventionally considered as separate but complementary basic techniques to tackle the impurity scattering effect. To compare with previous studies from the gauge-invariant kinetic equation approach [F. Yang and M. W. Wu, Phys. Rev. B **98**, 094507 (2018); **102**, 144508 (2020)], we analytically perform a diagrammatic formulation of the impurity scattering in superconductors, with both transport and collective Higgs modes studied, to fill the long missing calculation of the Kubo current-current correlation in superconductors with impurity scattering and resolve the controversy (whether the impurity scattering can lead to the damping of Higgs mode) between the gauge-invariant kinetic equation and Eilenberger equation. For transport behavior, through a special unitary transformation that is equivalent to the Wilson-line technique for the diamagnetic response, we derive the Meissner-supercurrent vertex. Then, by formulating the supercurrent-supercurrent correlation with Born and vertex corrections from impurity scattering, we recover the previously revealed microscopic momentum-relaxation rate of superfluid by gauge-invariant kinetic equation. This rate is finite only when the superconducting velocity is larger than a threshold, at which the normal fluid emerges and causes friction with the superfluid current, similar to Landau's superfluid theory of liquid helium. This derivation also provides a physical understanding of the relaxation-time approximation in the previous diagrammatic formulation in the literature, which leads to the friction resistance of the Meissner supercurrent. For the collective Higgs mode, we calculate the amplitude-amplitude correlation with Born and vertex corrections from impurity scattering. The vertex correction, which only emerges at the nonequilibrium case, leads to a Higgs-mode damping, whereas the Born correction that is equivalent to equilibrium self-energy makes no contribution due to the Anderson theorem. This induced damping agrees with the analysis through the Heisenberg equation of motion and is also exactly the same as the one obtained from the gauge-invariant kinetic equation.

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I. INTRODUCTION

The impurity scattering effect has attracted much attention in the field of superconductivity. On one hand, the stationary magnetic-flux expulsion due to the generated diamagnetic supercurrent (Meissner effect) [1,2], as well as the low-frequency optical conductivity described by the phenomenological two-fluid model [3,4], are characteristic transport properties of superconductors, among which elucidating the impurity scattering effect is essential to understand the superconductivity/resistivity phenomena. On the other hand, recently, inspired by nonlinear optical experiments in the THz regime [5–11], a great deal of effort has been devoted to the collective gapful Higgs mode, which describes the amplitude fluctuation of the superconducting order parameter [12–21]. Being charge neutral, this collective excitation does not manifest itself in the linear optical response, but can be generated in the second-order one at clean limit [22], leading to an experimentally observable fluctuation of superfluid density [22]. The damping mechanism of the Higgs mode after excitation then stimulated a lot of interest [10,23].

Theoretically, two kinds of schemes have been developed in the literature to formulate the impurity scattering effect, including the diagrammatic formalism and transport equation, which are conventionally considered as separate but complementary basic techniques as a crosscheck, as demonstrated in normal metals [24]. Nevertheless, in superconductors, the relationship of the two techniques has not been well developed in the literature for decades.

The formulation within the diagrammatic formalism requires the inevitable calculation of the vertex correction by impurity scattering [25,26], which is hard to tackle in superconductors. Specifically, it has been established [27] that superconductors with the small (large) mean-free path l in comparison with the skin depth δ lie in the normal (anomalous) skin-effect region and exhibit the London-type/local (Pippard-type/nonlocal) electromagnetic response. The linear electromagnetic responses of superconductors in the anomalous- and normal-skin-effect regions were discussed by Mattis and Bardeen [28] as well as Abrikosov *et al.* [26], based on the current-current correlation with the impurity scattering. To handle the scattering effect, the Mattis-Bardeen

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theory introduces a phenomenological constant scattering factor [28], which is similar to the relaxation-time approximation. Whereas Abrikosov *et al.* [26] applied an approximation that assumes an isotropic Green's function in consideration of a dirty case ($l \ll \xi$, with ξ being the coherence length) to integrate over the momentum variable of pairing electrons to simplify the vertex-correction calculation. Both approximations then drop the microscopic scattering process. Interestingly, both Mattis-Bardeen [28,29] and Abrikosov-Gorkov [26] theoretical descriptions in the diamagnetic response derive a penetration depth $\lambda = \lambda_c\sqrt{\xi/l}$ at the dirty limit, with λ_c being the clean-limit result. By using the relaxation-time approximation, this dependence was later phenomenologically extended by Tinkham [30] to a general form $\lambda = \lambda_c\sqrt{1+\xi/l}$ between clean and dirty cases, in good agreement with the experiments [31–35]. Nevertheless, as a direct consequence of this dependence that is derived from the current-current correlation, the Meissner supercurrent, which should be nonviscous, experiences a friction resistance by scattering. The physical origin of this resistance becomes untraceable due to the absent microscopic scattering process. Moreover, these theoretical descriptions also fail to recover the two-fluid model, which requires a microscopic distinction of the pairing (superfluid) and unpairing (normal fluid) electrons [3,36,37]. In contrast to the transport behavior, the diagrammatic formulation of the impurity scattering effect on Higgs mode remains stagnant so far. While the calculation of the amplitude-amplitude correlation at the clean case successfully gives the Higgs-mode energy spectrum in the long-wave limit [38–42], it is complicated to formulate the corresponding vertex correction by the impurity scattering.

The transport-equation approach with microscopic scattering can naturally contain and easily handle the calculation of the vertex correction by scattering, as demonstrated in normal metals [24,43]. In superconductors, three kinds of transport equations that construct the microscopic scattering have been developed in the literature, including the semiclassical Boltzmann equation of quasiparticles [44–46], quasiclassical Eilenberger equation [47–51], and gauge-invariant kinetic equation (GIKE) [52–55]. The semiclassical Boltzmann equation as an early stage of works only includes the quasiparticle dynamics but fails to contain the superfluid dynamics [44–46].

The Eilenberger equation [47,50,51] is derived from the basic Gorkov equation [26] of the τ_3-Gree's function $G_3(x,x') = -i\tau_3\langle\hat{T}\psi(x)\psi^\dagger(x')\rangle$ through the quasiclassical approximation which performs an integration over kinetic-energy variable. Here, τ_i denotes the Pauli matrices in Nambu space. This approach at the free case can successfully describe the Higgs-mode energy spectrum [56] and discuss topics like the proximity effect in multilayer junctions [48,49] as well as vortex dynamics [57–60] and unconventional superconductivity [61–64]. While concerning the electromagnetic response, the gauge invariance is lost during the derivation [65], leading to an incomplete electromagnetic effect. As a consequence, the Eilenberger equation only keeps the drive effect of the vector potential [67], making it well tailored to handle the diamagnetic response (i.e., derive the Ginzburg-Landau equation as well as the Meissner supercurrent [50]) and gives a finite Higgs-mode generation in the second optical response at the clean limit [22,68]. But the drive effect by scalar potential and all density-related electromagnetic effects are generically dropped out [22,67].

Focusing on the scattering effect, the Eilenberger equation contains the specific quasiclassical microscopic scattering integral [47–51,56]. In the diamagnetic response, the derived supercurrent from this approach also experiences a friction resistance [47,51,56]. Particularly, in the Usadel equation [69], which is a dirty-limit case of the Eilenberger equation, the induced supercurrent is directly proportional to the diffusive coefficient, consistent with the Mattis-Bardeen [28,29] and Abrikosov-Gorkov [26] theoretical descriptions mentioned above. Nevertheless, elucidating the origin of this friction resistance has long been overlooked. As for the collective excitation, with impurities, it is reported [56,70] that the derived Higgs-mode energy spectrum in the Eilenberger equation is free from the scattering influence, i.e., the impurity scattering does not cause the damping of the Higgs mode. In one view in the literature, the Higgs mode as the gap fluctuation is insensitive to disorder, as the Anderson theorem [71] reveals a vanishing renormalization by impurity self-energy on equilibrium s-wave gap [72–75] in consideration of the time-reversal-partner pairing. Very recently, this viewpoint has been challenged [54]. The key point lies in the fact that the Higgs mode is a nonequilibrium excitation which breaks the time-translational symmetry. Thus, applying the Anderson theorem to the nonequilibrium case is unsuitable. In this circumstance, considering the fact that the Higgs-mode excitation $\delta|\Delta|\tau_1$ and electron-impurity interaction $V(\mathbf{r})\tau_3$ are noncommutative in Nambu space, one immediately concludes that the nonequilibrium Higgs mode experiences a finite impurity influence according to the Heisenberg equation of motion. This analysis is then in sharp contrast to the derivation from the Eilenberger equation [56] mentioned above.

The GIKE [52,53] is derived from the Gorkov equation [26] of the τ_0-Green's function $G_0(x,x') = -i\langle\hat{T}\psi(x)\psi^\dagger(x')\rangle$ within an equal-time scheme [76,77]. To retain the gauge invariance, the gauge-invariant τ_0-Green's function is constructed through the Wilson line [66]. Then, the complete electromagnetic effects are included [52] and the charge conservation is naturally satisfied [53], making this approach capable of formulating both magnetic and optical responses in linear and nonlinear regimes. The well-known clean-limit results, such as the Ginzburg-Landau equation and Meissner supercurrent in the diamagnetic response and the low-frequency optical conductivity captured by the two-fluid model [52] as well as the linear electromagnetic responses of the collective phase and Higgs modes [53], can be directly derived from this microscopic approach. Very recently, in the second optical response at the clean limit, the derived finite Higgs-mode generation and vanishing charge-density fluctuation from GIKE [53] are exactly recovered from the basic path-integral approach [22].

Thanks to the equal-time scheme [76,77], the microscopic scattering in superconductors, which is hard to tackle within the diagrammatic formalism, becomes easy to handle within the GIKE. From this approach, not only the previously revealed phenomenological dependence of the penetration depth on mean-free path by Tinkham is recovered [52] but also the disorder-induced damping of Higgs mode is revealed [54]. Specifically, it is analytically demonstrated [52] that the generated Meissner supercurrent in diamagnetic response becomes viscous only when the superconducting velocity is larger than a threshold at which the normal fluid emerges, similar to Landau's theory for the emerged fluid viscosity in bosonic liquid helium at larger velocity [78]. The emergence of the viscous superfluid in superconductors arises from the friction between normal-fluid and superfluid currents due to the microscopic scattering [52]. A three-fluid model consisting of normal fluid as well as viscous and nonviscous superfluids is then proposed [52,55]. As for the damping of Higgs mode, it is found [54] that the impurity scattering leads to a fast exponential decay, which arises from the noncommutation relation between Higgs mode and electron-impurity interaction. This damping then agrees with the analysis through the Heisenberg equation of motion mentioned above, but is in contrast to the previous derivation from the Eilenberger equation [56].

In the present paper, to achieve a separate but complementary basic technique to compare with GIKE as a crosscheck, we try to apply the diagrammatic formulation of the impurity scattering in superconductors to fill the long missing calculation of the Kubo current-current correlation in superconductors with the impurity scattering in the textbook and resolve the controversy (whether the impurity scattering can lead to the damping of Higgs mode) between GIKE [54] and the Eilenberger equation [56] mentioned above. Specifically, for transport behavior in the diamagnetic response, because of the Meissner effect [1,2], it is shown that the density vertex τ_3 [79] in the conventional kinematical momentum operator $\hat{\mathbf{p}} - e\mathbf{A}\tau_3$ leads to a non-gauge-invariant current after the scattering treatment/correction on the current-current correlation. To eliminate this unphysical current, we apply a special unitary transformation that is equivalent to the Wilson-line technique for the diamagnetic response, and obtain the Meissner-supercurrent vertex. Then, by further calculating the supercurrent-supercurrent correlation with Born and vertex corrections from the impurity scattering, the microscopic momentum-relaxation rate of superfluid, which is exactly same as the one from GIKE [52,80], is derived. This rate becomes finite only when the superconducting velocity is larger than a threshold, at which the normal fluid emerges and causes the friction with superfluid current. Then, the three-fluid model proposed in Ref. [52] is recovered. Moreover, this derivation also provides a physical understanding of the relaxation-time approximation in the previous diagrammatic formulation [28–30], which leads to the friction resistance of the Meissner supercurrent as mentioned above. Furthermore, through the Wilson-line technique, a gauge-invariant Hamiltonian that explicitly distinguishes the Meissner effect and electric-field drive effect as well as the Josephson voltage effect is proposed.

As for the collective Higgs mode, we perform an analytical calculation of the amplitude-amplitude correlation with the Born and vertex corrections from impurity scattering. It is found that the vertex correction leads to a fast exponential damping of Higgs mode, whereas the Born correction that is equivalent to equilibrium impurity self-energy makes no contribution because of the Anderson theorem [71–75]. This induced damping by impurity scattering is exactly same as the one obtained from GIKE and agrees with analysis through the Heisenberg equation of motion mentioned above, in contrast to the previous derivation by the Eilenberger equation [56]. The revealed lifetime of the Higgs mode by impurity scattering provides a possible origin for the experimentally observed broadening of the resonance signal [7,8] as well as the damping after optical excitation [5,6] of the Higgs mode. In addition, as pointed out in Ref. [54], the damping by impurity can cause a phase shift in the optical signal of the Higgs mode, which exhibits a π jump at the resonance frequency and hence provides a very clear feature for further experimental detection.

II. MODEL

In this section, we first introduce the Hamiltonian and action of superconductors in the presence of the superconducting momentum. Then, based on the basic path-integral approach, we present the diagrammatic formalism to investigate the scattering effects on nonequilibrium property in superconductors.

A. Hamiltonian and action

It is well-known in superconductors that in the stationary magnetic response with a vector potential $\mathbf{A}$, a supercurrent is driven by superconducting momentum $\mathbf{p}_s = -e\mathbf{A}$ [26]. The Bogoliubov-de Gennes Hamiltonian of the conventional s-wave superconducting states in the presence of the superconducting momentum reads [26]

$$H = \int d\mathbf{x}\psi^{\dagger}(x)[\xi_{\hat{\mathbf{p}}+\mathbf{p}_s\tau_3}\tau_3 + \Delta_0\tau_1 + V(x)\tau_3]\psi(x), \quad (1)$$

where $\psi(x) = [\psi_{\uparrow}(x), \psi_{\downarrow}^{\dagger}(x)]^T$ represents the field operator in Nambu space with $x = (x_0, \mathbf{x})$ being the space-time four-vector; the momentum operator $\hat{\mathbf{p}} = -i\hbar\nabla$; $\xi_{\hat{\mathbf{p}}} = \hat{\mathbf{p}}^2/(2m) - \mu$ with m denoting the effective mass and μ being the chemical potential; Δ_0 and $V(x)$ denote the equilibrium gap and impurity potential, respectively.

Based on the Hamiltonian above, the action of superconductors after Hubbard-Stratonovich transformation is written as [79]

$$S = \int dx\bigg\{\sum_{s=\uparrow,\downarrow}\psi_s^*(x)[i\partial_{x_0} - \xi_{\hat{\mathbf{p}}+\mathbf{p}_s} - V(x)]\psi_s(x) - \psi^{\dagger}(x)\Delta_0\tau_1\psi(x) - \frac{\Delta_0^2}{g}\bigg\}, \quad (2)$$

which in Nambu space becomes

$$S = \int dx\bigg\{\psi^{\dagger}(x)\big[G_0^{-1}(\hat{p}) - V(x)\tau_3\big]\psi(x) - \frac{\eta_f p_s^2}{2m} - \eta_f V(x) - \frac{|\Delta(x)|^2}{g}\bigg\}. \quad (3)$$

Here, g denotes the BCS pairing potential and $\eta_f = \sum_{\mathbf{k}} 1$ emerges because of the anticommutation of the Fermi field; the Green's function $G_0^{-1}(\hat{p}) = i\partial_{x_0} - \mathbf{p}_s \cdot \mathbf{v}_{\hat{\mathbf{p}}} - (\xi_{\hat{\mathbf{p}}} + \frac{p_s^2}{2m})\tau_3 - \Delta_0\tau_1$ with $\mathbf{v}_{\hat{\mathbf{p}}} = \hat{\mathbf{p}}/m$ standing for the group velocity.

