Attractive multicomponent Gaudin-Yang model: Three roads to the energy gap

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We analytically determine the energy gap at weak coupling in the attractive multicomponent Gaudin-Yang model, an integrable model which describes interacting fermions in one dimension with κ components. We use three different methods. The first one is based on a direct analysis of the Bethe ansatz equations. The second method uses the theory of resurgence and the large order behavior of the perturbative series for the ground state energy. The third method is based on a renormalization group analysis. The three methods lead to the same answer, providing in this way a nontrivial test of the ideas of resurgence and renormalons as applied to nonrelativistic many-body systems.

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I. INTRODUCTION

One of the most important nonperturbative effects in quantum theory is the energy gap of many-fermion systems with an attractive interaction. This gap, which is exponentially small in the coupling constant, is a universal feature of these systems, and it is at the origin of conventional superconductivity. However, being a nonperturbative effect, it is not easy to compute. One possibility is to use Bardeen-Cooper-Schrieffer-like mean field theory, which provides an approximate expression for the gap. Another possibility is to use renormalization group (RG) methods. In exactly solvable models, one can often calculate the gap exactly, and this provides a useful test of approximate methods.

Recently, it has been pointed out that the leading behavior of the energy gap at weak coupling can be obtained from the large order behavior of the perturbative expansion for the ground state energy [1–3]. This is an example of the general connection between perturbative series and nonperturbative effects pointed out in quantum mechanics in Refs. [4,5]. The relationship between perturbative and nonperturbative sectors has evolved into a general framework to understand nonperturbative effects in mathematics and physics, sometimes called the *theory of resurgence* (see Refs. [6–9] for reviews). In the case of many-fermion systems with an attractive interaction, it has been argued in Refs. [1,2] that the energy gap is structurally very similar to a renormalon effect [10] in an asymptotically free theory. Therefore, one can use renormalon techniques, like all-order calculations based on particular families of diagrams, to obtain information on the energy gap. These ideas were tested in two integrable models: the Gaudin-Yang model [1,2] and the one-dimensional (1D) Hubbard model [3], in the case of attractive fermions with two components.

In this paper, we consider the Gaudin-Yang model with κ components and $SU(\kappa)$ symmetry, which was briefly addressed in Ref. [1]. This model is integrable [11,12] and has many interesting features. First of all, it might be relevant to the study of ultracold atoms with higher hyperfine spin in 1D traps (see, e.g., Refs. [13,14] and references therein). In addition, it displays qualitative phenomena: the ground state consists of bound states of κ elementary fermions, which generalize the familiar Cooper pairs occurring when $\kappa = 2$ (when $\kappa = 3$, these bound states are sometimes called *trions*). More generally, one finds bound states of $1 \leqslant n \leqslant \kappa$ fermions, leading to a rich phase structure. From a more theoretical point of view, this model might be an interesting testing ground for approximations based on a large number of components (large N).

Here, we are interested on the nonperturbative aspects of the model, and for this reason, we will focus on its energy gap, in the weak coupling regime. In principle, the energy gap can be determined from the Bethe ansatz (BA) solution, as pointed out in Ref. [15] in the case of $\kappa = 2$. However, an analytic calculation at weak coupling has not been performed for $\kappa > 2$ since it requires a detailed study of the BA equations like what was done in Ref. [1]. Our first result is then a formula for the energy gap, at next-to-leading order in the coupling constant, including the precise, κ-dependent prefactor.

According to the conjecture of Ref. [1], we expect the energy gap to control the large order behavior of the perturbative series for the ground state energy. Such a connection was established numerically in Ref. [1] for $\kappa = 2$, and we check in detail that this connection persists for general κ. This provides a precision test of the ideas of resurgence since the growth of perturbation theory at next-to-leading order in the number of loops predicts the dependence of the gap on the coupling at next-to-leading order. In fact, in the case of general κ, we first found this dependence by looking at the large order behavior of the perturbative series, and only later, we verified it with the BA calculation presented here.

As discussed in Refs. [1,3], the nonperturbative scale leading to the energy gap can be regarded as a renormalon effect.

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A renormalon effect is a nonperturbative phenomenon manifested in the Feynman diagrams which contribute to the large order behavior of the perturbative series. By definition, it appears as a family of diagrams which diverges factorially after integration over the momenta (this contrasts with instantons, which relate to the factorially large number of diagrams at each order). As shown in Ref. [1], the ring diagrams dominating at large κ are renormalon diagrams, and they lead to the right value for the leading order dependence of the gap on the coupling constant. However, we check explicitly that they fail to capture the next-to-leading dependence, which is to be expected since this dependence is subleading in the $1/\kappa$ expansion. It is well known however that, in asymptotically free quantum field theories, the coupling constant dependence of the nonperturbative scale can be determined by a RG analysis. The leading, exponential dependence of the nonperturbative scale is a one-loop effect, while the next-to-leading dependence requires knowledge of the beta function at two loops (see, e.g., Ref. [10]). In many-fermion systems, a similar argument linking the energy gap to a RG analysis was presented by Larkin and Sak [16], again in the case $\kappa = 2$. In view of this connection, the results that we have obtained for the gap predict the form of the two-loop beta function of the Gaudin-Yang model, as a function of κ. We verify this prediction by a direct calculation with RG techniques.

The agreement between these three answers provides a further test of the idea put forward in Refs. [1–3] that the energy gap in interacting many-fermion systems can be understood by using the theory of resurgence and the physics of renormalons.

The paper is organized as follows. In Sec. II, we review the multicomponent Gaudin-Yang model and its BA solution. In Sec. III, we calculate the energy gap from the BA equation at weak coupling, extending the results of Ref. [15] to the multicomponent case. In Sec. IV, we study the large order behavior of the perturbative series, and we show that it reproduces correctly the weak-coupling behavior of the energy gap, in agreement with the conjecture in Ref. [1]. In Sec. V, we compute the beta function of the model by using the RG at two loops, and derive the expression for the gap. Finally, in Sec. VI, we present some conclusions and prospects for future work.

There are in addition two Appendixes. In the first one, we show that ring diagrams lead to an approximate expression for the gap which is correct to leading order in the coupling constant but not to next-to-leading order. In the second Appendix, we show that the relativistic model obtained in Sec. V by using the approach of Refs. [16,17] is closely related to the chiral Gross-Neveu model and leads to the same beta function up to two loops.

II. MULTICOMPONENT GAUDIN-YANG MODEL AND ITS BA SOLUTION

The Hamiltonian for the Gaudin-Yang model is given by

$$H = -\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} - 2c \sum_{1\leqslant i<j\leqslant N} \delta(x_i - x_j). \tag{1}$$

We will consider the case of an attractive interaction, which corresponds to a positive coupling constant $c > 0$. We also consider the multicomponent case, so that each fermion has κ possible internal states $|1\rangle, \dots, |\kappa\rangle$. The number of fermions in the ith internal state $|i\rangle$ will be denoted by N^i. We will choose the labels of the states, $i = 1, \dots, \kappa$, in such a way that the numbers of particles are ordered as $N^1 \geqslant N^2 \geqslant \cdots \geqslant N^\kappa$.

The eigenvalue problem for this many-body system can be solved with the BA. We consider the system in an interval of length L, and we impose periodic boundary conditions. In the case of $\kappa = 2$, the solution was obtained by Gaudin [18] and Yang [19]. The generalization to arbitrary $\kappa > 2$ is due to Sutherland [11] and Takahashi [12]. The solution can be characterized by a system of nested BA equations. To write down these equations, we introduce

$$M_i = \sum_{j=i}^{\kappa-1} N^{j+1}. \tag{2}$$

Then the equations read

$$\begin{aligned} \exp(ik_iL) &= \prod_{\alpha=1}^{M_1} \frac{k_i - \lambda_\alpha^{(1)} + ic'}{k_i - \lambda_\alpha^{(1)} - ic'}, \\ i &= 1, \dots, M_0, \end{aligned} \tag{3}$$

$$\begin{aligned} \prod_{\eta=1}^{M_l} \frac{\lambda_\alpha^{(l)} - \lambda_\eta^{(l)} + 2ic'}{\lambda_\alpha^{(l)} - \lambda_\eta^{(l)} - 2ic'} &= -\prod_{\beta=1}^{M_{l-1}} \frac{\lambda_\alpha^{(l)} - \lambda_\beta^{(l-1)} + ic'}{\lambda_\alpha^{(l)} - \lambda_\beta^{(l-1)} - ic'} \\ &\quad \times \prod_{\delta=1}^{M_{l+1}} \frac{\lambda_\alpha^{(l)} - \lambda_\delta^{(l+1)} + ic'}{\lambda_\alpha^{(l)} - \lambda_\delta^{(l+1)} - ic'}, \\ \alpha &= 1, \dots, M_l, \\ l &= 1, \dots, \kappa - 1, \end{aligned} \tag{4}$$

where $c' = c/2$. The quasimomenta k_j appearing in Eq. (4) determine the energy eigenvalues through

$$E = \sum_{j=1}^{N} k_j^2, \tag{5}$$

while the Bethe roots $\lambda_\alpha^{(l)}$ are auxiliary variables.

The solutions k_j to the BA equations form strings in the complex plane, corresponding to bound states of m particles, where $1 \leqslant m \leqslant \kappa$. The number of bound states with m particles N_m is related to the numbers of particles in the ith state N^i by

$$N_m = N^m - N^{m+1}, \quad m = 1, \dots, \kappa - 1, \tag{6}$$

and $N_\kappa = N^\kappa$. The strings of quasimomenta, corresponding to a bound state of m particles labeled by $j = 1, \dots, N_m$, have the form:

$$\begin{aligned} k_j^{m,q} &= \lambda_j^m + i(m + 1 - 2q)c' + \mathcal{O}[\exp(-L)], \\ q &= 1, \dots, m. \end{aligned} \tag{7}$$

For each set of these k's, one has a set of $m-l$ complex roots $\lambda_j^{(l)m,q}$ at the levels $l=1,\ldots,m-1$, with the form:

$$\lambda_j^{(l)m,q} = \lambda_j^m + i(m-l+1-2q)c' + \mathcal{O}[\exp(-L)],$$
$$q=1,\ldots,m-l, \qquad l=1,\ldots,m-1. \tag{8}$$

All the roots associated with the jth bound state of size m, i.e., all $k_j^{m,q}$ and $\lambda_j^{(l)m,q}$, share the same real part λ_j^m, which corresponds to the unique real root at level $m-1$. These roots characterize the eigenstate made out of N_m bound states of size $1 \leqslant m \leqslant \kappa$. They can be found from the following approximate version of the BA equations, which is correct up to exponentially small corrections in L (see, e.g., Ref. [20]):

$$m\lambda_j^m L = 2\pi K_j^m + \sum_{p=1}^{m-1}\sum_{q=p}^{\kappa}\sum_{l=1}^{N_q} 2\tan^{-1}\left[\frac{\lambda_j^m - \lambda_l^q}{(q+m-2p)c'}\right]$$
$$+ \sum_{q=m+1}^{\kappa}\sum_{l=1}^{N_q} 2\tan^{-1}\left[\frac{\lambda_j^m - \lambda_l^q}{(q-m)c'}\right],$$
$$m=1,\ldots,\kappa \qquad j=1,\ldots,N_m. \tag{9}$$

In these equations,

$$K_j^m = -\frac{N_m - 1}{2} + j - 1. \tag{10}$$

In terms of the roots λ_j^m, the energy of such a state is given by

$$E(N_1,\ldots,N_\kappa) = \sum_{m=1}^{\kappa}\sum_{j=1}^{N_m} m\left[\left(\lambda_j^m\right)^2 - \frac{(m^2-1)c^2}{12}\right]. \tag{11}$$

The ground state of the system is found when all N fermions are in bound states of κ particles, which correspond to the 1D fully antisymmetric, or singlet, representation of the $\mathfrak{su}(N)$ algebra (in the $\kappa = 2$ case, these are the Cooper pairs). In that case, $N = \kappa N_\kappa$, and Eq. (9) reduces to

$$\kappa\lambda_j^\kappa L = 2\pi K_j^\kappa + \sum_{l=1}^{N_\kappa}\sum_{p=1}^{m-1} 2\tan^{-1}\left[\frac{\lambda_j^\kappa - \lambda_l^\kappa}{(2\kappa-2p)c'}\right]. \tag{12}$$

In the thermodynamic limit

$$L \to \infty, \qquad N \to \infty, \qquad \frac{N}{L} = n, \tag{13}$$

the position of the roots becomes a continuous variable $\lambda_j^\kappa \to \lambda$. The state number $K_j^\kappa \to K(\lambda)$ gives rise to a state density function $f(\lambda) = L^{-1}dK(\lambda)/d\lambda$. Taking a derivative of Eq. (12) with respect to λ, we find

$$\frac{\kappa}{2\pi} = f(\lambda) + \frac{1}{2\pi}\int_{-Q}^{Q} d\lambda' f(\lambda')\sum_{p=1}^{\kappa-1}\frac{2pc}{(pc)^2 + (\lambda-\lambda')^2}, \tag{14}$$

where Q is implicitly defined through

$$\int_{-Q}^{Q} f(\lambda)\, d\lambda = \frac{n}{\kappa}. \tag{15}$$

The ground state energy per unit length is then given by

$$E = \kappa\int_{-Q}^{Q}\left(\lambda^2 - \frac{\kappa^2-1}{12}c^2\right) f(\lambda)d\lambda. \tag{16}$$

These integral equations were found in Ref. [12].

It is convenient to change variables as

$$\theta = \frac{\lambda}{c}, \quad B = \frac{Q}{c}, \quad \rho(\theta) = \pi f(\lambda). \tag{17}$$

In these variables, the integral in Eq. (14) characterizing the ground state reads

$$\rho(\theta) + \int_{-B}^{B} d\theta' K(\theta-\theta')\rho(\theta') = \frac{\kappa}{2}, \tag{18}$$

where the kernel can be written in terms of the digamma function as follows:

$$K(\theta) = \frac{1}{2\pi}[\psi(\kappa+i\theta) + \psi(\kappa-i\theta)$$
$$- \psi(1-i\theta) - \psi(1+i\theta)]. \tag{19}$$

The integral in Eq. (18) was studied in Refs. [1,2] with the techniques developed in Refs. [21,22]. Let us introduce the dimensionless coupling:

$$\gamma = \frac{c}{n}. \tag{20}$$

Then from the normalization of the ground state distribution function:

$$\frac{1}{\pi}\int_{-B}^{B}\rho(\theta)d\theta = \frac{1}{\kappa\gamma}, \tag{21}$$

one finds the following weak coupling expansion for B:

$$B = \frac{\pi}{\gamma\kappa} + \frac{\kappa}{2\pi}\log(\kappa)$$
$$- \frac{\kappa-1}{2\pi}\left[\log\left(\frac{4\pi^2}{\gamma\kappa}\right) + 1\right] + \mathcal{O}(\gamma). \tag{22}$$

III. ENERGY GAP FROM THE BA

The BA solution summarized in the previous section makes it possible to calculate the energy gap of the model. In the case of the Gaudin-Yang model with $\kappa = 2$ components, the gap was calculated in this way by Krivnov and Ovchinnikov [15] (see also Ref. [23]). We will now extend this calculation to the case of arbitrary κ.