The Fourier component of the Green's function in Matsubara representation is given by [26]

$$G_0(p) = \frac{ip_n - \mathbf{p}_s\cdot\mathbf{v_k} + \xi_\mathbf{k}\tau_3 + p_s^2\tau_3/(2m) + \Delta_0\tau_1}{(ip_n - E_\mathbf{k}^+)(ip_n - E_\mathbf{k}^-)}, \quad (4)$$

where the four-vector momentum $p = (p_n, \mathbf{k})$ with $p_n = (2n+1)\pi T$ being the Matsubara frequency; the quasiparticle energy spectra read

$$E_\mathbf{k}^\pm = \mathbf{p}_s\cdot\mathbf{v_k} \pm E_\mathbf{k}, \quad (5)$$

with $E_\mathbf{k} = \sqrt{[\xi_\mathbf{k} + p_s^2/(2m)]^2 + \Delta_0^2}$.

It is noted that the term $\mathbf{p}_s\cdot\mathbf{v_k}$ in the equilibrium Green's function denotes the Doppler shift [52,80–87], which causes a tilted quasiparticle energy spectrum and hence markedly influences the superconducting anomalous correlation. Specifically, the gap equation reads [26]

$$\bar{\mathrm{T}}\mathrm{r}[G_0(p)\tau_1] = \sum_\mathbf{k} 2\Delta_0 F_\mathbf{k} = -\Delta_0/g, \quad (6)$$

where the anomalous correlation $F_\mathbf{k}$ is written as

$$F_\mathbf{k} = \frac{f(E_\mathbf{k}^+) - f(E_\mathbf{k}^-)}{2E_\mathbf{k}}. \quad (7)$$

In momentum space, the anomalous correlation $F_\mathbf{k}$ vanishes in regions with $|\mathbf{p}_s\cdot\mathbf{v_k}| > E_\mathbf{k}$ where the quasielectron energy $E_\mathbf{k}^+ < 0$ or quasihole energy $E_\mathbf{k}^- > 0$, but remains finite in the regions with $|\mathbf{p}_s\cdot\mathbf{v_k}| < E_\mathbf{k}$. Following the idea of the Fulde-Ferrell-Larkin-Ovchinnikov state in conventional superconductors [36,37], regions with nonzero and vanishing anomalous correlations are referred to as the pairing and unpairing regions, respectively. Particles in the pairing region contribute to the gap as a superfluid, whereas particles in the unpairing region no longer participate in the pairing and behave like normal ones, leading to the emergence of normal fluid [36,52]. However, in the discussion of the scattering effect, the essential Doppler shift term was approximately neglected in previous works [26,28,29] and has long been overlooked in the literature. In the present work, we sublate this approximation by keeping the Doppler-shift term in the Green's function.

B. Diagrammatic formalism

In this part, through the path-integral approach, we present the diagrammatic formalism to calculate the impurity scattering on nonequilibrium properties. For transport behavior in the diamagnetic response of superconductors, a supercurrent $\mathbf{j}$ is driven by a superconducting momentum $\mathbf{p}_s = -e\mathbf{A}$. At the clean limit, $\mathbf{j} = -e^2 n_s\mathbf{A}/m$, with n_s being the superfluid density, and the penetration depth then reads $\lambda_c = \sqrt{m/(4\pi e^2 n_s)}$ [26]. Nevertheless, with impurities, considering the friction resistance of the supercurrent mentioned in the Introduction as well as the role of the Doppler shift mentioned above, a self-consistent equation of motion of the superconducting current is required. In this circumstance, following the technique of applying the test charge in the Coulomb screening calculation [24], we consider a test nonequilibrium variation $\delta\mathbf{p}_s(x) = -e\delta\mathbf{A}$ on top of the uniform $\mathbf{p}_s = -e\mathbf{A}$, which leads to a nonequilibrium variation of the superconducting current $[\mathbf{j}\to\mathbf{j}+\delta\mathbf{j}(x)]$. Then, by deriving the linear response of $\delta\mathbf{j}$ to $\delta\mathbf{p}_s$, one equivalently obtains the self-consistent equation of motion of the superconducting current.

As for the Higgs mode (i.e., nonequilibrium gap fluctuation $\delta|\Delta(x)|$), its equation of motion $(\partial_t^2 - \omega_H^2)\delta|\Delta| = 0$ at the clean and low-frequency case, showing a gapful energy spectrum $\omega_H = 2\Delta_0$ in the long-wave limit, has been revealed by various theoretical approaches in the literature [12,16–22,38–42,53–56]. To discuss the damping, one also needs to derive the equation of motion of $\delta|\Delta|$ in the presence of the scattering.

To start, we begin with a general self-energy $\Sigma_\delta(x, p)$ by nonequilibrium variation. The action including this nonequilibrium self-energy is written as

$$S = \int dx\psi^\dagger(x)\big[G_0^{-1}(\hat{p}) - \Sigma_\delta(x,p) - V(x)\tau_3\big]\psi(x) - \int dx\bigg[\eta_f\Sigma_{\delta3} + \frac{\eta_f p_s^2}{2m} + \eta_f V(x) + \frac{(\Delta_0+\delta|\Delta|)^2}{g}\bigg], \quad (8)$$

where $\Sigma_{\delta i}$ denotes the τ_i component of Σ_δ. Through the integration over the Fermi field within the path-integral approach, one obtains the effective action,

$$\begin{aligned} S = &\int dx\bigg[\bar{\mathrm{T}}\mathrm{r}\ln\big[G_0^{-1} - \Sigma_\delta - V\tau_3\big] - \frac{(\Delta_0+\delta|\Delta|)^2}{g} - \eta_f \\ &\times\bigg(\Sigma_{\delta3} + V + \frac{\eta_f p_s^2}{2m}\bigg)\bigg] \\ &\times\int dx\bigg[\bar{\mathrm{T}}\mathrm{r}\ln G_0^{-1} - \eta_f\frac{p_s^2}{2m} - nV \\ &- \frac{(\Delta_0+\delta|\Delta|)^2}{g} - \eta_f\Sigma_{\delta3} \\ &- \bar{\mathrm{T}}\mathrm{r}(G_0\Sigma_\delta) - \sum_{n=2}^\infty\frac{1}{n}\bar{\mathrm{T}}\mathrm{r}\{[G_0(\Sigma_\delta+V\tau_3)]^n\}\bigg], \end{aligned} \quad (9)$$

where we have used $\bar{\mathrm{T}}\mathrm{r}[G_0\tau_3] + \eta_f = n$ (refer to Appendix A), with n denoting the charge density.

The equilibrium part in the effective action above reads

$$\begin{aligned} S_0 = \int dx\bigg\{&\sum_{p_n,\mathbf{k}}\ln(ip_n - E_\mathbf{k}^+)(ip_n - E_\mathbf{k}^-) - \bigg(\frac{\eta_f p_s^2}{2m} + \frac{\Delta_0^2}{g}\bigg) \\ &- \sum_{n=2}^\infty\frac{1}{n}\bar{\mathrm{T}}\mathrm{r}[(G_0V\tau_3)^n]\bigg\}. \end{aligned} \quad (10)$$

It is noted that the last term on the right-hand side of above equation denotes the equilibrium impurity self-energy [72–75], which in principle can cause renormalization on the equilibrium parameters, such as effective mass, chemical potential (charge density), and superconducting momentum, as well as gap [24].

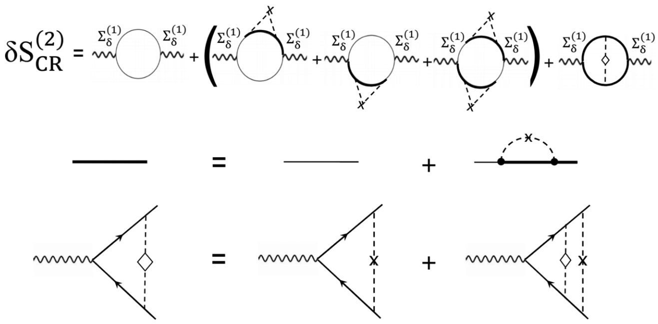

FIG. 1. Diagrammatic formalism for $\delta S^{(2)}_{\rm CR}$ (i.e., $\Sigma^{(1)}_{\delta}$-$\Sigma^{(1)}_{\delta}$ correlation). On the right-hand side of the equation of $\delta S^{(2)}_{\rm CR}$, the first diagram denotes the bare $\Sigma^{(1)}_{\delta}$-$\Sigma^{(1)}_{\delta}$ correlation; the second and third diagrams represent the Born and vertex corrections by impurity scattering, respectively. In the figure, the dashed lines with cross and diamond represent the impurity interaction and ladder diagram of impurity scattering, respectively; the wavy line is associated with the nonequilibrium self-energy $\Sigma^{(1)}_{\delta}$; the thin and thick solid lines denote the bare and renormalized Green's functions, respectively.

The nonequilibrium part in the effective action reads

$$\delta S = -\int dx\bigg[\sum_{n=2}^{\infty}\frac{1}{n}\bar{\rm Tr}([G_0(\Sigma_\delta+V\tau_3)]^n-(G_0V\tau_3)^n)$$
$$+\bar{\rm Tr}(G_0\Sigma_\delta)+\eta_f\Sigma_{\delta3}+\frac{2\Delta_0\delta|\Delta|+\delta|\Delta|^2}{g}\bigg]. \quad (11)$$

In principle, one only needs to consider the linear response of the weak nonequilibrium variation, i.e., keep up to the second order of the weak nonequilibrium variation in the nonequilibrium action. Then, by expanding the nonequilibrium self-energy as $\Sigma_\delta=\Sigma^{(1)}_\delta+\Sigma^{(2)}_\delta$ with $\Sigma^{(1)}_\delta$ and $\Sigma^{(2)}_\delta$ denoting the parts from the linear and second orders of the variation, respectively, the nonequilibrium action in Eq. (11) becomes

$$\delta S^{(2)}=\delta S^{(2)}_{\rm CR}+\delta S^{(2)}_{\rm VT}-\int dx\bigg(\eta_f\Sigma^{(2)}_{\delta3}+\frac{\delta|\Delta|^2}{g}\bigg), \quad (12)$$

with the contribution of the $\Sigma^{(1)}_{\delta}$-$\Sigma^{(1)}_{\delta}$ correlation,

$$\delta S^{(2)}_{\rm CR}=-\frac{1}{2}\int dx\bar{\rm Tr}\big[\Sigma^{(1)}_\delta G_0\Sigma^{(1)}_\delta G_0+2(G_0V\tau_3)^2\big(G_0\Sigma^{(1)}_\delta\big)^2$$
$$+G_0\Sigma^{(1)}_\delta G_0V\tau_3G_0\Sigma^{(1)}_\delta G_0V\tau_3+O(V^{n>2})\big], \quad (13)$$

as well as a direct $\Sigma^{(2)}_{\delta}$-vertex contribution:

$$\delta S^{(2)}_{\rm VT}=-\int dx\bar{\rm Tr}\big[\big(G_0\Sigma^{(2)}_\delta\big)+G_0\Sigma^{(2)}_\delta(G_0V\tau_3)^2+O(V^{n>2})\big]. \quad (14)$$

Consequently, from the nonequilibrium action $\delta S^{(2)}$, by determining the corresponding nonequilibrium self-energy, one can derive the property of the nonequilibrium variation as well as the related scattering effect. Specifically, the $\Sigma^{(1)}_{\delta}$-$\Sigma^{(1)}_{\delta}$ correlation $\delta S^{(2)}_{\rm CR}$ in Eq. (13) is illustrated in Fig. 1 by a connected Feynman diagram of the correlation. Corresponding to Fig. 1, on the right-hand side of Eq. (13), the first term denotes the bare $\Sigma^{(1)}_{\delta}$-$\Sigma^{(1)}_{\delta}$ correlation; the second and third terms represent the Born and vertex corrections by impurity scattering [24], respectively. Whereas the $\Sigma^{(2)}_{\delta}$-vertex contribution $\delta S^{(2)}_{\rm VT}$ in Eq. (14) is illustrated by the Feynman diagram in Fig. 2 with a renormalized bubble. As seen from the figure, the impurity interaction in $\delta S^{(2)}_{\rm VT}$ makes no contribution to the nonequilibrium property and only provides the renormalization to the fermion bubble, which is same as the one by the equilibrium impurity self-energy.

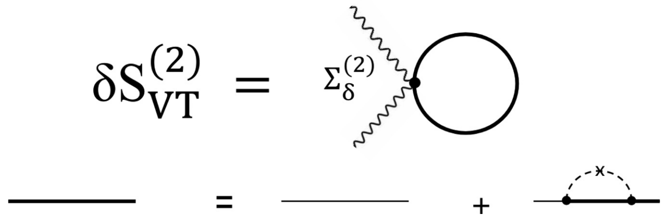

FIG. 2. Diagrammatic formalism for $\delta S^{(2)}_{\rm VT}$ (i.e., contribution directly from $\Sigma^{(2)}_{\delta}$ vertex). In the figure, the thin and thick solid lines denote the bare and renormalized Green's function, respectively; the wavy line and dashed line with cross are associated with the nonequilibrium self-energy $\Sigma^{(2)}_{\delta}$ and impurity interaction, respectively.

III. TRANSPORT BEHAVIOR

In this section, we focus on the transport behavior in the diamagnetic response of superconductors. Physically, the current is conventionally expressed as $\mathbf{j}=\langle\psi|\mathbf{\Pi}|\psi\rangle/m$, with the kinematical momentum operator $\hat{\mathbf{\Pi}}=\hat{\mathbf{p}}-e\mathbf{A}\tau_3$. Among this expression, the current vertex $\hat{\mathbf{p}}/m$ drives a current $\mathbf{j}_d$ through the current-current correlation within the path-integral approach [22,42] or Kubo formula [24]. The $e\mathbf{A}\tau_3/m$ part is related to the density vertex τ_3 [79] and directly pumps a current $\mathbf{j}_p=-e^2n\mathbf{A}/m$, which is considered as an unphysical non-gauge-invariant current in the literature [24,26]. In normal metals, at the stationary case, the drive current $\mathbf{j}_d$ exactly cancels the pump current $\mathbf{j}_p$, and hence, the total current vanishes as it should be, since the stationary magnetic vector potential can not drive the normal-state current. Whereas in superconductors, only a part of $\mathbf{j}_d$ cancels $\mathbf{j}_p$ [22], and then the diamagnetic superfluid current $\mathbf{j}=-en_s\mathbf{A}/m$ emerges in the remaining part of the drive current $\mathbf{j}_d$.

For the convenience of analysis and understanding, we first derive the equilibrium supercurrent $\mathbf{j}=en_s\mathbf{p}_s/m$ and hence superfluid density n_s from the equilibrium action S_0 [Eq. (10)] at the clean case. After that, with impurities, we discuss the nonequilibrium transport behavior by considering a nonequilibrium variation $\delta\mathbf{p}_s$ generated by $e\delta\mathbf{A}$. It is shown that in the calculation based on kinematical momentum operator $\delta\hat{\mathbf{\Pi}}=\hat{\mathbf{p}}-e\delta\mathbf{A}\tau_3$, the scattering treatment/correction leads to a mismatch in the cancellation process of the non-gauge-invariant pump current, because of the non-commutative $[\hat{\mathbf{p}},V(\mathbf{r})\tau_3]$ and commutative $[e\delta\mathbf{A}\tau_3,V(\mathbf{r})\tau_3]$. As a consequence, an unphysical current emerges. To fix this issue, we apply a special unitary transformation to eliminate the non-gauge-invariant current vertex $e\delta\mathbf{A}\tau_3/m$ and obtain the Meissner supercurrent vertex. Then, by performing an analytical calculation of the supercurrent-supercurrent correlation with Born and vertex corrections from the impurity scattering, the microscopic momentum-relaxation rate of superfluid is obtained to compare with GIKE and explain the relaxation-time approximation in the previous diagrammatic formulations [28–30] as well as the friction resistance of the Meissner supercurrent revealed in the previous works [26,28–30,47,51,56,69]. It is further proved that the applied unitary transformation is actually equivalent to the Wilson-line technique [66] for the diamagnetic response.