To find the energy gap, one must identify the first excited state, which involves breaking one of the bound states with κ fermions in the ground state. From Eq. (11), we can see that the binding energy of a bound state of m particles is

$$E_m \sim -\left[m\frac{(m^2-1)c^2}{12}\right]. \tag{23}$$

Thus, the first excited state, at weak coupling, consists of a free fermion (i.e., a 1 bound state) and a bound state of $\kappa - 1$ fermions. The energy gap is given by

$$\Delta_\kappa = E(1,0,\ldots,0,1,N_\kappa - 1) - E(0,0,\ldots,0,0,N_\kappa), \tag{24}$$

which we will compute in the thermodynamic limit in Eq. (13). We can do this by perturbing the ground state problem. In the ground state, we have N_κ bound states characterized by the Bethe roots λ_j^κ, which satisfy Eq. (12). In the first excited state, we have

$$\lambda_1^1 \equiv k, \quad \lambda_1^{\kappa-1} \equiv \Lambda, \quad \bar{\lambda}_j^\kappa \equiv \lambda_j^\kappa + \frac{\xi_j}{L},$$
$$\bar{K}_j^\kappa = K_j^\kappa + \frac{1}{2}. \tag{25}$$

The perturbed BA equations in Eq. (9) become

$$\kappa\left(L\lambda_j^\kappa + \xi_j\right) = 2\pi K_j^\kappa + \pi + 2\tan^{-1}\left[\frac{\bar{\lambda}_j^\kappa - k}{(\kappa-1)c'}\right] + \sum_{p=1}^{\kappa-1} 2\tan^{-1}\left[\frac{\bar{\lambda}_j^\kappa - \Lambda}{(2p-1)c'}\right] + \sum_{p=1}^{\kappa-1}\sum_{l=1}^{N_\kappa-1} 2\tan^{-1}\left(\frac{\bar{\lambda}_j^\kappa - \bar{\lambda}_l^\kappa}{2pc'}\right), \tag{26}$$

$$Lk = 2\tan^{-1}\left[\frac{k-\Lambda}{(\kappa-2)c'}\right] + \sum_{l=1}^{N_\kappa-1} 2\tan^{-1}\left[\frac{k-\bar{\lambda}_l^\kappa}{(\kappa-1)c'}\right], \tag{27}$$

$$(\kappa-1)L\Lambda = 2\tan^{-1}\left[\frac{\Lambda-k}{(\kappa-2)c'}\right] + \sum_{l=1}^{N_\kappa-1}\sum_{p=1}^{\kappa-1} 2\tan^{-1}\left[\frac{\Lambda-\bar{\lambda}_l^\kappa}{(2\kappa-2p-1)c'}\right]. \tag{28}$$

The last two equations are easy to solve. In the thermodynamic limit, one has

$$k = 2\int_{-Q}^{Q} d\lambda f(\lambda)\tan^{-1}\left[\frac{k-\lambda}{(\kappa-1)c'}\right] + \mathcal{O}\left(\frac{1}{L}\right), \tag{29}$$

$$\Lambda = \frac{2}{\kappa-1}\sum_{p=1}^{\kappa-1}\int_{-Q}^{Q} d\lambda f(\lambda)\tan^{-1}\left[\frac{\Lambda-\lambda}{(2\kappa-2p-1)c'}\right] + \mathcal{O}\left(\frac{1}{L}\right). \tag{30}$$

Since $\tan^{-1}$ is odd and f is even, $k = \Lambda = 0$ solves these equations.

Let us now consider Eq. (26). Its last term can expanded as

$$2\tan^{-1}\left(\frac{\bar{\lambda}_j^\kappa - \bar{\lambda}_l^\kappa}{2pc'}\right) \sim 2\tan^{-1}\left(\frac{\lambda_j^\kappa - \lambda_l^\kappa}{2pc'}\right) + \frac{4pc'}{L}\frac{\xi_j - \xi_l}{(2pc')^2 + \left(\lambda_j^\kappa - \lambda_l^\kappa\right)^2} + \mathcal{O}\left(\frac{1}{L^2}\right). \tag{31}$$

We need to keep the $\mathcal{O}(L^{-1})$ term because the sum over l is of the order of $N_\kappa \propto L$. Putting together Eqs. (26) and (31), then subtracting Eq. (12), we find

$$\kappa\xi_j = \pi + 2\tan^{-1}\left[\frac{\lambda_j^\kappa - k}{(\kappa-1)c'}\right] + \sum_{p=1}^{\kappa-1} 2\tan^{-1}\left[\frac{\lambda_j^\kappa - \Lambda}{(2p-1)c'}\right] - \sum_{p=1}^{\kappa-1} 2\tan^{-1}\left(\frac{\lambda_j^\kappa - \lambda_{N_\kappa}^\kappa}{2pc'}\right) + \sum_{l=1}^{N_\kappa-1}\sum_{p=1}^{\kappa-1}\frac{4pc'}{L}\frac{\xi_j - \xi_l}{(2pc')^2 + \left(\lambda_j^\kappa - \lambda_l^\kappa\right)^2}. \tag{32}$$

The continuum limit follows naturally,

$$\kappa\xi(\lambda) = \pi + 2\tan^{-1}\left[\frac{\lambda}{(\kappa-1)c'}\right] + \sum_{p=1}^{\kappa-1} 2\tan^{-1}\left[\frac{\lambda}{(2p-1)c'}\right] - \sum_{p=1}^{\kappa-1} 2\tan^{-1}\left(\frac{\lambda - Q}{2pc'}\right) + \sum_{p=1}^{\kappa-1}\int_{-Q}^{Q} d\lambda' f(\lambda')\frac{2pc[\xi(\lambda) - \xi(\lambda')]}{(pc)^2 + (\lambda-\lambda')^2}. \tag{33}$$

The last term on the right-hand side can be simplified by using Eq. (14), and one finds

$$2\pi f(\lambda)\xi(\lambda) = \pi + 2\tan^{-1}\left[\frac{\lambda}{(\kappa-1)c'}\right] + \sum_{p=1}^{\kappa-1} 2\tan^{-1}\left[\frac{\lambda}{(2p-1)c'}\right] - \sum_{p=1}^{\kappa-1} 2\tan^{-1}\left(\frac{\lambda - Q}{2pc'}\right) - \sum_{p=1}^{\kappa-1}\int_{-Q}^{Q} d\lambda' f(\lambda')\xi(\lambda')\frac{2pc}{(pc)^2 + (\lambda-\lambda')^2}. \tag{34}$$

We change variables to $\theta = \lambda/c$, $B = Q/c$, and introduce the distribution $\Psi(\theta) = f(\lambda)\xi(\lambda)$. We find the following integral equation for $\Psi(\theta)$:

$$2\pi\Psi(\theta) = \pi + 2\tan^{-1}\left(\frac{2\theta}{\kappa-1}\right) + \sum_{p=1}^{\kappa-1} 2\tan^{-1}\left(\frac{2\theta}{2p-1}\right) - \sum_{p=1}^{\kappa-1} 2\tan^{-1}\left(\frac{\theta - B}{p}\right) - \sum_{p=1}^{\kappa-1}\int_{-B}^{B} d\theta'\Psi(\theta')\frac{2p}{p^2 + (\theta-\theta')^2}. \tag{35}$$

This generalizes a similar equation in Ref. [15] for $\kappa = 2$ to arbitrary κ.

The energy gap is given by

$$\begin{aligned}\Delta_\kappa &= k^2 + (\kappa-1)\left[\Lambda^2 - \frac{(\kappa-1)^2-1}{12}c^2\right]\\ &+ \sum_{j=1}^{N_\kappa-1}\kappa\left[\left(\bar{\lambda}_j^\kappa\right)^2 - \frac{\kappa^2-1}{12}c^2\right]\\ &- \sum_{j=1}^{N_\kappa}\kappa\left(\lambda_j^2 - \frac{\kappa^2-1}{12}c^2\right),\end{aligned} \tag{36}$$

and in the thermodynamic limit, we find

$$\Delta_\kappa = -\kappa c^2 B^2 + \frac{\kappa(\kappa-1)}{4}c^2 + 2\kappa c^2\int_{-B}^{B}\theta\Psi(\theta)d\theta. \tag{37}$$

To tackle the integral in Eq. (35), it is convenient to antisymmetrize it, as in Ref. [15]. We define the odd function:

$$h(\theta) = \frac{\Psi(\theta)-\Psi(-\theta)}{2} - \frac{\mathrm{sgn}(\theta)}{2}. \tag{38}$$

If we consider that

$$\begin{aligned}\int_{-B}^{B} d\theta' \frac{p\,\mathrm{sgn}(\theta')}{p^2+(\theta-\theta')^2} &= 2\tan^{-1}\left(\frac{\theta}{p}\right) - \tan^{-1}\left(\frac{\theta+B}{p}\right)\\ &- \tan^{-1}\left(\frac{\theta-B}{p}\right),\end{aligned} \tag{39}$$

we find that the sum of Eq. (35) with its reflection yields

$$\begin{aligned}h(\theta) &+ \frac{1}{2\pi}\int_{-B}^{B} d\theta' K(\theta-\theta')h(\theta')\\ &= \tau_0(\theta) - \frac{1}{2}\mathrm{sgn}(\theta),\end{aligned} \tag{40}$$

$$\begin{aligned}\tau_0(\theta) &= \frac{1}{\pi}\tan^{-1}\left(\frac{2\theta}{\kappa-1}\right) + \frac{1}{\pi}\sum_{p=1}^{\kappa-1}\tan^{-1}\left(\frac{2\theta}{2p-1}\right)\\ &- \frac{1}{\pi}\sum_{p=1}^{\kappa-1}\tan^{-1}\left(\frac{\theta}{p}\right),\end{aligned} \tag{41}$$

where the kernel $K(\theta)$ is given in Eq. (19). The energy gap has a simple expression in terms of the function $h(\theta)$:

$$\frac{\Delta_\kappa}{c^2} = -4\int_B^\infty \theta h(\theta)d\theta. \tag{42}$$

At large B, the integral in Eq. (41) can be solved with the techniques introduced in Refs. [24,25]. First, we consider Eq. (41) in the strict limit $B \to \infty$, which defines the function $h_0(\theta)$ through the integral equation:

$$\begin{aligned}h_0(\theta) &+ \frac{1}{2\pi}\int_{-\infty}^{\infty} d\theta' K(\theta-\theta')h_0(\theta')\\ &= \tau_0(\theta) - \frac{1}{2}\mathrm{sgn}(\theta),\end{aligned} \tag{43}$$

This equation can be solved by Fourier transform, leading to

$$\begin{aligned}\tilde{h}_0(\omega) &= -\frac{i}{\omega}\frac{1-\exp\left(-\frac{|\omega|}{2}\right)}{1+\exp\left(-\frac{\kappa|\omega|}{2}\right)}\left\{1-\exp\left[-\frac{(\kappa-1)|\omega|}{2}\right]\right\}\\ &= -\frac{i}{\pi}\sum_{n=-\infty}^{\infty}\frac{\sin\left[\frac{2\pi}{\kappa}\left(n-\frac{1}{2}\right)\right]}{\left(n-\frac{1}{2}\right)\left[\omega-\frac{4\pi i}{\kappa}\left(n-\frac{1}{2}\right)\right]}.\end{aligned} \tag{44}$$

From this representation, one can invert the Fourier transform:

$$h_0(\theta) = -\frac{1}{\pi}\tan^{-1}\left[\frac{\sin\left(\frac{\pi}{\kappa}\right)}{\sinh\left(\frac{2\pi\theta}{\kappa}\right)}\right], \tag{45}$$

and one has, at large B:

$$\begin{aligned}h_0(\theta+B) &\sim -\frac{2}{\pi}\sin\left(\frac{\pi}{\kappa}\right)\exp\left[-\frac{2\pi}{\kappa}(\theta+B)\right]\\ &+ \mathcal{O}\left[\exp\left(-\frac{4\pi}{\kappa}B\right)\right].\end{aligned} \tag{46}$$

However, h_0 is not a good enough approximation of h to calculate the gap. Following Refs. [24,25], we have that $h(\theta+B) \approx r(\theta)$, where $r(\theta)$ satisfies the integral equation:

$$\begin{aligned}r(\theta) &= -\frac{2}{\pi}\sin\left(\frac{\pi}{\kappa}\right)\exp\left[-\frac{2\pi}{\kappa}(\theta+B)\right]\\ &+ \int_0^\infty d\theta' R(\theta-\theta')r(\theta').\end{aligned} \tag{47}$$

The first term on the right-hand side of Eq. (47) is the approximate form of $h_0(\theta+B)$ found in Eq. (46), and the kernel is given by

$$R(\theta) = \frac{1}{2\pi}\int_{\mathbb{R}} d\omega \frac{\exp(i\omega\theta)\tilde{K}(\omega)}{1+\tilde{K}(\omega)}, \tag{48}$$

where

$$\begin{aligned}\tilde{K}(\omega) &= \int_{\mathbb{R}} d\theta \exp(i\omega\theta)K(\theta)\\ &= \frac{\exp(-|\omega|)-\exp(-\kappa|\omega|)}{1-\exp(-|\omega|)}.\end{aligned} \tag{49}$$

One can now use Wiener-Hopf techniques to obtain the Fourier transform of $r(\theta)$:

$$\begin{aligned}\mathcal{F}_+(\omega) &= \int_0^\infty d\theta \exp(i\omega\theta)r(\theta)\\ &= -\frac{2}{\pi}\sin\left(\frac{\pi}{\kappa}\right)\exp\left(-\frac{2\pi}{\kappa}B\right)\frac{G_+(\omega)G_+\left(\frac{2\pi i}{\kappa}\right)}{\frac{2\pi}{\kappa}-i\omega},\end{aligned} \tag{50}$$

where

$$\begin{aligned}G_+(\omega) &= \sqrt{\kappa}\frac{\Gamma\left(1-\frac{i\omega}{2\pi}\right)}{\Gamma\left(1-\frac{i\kappa\omega}{2\pi}\right)}\\ &\times \exp\left\{\frac{i\omega\left[\log\left(-\frac{i\omega}{2\pi}\right)-1\right]}{2\pi}\right.\\ &\left.- \frac{i\kappa\omega\left[\log\left(-\frac{i\kappa\omega}{2\pi}\right)-1\right]}{2\pi}\right\}.\end{aligned} \tag{51}$$

In terms of $r(\theta)$, the energy gap at large B, which corresponds to weak coupling, is then given by

$$\begin{aligned}\frac{\Delta_\kappa}{c^2} &\approx -4B\int_0^\infty r(\theta)d\theta = -4B\mathcal{F}_+(0)\\ &= \frac{8B}{\pi}\sin\left(\frac{\pi}{\kappa}\right)\frac{\exp\left(\frac{1}{\kappa}-1\right)\kappa^{\frac{1}{\kappa}+1}\Gamma\left(\frac{1}{\kappa}\right)}{2\pi}\\ &\quad\times\exp\left(-\frac{2\pi}{\kappa}B\right).\end{aligned}\tag{52}$$

This constant overall factor can be tested by numerically solving the integral in Eq. (35), which we have done for $\kappa = 2, 3, 4, 7, 8$. The result in Eq. (52) generalizes the calculation of Ref. [15] to arbitrary κ.

It is convenient to express the result in Eq. (52) in terms of γ. This last step is nontrivial, and in the calculation in Ref. [15] for $\kappa = 2$, it involved a constant which had to be determined numerically. An analytic expression for this constant, leading to a complete answer for $\kappa = 2$ at next-to-leading order in γ, was obtained in Ref. [16] by an indirect argument and later confirmed in Ref. [26] (see Ref. [27]). In the case of general κ, the methods developed in Refs. [1,2] lead to the explicit expression in Eq. (22), which make it possible to obtain the analytic form of the answer for arbitrary κ. By using that result, we can finally write

$$\frac{\Delta_\kappa}{E_\mathrm{F}} \approx \left(\frac{\kappa}{2\pi}\right)^{2/\kappa}\frac{64}{\kappa^2\Gamma\left(1-\frac{1}{\kappa}\right)}\gamma^{1/\kappa}\exp\left(-\frac{2\pi^2}{\kappa^2}\frac{1}{\gamma}\right),\tag{53}$$

where

$$E_\mathrm{F} = \frac{\pi n^2}{4}\tag{54}$$

is the Fermi energy of the free 1D Fermi gas. This expression should be understood as the leading asymptotic behavior of the gap as $\gamma \to 0$. It can be easily checked that, when $\kappa = 2$, Eq. (53) agrees with the results in Refs. [16,26,27].