A. Equilibrium transport property

From the equilibrium action in Eq. (10) at the clean case, the supercurrent is given by

$$\mathbf{j} = -e\partial_{\mathbf{p}_s} S_0 = \mathbf{j}_d + \mathbf{j}_p, \tag{15}$$

where

$$\begin{aligned}\mathbf{j}_p &= -\frac{2e\mathbf{p}_s}{m}\frac{\partial S_0}{\partial\left(p_s^2/m\right)} \\ &= \sum_{p_n,\mathbf{k}} \frac{2e\mathbf{p}_s\left(\xi_{\mathbf{k}} + p_s^2/2m\right)}{m(ip_n - E_{\mathbf{k}}^+)(ip_n - E_{\mathbf{k}}^-)} + \frac{\eta_f e\mathbf{p}_s}{m} = \frac{en\mathbf{p}_s}{m}\end{aligned} \tag{16}$$

and

$$\begin{aligned}\mathbf{j}_d &= -e\mathbf{v}_{\mathbf{k}}\frac{\partial S_0}{\partial(\mathbf{v}_{\mathbf{k}}\cdot\mathbf{p}_s)} = \sum_{p_n,\mathbf{k}} \frac{2e\mathbf{v}_{\mathbf{k}}(ip_n - \mathbf{v}_{\mathbf{k}}\cdot\mathbf{p}_s)}{(ip_n - E_{\mathbf{k}}^+)(ip_n - E_{\mathbf{k}}^-)} \\ &= \frac{2ek_F^2\mathbf{p}_s}{3m^2}\sum_{\mathbf{k}} \partial_{E_{\mathbf{k}}}(E_{\mathbf{k}}F_{\mathbf{k}}) \\ &= \frac{2ek_F^2\mathbf{p}_s}{3m^2}\sum_{\mathbf{k}} \left[\frac{\Delta_0^2}{E_{\mathbf{k}}}\partial_{E_{\mathbf{k}}}F_{\mathbf{k}} + \partial_{\xi_{\mathbf{k}}}(\xi_{\mathbf{k}}F_{\mathbf{k}})\right] \\ &= \frac{en_s\mathbf{p}_s}{m} - \frac{en\mathbf{p}_s}{m}.\end{aligned} \tag{17}$$

Here, the superfluid density n_s is given by

$$n_s = \frac{2k_F^2}{3m}\sum_{\mathbf{k}} \frac{\Delta_0^2}{E_{\mathbf{k}}}\partial_{E_{\mathbf{k}}}F_{\mathbf{k}}, \tag{18}$$

which is exactly same as the results obtained in previous works [26,42,52,80] by various approaches.

It is noted that $\mathbf{j}_d$ is associated with the drive current mentioned above, since it arises from the second order of the current-vertex-related term $\mathbf{p}_s \cdot \mathbf{v}_{\mathbf{k}}$, whereas $\mathbf{j}_p$ comes from the density-vertex-related term $\mathbf{p}_s^2\tau_3/(2m)$ and corresponds to the non-gauge-invariant pump current. Then, it is clearly seen that in normal metals with the vanishing superfluid density ($n_s = 0$), the drive current $\mathbf{j}_d$ exactly cancels the pump current $\mathbf{j}_p$, and hence the total current vanishes, whereas in superconductors, only the second term in $\mathbf{j}_d$ [Eq. (17)] cancels $\mathbf{j}_p$ [22], and then, the superfluid current $\mathbf{j} = -en_s\mathbf{A}/m$ emerges in the remaining part [first term in Eq. (17)] of $\mathbf{j}_d$.

B. Issue of gauge-invariance breaking in the conventional current-current correlation

We next discuss the nonequilibrium property. Based on the Hamiltonian in Eq. (1), considering a variation of the superconducting momentum, i.e., $\mathbf{p}_s \to \mathbf{p}_s + \delta\mathbf{p}_s(x)$, the nonequilibrium self-energy reads

$$\Sigma_\delta = \delta\mathbf{p}_s\cdot\mathbf{v}_{\hat{\mathbf{p}}} + \delta\mathbf{p}_s\cdot\mathbf{v}_{\mathbf{p}_s}\tau_3 + \frac{\delta p_s^2\tau_3}{2m} \approx \delta\mathbf{p}_s\cdot\mathbf{v}_{\hat{\mathbf{p}}} + \frac{\delta p_s^2\tau_3}{2m}, \tag{19}$$

in which we have neglected the term $\delta\mathbf{p}_s\cdot\mathbf{v}_{\mathbf{p}_s}\tau_3$ in comparison to $\delta\mathbf{p}_s\cdot\mathbf{v}_{\hat{\mathbf{p}}}$ since $p_s \ll k_F$ in conventional superconductors.

One then has the current-vertex-related term $\Sigma_\delta^{(1)} = \delta\mathbf{p}_s\cdot\mathbf{v}_{\hat{\mathbf{p}}}$ and density-vertex-related one $\Sigma_\delta^{(2)} = \delta p_s^2\tau_3/(2m)$, which therefore contribute to the drive $\delta\mathbf{j}_d$ and pump $\delta\mathbf{j}_p$ currents through the corresponding current-current correlation in $S_{\rm CR}^{(2)}$ [Eq. (13)/Fig. 1] and density-vertex contribution in $S_{\rm VT}^{(2)}$ [Eq. (14)/Fig. 2], respectively. Nevertheless, as pointed out in Sec. II B, the correlation contribution $S_{\rm CR}^{(2)}$ experiences the Born and vertex corrections by impurity scattering, and hence the drive current $\delta\mathbf{j}_d$ experiences the scattering influence, whereas the impurity interaction in the vertex contribution $S_{\rm VT}^{(2)}$ makes no contribution to the nonequilibrium property, except for the normalization to the corresponding vertex. For the density vertex in this circumstance, the renormalization on charge density has been revealed to vanish in the literature [72–75]. Therefore, one has the pump current $\delta\mathbf{j}_p = en\delta\mathbf{p}_s/m$ free from the scattering influence.

As mentioned above, the non-gauge-invariant pump current $\delta\mathbf{j}_p$ needs to be canceled by the corresponding charge-density part in the drive current $\delta\mathbf{j}_d$, so only the contribution of the superfluid density retains in the diamagnetic response. Nevertheless, with impurities, $\delta\mathbf{j}_d$ experiences the scattering influence but $\delta\mathbf{j}_p$ does not, directly leading to a mismatch in the cancellation process. This mismatch arises from the breaking of the gauge invariance by scattering treatment. Specifically, in the diamagnetic response, the prerequisite for $\delta\mathbf{j}_p$ to exactly cancel the corresponding charge-density part in $\delta\mathbf{j}_d$ requires a gauge-invariant expected value $\langle\psi|\hat{\mathbf{p}} - e\delta\mathbf{A}\tau_3|\psi\rangle/m$ of current. But due to the noncommutative $[\hat{\mathbf{p}}, V(\mathbf{r})\tau_3]$ and commutative $[e\delta\mathbf{A}\tau_3, V(\mathbf{r})\tau_3]$, the scattering treatment only plays a role in $\langle\psi|\hat{\mathbf{p}}|\psi\rangle$ but makes zero influence on $\langle\psi|e\delta\mathbf{A}\tau_3|\psi\rangle$, leading to the gauge-invariance breaking of the expected value of current.

To solve this issue, Abrikosov *et al.* applied an approximation [26] that assumes an isotropic Green's function at the dirty case to first integrate over the momentum variable as mentioned in the Introduction. Then, one can distinguish the contributions from superfluid density n_s and total charge density n in $\delta\mathbf{j}_d$, and eliminate the scattering effect in the later contribution to cancel the non-gauge-invariant pump current $\delta\mathbf{j}_p$, whereas the transport-equation formalism [52,65] applies the Wilson-line [66] technique. This technique by constructing the gauge-invariant basis $|\psi_g\rangle$ leads to a gauge-invariant current vertex $\hat{\mathbf{j}}_g$, and the non-gauge-invariant current part $e\delta\mathbf{A}\tau_3/m$ naturally vanishes.

C. Supercurrent-supercurrent correlation

In this section, we apply a special unitary transformation to eliminate the non-gauge-invariant current vertex $e\delta\mathbf{A}\tau_3/m$ and obtain the Meissner supercurrent vertex. Then, one can calculate the supercurrent-supercurrent correlation with Born and vertex corrections from the impurity scattering and obtain the microscopic momentum-relaxation rate of the superfluid.

Specifically, from Eq. (2), considering a variation $\delta\mathbf{p}_s$ of the superconducting momentum, the action is written as

$$S = \int dx\Big\{\sum_{s=\uparrow,\downarrow}\psi_s^*(x)[i\partial_{x_0}-\xi_{\hat{\mathbf{p}}+\mathbf{p}_s+\delta\mathbf{p}_s}-V(x)]\psi_s(x) - \psi^\dagger(x)\Delta_0\tau_1\psi(x)-\frac{\Delta_0^2}{g}\Big\}. \quad (20)$$

Applying the unitary transformation

$$\psi(x)\to\exp\Big[i\tau_3\int_0^{\mathbf{x}}\delta\mathbf{p}_s(\mathbf{x}')\cdot d\mathbf{x}'\Big]\psi(x), \quad (21)$$

the action becomes

$$S = \int dx\psi_g^\dagger(x)\Big[i\partial_{x_0}-\xi_{\hat{\mathbf{p}}+\mathbf{p}_s\tau_3}\tau_3-V(x)\tau_3-\Delta_0\tau_+ \times\exp\Big(2i\int_0^{\mathbf{x}}\delta\mathbf{p}_s\cdot d\mathbf{x}'\Big)-\Delta_0\tau_- \times\exp\Big(-2i\int_0^{\mathbf{x}}\delta\mathbf{p}_s\cdot d\mathbf{x}'\Big)\Big]\psi_g(x) - \int dx\Big[\frac{\Delta_0^2}{g}+\eta_f V(x)\Big]. \quad (22)$$

Here, we only focus on the stationary diamagnetic response and neglect the electric-field effect.

Consequently, from the action in Eq. (22), for the small variation, one finds the nonequilibrium self-energy:

$$\Sigma_\delta = -2\Delta_0\tau_2\int_0^{\mathbf{x}}\delta\mathbf{p}_s\cdot\mathbf{dx}'. \quad (23)$$

Then, in comparison to Eq. (19) based on the conventional current vertex, the density-vertex-related term $\delta p_s^2\tau_3/m$ that is related to the non-gauge-invariant pump current vanishes in Eq. (23), and hence there is no vertex contribution $\delta S^{(2)}_{\rm VT}$. Particularly, it is noted that at the long-wave limit, the derived self-energy in Eq. (23) becomes

$$\Sigma_\delta^{(1)} = -2\Delta_0\tau_2\delta\mathbf{p}_s\cdot\hat{\mathbf{x}} = 2i\Delta_0\tau_2\delta\mathbf{p}_s\cdot\partial_{\hat{\mathbf{k}}}, \quad (24)$$

which describes the drive effect by $\delta\mathbf{p}_s$ in the diamagnetic response to generate the Meissner supercurrent [26]. We therefore refer to $i2\Delta_0\tau_2\partial_{\hat{\mathbf{k}}}$ as the Meissner-supercurrent vertex.

Consequently, with the Meissner-supercurrent vertex [Eq. (24)], one can derive the supercurrent-supercurrent correlation $\delta S^{(2)}_{\rm CR}$ [Eq. (13)/Fig. 1] with the Born and vertex corrections by impurity scattering. By assuming an adiabatic variation $\delta\mathbf{p}_s(x)=\delta\mathbf{p}_s e^{iqz+0^+x_0}$ with z being the spatial coordinate along the direction perpendicular to the surface, at weak impurity interaction, after the summation of the Matsubara frequency, the current $\delta\mathbf{j}=-e\partial_{\delta\mathbf{p}_s^*(x)}\delta S^{(2)}=-e\partial_{\delta\mathbf{p}_s}\delta S^{(2)}/2$ is derived as (refer to Appendix B)

$$\delta\mathbf{j} = \sum_{\mathbf{k}}e\mathbf{v}_{\mathbf{k}}\Big\{\rho_{\mathbf{k}}+\frac{n_i\pi}{i\zeta_{\mathbf{k}}v_Fq}\sum_{\mathbf{k}'}|V_{\mathbf{k}\mathbf{k}'}|^2(\rho_{\mathbf{k}}-\rho_{\mathbf{k}'}) \times\Big[\sum_{\eta=\pm}\big(e^-_{\mathbf{k}\mathbf{k}'}\delta(E^\eta_{\mathbf{k}}-E^\eta_{\mathbf{k}'})+e^+_{\mathbf{k}\mathbf{k}'}\delta(E^\eta_{\mathbf{k}}-E^{-\eta}_{\mathbf{k}'})\big)\Big]\Big\}, \quad (25)$$

with $\rho_{\mathbf{k}}=(\mathbf{v}_{\mathbf{k}}\cdot\delta\mathbf{p}_s)\frac{2\Delta_0^2}{E_{\mathbf{k}}}\partial_{E_{\mathbf{k}}}F_{\mathbf{k}}$ and $e^\pm_{\mathbf{k}\mathbf{k}'}=\frac{1}{2}(1\pm\frac{\Delta_0^2}{E_{\mathbf{k}}E_{\mathbf{k}'}})$; $\zeta_{\mathbf{k}}$ represents a coefficient (refer to Appendix B). It is noted that $\zeta_{\mathbf{k}}v_Fq$ is a diffusive pole, which emerges at the stationary diffusion case. The first and second terms on the right-hand side of Eq. (25) represent the source and scattering terms, respectively, both of which exactly recover the ones from GIKE [52,80].

Further considering the fact that $(\rho_{\mathbf{k}}-\rho_{\mathbf{k}'})[\delta(E^+_{\mathbf{k}}-E^+_{\mathbf{k}'})+\delta(E^-_{\mathbf{k}}-E^-_{\mathbf{k}'})]$ in the scattering term of Eq. (25) vanishes around the Fermi surface, the current becomes

$$\delta\mathbf{j} = \sum_{\mathbf{k}}e\mathbf{v}_{\mathbf{k}}\Big\{\rho_{\mathbf{k}}+\frac{n_i\pi}{i\zeta_{\mathbf{k}}v_Fq}\sum_{\mathbf{k}'}|V_{\mathbf{k}\mathbf{k}'}|^2(\rho_{\mathbf{k}}-\rho_{\mathbf{k}'}) \times\Big[\sum_{\eta=\pm}e^+_{\mathbf{k}\mathbf{k}'}\delta(E^\eta_{\mathbf{k}}-E^{-\eta}_{\mathbf{k}'})\Big]\Big\}. \quad (26)$$

As pointed out in Refs. [52,80], the scattering term in above equation is finite only at the emergence of the normal fluid, which requires $|\mathbf{v}_{\mathbf{k}}\cdot\mathbf{p}_s|>E_{\mathbf{k}}$ as mentioned in Sec. II A, whereas this condition requires a threshold $p_L=\Delta_0/v_F$ for the superconducting momentum p_s to exceed. Therefore, at $p_s<\Delta_0/v_F$, one has

$$\delta\mathbf{j} = \frac{en_s\delta\mathbf{p}_s}{m}, \quad (27)$$

which is free from the diffusive influence by impurity scattering, showing the superconductivity phenomenon.