The energy gap determines the fundamental nonperturbative scale of the theory. It is exponentially small in γ, and its prefactor scales with γ like $\gamma^{1/\kappa}$. We will now see how the main features of this result can be obtained from two different approaches: the behavior of perturbation theory at large order and a RG analysis.

IV. ENERGY GAP FROM LARGE ORDER BEHAVIOR

The energy gap in Eq. (53) is clearly a nonperturbative effect. It has been known for a long time that nonperturbative effects in quantum physics can often be extracted from the large order behavior of the perturbative series (see, e.g., Ref. [28] for a textbook exposition and Ref. [29] for a collection of articles on the subject). Let us suppose that we have a perturbative series of the form:

$$\varphi(z) = \sum_{k\geqslant 0} a_k z^k.\tag{55}$$

Here, z is the (small) coupling constant of the problem. In most examples in quantum theory, the coefficients a_k grow factorially with k. More precisely, we have

$$a_k \sim \frac{\mu_0}{2\pi}A^{-k-b}\Gamma(k+b), \qquad k \gg 1,\tag{56}$$

where A, b, and μ_0 are parameters that characterize the growth of perturbation theory at next-to-leading order in $1/k$. This growth leads to an exponentially small, nonperturbative effect of the form:

$$\mu_0 z^{-b}\exp\left(-\frac{A}{z}\right), \quad z \to 0.\tag{57}$$

Therefore, the parameters in the factorial growth in Eq. (56) determine the strength of the nonperturbative effect. In real examples, these parameters can be extracted numerically from the growth of the perturbative series and then compared with expectations about the presence of nonperturbative effects. Particularly important are A and b since they determine the leading dependence of the nonperturbative effect on the coupling constant z. In general, there is a minimal nonperturbative scale in the problem:

$$\Lambda(z) = z^{-b}\exp\left(-\frac{A}{z}\right),\tag{58}$$

and a generic nonperturbative effect scales at small z as $\Lambda^d(z)$, where d is often an integer. We note that Eq. (58) is often the leading approximation to the full answer, and it multiplies a power series in z.

An illustrative example of the considerations above is the double-well potential in 1D quantum mechanics, of the form:

$$V(x) = \frac{x^2}{2}(1 + xg^{1/2})^2.\tag{59}$$

Here, g can be regarded as a coupling constant, and the energy levels can be computed as formal power series in g by using standard stationary perturbation theory. In this potential, the energy gap, i.e., the difference between the ground state energy and the first excited state, is purely nonperturbative in g. At leading order, it is given by the scale:

$$\Lambda(g) = g^{-1/2}\exp\left(-\frac{1}{6g}\right),\tag{60}$$

and it is due to tunneling between the two classical vacua (in the language of instantons, this is a one-instanton effect). One way to extract this scale is to look at the larger order behavior of the perturbative series for the ground-state energy. Its coefficients grow as [30,31]

$$a_k \sim \Gamma(k+1)3^{k+1}, \quad k \gg 1,\tag{61}$$

so they lead to a nonperturbative scale which is the square $\Lambda^2(g)$ of the minimal scale in Eq. (60).

It was conjectured in Ref. [1] that precisely this phenomenon occurs in Fermi systems with an attractive interaction: the large order behavior of the perturbative series for the ground-state energy leads to a nonperturbative scale which is the square of the scale appearing in the energy gap. This was verified for the Gaudin-Yang model with $\kappa = 2$ components. We will now provide evidence for the same phenomenon in the multicomponent case. This will determine a minimal nonperturbative scale:

$$\Lambda(\gamma) = \gamma^{1/\kappa}\exp\left(-\frac{2\pi^2}{\kappa^2\gamma}\right),\tag{62}$$

in agreement with Eq. (53).

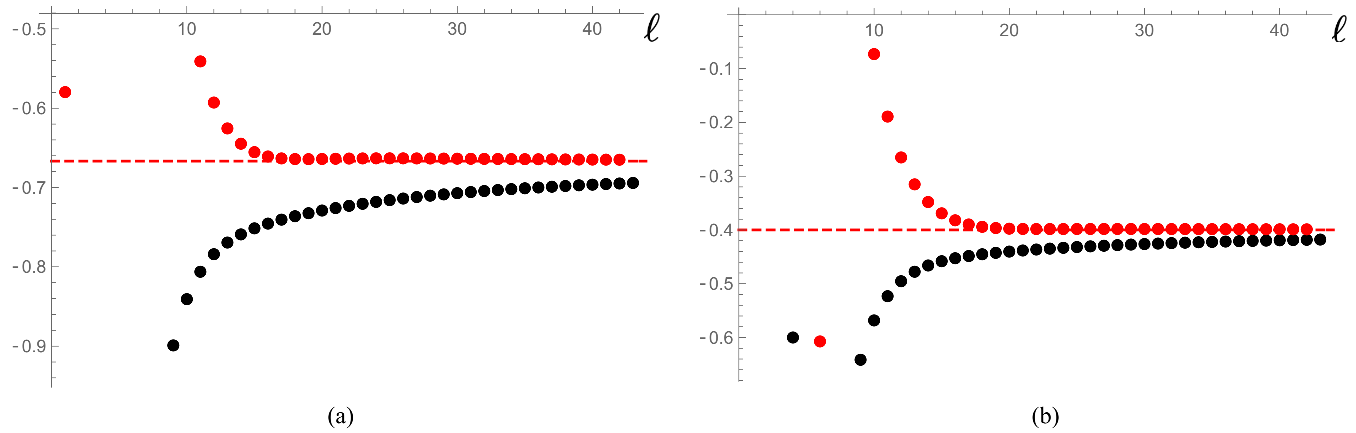

FIG. 1. The sequence t_ℓ from Eq. (70) in black dots and its first Richardson transform in red dots, for two values of κ. The horizontal dashed line is the expected value $b(\kappa) = -2/\kappa$. (a) Sequence t_ℓ for $\kappa = 3$ and (b) Sequence t_ℓ for $\kappa = 5$.

Following Ref. [1], it is useful to introduce the 't Hooft-like coupling:

$$\lambda = \left(\frac{\kappa}{2}\right)^2 \gamma, \tag{63}$$

and the rescaled ground energy density:

$$e(\lambda;\kappa) = \frac{1}{4}\frac{\frac{E}{\kappa}}{\left(\frac{n}{\kappa}\right)^3}, \tag{64}$$

where E is given in Eq. (16) in terms of the BA solution. This function has the perturbative expansion:

$$e(\lambda;\kappa) = \sum_{\ell \geqslant 0} c_\ell(\kappa)\lambda^\ell. \tag{65}$$

The coefficients $c_\ell(\kappa)$ can be computed systematically by using the algorithm presented in Ref. [1]. One finds, for the very first orders:

$$\begin{aligned} c_0 &= \frac{\pi^2}{12}, \quad c_1 = \Delta - 1, \quad c_2 = \frac{1}{3} - \frac{\Delta}{3}, \\ c_3 &= \frac{4\Delta(\Delta-1)\zeta(3)}{\pi^4}, \\ c_4 &= -\frac{12\Delta(\Delta-1)^2\zeta(3)}{\pi^6}, \end{aligned} \tag{66}$$

where we have denoted

$$\Delta = \frac{1}{\kappa}. \tag{67}$$

We have computed the first 45 coefficients in Eq. (65), which turn out to be sufficient to numerically study the large order behavior of the sequence $c_\ell(\kappa)$. We find

$$c_\ell(\kappa) \sim A^{-\ell - b(\kappa)}\Gamma[\ell + b(\kappa)], \tag{68}$$

where

$$A = \pi^2, \quad b(\kappa) = -\frac{2}{\kappa}. \tag{69}$$

The numerical procedure to extract these numbers is standard (see, e.g., Ref. [32]). For example, to determine $b(\kappa)$, we consider the sequence:

$$t_\ell = \frac{A c_{\ell+1}}{c_\ell} - \ell, \quad \ell \geqslant 0, \tag{70}$$

which should approach $b(\kappa)$ as $\ell \gg 1$. The convergence of the sequence to the expected value can be accelerated with Richardson transforms. Examples of these numerical determinations are shown in Fig. 1. If we now consider that the expansion in Eq. (65) is done in the coupling λ, and we go back to the coupling γ, we find that the large order growth leads to the nonperturbative scale:

$$\Lambda^2(\gamma) = \gamma^{2/\kappa} \exp\left(-\frac{4\pi^2}{\kappa^2\gamma}\right), \tag{71}$$

which is precisely the square of Eq. (62).

As explained in Ref. [1], the factorial growth of perturbation theory is due to a renormalon effect [10], and ring diagrams explain the exponential dependence in Eq. (71). However, as we check explicitly in Appendix A, the prefactor $\gamma^{2/\kappa}$ is subleading in $1/\kappa$ and cannot be explained by ring diagrams only.

V. ENERGY GAP FROM RG

As it is well known, in relativistic asymptotically free theories, the coupling dependence of the nonperturbative scale can be determined, at weak coupling, by a RG analysis. The argument is very simple. Let us assume that we have a running coupling constant $g(\mu)$, depending on a scale μ, and satisfying a RG equation of the form:

$$\mu\frac{dg}{d\mu} = \beta(g) = \beta_0 g^2 + \beta_1 g^3 + \cdots, \tag{72}$$

where $\beta_0 < 0$. Then the following quantity:

$$\begin{aligned} \mathcal{I}(g) &= \mu\, g(\mu)^{\beta_1/\beta_0^2} \exp\left[\frac{1}{\beta_0 g(\mu)}\right] \\ &\times \exp\left\{-\int_{g_*}^{g(\mu)}\left[\frac{1}{\beta(\overline{g})} - \frac{1}{\beta_0\overline{g}^2} + \frac{\beta_1}{\beta_0^2\overline{g}}\right]d\overline{g}\right\}, \end{aligned} \tag{73}$$

is invariant under the RG flow, i.e., it is independent of the scale μ. Here, $g_\star$ is an arbitrary value, which is equivalent to the freedom of multiplying $\mathcal{I}$ by an arbitrary μ-independent constant. Since $\beta_0 < 0$, $\mathcal{I}(g)$ is an exponentially small quantity in the coupling constant, and it can be regarded as the all-orders generalization of the minimal nonperturbative scale in Eq. (58) for these theories. We note that A in Eq. (58) is essentially given by the inverse of the first coefficient of the beta function, while b involves the first two coefficients β_0, β_1. It was pointed out by Parisi [33] that the nonperturbative ambiguities due to renormalons are given by integer powers of the scale in Eq. (73), and he conjectured that they govern the large order behavior of the corresponding perturbative series.

There are many structural similarities between many-fermion systems with an attractive interaction and asymptotically free field theories. One could then use the RG equations to determine the coupling constant dependence of nonperturbative quantities. In the case of 1D Fermi systems, this was pointed out by Larkin and Sak [16]. They determined the energy gap in the Gaudin-Yang model with $\kappa = 2$ from the RG equations of Ref. [34].

In this section, we determine the RG equations for the Gaudin-Yang model with arbitrary κ, and we rederive the nonperturbative scale in Eq. (62). This shows that the connection between nonperturbative effects, RG equations, and large order behavior in asymptotically free, relativistic field theories, also holds in this 1D many-body model. As in Ref. [16], we will use the RG approach of Ref. [34], which we will call multiplicative renormalization (see Ref. [17] for a review).

As is well known, the first step in the multiplicative renormalization procedure in 1D is to linearize the dispersion relation near the Fermi surface. We start with a free Hamiltonian:

$$H_0 = \sum_{k,\alpha} \epsilon_k c^\dagger_{k,\alpha} c_{k,\alpha}, \tag{74}$$

where $\alpha = 1, \ldots, \kappa$. We focus our attention on energies around $k = \pm k_{\rm F}$ and integrate out modes with $|k - k_{\rm F}| \gg k_0$ for some cutoff $k_0 \ll k_{\rm F}$. This leads to a Hamiltonian of the form:

$$\begin{aligned} H_0 = &\sum_{k,\alpha} v_{\rm F}(k - k_{\rm F}) a^\dagger_{k,\alpha} a_{k,\alpha} \\ &+ \sum_{k,\alpha} v_{\rm F}(-k - k_{\rm F}) b^\dagger_{k,\alpha} b_{k,\alpha}, \end{aligned} \tag{75}$$

where a and b are annihilation operators for right- and left-moving particles, respectively, and $v_{\rm F}$ is the Fermi velocity. The energy bandwidth associated with the cutoff k_0 is given by

$$E_0 = 2 v_{\rm F} k_0. \tag{76}$$

We also define the free Green's function for right/left movers as

$$G_\pm(k, i\omega) = \frac{1}{i\omega \mp k + k_{\rm F}}. \tag{77}$$

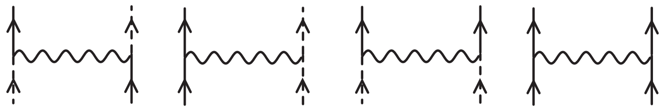

FIG. 2. The allowed couplings g_1, g_2, g_3, and g_4, from left to right. They mix right-moving (continuous line) and left-moving (dashed line) particles. We only consider g_1 and g_2, the leftmost couplings. The arrows are taken to be implicit in other diagrams.

We can now add interactions which are diagrammatically illustrated in Fig. 2:

$$\begin{aligned} H_I = &\sum_{k_1,k_2,k_3,k_4} \sum_{\alpha,\beta}^{\kappa} \delta(k_1 + k_2 - k_3 - k_4) \\ &\times [g_1 b^\dagger_{k_1,\alpha} a^\dagger_{k_2,\beta} a_{k_3,\alpha} b_{k_4,\beta} + g_2 b^\dagger_{k_1,\alpha} a^\dagger_{k_2,\beta} b_{k_3,\alpha} a_{k_4,\beta} \\ &+ g_3 (a^\dagger_{k_1,\alpha} a^\dagger_{k_2,\beta} b_{k_3,\alpha} b_{k_4,\beta} + b^\dagger_{k_1,\alpha} b^\dagger_{k_2,\beta} a_{k_3,\alpha} a_{k_4,\beta}) \\ &+ g_4 (a^\dagger_{k_1,\alpha} a^\dagger_{k_2,\beta} a_{k_3,\alpha} a_{k_4,\beta} \\ &+ b^\dagger_{k_1,\alpha} b^\dagger_{k_2,\beta} b_{k_3,\alpha} b_{k_4,\beta})]. \end{aligned} \tag{78}$$

Very often, the couplings g_i are split into $g_{i\perp}$ and $g_{i\parallel}$ for particles with different/identical spin. In the Gaudin-Yang case, the g_3 interaction, which corresponds to Umklapp scattering, is not allowed. In the present scheme of bandwidth cutoff, the g_4 process does not contribute. We will then focus on the couplings $g_{1,2}$.