Whereas for $p_s>\Delta_0/v_F$, as pointed out in Ref. [52,80], the current is captured by a three-fluid (normal fluid as well as viscous and nonviscous superfluids) model and can be divided into three parts:

$$\delta\mathbf{j} = \delta\mathbf{j}_{\rm vs}+\delta\mathbf{j}_{\rm nvs}+\delta\mathbf{j}_{\rm n}, \quad (28)$$

with

$$\delta\mathbf{j}_{\rm vs} = \sum_{\mathbf{k}\in{\rm P_v}}e\mathbf{v}_{\mathbf{k}}\rho_{\mathbf{k}}\Big(1+\frac{\Gamma_{\mathbf{k}}}{i\zeta_{\mathbf{k}}v_Fq}\Big), \quad (29)$$

$$\delta\mathbf{j}_{\rm nvs} = \sum_{\mathbf{k}\in{\rm P_{nv}}}e\mathbf{v}_{\mathbf{k}}\rho_{\mathbf{k}}, \quad (30)$$

$$\delta\mathbf{j}_{\rm n} = -\sum_{\mathbf{k}\in U}\frac{e\mathbf{v}_{\mathbf{k}}}{i\zeta_{\mathbf{k}}v_Fq}\sum_{\mathbf{k}'\in{\rm P_v}}\rho_{\mathbf{k}'}D_{\mathbf{k}\mathbf{k}'}, \quad (31)$$

and

$$\Gamma_{\mathbf{k}} = n_i\pi\sum_{\mathbf{k}'\in U}|V_{\mathbf{k}\mathbf{k}'}|^2e^+_{\mathbf{k}\mathbf{k}'}\Big[\sum_{\eta=\pm}\delta(E^\eta_{\mathbf{k}}-E^{-\eta}_{\mathbf{k}'})\Big], \quad (32)$$

$$D_{\mathbf{k}\mathbf{k}'} = n_i\pi|V_{\mathbf{k}\mathbf{k}'}|^2\Big[\sum_{\eta=\pm}e^+_{\mathbf{k}\mathbf{k}'}\delta(E^\eta_{\mathbf{k}}-E^{-\eta}_{\mathbf{k}'})\Big]. \quad (33)$$

Here, n_i denotes the impurity density and $N(0)$ represents the density of states; $\Gamma_{\mathbf{k}}$ stands for the microscopic momentum-relaxation rate of superfluid; $D_{\mathbf{k}\mathbf{k}'}$ represents the microscopic friction rate between superfluid and normal fluid; $\rm P_{nv}$ denotes the nonviscous pairing regions in momentum space with finite $F_{\mathbf{k}}$ but zero $\Gamma_{\mathbf{k}}$; $\rm P_v$ represents the viscous pairing regions with both finite $F_{\mathbf{k}}$ and $\Gamma_{\mathbf{k}}$; U stands for the unpairing region with vanishing $F_{\mathbf{k}}$.

Specifically, in the source term on the right-hand side of Eq. (26), only particles in the pairing regions with nonzero anomalous correlation $F_{\mathbf{k}}$ are driven by $\delta\mathbf{p}_s$ to contribute to the current. For the $\mathbf{k}$ particle lying in the pairing region, one has $E^+_{\mathbf{k}} > 0$ and $E^-_{\mathbf{k}} < 0$ as mentioned in Sec. II A. In this circumstance, in the scattering term, once the energy conservation cannot be satisfied for any $\mathbf{k}'$, the $\mathbf{k}$ particle is free from the momentum-relaxation scattering, and one therefore gets the nonviscous superfluid and hence the current $\mathbf{j}_{\rm nvs}$ in Eq. (30). But once the energy conservation is satisfied, to give rise to the nonzero scattering term, one finds $E^-_{\mathbf{k}'} > 0$ by $\delta(E^+_{\mathbf{k}} - E^-_{\mathbf{k}'})$ or $E^+_{\mathbf{k}'} < 0$ by $\delta(E^-_{\mathbf{k}} - E^+_{\mathbf{k}'})$, and hence the $\mathbf{k}'$ particle lies in the unpairing region (normal fluid) with vanishing $F_{\mathbf{k}'}$ and hence $\rho_{\mathbf{k}'}$. This scattering between particles in pairing and unpairing regions behaves like the friction between superfluid and normal fluid, leading to the viscous superfluid and the current $\mathbf{j}_{\rm vs}$ in Eq. (29).

It is noted that for particles in the unpairing regions (normal fluid), although the source term in Eq. (26) is zero as it should be, but due to the friction mentioned above, the scattering term is finite. Specifically, if the $\mathbf{k}$ particle lies in the unpairing region with $E^-_{\mathbf{k}} > 0$ or $E^+_{\mathbf{k}} < 0$, according to the energy conservation in the scattering term, the $\mathbf{k}'$ particle can lie in both viscous pairing and unpairing regions, i.e., the particles from the normal fluid experience the scattering from those in both viscous superfluid and normal fluid. The scattering between particles in normal fluid is natural but makes zero contribution to the current as $\rho_{\mathbf{k}\in U} = \rho_{\mathbf{k}'\in U} = 0$ in the scattering term. But through the friction drag with the viscous superfluid current, a normal-fluid current is induced in Eq. (31).

Based on the analysis above, the total current at $p_s > \Delta_0/v_F$ can be rewritten as

$$\delta\mathbf{j} = \frac{en_s^{\rm eff}\delta\mathbf{p}_s}{m} + \frac{\Gamma_c^{\rm eff}}{iv_Fq}\frac{en_s^{\rm eff}\delta\mathbf{p}_s}{m}, \tag{34}$$

with the effective superfluid density,

$$n_s = \sum_{\mathbf{k}\in {\rm P_{nv}+P_v}} \frac{2k_F^2\cos\theta_{\mathbf{k}}^2}{m}\frac{\Delta_0^2}{E_{\mathbf{k}}}\partial_{E_{\mathbf{k}}}F_{\mathbf{k}}, \tag{35}$$

and effective current-relaxation rate:

$$\Gamma_c^{\rm eff} = \frac{\sum_{\mathbf{k}\in {\rm P_v}}\mathbf{v}_{\mathbf{k}}\rho_{\mathbf{k}}\Gamma_{\mathbf{k}}/\zeta_{\mathbf{k}} - \sum_{\mathbf{k}\in {\rm U},\mathbf{k}'\in {\rm P_v}}\mathbf{v}_{\mathbf{k}}\rho_{\mathbf{k}'}D_{\mathbf{k}\mathbf{k}'}/\zeta_{\mathbf{k}}}{\sum_{\mathbf{k}\in {\rm P_{nv}+P_v}}\mathbf{v}_{\mathbf{k}}\rho_{\mathbf{k}}}. \tag{36}$$

We then obtain the equation of motion of the supercurrent with the influence of the scattering. Then, in real space, Eq. (34) becomes a diffusive equation:

$$\partial_z\delta\mathbf{j} = \partial_z\delta\mathbf{p}_s\frac{en_s^{\rm eff}}{m} + \frac{\Gamma_c^{\rm eff}}{v_F}\frac{en_s^{\rm eff}\delta\mathbf{p}_s}{m}, \tag{37}$$

which in consideration of the fact $\partial_z\delta\mathbf{p}_s = \partial_z\mathbf{p}_s$ is equivalent to

$$\partial_z^2\mathbf{j} = \partial_z^2\mathbf{p}_s\frac{en_s^{\rm eff}}{m} + \frac{\Gamma_c^{\rm eff}}{v_F}\frac{en_s^{\rm eff}\partial_z\mathbf{p}_s}{m}, \tag{38}$$

Consequently, we arrive at a self-consistent equation of motion of the superconducting momentum/current with the influence of the scattering. Clearly, the second term on the right-hand side of the above equation denotes the friction resistance of the supercurrent. As mentioned above, this resistance [$\Gamma_{\mathbf{k}}$ in Eq. (32) and $D_{\mathbf{k}}$ in Eq. (33)] is nonzero only with the emergence of the normal fluid at $p_s > \Delta_0/v_F$. Therefore, as pointed out in Ref. [52], the friction resistance of the Meissner supercurrent in the diamagnetic response emerges only when the superconducting velocity is larger than a threshold at which the normal fluid emerges, similar to Landau's theory for the emerged fluid viscosity in bosonic liquid helium at larger velocity [78].

Together with the Maxwell equation, the penetration depth from Eq. (38) is derived as

$$\lambda = \lambda_c\sqrt{\frac{1}{1-\frac{\xi}{l}}} \approx \lambda_c\sqrt{1+\frac{\xi}{l}}, \tag{39}$$

where the coherence length $\xi = \lambda/\kappa$ as well as the mean-free path $l = v_F/(\kappa\Gamma_c^{\rm eff})$ and clean-limit penetration depth $\lambda_c = \sqrt{m/(4\pi e^2 n_s^{\rm eff})}$, with κ being the Ginzburg-Landau parameter. Then, the previously revealed phenomenological dependence of the penetration depth on mean-free path by Tinkham [30] is recovered within the diagrammatic formalism at the weak scattering case—same as the formulation within the GIKE [52].

1. Role of Doppler shift

It is noted that the Doppler shift plays two important roles in the derivation/results above. On one hand, as mentioned in Sec. II A, it leads to the generation of the normal fluid at $p_s > \Delta_0/v_F$. On the other hand, it guarantees the vanishing intraband scattering in Eq. (25) by the $(\rho_{\mathbf{k}} - \rho_{\mathbf{k}'})\delta(E^\eta_{\mathbf{k}} - E^\eta_{\mathbf{k}'})$ part around Fermi surface, and hence only the interband scattering by the $(\rho_{\mathbf{k}} - \rho_{\mathbf{k}'})\delta(E^\eta_{\mathbf{k}} - E^{-\eta}_{\mathbf{k}'})$ part retains. This interband scattering occurs only when p_s is larger than the threshold Δ_0/v_F [52], and hence the momentum relaxation of the superfluid current emerges only at $p_s > \Delta_0/v_F$ (where the normal fluid emerges). However, in the previous formulation of the scattering in superconductors [26], the Doppler shift was approximately neglected. As a consequence, the normal fluid dynamics is absent. Most importantly, in this circumstance, the interband scattering in Eq. (25) by the $(\rho_{\mathbf{k}} - \rho_{\mathbf{k}'})\delta(E^\eta_{\mathbf{k}} - E^{-\eta}_{\mathbf{k}'}) = (\rho_{\mathbf{k}} - \rho_{\mathbf{k}'})\delta(\sqrt{\Delta_0^2+\xi_k^2} + \sqrt{\Delta_0^2+\xi_{k'}^2})$ part is forbidden, but the interband scattering by the $(\rho_{\mathbf{k}} - \rho_{\mathbf{k}'})\delta(E^\eta_{\mathbf{k}} - E^\eta_{\mathbf{k}'}) = (\rho_{\mathbf{k}} - \rho_{\mathbf{k}'})\delta(\sqrt{\Delta_0^2+\xi_k^2} - \sqrt{\Delta_0^2+\xi_{k'}^2})$ part is always finite around Fermi surface. Consequently, the superfluid current in the diamagnetic response always experiences the friction resistance by impurity scattering as the theoretical descriptions in the previous works [26,47,51,56,69] revealed, in contrast to the superconductivity phenomenon.

D. A gauge-invariant description

In the previous part, within the diagrammatic formalism, by applying the unitary transformation in Eq. (21), the non-gauge-invariant-current vertex $e\mathbf{A}\tau_3/m$, i.e., the issue of gauge-invariance breaking by scattering treatment as mentioned in Sec. III B, is eliminated. To understand this unitary transformation, following the treatment within the transport-equation formalism [52,65], we next apply the Wilson-line [66] technique to construct the gauge-invariant field operator and provide a gauge-invariant description to handle the nonequilibrium transport property in superconductors.

We begin with the action in consideration of a four-vector variation $e\delta A_\mu = (e\delta\phi, e\delta\mathbf{A})$ of electromagnetic potential:

$$S = \int dx \sum_{s=\uparrow,\downarrow} \psi_s^*(x)[i\partial_{x_0} - \xi_{\hat{\mathbf{p}}+\mathbf{p}_s-e\delta\mathbf{A}} - e\delta\phi(x)]\psi_s(x) - \int dx\left[\psi^\dagger(x)\hat{\Delta}(x)\psi(x) + \frac{|\Delta(x)|^2}{g}\right] = \int dx\left[\psi^\dagger(x)\hat{G}^{-1}\psi(x) - \eta_f\frac{(\mathbf{p}_s - e\delta\mathbf{A})^2}{2m} - \frac{|\Delta(x)|^2}{g}\right]. \quad (40)$$

Here, $\hat{G}^{-1} = i\partial_{x_0} - \xi_{\hat{\mathbf{p}}+\mathbf{p}_s\tau_3-e\delta\mathbf{A}\tau_3}\tau_3 - e\delta\phi(x)\tau_3 - \hat{\Delta}(x)$; $\hat{\Delta}(x) = \Delta(x)\tau_+ + \Delta^*(x)\tau_-$, where the superconducting order parameter $\Delta(x) = [\Delta_0 + \delta|\Delta(x)|]e^{i\delta\theta(x)}$ with Δ_0 and $\delta|\Delta(x)|$ as well as $\delta\theta(x)$ denoting the equilibrium gap and nonequilibrium Higgs mode [12–21] as well as the superconducting phase fluctuation [12,79,88–96], respectively. In addition, we expand the scalar potential as

$$\delta\phi(x) = \delta\phi_0(x_0) + \int_0^{\mathbf{x}} \nabla_{\mathbf{x}'}\delta\phi(x_0, \mathbf{x}')d\mathbf{x}' \quad (41)$$

to distinguish the Josephson voltage effect [97] by $\delta\phi_0(x_0)$ and electric-field drive effect by $\nabla_{\mathbf{x}}\delta\phi(x_0, \mathbf{x})$.

It is noted that under a gauge transformation,

$$\psi(x) \to e^{i\tau_3\chi(x)}\psi(x), \quad (42)$$

the action in Eq. (40) satisfies the gauge structure in superconductors revealed by Nambu [88,95],

$$e\delta A_\mu \to e\delta A_\mu - \partial_\mu\chi(x), \quad (43)$$

$$\delta\theta(x) \to \delta\theta(x) + 2\chi(x), \quad (44)$$

where the four-vector $\partial_\mu = (\partial_{x_0}, -\nabla)$.

Under the gauge transformations in Eqs. (42)–(44), the conventional Wilson-line [66] technique to construct gauge-invariant field operator $\psi_g = e^{i\tau_3 P\int_0^x dx^\mu e\delta A_\mu}\psi$ is difficult to handle for deriving the gauge-invariant kernel $\hat{G}^{-1}$ and performing the further calculation within the diagrammatic formalism. To simplify the formulation, we restrict the gauge-transformation function $\chi(x)$ to depend on either a spatial coordinate or time coordinate. Then, one can apply a simplified Wilson-line technique to construct the gauge-invariant field operator:

$$\psi_g(x) = \exp\left[i\tau_3 e\left(\int_0^{x_0}\delta\phi_0 dx_0' - \int_0^{\mathbf{x}}\delta\mathbf{A}\cdot d\mathbf{x}'\right)\right]\psi(x). \quad (45)$$

Consequently, on basis of the gauge-invariant $\psi_g(x)$, the action in Eq. (40) becomes

$$S = \int dx\left[\psi_g^\dagger(x)\hat{G}_g^{-1}(x)\psi_g(x) - \frac{|\Delta(x)|^2}{g}\right], \quad (46)$$

with the Green's-function kernel:

$$\hat{G}_g^{-1}(x) = i\partial_{x_0} - \xi_{\hat{\mathbf{p}}+\mathbf{p}_s\tau_3}\tau_3 - \int_0^{\mathbf{x}} e\mathbf{E}\cdot d\mathbf{x}' - |\Delta|\tau_+\exp\left(i\delta\theta + 2ie\int_0^{x_0}\delta\phi_0 dx_0' - 2ie\int_0^{\mathbf{x}}\delta\mathbf{A}\cdot d\mathbf{x}'\right) - |\Delta|\tau_-\exp\left(-i\delta\theta - 2ie\int_0^{x_0}\delta\phi_0 dx_0' + 2ie\int_0^{\mathbf{x}}\delta\mathbf{A}\cdot d\mathbf{x}'\right). \quad (47)$$

Here, $\mathbf{E} = -\nabla_{\mathbf{x}}\delta\phi - \partial_{x_0}\delta\mathbf{A}$ denotes the gauge-invariant electric field. It is noted that $\hat{G}_g^{-1}(x)$ is directly gauge invariant under the gauge transformations in Eqs. (43) and (44). Particularly, in the derived $\hat{G}_g^{-1}(x)$ via Wilson-line technique, there is no non-gauge-invariant-current (density-vertex-related) term, similar to the derivation within the transport-equation formalism [52], whereas in diamagnetic response, the Wilson-line technique in Eq. (45) reduces to the unitary transformation in Eq. (21), and hence the derivation applying this unitary transformation in Sec. III C avoids the issue of gauge-invariance breaking by the scattering treatment mentioned in Sec. III B.