The procedure of multiplicative renormalization is based on comparing Green's functions and vertex functions at different values of the cutoffs. The working hypothesis is that, once the coupling constants are appropriately adjusted, these functions differ in a multiplicative factor only. We have, for the Green's functions,

$$G(k, \omega, g_i', E_0') = z\left(\frac{E_0'}{E_0}, g_i\right) G(k, \omega, g_i, E_0). \tag{79}$$

The vertex or four-point functions are associated with the couplings and related through the equation:

$$\begin{aligned} \Gamma_i'(\{k, \omega\}, g_i', E_0') = z_i^{-1}\left(\frac{E_0'}{E_0}, g_i\right) \Gamma_i(\{k, \omega\}, g_i, E_0), \\ i = 1, 2, \end{aligned} \tag{80}$$

where $\{k, \omega\}$ denote the four different momenta and frequencies appearing in the vertex. This leads to the following renormalization of the coupling constant:

$$g_i' = \frac{z_i\left(\frac{E_0'}{E_0}, g_i\right)}{z^2\left(\frac{E_0'}{E_0}, g_i\right)} g_i, \tag{81}$$

and to the beta functions:

$$\begin{aligned} \beta_i = \frac{d g_i(\mu)}{d \log \mu} = \frac{d}{d \log \mu} \left(\frac{z_i(\mu, g_i)}{z^2(\mu, g_i)}\right)\bigg|_{\mu=1} g_i, \\ i = 1, 2, \end{aligned} \tag{82}$$

where $\mu = E_0'/E_0$.

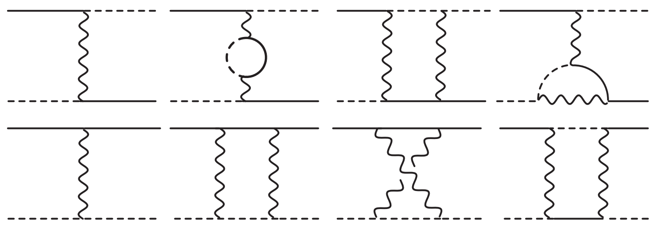

FIG. 3. One loop correction to g_1 (top) and g_2 (bottom). The top rightmost diagrams have a multiplicity of 2 since we can pick the lines to be ingoing/outgoing in two distinct ways.

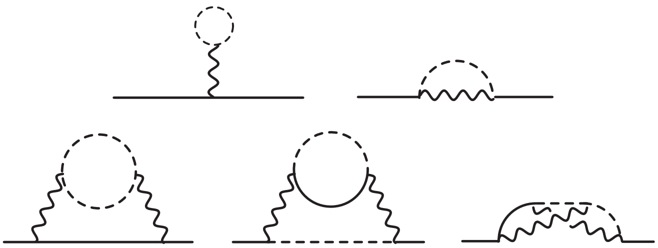

FIG. 4. Corrections to the right-moving self-energy. Note that one-loop contributions are zero. The left-moving self-energy is identical.

At one loop, the procedure is rather simple since there are no corrections to the self energy. To compute the scaling of $\Gamma_{1,2}$, we must first assign a set of external momenta and frequencies. Since we are ultimately interested in how the couplings vary with the scale, we can choose one of the external parameters to play the role of probe scale. As usual in renormalization, we work under the assumption that we probe energies far below the cutoff. We choose, following Ref. [17], ω, though one could just as well pick k or even the inverse temperature β. A convenient choice of external parameters is proposed in Ref. [17]. We set the momenta of right/left movers at the Fermi points $\pm k_{\rm F}$, respectively. The incoming right-moving particle has an energy of $3i\omega/2$, while the incoming left-moving particle has an energy of $-i\omega/2$, and both outgoing particles have the energy $i\omega/2$.

At one loop, one has the diagrams shown in Fig. 3. There are only two types of loop integrals, which correspond to the so-called Cooper and Peierls channels, denoted by $J_{\rm C}(\omega)$ and $J_{\rm P}(\omega)$, respectively. We can write at first order

$$g_1\Gamma_1(i\omega, g_i, E_0) = g_1 - \left[2g_1g_2J_{\rm C}(\omega) + \left(2g_2g_1 - \kappa g_1^2\right)J_{\rm P}(\omega)\right] + \cdots, \tag{83}$$

$$g_2\Gamma_2(i\omega, g_i, E_0) = g_1 - \left[\left(g_1^2 + g_2^2\right)J_{\rm C}(\omega) + g_2^2 J_{\rm P}(\omega)\right] + \cdots. \tag{84}$$

We take $\omega \ll E_0$ to single out the leading logarithmic dependence, and we find

$$J_{\rm C}(\omega) = \int_{k_{\rm F}-k_0}^{k_{\rm F}+k_0} \frac{dq}{2\pi}\int_{-\infty}^{\infty}\frac{d\omega'}{2\pi} G_+(q, i\omega') \times G_-(-q, i\omega - i\omega') \approx -\frac{1}{2\pi v_{\rm F}}\log\left(\frac{\omega}{E_0}\right), \tag{85}$$

$$J_{\rm P}(\omega) = \int_{-k_0}^{+k_0}\frac{dq}{2\pi}\int_{-\infty}^{\infty}\frac{d\omega'}{2\pi} G_+(q + k_{\rm F}, i\omega') \times G_-(q - k_{\rm F}, i\omega' - i\omega) = -J_{\rm C}(\omega). \tag{86}$$

The vertices are

$$g_1\Gamma_1(i\omega, g_i, E_0) = g_1 + \frac{1}{2\pi v_{\rm F}}\left(\kappa g_1^2\right)\log\left(\frac{\omega}{E_0}\right) + \cdots, \tag{87}$$

$$g_2\Gamma_2(i\omega, g_i, E_0) = g_2 + \frac{1}{2\pi v_{\rm F}}\left(g_1^2\right)\log\left(\frac{\omega}{E_0}\right) + \cdots. \tag{88}$$

From the definition in Eq. (80), we read

$$z_1\left(\frac{E_0'}{E_0}, g_i\right) = \frac{\Gamma_1(\omega, g_i, E_0)}{\Gamma_1(\omega, g_i', E_0')} = 1 + \frac{1}{2\pi v_{\rm F}}(\kappa g_1)\log\left(\frac{E_0'}{E_0}\right) + \cdots, \tag{89}$$

$$z_2\left(\frac{E_0'}{E_0}, g_i\right) = \frac{\Gamma_2(\omega, g_i, E_0)}{\Gamma_2(\omega, g_i', E_0')} = 1 + \frac{1}{2\pi v_{\rm F}}\left(\frac{g_1^2}{g_2}\right)\log\left(\frac{E_0'}{E_0}\right) + \cdots, \tag{90}$$

where we use $g_i' \approx g_i + \mathcal{O}(g^2)$. These results are independent of ω, as required by the multiplicative renormalization hypothesis. At one loop, we find, by using Eq. (81),

$$g_1' = g_1 + \frac{g_1^2\kappa}{2\pi v_{\rm F}}\log\left(\frac{E_0'}{E_0}\right) + \cdots,$$
$$g_2' = g_2 + \frac{g_1^2}{2\pi v_F}\log\left(\frac{E_0'}{E_0}\right) + \cdots. \tag{91}$$

The calculation at two loops is more involved. For the self-energy, we have the diagrams in Fig. 4. We take the inflowing momentum and energy to be $k_{\rm F} + k$ and $i\omega$, respectively, and we find

$$G(k, i\omega, g_i, E_0) = \left[1 + \left(\kappa g_1^2 + \kappa g_2^2 - 2g_1g_2\right)\frac{\log\left(\frac{\omega}{E_0}\right)}{8\pi^2 v_{\rm F}^2} + \cdots\right] \times G_+(k, i\omega). \tag{92}$$