Furthermore, it is established that superconductors can directly respond to vector potential $\mathbf{A}$ (Meissner effect/Ginzburg-Landau kinetic term) in addition to the electric field $\mathbf{E} = -\nabla_{\mathbf{R}}\phi - \partial_t\mathbf{A}$, differing from normal metals that solely respond to electric field, whereas the conventional calculation with the vector potential alone is hard to distinguish these two effects in superconductors. Therefore, $\hat{G}_g^{-1}(x)$ in Eq. (47) provides an efficient Lagrangian/Hamiltonian kernel, which explicitly distinguishes the drive effect by the electric field $e\mathbf{E}$ and the Meissner effect driven by effective vector potential [26],

$$i\delta\theta - 2ie\int_0^{\mathbf{x}}\delta\mathbf{A}\cdot d\mathbf{x}' = 2i\int_0^{\mathbf{x}}(\nabla_{\mathbf{x}}\delta\theta/2 - e\delta\mathbf{A})d\mathbf{x}', \quad (48)$$

as well as the Josephson effect induced by effective electric voltage [97],

$$i\delta\theta + 2ie\int_0^{x_0}\delta\phi_0 dx_0' = 2i\int_0^{x_0}(\partial_{x_0}\delta\theta/2 + e\delta\phi_0), \quad (49)$$

and all these characteristic effects manifest themselves in a gauge-invariant description. One therefore expects a wide application of this kernel to study the mesoscopic physics in superconductors as well as more diagrammatic-formalism and transport-equation investigations.

IV. HIGGS MODE

We next focus on the Higgs mode. Based on the BCS Hamiltonian in Eq. (1), considering a variation of the superconducting gap (i.e., Higgs mode), the nonequilibrium self-energy is derived as $\Sigma_\delta(x) = \delta|\Delta(x)|\tau_1$. In this circumstance, $\delta S^{(2)}_{\rm CR}$ in Eq. (13) denotes the contribution from the amplitude-amplitude correlation with the Born and vertex corrections by impurity scattering.

For the free case, we take zero superconducting momentum (i.e., $\mathbf{p}_s = 0$) and hence vanishing Doppler shift. In this situation, the Green's function in Eq. (4) becomes the standard BCS one [26], similar to the previous formulations of the Higgs mode at the clean limit in the literature [12,16–22, 38–42,53–56]. Then, in center-of-mass frequency-momentum space $[x = (x_0, \mathbf{x}) \to q = (\Omega, \mathbf{q})]$, at weak impurity interaction, and long-wave limit ($\mathbf{q} = 0$), after the summation of the Matsubara frequency, one has (refer to Appendix C)

$$\delta S^{(2)} = \int d\Omega |\delta\Delta|^2 \{[(2\Delta_0)^2 - \Omega^2]\beta_H - 2i\Omega\Gamma_H\}, \quad (50)$$

with $\Gamma_H = 2n_i\pi \sum_{\mathbf{k}\mathbf{k}'} |V_{\mathbf{k}\mathbf{k}'}|^2 \frac{\Delta_0^2 \xi_{\mathbf{k}}^2 F_{\mathbf{k}}}{E_{\mathbf{k}}^2 (4E_{\mathbf{k}}^2 - \Omega^2)^2} \delta(E_{\mathbf{k}} - E_{\mathbf{k}'}) = \sum_{\mathbf{k}} \gamma_k \frac{4\xi_{\mathbf{k}}^2 F_{\mathbf{k}}}{(4E_{\mathbf{k}}^2 - \Omega^2)^2}$ and $\beta_H = \sum_{\mathbf{k}} \frac{F_{\mathbf{k}}}{4E_{\mathbf{k}}^2 - \Omega^2}$. Here, $\gamma_k = 2n_i\pi N(0)\Delta_0^2/(|\xi_k|E_k)\int \frac{d\Omega_{\mathbf{k}'}}{8\pi}|V_{\mathbf{k}_F - \mathbf{k}'_F}|^2$ and the anomalous correlation becomes $F_{\mathbf{k}} = \frac{f(E_{\mathbf{k}}) - f(-E_{\mathbf{k}})}{2E_{\mathbf{k}}}$. It is noted that the emerged term $2i\Omega\Gamma_H$ (second term) in Eq. (50) by impurity scattering is proportional to Ω, suggesting that the impurity scattering effect on the Higgs mode is a nonequilibrium property with the time-translational-symmetry breaking. Actually, it is pointed out that this scattering part arises from the vertex correction solely, whereas the Born correction makes no contribution at all (refer to Appendix C). This is because the Born correction is equivalent to the renormalization of the equilibrium impurity self-energy and hence vanishes according to the Anderson theorem [71–75], whereas the vertex correction that only emerges at nonequilibrium case breaks the time-translational symmetry, as mentioned in the Introduction, and hence, makes a finite contribution to the Higgs-mode damping.

Furthermore, from $\partial_{\delta|\Delta|}\delta S^{(2)} = 0$, the equation of motion of the Higgs mode is given by

$$\beta_H[(2\Delta_0)^2 - \Omega^2 - 2i\Omega\Gamma_H/\beta_H]\delta|\Delta| = 0. \quad (51)$$

It is noted that this equation at the clean limit ($\Gamma_H = 0$) reduces to the previously revealed one by various theoretical approaches in the literature [12,16–22,38–42,53–56], showing a gapful energy spectrum $\omega_H = 2\Delta_0$.

At low frequency $\Omega \ll 2\Delta_0$, one can consider that the coefficients β_H and Γ_H are independent on Ω. Then, the equation of motion in Eq. (51) becomes

$$\left[\partial_t^2 + (2\Delta_0)^2 - 2\Gamma_H/\beta_H \partial_t\right]\delta|\Delta| = 0, \quad (52)$$

which is a typical one of the damped oscillator (i.e., damped Klein-Gordon equation in the field theory) as it should be since the Higgs mode represents the radial collective excitation in the Mexican-hat potential of free energy [12]. Moreover, at the clean limit, this equation of motion, written in time space, exactly recovers the one derived from the superconducting Ginzburg-Landau Lagrangian by considering an amplitude variation of the Landau order parameter [20,22].

Whereas in the presence of the external optical excitation at THz regime, the equation of motion of the Higgs mode in Eq. (51) is rewritten as

$$\beta_H[(2\Delta_0)^2 - \Omega^2 - 2i\Omega\Gamma_H/\beta_H]\delta|\Delta| = Q(\Omega), \quad (53)$$

where $Q(\Omega)$ represents the response function determined by the external excitation/field. The left-hand side of this equation derived in the present paper, i.e., the spectral function of Higgs mode, is an intrinsic character of the system, irrelevant of the external probe, and can therefore manifest itself in the out-of-equilibrium properties.

On one hand, for the continuous-wave single-frequency excitation, in the second-order response, the response function is given by $Q(\Omega) = Q_0\delta(\Omega - 2\omega)$. Then, at weak scattering, from Eq. (53), the magnitude of the second-order response of the Higgs mode $\frac{Q_0}{\sqrt{\beta_H^2[4\Delta_0^2 - (2\omega)^2]^2 + (4\omega\Gamma_H)^2}}$ exhibits a resonant peak at $2\omega = 2\Delta_0$ with a *broadening* caused by the impurity scattering, consistent with the experimental findings from the third-harmonic signal [7,8]. It is also noted that the existence of the impurity scattering results in an imaginary part in the second-order response of the Higgs mode, and hence leads to a phase shift ϕ_H in this signal. Particularly, this phase shift $\phi_H \propto \arctan[(\omega^2 - \Delta_0^2)^{-1}]$ at weak scattering, and hence, exhibits a significant π-jump at $\omega = \Delta_0$, as predicted in Ref. [54] by numerical calculation from the GIKE, providing a very clear feature for the experimental detection. Recently, a π jump of the phase shift has been experimentally observed at $\omega = \Delta_0$ in the second-order optical response of the disordered high-T_c cuprate-based superconductors [10]. The origin of this jump is still controversial, whereas it is suggested that the π-jump of the phase shift in the second-order optical response can also be realized in the conventional superconductors due to the scattering effect.

On the other hand, for the long-time dynamic after the excitation, the response function can be approximately treated as a short pulse, and one has $Q(t) = Q_0\delta(t)$ in time space with the Fourier component $Q(\Omega) = Q_0$ in frequency space. Then, from Eq. (51), with the consideration of the frequency dependence of β_H and Γ_H in the frequency integral, the long-time dynamic of the Higgs mode after excitation behaves as (refer to Appendix D)

$$\delta|\Delta(t)| \sim \frac{\cos(2\Delta_0 t)e^{-\bar{\gamma}t}}{\sqrt{\Delta_0 t}}, \quad (54)$$

where $\bar{\gamma}$ is the average of γ_k in the momentum space. It is noted that this solution at clean case $\delta|\Delta(t)| \sim \frac{\cos(2\Delta_0 t)}{\sqrt{\Delta_0 t}}$ has been derived from the equation of motion above in the literature through various approaches and referred to as coherent BCS oscillatory decay [13–15,19,98], whereas in contrast to the coherent BCS oscillatory decay at clean limit, the impurity scattering leads to the faster exponential decay. The derived damping rate $\bar{\gamma}$ is exactly same as the one obtained from GIKE through both analytical and numerical calculations [54]. The oscillatory decay behavior with oscillating frequency at the Higgs-mode energy $2\Delta_0$ and fast exponential decay in Eq. (54) also agrees with the experimental findings from the pump-probe measurement with a short pulse [5,6], whereas it is suggested that the damping rate of the Higgs mode increases by increasing the impurity density, providing a possible scheme for the experimental detection by measuring the impurity-density dependence of the Higgs-mode damping rate or resonance broadening.

The induced damping of the Higgs mode by impurity scattering agrees with the analysis through the Heisenberg equation of motion as mentioned in the Introduction, since the Higgs-mode excitation and electron-impurity interaction are noncommutative in Nambu space, whereas as mentioned in the Introduction, the previous derivation of the Higgs mode within the Eilenberger equation [56] fails to give this damping and derives a Higgs-mode energy spectrum that is free from the scattering influence and hence insensitive to impurities, against the results from GIKE [54]. The present paper through the diagrammatic formulation exactly helps to resolve this controversy. Actually, this is because the microscopic scattering integral in the Eilenberger equation is incomplete. As proved in Ref. [67], because of the quasiclassical approximation on the τ_3-Green's function, the scattering integral in the Eilenberger equation only involves the anisotropic part of the Green's function that is related to the transport property, but generically drops the isotropic one which determines the Higgs mode [67].

V. SUMMARY

In summary, we have analytically performed a diagrammatic formulation of the impurity scattering in superconductors in the present paper to compare with previous studies through GIKE [52,54]. Theoretically, compared to the transport equation approach, the diagrammatic formalism provides a basic and complementary technique to tackle the impurity scattering effect for nonequilibrium properties as a crosscheck. However, as pointed out in the Introduction, in superconductors, the diagrammatic formulation has not been rigorously developed in the literature due to the inevitable calculation of the vertex correction by impurity scattering. Various approximations [26,28–30] that in fact drop the microscopic scattering process have been taken in the literature to handle the scattering effect in transport properties. Particularly, the revealed Meissner supercurrent, which should be nonviscous, experiences a friction resistance by scattering in those previous works [26,28–34,47,51,56,69]. As for the Higgs mode, even with the growing experimental evidence of its damping, the diagrammatic formulation of amplitude-amplitude correlation with vertex correction by impurities is still absent in the literature, not to mention that the Eilenberger equation [56] and GIKE [54] revealed opposite conclusions about the disorder effect on the Higgs mode. Consequently, all of these call for a revisit of the impurity scattering within the diagrammatic formalism. We have therefore performed the diagrammatic formulations of both transport behavior and collective Higgs mode with impurities in the present paper.

For transport behavior, we tend to fill the long missing rigorous textbook calculation of the Kubo current-current correlation with impurity scattering in superconductors. Specifically, in the diamagnetic response, within the conventional calculation based on kinematical momentum operator $\hat{\mathbf{\Pi}} = \hat{\mathbf{p}} - e\mathbf{A}\tau_3$, it is shown that a non-gauge-invariant current emerges after the scattering treatment/correction in the current-current correlation. To resolve this issue of the gauge invariance breaking, we apply a special unitary transformation, and obtain the Meissner-supercurrent vertex. Then, the supercurrent-supercurrent correlation with the Born and vertex corrections from impurity scattering is formulated. Particularly, in contrast to previous works [26] in the literature that overlooked the Doppler shift, we keep this effect in the quasiparticle energy spectra.

Then, the present diagrammatic formulation of the impurity scattering effect in transport properties confirms the previously revealed microscopic momentum-relaxation rate of the superfluid and the current captured by the three-fluid (normal fluid as well as viscous and nonviscous superfluids) model [52,80]. The momentum-relaxation rate of the superfluid is finite only when the superconducting momentum is larger than a threshold Δ_0/v_F, at which the normal fluid emerges and causes the friction with the superfluid current, similar to Landau's superfluid theory of bosonic liquid helium [78]. This derivation uncovers the physics behind the relaxation-time approximation in previous diagrammatic formulations [28–30], which leads to the theoretically revealed [26,28–30,47,51,56,69] and experimentally confirmed [31–35] friction resistance of the Meissner supercurrent.

It is also pointed out that the Doppler shift is essential in theoretical calculations to handle the scattering effect, as it guarantees the vanishing momentum-relaxation rate of superfluid at small superconducting velocity. The Doppler shift that emerges with the presence of the superconducting momentum in transport behavior causes a tilted quasiparticle energy spectrum and hence markedly influences the superconducting anomalous correlation, as demonstrated/established in the previous works [52,80–87]. However, in the discussion of the scattering effect, this effect has long been totally overlooked in the literature. Thus, in previous theoretical descriptions [26,47,51,56,69] that overlooked this effect, the derived superfluid current always experiences the friction resistance by impurity scattering as a consequence, in contrast to the superconductivity phenomenon.

Furthermore, using the Wilson-line technique, a gauge-invariant Hamiltonian that explicitly distinguishes the Meissner effect and electric-field drive effect as well as the Josephson voltage effect is proposed. This kernel helps us to understand the unitary transformation applied in the present paper, which eliminates the issue of gauge-invariance breaking by scattering treatment and gives the supercurrent vertex in the diagrammatic formalism. We expect a wide application of this kernel to study the mesoscopic physics in superconductors as well as more diagrammatic-formalism and transport-equation investigations.

As for the collective Higgs mode, the present diagrammatic formulation with impurities confirms the Higgs-mode damping mechanism caused by impurity scattering, which not only agrees with the analysis through the Heisenberg equation of motion but also directly helps to resolve the current controversy between GIKE [54] and the Eilenberger equation [56] in the literature. Specifically, we calculate the amplitude-amplitude correlation with the Born and vertex corrections from impurity scattering. The vertex correction, which only emerges at the nonequilibrium case with time-translational-symmetry breaking, leads to a fast exponential damping of the Higgs mode, whereas the Born correction that is equivalent to equilibrium impurity self-energy makes no contribution because of the Anderson theorem [71–75]. The derived damping by impurity scattering from the diagrammatic formalism exactly recovers the one from GIKE [54] and agrees with the analysis through the Heisenberg equation of motion, but is in contrast to the vanishing one obtained in the Eilenberger equation [56]. The reason leading to missing damping is due to the generically incomplete scattering integral in the Eilenberger equation [67]. The lifetime of the Higgs mode due to the impurity scattering provides a possible origin for the experimentally observed broadening of the resonance signal [7,8] as well as the damping after optical excitation [5,6] of the Higgs mode. Moreover, as pointed out in Ref. [54], the damping by impurities can cause a phase shift in the optical signal of Higgs mode, which exhibits a π jump at the resonance frequency and hence provides a very clear feature for further experimental detection. In addition, the damping rate of the Higgs mode increases by increasing the impurity density, providing a possible scheme for the experimental detection by measuring the impurity-density dependence of the Higgs-mode damping rate or resonance broadening.

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APPENDIX A: DERIVATION OF CHARGE DENSITY

In this Appendix, we present the derivation of the charge density. With the density vertex τ_3 in Nambu space [79], in the effective nonequilibrium action in Eq. (12), the contribution from the density-vertex-related part of the nonequilibrium self-energy reads $\{\bar{\mathrm{T}}\mathrm{r}[G_0(p)\tau_3]+\eta_f\}\Sigma_{\delta 3}$. In this contribution, substituting the Green's function in Eq. (4), one finds the prefactor,

$$\bar{\mathrm{T}}\mathrm{r}[G_0(p)\tau_3]+\eta_f=\sum_{p_n,\mathbf{k}}\frac{2\big(\xi_{\mathbf{k}}+p_s^2/2m\big)}{m(ip_n-E_{\mathbf{k}}^+)(ip_n-E_{\mathbf{k}}^-)}+\eta_f=\sum_{\mathbf{k}}\bigg[1+2\bigg(\xi_{\mathbf{k}}+\frac{p_s^2}{2m}\bigg)F_{\mathbf{k}}\bigg]=-\frac{2k_F^2}{3m}\sum_{\mathbf{k}}\partial_{\xi_{\mathbf{k}}}(\xi_{\mathbf{k}}F_{\mathbf{k}})\approx\frac{2k_F^2N(0)}{3m},\tag{A1}$$

which is exactly the charge density n.

APPENDIX B: DERIVATION OF SUPERCURRENT-SUPERCURRENT CORRELATION

In this Appendix, we derive the supercurrent-supercurrent correlation with Born and vertex corrections from the impurity scattering. Specifically, for transport behavior in the diamagnetic response, substituting the derived self-energy in Eq. (24) that is related to the Meissner-supercurrent vertex, the supercurrent-supercurrent correlation $S_{\mathrm{CR}}^{(2)}$ [Eq. (13)/Fig. 1] is written as

$$\delta S_{\mathrm{CR}}^{(2)}=2\Delta_0^2\int dx\bar{\mathrm{T}}\mathrm{r}\big[\tau_2(\delta\mathbf{p}_s\cdot\mathbf{v}_{\hat{\mathbf{k}}})\partial_{\xi_{\mathbf{k}}}G_0\tau_2(\delta\mathbf{p}_s\cdot\mathbf{v}_{\hat{\mathbf{k}}})\partial_{\xi_{\mathbf{k}}}G_0+2(G_0V\tau_3)^2(G_0\tau_2\delta\mathbf{p}_s\cdot\mathbf{v}_{\hat{\mathbf{k}}}\partial_{\xi_{\mathbf{k}}})^2$$
$$+G_0\tau_2(\delta\mathbf{p}_s\cdot\mathbf{v}_{\hat{\mathbf{k}}})\partial_{\xi_{\mathbf{k}}}G_0V\tau_3G_0\tau_2(\delta\mathbf{p}_s\cdot\mathbf{v}_{\hat{\mathbf{k}}})\partial_{\xi_{\mathbf{k}}}G_0V\tau_3\big]=I_{\mathrm{ba}}+I_{\mathrm{bc}}+I_{\mathrm{vc}},\tag{B1}$$

with the bare supercurrent-supercurrent correlation I_{ba} as well as Born I_{Bc} and vertex I_{vc} corrections written as

$$I_{\mathrm{ba}}=2\Delta_0^2\sum_{ip_n,\mathbf{k}}[(\delta\mathbf{p}_s\cdot\mathbf{v}_{\mathbf{k}})^2\mathrm{Tr}[\tau_2\partial_{\xi_{\mathbf{k}}}G_0(ip_n^+,\mathbf{k}^+)\tau_2\partial_{\xi_{\mathbf{k}}}G_0(ip_n,\mathbf{k})],\tag{B2}$$

$$I_{\mathrm{Bc}}=-2\Delta_0^2\sum_{ip_n,\mathbf{kk}'}(\delta\mathbf{p}_s\cdot\mathbf{v}_{\mathbf{k}})^2n_i|V_{\mathbf{kk}'}|^2\{\mathrm{Tr}[\partial_{\xi_{\mathbf{k}}}G_0(ip_n,\mathbf{k})\tau_2G_0(ip_n^+,\mathbf{k}^+)\tau_2\partial_{\xi_{\mathbf{k}}}G_0(ip_n,\mathbf{k})\tau_3G(ip_n,\mathbf{k}')\tau_3]+(p^+\to p^-)\},\tag{B3}$$

$$I_{\mathrm{vc}}=-2\Delta_0^2\sum_{ip_n,\mathbf{kk}'}(\delta\mathbf{p}_s\cdot\mathbf{v}_{\mathbf{k}})(\delta\mathbf{p}_s\cdot\mathbf{v}_{\mathbf{k}'})n_i|V_{\mathbf{kk}'}|^2\mathrm{Tr}[\partial_{\xi_{\mathbf{k}}}G_0(ip_n,\mathbf{k})\tau_2G_0(ip_n^+,\mathbf{k}^+)\tau_3G_0(ip_n^+,\mathbf{k}'^+)\tau_2\partial_{\xi_{\mathbf{k}'}}G_0(ip_n,\mathbf{k}')\tau_3].\tag{B4}$$

Here, we have kept up to the second order of the impurity interaction by considering the case of weak impurity scattering. Here, $p^\pm=(ip_n^\pm,\mathbf{k}^\pm)=(ip_n\pm i0^+,\mathbf{k}\pm\mathbf{q})$.

With the Green's function in Eq. (4), around the Fermi surface, one has $\partial_{\xi_{\mathbf{k}}}G_0(ip_n,\mathbf{k})\approx\tau_3/\Lambda_{\mathbf{k}}(ip_n)$ with $\Lambda_{\mathbf{k}}(ip_n)=(ip_n-E_{\mathbf{k}}^+)(ip_n-E_{\mathbf{k}}^-)$. Then, after the summation of Matsubara frequency, the bare supercurrent-supercurrent correlation [Eq. (B2)] reads

$$I_{\mathrm{ba}}=-\sum_{ip_n,\mathbf{k}}\frac{2\Delta_0^22(\delta\mathbf{p}_s\cdot\mathbf{v}_{\mathbf{k}})^2}{\Lambda_{\mathbf{k}}(ip_n)\Lambda_{\mathbf{k}^+}(ip_n^+)}\approx-2\Delta_0^2\sum_{ip_n,\mathbf{k}}\frac{2(\delta\mathbf{p}_s\cdot\mathbf{v}_{\mathbf{k}})^2}{\Lambda_{\mathbf{k}}^2(ip_n)}=-\sum_{\mathbf{k}}(\delta\mathbf{p}_s\cdot\mathbf{v}_{\mathbf{k}})^2\frac{2\Delta_0^2}{E_{\mathbf{k}}}\partial_{E_{\mathbf{k}}}F_{\mathbf{k}}.\tag{B5}$$

Moreover, using the fact $\frac{1}{\Lambda_{\mathbf{k}}(ip_n)}=\frac{1}{2E_{\mathbf{k}}}\sum_{\eta=\pm}\frac{\eta}{ip_n-E_{\mathbf{k}}^\eta}$, the Born [Eq. (B3)] and vertex [Eq. (B4)] corrections by impurity scattering are given by

$$I_{\mathrm{Bc}}=-4\Delta_0^2\sum_{ip_n,\mathbf{kk}'}(\delta\mathbf{p}_s\cdot\mathbf{v}_{\mathbf{k}})^2n_i|V_{\mathbf{kk}'}|^2\bigg[\frac{(ip_n^+-\mathbf{v}_{\mathbf{k}^+}\cdot\mathbf{p}_s)(ip_n-\mathbf{v}_{\mathbf{k}'}\cdot\mathbf{p}_s)-\Delta_0^2}{\Lambda_{\mathbf{k}}^2(ip_n)\Lambda_{\mathbf{k}^+}(ip_n^+)\Lambda_{\mathbf{k}'}(ip_n)}+(p^+\to p^-)\bigg]$$
$$=4\Delta_0^2\sum_{ip_n,\mathbf{kk}',\eta\lambda}(\delta\mathbf{p}_s\cdot\mathbf{v}_{\mathbf{k}})^2n_i|V_{\mathbf{kk}'}|^2\bigg[\frac{\Delta_0^2-(ip_n^+-\mathbf{v}_{\mathbf{k}^+}\cdot\mathbf{p}_s)(ip_n-\mathbf{v}_{\mathbf{k}'}\cdot\mathbf{p}_s)}{4E_{\mathbf{k}'}E_{\mathbf{k}^+}\Lambda_{\mathbf{k}}^2(ip_n)}\frac{\eta}{ip_n^+-E_{\mathbf{k}^+}^\eta}\frac{\lambda}{ip_n-E_{\mathbf{k}'}^\lambda}+(p^+\to p^-)\bigg],\tag{B6}$$

$$I_{\rm vc} = 4\Delta_0^2 \sum_{ip_n,\mathbf{kk}',\eta\lambda} (\delta\mathbf{p}_s\cdot\mathbf{v}_\mathbf{k})(\delta\mathbf{p}_s\cdot\mathbf{v}_{\mathbf{k}'})n_i|V_{\mathbf{kk}'}|^2 \frac{(ip_n^+ - \mathbf{v}_{\mathbf{k}^+}\cdot\mathbf{p}_s)(ip_n^+ - \mathbf{v}_{\mathbf{k}'^+}\cdot\mathbf{p}_s) - \Delta_0^2}{\Lambda_{\mathbf{k}^+}(ip_n^+)\Lambda_{\mathbf{k}'}(ip_n)\Lambda_\mathbf{k}(ip_n)\Lambda_{\mathbf{k}'^+}(ip_n^+)}$$
$$= 4\Delta_0^2 \sum_{ip_n,\mathbf{kk}'} (\delta\mathbf{p}_s\cdot\mathbf{v}_\mathbf{k})(\delta\mathbf{p}_s\cdot\mathbf{v}_{\mathbf{k}'})n_i|V_{\mathbf{kk}'}|^2 \frac{(ip_n^+ - \mathbf{v}_{\mathbf{k}^+}\cdot\mathbf{p}_s)(ip_n^+ - \mathbf{v}_{\mathbf{k}'^+}\cdot\mathbf{p}_s) - \Delta_0^2}{4E_{\mathbf{k}^+}E_{\mathbf{k}'^+}\Lambda_{\mathbf{k}'}(ip_n)\Lambda_\mathbf{k}(ip_n)} \frac{\eta}{ip_n^+ - E_{\mathbf{k}^+}^\eta}\frac{\lambda}{ip_n^+ - E_{\mathbf{k}'^+}^\lambda}. \quad \text{(B7)}$$

Further considering the imaginary (i.e., scattering) parts of the Born and vertex corrections, through the summation of the Matsubara frequency, one has

$$I_{\rm Bc} = -i\pi 4\Delta_0^2 \sum_{ip_n,\mathbf{kk}',\eta'\eta\lambda} (\delta\mathbf{p}_s\cdot\mathbf{v}_\mathbf{k})^2 n_i|V_{\mathbf{kk}'}|^2\eta'\eta\lambda\left[\frac{\Delta_0^2 - (ip_n - \mathbf{v}_{\mathbf{k}^+}\cdot\mathbf{p}_s)(ip_n - \mathbf{v}_{\mathbf{k}'}\cdot\mathbf{p}_s)}{4E_{\mathbf{k}'}E_{\mathbf{k}^+}2E_\mathbf{k}\Lambda_\mathbf{k}(ip_n)\big(ip_n - E_\mathbf{k}^{\eta'}\big)}\frac{\delta\big(ip_n - E_{\mathbf{k}^+}^\eta\big)}{ip_n - E_{\mathbf{k}'}^\lambda} - (\mathbf{k}^+\to\mathbf{k}^-)\right]$$
$$= -i\pi 4\Delta_0^2 \sum_{\mathbf{kk}',\eta\eta'\lambda} (\delta\mathbf{p}_s\cdot\mathbf{v}_\mathbf{k})^2 n_i|V_{\mathbf{kk}'}|^2\eta\eta'\lambda\left[\frac{\Delta_0^2 - \big(E_{\mathbf{k}'}^\lambda - \mathbf{v}_{\mathbf{k}^+}\cdot\mathbf{p}_s\big)\lambda E_{\mathbf{k}'}}{4E_{\mathbf{k}'}E_{\mathbf{k}^+}\Lambda_\mathbf{k}\big(E_{\mathbf{k}'}^\lambda\big)\big(E_{\mathbf{k}'}^\lambda - E_\mathbf{k}^{\eta'}\big)}\frac{f\big(E_{\mathbf{k}'}^\lambda\big)\delta\big(E_{\mathbf{k}'}^\lambda - E_{\mathbf{k}^+}^\eta\big)}{2E_\mathbf{k}} - (\mathbf{k}^+\to\mathbf{k}^-)\right]$$
$$= i\pi 4\Delta_0^2 \sum_{\mathbf{kk}',\eta\eta'\lambda} (\delta\mathbf{p}_s\cdot\mathbf{v}_\mathbf{k})^2 \frac{n_i|V_{\mathbf{kk}'}|^2}{4E_\mathbf{k}}\eta'\left[e_{\mathbf{k}^+\mathbf{k}'}^{-\eta\lambda}\frac{f\big(E_{\mathbf{k}^+}^\eta\big)\delta\big(E_{\mathbf{k}'}^\lambda - E_{\mathbf{k}^+}^\eta\big)}{\Lambda_\mathbf{k}\big(E_{\mathbf{k}^+}^\eta\big)\big(E_{\mathbf{k}^+}^\eta - E_\mathbf{k}^{\eta'}\big)} - (\mathbf{k}^+\to\mathbf{k}^-)\right] \quad \text{(B8)}$$

and

$$I_{\rm vc} = i\pi 4\Delta_0^2 \sum_{\eta\lambda,ip_n\mathbf{kk}'} \eta\lambda(\delta\mathbf{p}_s\cdot\mathbf{v}_\mathbf{k})(\delta\mathbf{p}_s\cdot\mathbf{v}_{\mathbf{k}'})n_i|V_{\mathbf{kk}'}|^2\frac{\Delta_0^2 - (ip_n - \mathbf{v}_{\mathbf{k}^+}\cdot\mathbf{p}_s)(ip_n - \mathbf{v}_{\mathbf{k}'^+}\cdot\mathbf{p}_s)}{4E_{\mathbf{k}^+}E_{\mathbf{k}'^+}\Lambda_{\mathbf{k}'}(ip_n)\Lambda_\mathbf{k}(ip_n)}\left[\frac{\delta\big(ip_n - E_{\mathbf{k}^+}^\eta\big)}{ip_n - E_{\mathbf{k}'^+}^\lambda} + \frac{\delta\big(ip_n - E_{\mathbf{k}'^+}^\lambda\big)}{ip_n - E_{\mathbf{k}^+}^\eta}\right]$$
$$= i\pi 4\Delta_0^2 \sum_{ip_n,\mathbf{kk}',\eta\lambda} \eta\lambda(\delta\mathbf{p}_s\cdot\mathbf{v}_\mathbf{k})(\delta\mathbf{p}_s\cdot\mathbf{v}_{\mathbf{k}'})n_i|V_{\mathbf{kk}'}|^2\frac{\Delta_0^2 - (ip_n - \mathbf{v}_{\mathbf{k}^+}\cdot\mathbf{p}_s)(ip_n - \mathbf{v}_{\mathbf{k}'^+}\cdot\mathbf{p}_s)}{4E_{\mathbf{k}^+}E_{\mathbf{k}'^+}\Lambda_{\mathbf{k}'}(ip_n)\Lambda_\mathbf{k}(ip_n)}\frac{2\delta\big(ip_n - E_{\mathbf{k}^+}^\eta\big)}{ip_n - E_{\mathbf{k}'^+}^\lambda}$$
$$= i\pi 8\Delta_0^2 \sum_{ip_n,\mathbf{kk}',\eta\lambda} (\delta\mathbf{p}_s\cdot\mathbf{v}_\mathbf{k})(\delta\mathbf{p}_s\cdot\mathbf{v}_{\mathbf{k}'})n_i|V_{\mathbf{kk}'}|^2\frac{\Delta_0^2 - (ip_n - \mathbf{v}_{\mathbf{k}^+}\cdot\mathbf{p}_s)(ip_n - \mathbf{v}_{\mathbf{k}'^+}\cdot\mathbf{p}_s)}{4E_{\mathbf{k}^+}E_{\mathbf{k}'^+}2E_{\mathbf{k}'}\Lambda_\mathbf{k}(ip_n)}\frac{\eta\lambda\lambda'\delta\big(ip_n - E_{\mathbf{k}^+}^\eta\big)}{\big(ip_n - E_{\mathbf{k}'^+}^\lambda\big)\big(ip_n - E_{\mathbf{k}'}^{\lambda'}\big)}$$
$$= -i\pi 8\Delta_0^2 \sum_{\mathbf{kk}',\eta\lambda\lambda'} \frac{(\delta\mathbf{p}_s\cdot\mathbf{v}_\mathbf{k})(\delta\mathbf{p}_s\cdot\mathbf{v}_{\mathbf{k}'})n_i|V_{\mathbf{kk}'}|^2}{\Lambda_\mathbf{k}\big(E_{\mathbf{k}^+}^\eta\big)}\left[\frac{\lambda' e_{\mathbf{k}^+\mathbf{k}'^+}^{-\eta\lambda}f\big(E_{\mathbf{k}'^+}^\lambda\big)\delta\big(E_{\mathbf{k}'^+}^\lambda - E_{\mathbf{k}^+}^\eta\big)}{4E_{\mathbf{k}'}\big(E_{\mathbf{k}'^+}^\lambda - E_{\mathbf{k}'}^{\lambda'}\big)} - \frac{\lambda e_{\mathbf{k}^+\mathbf{k}'^+}^{-\eta\lambda'}f\big(E_{\mathbf{k}'}^{\lambda'}\big)\delta\big(E_{\mathbf{k}'}^{\lambda'} - E_{\mathbf{k}^+}^\eta\big)}{4E_{\mathbf{k}'}\big(E_{\mathbf{k}'^+}^\lambda - E_{\mathbf{k}'}^{\lambda'}\big)}\right]$$
$$= -i\pi 8\Delta_0^2 \sum_{\mathbf{kk}',\eta\lambda\lambda'} \frac{(\delta\mathbf{p}_s\cdot\mathbf{v}_\mathbf{k})(\delta\mathbf{p}_s\cdot\mathbf{v}_{\mathbf{k}'})n_i|V_{\mathbf{kk}'}|^2}{\Lambda_\mathbf{k}\big(E_{\mathbf{k}^+}^\eta\big)}e_{\mathbf{k}^+\mathbf{k}'^+}^{-\eta\lambda}\lambda'\left[\frac{f\big(E_{\mathbf{k}'^+}^\lambda\big)\delta\big(E_{\mathbf{k}'^+}^\lambda - E_{\mathbf{k}^+}^\eta\big)}{4E_{\mathbf{k}'}\big(E_{\mathbf{k}'^+}^\lambda - E_{\mathbf{k}'}^{\lambda'}\big)} - \frac{f\big(E_{\mathbf{k}'}^\lambda\big)\delta\big(E_{\mathbf{k}'}^\lambda - E_{\mathbf{k}^+}^\eta\big)}{4E_{\mathbf{k}'}\big(E_{\mathbf{k}'^+}^{\lambda'} - E_{\mathbf{k}'}^\lambda\big)}\right], \quad \text{(B9)}$$

in which we have used the fact $\delta(E_{\mathbf{k}^+}^{\eta'} - E_\mathbf{k}^\eta) \equiv 0$.

For the long-wave case, the leading contribution in Eq. (B8) comes from the $\eta = \eta'$ part, and the Born correction becomes

$$I_{\rm Bc} \approx i\pi 4\Delta_0^2 \sum_{\mathbf{kk}',\eta\lambda} (\delta\mathbf{p}_s\cdot\mathbf{v}_\mathbf{k})^2\frac{n_i|V_{\mathbf{kk}'}|^2}{4E_\mathbf{k}}\eta\left[\frac{e_{\mathbf{kk}'}^{-\eta\lambda}f\big(E_{\mathbf{k}^+}^\eta\big)}{\big(E_{\mathbf{k}^+}^\eta - E_\mathbf{k}^\eta\big)^2\big(E_{\mathbf{k}^+}^\eta - E_\mathbf{k}^{-\eta}\big)} - (\mathbf{k}^+\to\mathbf{k}^-)\right]\delta\big(E_{\mathbf{k}'}^\lambda - E_\mathbf{k}^\eta\big)$$
$$\approx i\pi 4\Delta_0^2 \sum_{\eta\lambda}\sum_{\mathbf{kk}'} (\delta\mathbf{p}_s\cdot\mathbf{v}_\mathbf{k})^2\frac{n_i|V_{\mathbf{kk}'}|^2}{4E_\mathbf{k}}\eta\left[\frac{e_{\mathbf{kk}'}^{-\eta\lambda}f\big(E_{\mathbf{k}^+}^\eta\big)}{\big(E_{\mathbf{k}^+}^\eta - E_\mathbf{k}^\eta\big)^2}\left(\frac{1}{2\eta E_\mathbf{k}} - \frac{E_{\mathbf{k}^+}^\eta - E_\mathbf{k}^\eta}{4E_\mathbf{k}^2}\right) - (\mathbf{k}^+\to\mathbf{k}^-)\right]\delta\big(E_{\mathbf{k}'}^\lambda - E_\mathbf{k}^\eta\big)$$
$$= i\pi 8\Delta_0^2 \sum_{\eta\lambda}\sum_{\mathbf{kk}'} (\delta\mathbf{p}_s\cdot\mathbf{v}_\mathbf{k})^2\frac{n_i|V_{\mathbf{kk}'}|^2}{4E_\mathbf{k}}\eta\left[\frac{e_{\mathbf{kk}'}^{-\eta\lambda}f\big(E_{\mathbf{k}^+}^\eta\big)}{\big(E_{\mathbf{k}^+}^\eta - E_\mathbf{k}^\eta\big)^2}\left(\frac{1}{2\eta E_\mathbf{k}} - \frac{E_{\mathbf{k}^+}^\eta - E_\mathbf{k}^\eta}{4E_\mathbf{k}^2}\right) - (\mathbf{k}^+\to\mathbf{k})\right]\delta\big(E_{\mathbf{k}'}^\lambda - E_\mathbf{k}^\eta\big)$$
$$\approx -\pi \sum_{\eta\lambda\mathbf{kk}'} (\delta\mathbf{p}_s\cdot\mathbf{v}_\mathbf{k})^2 n_i|V_{\mathbf{kk}'}|^2 e_{\mathbf{kk}'}^{-\eta\lambda}\frac{2\Delta_0^2}{E_\mathbf{k}}\partial_{E_\mathbf{k}}\left[\frac{\eta f\big(E_\mathbf{k}^\eta\big)}{E_\mathbf{k}}\right]\frac{\delta\big(E_{\mathbf{k}'}^\lambda - E_\mathbf{k}^\eta\big)}{i\zeta_\mathbf{k} v_F q}, \quad \text{(B10)}$$

whereas in Eq. (B9), both $\lambda' = \lambda$ and $\lambda' = -\lambda$ parts play an important role, and the vertex correction reads

$$I_{\rm vc} \approx -i\pi 8\Delta_0^2 \sum_{\mathbf{kk}',\eta\lambda} \frac{(\delta\mathbf{p}_s\cdot\mathbf{v_k})(\delta\mathbf{p}_s\cdot\mathbf{v_{k'}})n_i|V_{\mathbf{kk}'}|^2}{\left(E^\eta_{\mathbf{k}^+}-E^\eta_{\mathbf{k}}\right)\left(E^\eta_{\mathbf{k}^+}-E^{-\eta}_{\mathbf{k}}\right)} e^{-\eta\lambda}_{\mathbf{kk}'}\left[\lambda\frac{e^{\eta\lambda}_{\mathbf{kk}'}}{4E_{\mathbf{k}'}}\frac{f\left(E^\lambda_{\mathbf{k}'^+}\right)-f\left(E^\lambda_{\mathbf{k}'}\right)}{E^\lambda_{\mathbf{k}'^+}-E^\lambda_{\mathbf{k}'}} - \frac{f\left(E^\lambda_{\mathbf{k}'^+}\right)}{4E^2_{\mathbf{k}'}}\right]\delta\left(E^\lambda_{\mathbf{k}'^+}-E^\eta_{\mathbf{k}}\right)$$

$$\approx \pi\Delta_0^2 \sum_{\mathbf{kk}',\eta\lambda} (\delta\mathbf{p}_s\cdot\mathbf{v_k})(\delta\mathbf{p}_s\cdot\mathbf{v_{k'}})n_i|V_{\mathbf{kk}'}|^2 e^{-\eta\lambda}_{\mathbf{kk}'}\frac{2\Delta_0^2}{E_\mathbf{k}}\partial_{E_{\mathbf{k}'}}\left[\frac{\eta f\left(E^\lambda_{\mathbf{k}'}\right)}{E_{\mathbf{k}'}}\right]\frac{\delta\left(E^\lambda_{\mathbf{k}'}-E^\eta_{\mathbf{k}}\right)}{i\zeta_\mathbf{k} v_F q}. \tag{B11}$$

Here, $\zeta_\mathbf{k} v_F q = 2\mathbf{v_q}\cdot\mathbf{p}_s + 4\xi_\mathbf{k}\mathbf{v_k}\cdot\mathbf{q}/E_\mathbf{k} + \eta(\mathbf{v_k}\cdot\mathbf{q})^2/E_\mathbf{k} \approx 2\mathbf{v_q}\cdot\mathbf{p}_s + 4\xi_\mathbf{k}\mathbf{v_k}\cdot\mathbf{q}/E_\mathbf{k} + (\mathbf{v_k}\cdot\mathbf{q})^2/E_\mathbf{k}$ at the long-wave case. Consequently, with Eqs. (B5) and (B10) as well as (B11), the supercurrent-supercurrent correlation with Born and vertex corrections by impurity scattering and hence the current in Eq. (25) are derived.

Moreover, it is noted that $\zeta_\mathbf{k} v_F q$ in the scattering contribution [Eqs. (B10) and (B11)] provides a diffusive pole that emerges at the stationary diffusion case, whereas for the nonstationary case at the long-wave limit [i.e., $\delta\mathbf{p}_s(x) = \delta\mathbf{p}_s e^{-i(\Omega+i0^+)x_0}$], with $p^\pm = (ip_n^\pm, \mathbf{k}^\pm) = [ip_n \pm (\Omega + i0^+), \mathbf{k}]$ in Eq. (B1), the diffusive pole $\zeta_\mathbf{k} v_F q$ in Eqs. (B10) and (B11) is replaced by $-\Omega$ after the derivation, and one finds a current,

$$\delta\mathbf{j} = \frac{e n_s^{\rm eff}\delta\mathbf{p}_s}{m} - \frac{\Gamma_c^{\rm eff}}{i\Omega}\frac{e n_s^{\rm eff}\delta\mathbf{p}_s}{m}, \tag{B12}$$

which is similar to the one described by the Drude model.

APPENDIX C: DERIVATION OF AMPLITUDE-AMPLITUDE CORRELATION

In this Appendix, we present the derivation of the amplitude-amplitude with Born and vertex corrections from the impurity scattering. Specifically, for the collective Higgs mode, substituting the derived self-energy $\Sigma_\delta^{(1)} = \delta|\Delta|\tau_1$, the amplitude-amplitude correlation $S^{(2)}_{\rm CR}$ [Eq. (13)/Fig. 1] is written as

$$\delta S^{(2)}_{\rm CR} = -\frac{1}{2}\int dx \bar{\rm T}{\rm r}[\delta|\Delta|\tau_1 G_0\delta|\Delta|\tau_1 G_0 + (G_0 V\tau_3)^2(G_0\delta|\Delta|\tau_1)^2 + G_0\delta|\Delta|\tau_1 G_0 V\tau_3 G_0\delta|\Delta|\tau_1 G_0 V\tau_3]$$
$$= -H_{\rm ba} - H_{\rm Bc} - H_{\rm vc}, \tag{C1}$$

where the bare amplitude-amplitude correlation $H_{\rm ba}$ as well as Born $H_{\rm Bc}$ and vertex $H_{\rm vc}$ corrections read

$$H_{\rm ba} = \frac{\delta|\Delta|^2}{2}\sum_{ip_n,\mathbf{k}}{\rm Tr}[\tau_1 G_0(ip_n^+,\mathbf{k})\tau_1 G_0(ip_n,\mathbf{k})] = \delta|\Delta|^2\sum_{ip_n,\mathbf{k}}\frac{ip_n^+ ip_n - \xi_\mathbf{k}^2 + \Delta_0^2}{\Lambda_\mathbf{k}(ip_n^+)\Lambda_\mathbf{k}(ip_n)}, \tag{C2}$$

$$H_{\rm Bc} = \frac{\delta|\Delta|^2}{2}\sum_{ip_n,\mathbf{kk}'} n_i|V_{\mathbf{kk}'}|^2\left\{{\rm Tr}[\tau_1 G_0(ip_n^+,\mathbf{k})\tau_1 G_0(ip_n,\mathbf{k})\tau_3 G(ip_n,\mathbf{k}')\tau_3 G_0(ip_n,\mathbf{k})] + (p^+\to p^-)\right\}$$
$$= \delta|\Delta|^2\sum_{ip_n,\mathbf{kk}'} n_i|V_{\mathbf{kk}'}|^2\left[\frac{\left(ip_n^+ ip_n - \xi_\mathbf{k}^2 + \Delta_0^2\right)\left[(ip_n)^2 - \Delta_0^2\right] + 2\xi_\mathbf{k}^2\Delta_0^2 + \Omega ip_n\xi_\mathbf{k}^2}{\Lambda^2_\mathbf{k}(ip_n)\Lambda_{\mathbf{k}'}(ip_n)\Lambda_\mathbf{k}(ip_n^+)} + (p^+\to p^-)\right], \tag{C3}$$

$$H_{\rm vc} = \frac{\delta|\Delta|^2}{2}\sum_{ip_n,\mathbf{kk}'} n_i|V_{\mathbf{kk}'}|^2{\rm Tr}[G_0(ip_n,\mathbf{k})\tau_1 G_0(ip_n^+,\mathbf{k}^+)\tau_3 G_0(ip_n^+,\mathbf{k}'^+)\tau_1 G_0(ip_n,\mathbf{k}')\tau_3]$$
$$= \delta|\Delta|^2\sum_{ip_n,\mathbf{kk}'} n_i|V_{\mathbf{kk}'}|^2\frac{\Delta_0^2(ip_n + ip_n^+)^2 - \left(ip_n ip_n^+ - E_\mathbf{k}^2 + 2\Delta_0^2\right)\left(ip_n ip_n^+ - E_{\mathbf{k}'}^2 + 2\Delta_0^2\right)}{\Lambda_\mathbf{k}(ip_n^+)\Lambda_{\mathbf{k}'}(ip_n)\Lambda_\mathbf{k}(ip_n)\Lambda_{\mathbf{k}'}(ip_n^+)}. \tag{C4}$$

Here, $p^\pm = (ip_n^\pm, \mathbf{k}^\pm) = [ip_n \pm (\Omega - i0^+), \mathbf{k}]$ and we have kept up to the second order of the impurity interaction by considering the case of weak impurity scattering.

After the summation of Matsubara frequency, the bare amplitude-amplitude correlation that has been well established in the literature [22,38–42] is written as

$$H_{\rm ba} = \delta|\Delta|^2\sum_{ip_n,\mathbf{k}}\frac{(ip_n^+)^2 + (ip_n)^2 - (ip_n^+ - ip_n)^2 - 2E_\mathbf{k}^2 + 4\Delta_0^2}{2\Lambda_\mathbf{k}(ip_n^+)\Lambda_\mathbf{k}(ip_n)} = \delta|\Delta|^2\sum_{ip_n,\mathbf{k}}\left[\frac{4\Delta_0^2 - \Omega^2}{2\Lambda_\mathbf{k}(ip_n^+)\Lambda_\mathbf{k}(ip_n)} + \frac{1}{\Lambda_\mathbf{k}(ip_n)}\right]$$
$$= \delta|\Delta|^2\left[\sum_{ip_n,\mathbf{k}}\frac{4\Delta_0^2 - \Omega^2}{2\Lambda_\mathbf{k}(ip_n^+)\Lambda_\mathbf{k}(ip_n)} - \frac{1}{g}\right] = \delta|\Delta|^2\left[\sum_\mathbf{k}\frac{(4\Delta_0^2 - \Omega^2)F_\mathbf{k}}{\Omega^2 - 4E_\mathbf{k}^2} - \frac{1}{g}\right]. \tag{C5}$$

It is noted that the last term in the above equation is canceled by the last term in Eq. (12), and then the previously established equation of motion of Higgs mode at the clean and low-frequency case [22,38–42] is recovered.

We then consider the imaginary (i.e., scattering) parts of the Born [Eq. (C3)] and vertex [Eq. (C4)] corrections by using the fact $\frac{1}{\Lambda_{\mathbf{k}}(ip_n^+)}=\frac{1}{2E_{\mathbf{k}}}\sum_{\eta=\pm}\frac{\eta}{ip_{n+}-i0^+-E_{\mathbf{k}}^{\eta}}$, where $ip_{n+}=ip_n+\Omega$.

Specifically, through the summation of the Matsubara frequency, one finds the imaginary part of the Born correction:

$$\begin{aligned}H_{\rm Bc}&=\delta|\Delta|^2 i\pi\sum_{\eta\lambda}\eta\lambda\sum_{ip_n,\mathbf{kk}'}n_i|V_{\mathbf{kk}'}|^2\bigg[\frac{(ip_{n+}ip_n-\xi_{\mathbf{k}}^2+\Delta_0^2)[(ip_n)^2-\Delta_0^2]+2\xi_{\mathbf{k}}^2\Delta_0^2+\Omega ip_n\xi_{\mathbf{k}}^2}{4E_{\mathbf{k}}E_{\mathbf{k}'}\Lambda_{\mathbf{k}}^2(ip_n)}\frac{\delta(ip_{n+}-E_{\mathbf{k}}^{\eta})}{ip_n-E_{\mathbf{k}'}^{\lambda}}-(\Omega\to-\Omega)\bigg]\\&=\delta|\Delta|^2 i\pi\sum_{\eta\lambda}\eta\lambda\sum_{\mathbf{kk}'}n_i|V_{\mathbf{kk}'}|^2\bigg[\frac{(2\Delta_0^2-\Omega\eta E_{\mathbf{k}})\xi_{\mathbf{k}'}^2+2\xi_{\mathbf{k}}^2\Delta_0^2+\Omega(\eta E_{\mathbf{k}}-\Omega)\xi_{\mathbf{k}}^2}{4E_{\mathbf{k}}E_{\mathbf{k}'}(E_{\mathbf{k}}+E_{\mathbf{k}'})^2}\frac{f(E_{\mathbf{k}'}^{\lambda})\delta(E_{\mathbf{k}'}^{\lambda}+\Omega-E_{\mathbf{k}}^{\eta})}{(E_{\mathbf{k}}-E_{\mathbf{k}'})^2}-(\Omega\to-\Omega)\bigg]\\&\approx\delta|\Delta|^2 i\pi\sum_{\eta}\sum_{\mathbf{kk}'}n_i|V_{\mathbf{kk}'}|^2\frac{(4\Delta_0^2-\Omega^2)\xi_{\mathbf{k}}^2}{4E_{\mathbf{k}}E_{\mathbf{k}'}(E_{\mathbf{k}}+E_{\mathbf{k}'})^2}\bigg[\frac{f(E_{\mathbf{k}'}^{\eta})\delta(E_{\mathbf{k}'}-E_{\mathbf{k}})}{\Omega^2}-\frac{f(E_{\mathbf{k}'}^{\eta})\delta(E_{\mathbf{k}'}-E_{\mathbf{k}})}{\Omega^2}\bigg]=0.\end{aligned}\tag{C6}$$

Consequently, the Born correction to the amplitude-amplitude correlation by impurity scattering vanishes. This is because the Born correction is equivalent to the renormalization of the equilibrium impurity self-energy whereas, according to the Anderson theorem [71], this renormalization does not influence the gap, and hence makes no contribution to the damping of the Higgs mode. Actually, the bare amplitude-amplitude correlation and Born correction together can be rewritten as

$$H_{\rm ba}+H_{\rm Bc}=\frac{\delta|\Delta|^2}{2}\sum_{ip_n,\mathbf{k}}{\rm Tr}[\tau_1\bar{G}(ip_n^+,\mathbf{k})\tau_1\bar{G}(ip_n,\mathbf{k})],\tag{C7}$$

with the renormalized Green's function $\bar{G}=G_0+G_0V\tau_3GV\tau_3G$ (thick solid line in Fig. 1) given by [72–75] $\bar{G}(ip_n,\mathbf{k})=[i\bar{p}_n+\xi_{\mathbf{k}}\tau_3+\bar{\Delta}_0\tau_1]/[(i\bar{p}_n)^2-\xi_{\mathbf{k}}^2-\bar{\Delta}_0^2]$. Here, $\bar{p}_n$ and $\bar{\Delta}_0$ denote the renormalized Matsubara frequency and gap by impurity self-energy, respectively. It has been revealed in the literature [72–75] that $\bar{p}_n/\bar{\Delta}_0=p_n/\Delta_0$, leading to a vanishing influence from the renormalization on gap equation (Anderson theorem [71]). Then, similar to the derivation of the bare amplitude-amplitude correlation [Eq. (C5)], Eq. (C7) is directly derived as

$$\begin{aligned}H_{\rm ba}+H_{\rm Bc}&=\delta|\Delta|^2\sum_{ip_n,\mathbf{k}}\frac{4\bar{\Delta}_0^2-\bar{\Omega}^2}{2(\xi_{\mathbf{k}}^2+\bar{\Delta}_0^2+\bar{p}_n^2)^2}=\delta|\Delta|^2\sum_{ip_n}\pi N(0)\frac{4\bar{\Delta}_0^2-\bar{\Omega}^2}{4(\bar{\Delta}_0^2+\bar{p}_n^2)^{3/2}}=\delta|\Delta|^2(4\Delta_0^2-\Omega^2)\sum_{ip_n}\frac{\pi N(0)\Delta_0/\bar{\Delta}_0}{4(\Delta_0^2+p_n^2)^{3/2}}\\&=\delta|\Delta|^2(4\Delta_0^2-\Omega^2)\sum_{ip_n,\mathbf{k}}\frac{\Delta_0/\bar{\Delta}_0}{2(\xi_{\mathbf{k}}^2+\Delta_0^2+p_n^2)^2},\end{aligned}\tag{C8}$$

in which there is no damping term of the Higgs mode. Clearly, the Born correction makes no contribution to the damping of Higgs mode.

The imaginary part of the vertex correction [Eq. (C4)] by impurity scattering after the summation of Matsubara frequency is written as

$$\begin{aligned}H_{\rm vc}&=\delta|\Delta|^2 i\pi\sum_{\eta\eta'\lambda}\sum_{ip_n,\mathbf{kk}'}n_i|V_{\mathbf{kk}'}|^2\eta'\frac{\Delta_0^2(2\lambda E_{\mathbf{k}'}-\Omega)^2-(2\Delta_0^2-\lambda E_{\mathbf{k}'}\Omega)^2}{8E_{\mathbf{k}}^3\Lambda_{\mathbf{k}'}(E_{\mathbf{k}'}^{\lambda}-\Omega)}\frac{2\eta\lambda\delta(ip_{n+}-E_{\mathbf{k}'}^{\lambda})}{(ip_{n+}-E_{\mathbf{k}}^{\eta})(ip_n-E_{\mathbf{k}}^{\eta'})}\\&=\delta|\Delta|^2 i\pi\sum_{\eta\eta'\lambda}\sum_{\mathbf{kk}'}n_i|V_{\mathbf{kk}'}|^2\eta'\eta\lambda\frac{(4\Delta_0^2-\Omega^2)\xi_{\mathbf{k}}^2}{4E_{\mathbf{k}}^3\Lambda_{\mathbf{k}'}(E_{\mathbf{k}'}^{\lambda}-\Omega)}\frac{f(E_{\mathbf{k}}^{\eta}-\Omega)\delta(E_{\mathbf{k}}^{\eta}-E_{\mathbf{k}'}^{\lambda})-f(E_{\mathbf{k}'}^{\lambda}-\Omega)\delta(E_{\mathbf{k}}^{\eta'}+\Omega-E_{\mathbf{k}'}^{\lambda})}{E_{\mathbf{k}}^{\eta}-\Omega-E_{\mathbf{k}}^{\eta'}}\\&\approx\delta|\Delta|^2 i\pi\sum_{\eta\eta'\lambda}\sum_{\mathbf{kk}'}n_i|V_{\mathbf{kk}'}|^2\eta'\eta\lambda\frac{(4\Delta_0^2-\Omega^2)\xi_{\mathbf{k}}^2}{4E_{\mathbf{k}}^3\Lambda_{\mathbf{k}'}(E_{\mathbf{k}'}^{\lambda}-\Omega)}\frac{f(E_{\mathbf{k}}^{\eta}-\Omega)\delta(E_{\mathbf{k}}^{\eta}-E_{\mathbf{k}'}^{\lambda})-f(E_{\mathbf{k}'}^{\lambda}-\Omega)\delta(E_{\mathbf{k}}^{\eta'}-E_{\mathbf{k}'}^{\lambda})}{E_{\mathbf{k}}^{\eta}-\Omega-E_{\mathbf{k}}^{\eta'}}.\end{aligned}\tag{C9}$$

The $\eta'=\eta$ part vanishes in above equation. Then, with $\eta'=-\eta$, one has

$$\begin{aligned}H_{\rm vc}&=\delta|\Delta|^2 i\pi\sum_{\eta\lambda}\sum_{\mathbf{kk}'}n_i|V_{\mathbf{kk}'}|^2\lambda\frac{(4\Delta_0^2-\Omega^2)\xi_{\mathbf{k}}^2}{4E_{\mathbf{k}}^3\Omega(\Omega-2\lambda E_{\mathbf{k}'})}\bigg[\frac{f(E_{\mathbf{k}}^{\eta}-\Omega)}{\Omega-2\eta E_{\mathbf{k}}^{\eta}}+\frac{f(E_{\mathbf{k}'}^{\lambda}-\Omega)}{\Omega+2\eta E_{\mathbf{k}}^{\eta}}\bigg]\delta(E_{\mathbf{k}}^{\eta}-E_{\mathbf{k}'}^{\lambda})\\&=\delta|\Delta|^2 i\pi\sum_{\eta}\sum_{\mathbf{kk}'}n_i|V_{\mathbf{kk}'}|^2\frac{(4\Delta_0^2-\Omega^2)\xi_{\mathbf{k}}^2}{2E_{\mathbf{k}}^3(\Omega-2\eta E_{\mathbf{k}})}\frac{\eta f(E_{\mathbf{k}}^{\eta})}{\Omega^2-4E_{\mathbf{k}}^2}\delta(E_{\mathbf{k}}-E_{\mathbf{k}'})\\&=\delta|\Delta|^2 i\Omega\pi\sum_{\mathbf{kk}'}n_i|V_{\mathbf{kk}'}|^2\frac{(4\Delta_0^2-\Omega^2)\xi_{\mathbf{k}}^2F_{\mathbf{k}}}{E_{\mathbf{k}}^2(4E_{\mathbf{k}}^2-\Omega^2)^2}\delta(E_{\mathbf{k}}-E_{\mathbf{k}'})-\delta|\Delta|^2 i\pi\sum_{\mathbf{kk}'}n_i|V_{\mathbf{kk}'}|^2\frac{(4\Delta_0^2-\Omega^2)\xi_{\mathbf{k}}^2}{E_{\mathbf{k}}^2(4E_{\mathbf{k}}^2-\Omega^2)^2}\delta(E_{\mathbf{k}}-E_{\mathbf{k}'})\\&\approx\delta|\Delta|^2 i\Omega\pi\sum_{\mathbf{kk}'}n_i|V_{\mathbf{kk}'}|^2\frac{4\Delta_0^2\xi_{\mathbf{k}}^2F_{\mathbf{k}}}{E_{\mathbf{k}}^2(4E_{\mathbf{k}}^2-\Omega^2)^2}\delta(E_{\mathbf{k}}-E_{\mathbf{k}'}).\end{aligned}\tag{C10}$$

Here, we have only kept the terms that are relevant to the Higgs-mode damping (i.e., linearly dependent on Ω). Consequently, with Eqs. (C5) and (C6) as well as (C10), from the nonequilibrium action $\delta S^{(2)} = -H_{\rm ba} - H_{\rm bc} - H_{\rm vc} - \delta|\Delta|^2/g$, Eq. (50) is derived.

APPENDIX D: DERIVATION OF EQ. (54)

In this Appendix, we derive the long-time dynamic of the Higgs mode [Eq. (54)] from the equation of motion in Eq. (51). In the presence of the external excitation, the equation of motion of the Higgs mode in Eq. (51) is rewritten as

$$\beta_H[(2\Delta_0)^2 - \Omega^2 - 2i\Omega\Gamma_H/\beta_H]\delta|\Delta| = Q(\Omega), \tag{D1}$$

where $Q(\Omega)$ represents the response function determined by the external excitation/field solely.

In consideration of the long-time dynamic after the excitation, the response function can be approximately treated as a short pulse, and then, one has $Q(t) = Q_0\delta(t)$ in time space with the Fourier component $Q(\Omega) = Q_0$ in frequency space. At weak scattering, Eq. (D1) becomes

$$\begin{aligned} Q_0 &= \delta|\Delta|\sum_{\mathbf{k}}\left\{\frac{[(2\Delta_0)^2-\Omega^2]F_{\mathbf{k}}}{4E_{\mathbf{k}}^2-\Omega^2} - \frac{2i\Omega\gamma_k 4\xi_{\mathbf{k}}^2F_{\mathbf{k}}}{\left(4E_{\mathbf{k}}^2-\Omega^2\right)^2}\right\} \approx \delta|\Delta|\sum_{\mathbf{k}}\left\{\frac{[(2\Delta_0)^2-\Omega^2]F_{\mathbf{k}}}{\Omega^2-4E_{\mathbf{k}}^2} - \frac{2i\Omega\gamma_kF_{\mathbf{k}}}{4E_{\mathbf{k}}^2-\Omega^2} + \frac{2i\Omega\gamma_k4\Delta_0^2F_{\mathbf{k}}}{\left(4E_{\mathbf{k}}^2-\Omega^2\right)^2}\right\} \\ &\approx \delta|\Delta|\sum_{\mathbf{k}}\frac{[(2\Delta_0)^2-(\Omega+i\gamma_k)^2]F_{\mathbf{k}}}{4E_{\mathbf{k}}^2-(\Omega+i\gamma_k)^2}. \end{aligned} \tag{D2}$$

Consequently, from the above equation, the temporal evolution of the Higgs mode at low temperature is given by

$$\delta|\Delta(t)| \sim \int\frac{d\Omega}{2\pi}\frac{Q_0e^{-i\Omega t}}{\sqrt{(2\Delta_0)^2-(\Omega+i\bar{\gamma})^2}} = e^{-\bar{\gamma}t}\int_{-\infty+i\bar{\gamma}}^{\infty+i\bar{\gamma}}\frac{d\Omega}{2\pi}\frac{Q_0e^{-i\Omega t}}{\sqrt{(2\Delta_0)^2-\Omega^2}}. \tag{D3}$$

It is noted that for the integrand in Eq. (D3), in the complex plane of Ω, there exist two branching points at $\Omega = \pm 2\Delta_0$. Then, similar to the previous work [15], after the standard construction of the closed contour, one obtains

$$\delta|\Delta(t)| \sim \pi e^{-\bar{\gamma}t}\frac{e^{2i\Delta_0t}+e^{-2i\Delta_0t}}{\sqrt{4\Delta_0t}} = \pi e^{-\bar{\gamma}t}\frac{\cos(2\Delta_0t)}{\sqrt{\Delta_0t}}. \tag{D4}$$

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