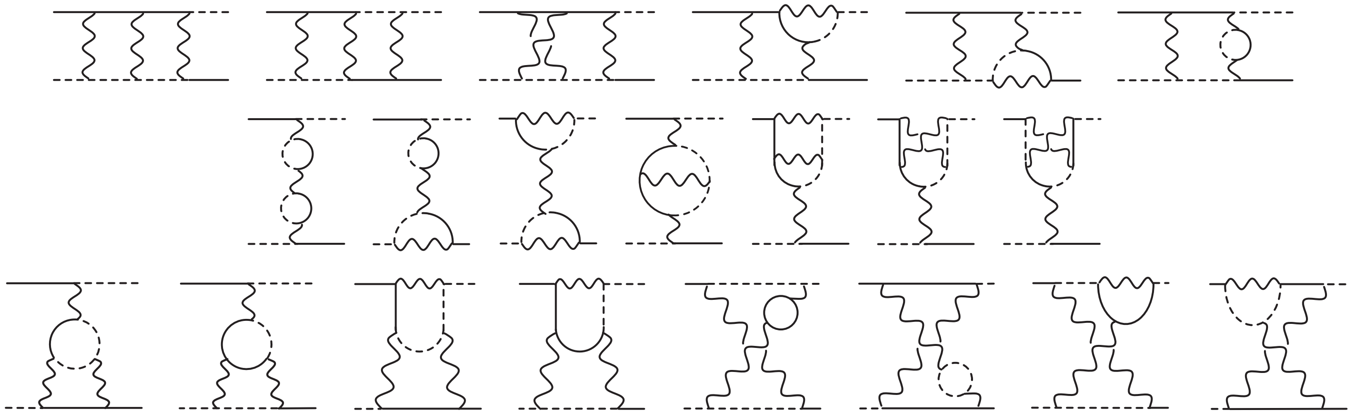

FIG. 5. Two-loop correction to g_1. Some diagrams which are distinct over choice of ingoing/outgoing legs have a multiplicity of 2; diagrams with a spin loop have a factor of $-\kappa$.

Due to Eq. (79), z is given by

$$z\left(\frac{E_0'}{E_0}, g_i\right) = \left[\frac{G(k_F, i\omega, g_i, E_0)}{G(k_F, i\omega, g_i', E_0')}\right]^{-1} = 1 - \left(\kappa g_1^2 + \kappa g_2^2 - 2g_1 g_2\right)\frac{\log\left(\frac{E_0'}{E_0}\right)}{8\pi^2 v_F^2} + \cdots. \tag{93}$$

At two loops, one finds far more diagrams for the vertices, as detailed in Figs. 5 and 6, which add up to

$$g_1\Gamma_1(i\omega, g_i, E_0) = g_1 + \frac{1}{2\pi v_F}\left(\kappa g_1^2\right)\log\left(\frac{\omega}{E_0}\right) + \frac{1}{4\pi^2 v_F^2}\left(2g_1^2 g_2 - \kappa g_2^2 g_1\right)\log\left(\frac{\omega}{E_0}\right) + \frac{1}{8\pi^2 v_F^2}\left(2\kappa^2 g_1^3\right)\log^2\left(\frac{\omega}{E_0}\right) + \cdots, \tag{94}$$

$$g_2\Gamma_2(i\omega, g_i, E_0) = g_2 + \frac{1}{2\pi v_F}\left(g_1^2\right)\log\left(\frac{\omega}{E_0}\right) + \frac{1}{4\pi^2 v_F^2}\left(g_1^3 - \kappa g_1^2 g_2 + 2g_2^2 g_1 - \kappa g_2^3\right) \times \log\left(\frac{\omega}{E_0}\right) + \frac{1}{8\pi^2 v_F^2}\left(2\kappa g_1^3\right)\log^2\left(\frac{\omega}{E_0}\right) + \cdots, \tag{95}$$

When $\kappa = 2$, the result above agrees with the calculation in Refs. [17,34]. From these results, we obtain

$$z_1\left(\frac{E_0'}{E_0}, g_i\right) = 1 + \frac{1}{2\pi v_F}(\kappa g_1)\log\left(\frac{E_0'}{E_0}\right) + \frac{1}{4\pi^2 v_F^2}\left(2g_1 g_2 - \kappa g_2^2\right)\log\left(\frac{E_0'}{E_0}\right) + \frac{1}{8\pi^2 v_F^2}\left(2\kappa^2 g_1^2\right)\log^2\left(\frac{E_0'}{E_0}\right) + \cdots, \tag{96}$$

Here, we must use the corrections in Eq. (91) to g_i' since they are crucial to cancel $\log(\omega)$ dependencies at order $\mathcal{O}(g^2)$. For z_2, we find

$$z_2\left(\frac{E_0'}{E_0}, g_i\right) = 1 + \frac{1}{2\pi v_F}\left(\frac{g_1^2}{g_2}\right)\log\left(\frac{E_0'}{E_0}\right) + \frac{1}{4\pi^2 v_F^2}\left(\frac{g_1^3}{g_2} - \kappa g_1^2 - \kappa g_2^2 + 2g_2 g_1\right) \times \log\left(\frac{E_0'}{E_0}\right) + \frac{1}{8\pi^2 v_F^2}\left(2\kappa\frac{g_1^3}{g_2}\right)\log^2\left(\frac{E_0'}{E_0}\right) + \cdots. \tag{97}$$

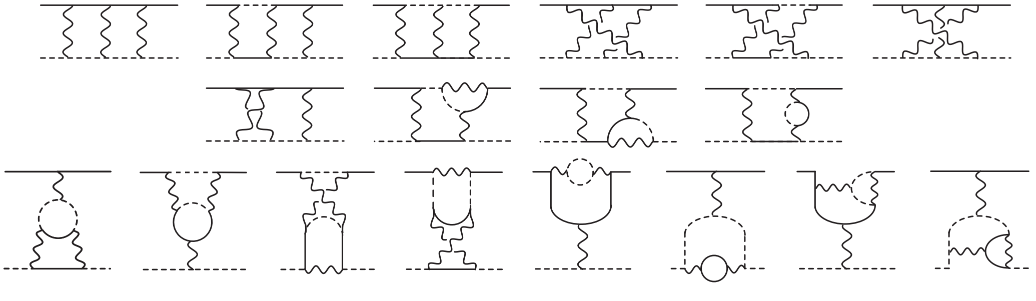

FIG. 6. Two-loop corrections to g_2. Some diagrams which are distinct over choice of ingoing/outgoing legs have a multiplicity of 2; diagrams with a spin loop have a factor of $-\kappa$.

The cancellation of $\log(\omega)$ for z_1, z_2 is a nontrivial check of the diagrammatic calculations.

Finally, by assembling the pieces and plugging them into Eq. (81), we find

$$g_1' = g_1 + \frac{g_1^2\kappa}{2\pi v_F}\log\left(\frac{E_0'}{E_0}\right) + \frac{g_1^3\kappa}{4\pi^2 v_F^2}\log\left(\frac{E_0'}{E_0}\right) + \frac{g_1^3\kappa^2}{4\pi^2 v_F^2}\log^2\left(\frac{E_0'}{E_0}\right) + \cdots, \tag{98}$$

$$g_2' = g_2 + \frac{g_1^2}{2\pi v_F}\log\left(\frac{E_0'}{E_0}\right) + \frac{g_1^3}{4\pi^2 v_F^2}\log\left(\frac{E_0'}{E_0}\right) + \frac{g_1^3\kappa}{4\pi^2 v_F^2}\log^2\left(\frac{E_0'}{E_0}\right) + \cdots. \tag{99}$$

We are now ready to calculate the beta functions for the couplings g_1, g_2. By using Eqs. (96), (97), and (93) in Eq. (82), we obtain

$$\beta_1 = \frac{\kappa}{2\pi v_F}g_1^2 + \frac{\kappa}{4\pi^2 v_F^2}g_1^3 + \cdots, \tag{100}$$

$$\beta_2 = \frac{1}{2\pi v_F}g_1^2 + \frac{1}{4\pi^2 v_F^2}g_1^3 + \cdots. \tag{101}$$

These beta functions agree with a similar calculation in the $SU(\kappa)$ Hubbard model in Ref. [35].

With all these results, we can now calculate the gap in the attractive regime $g_1 = -2c < 0$. We introduce

$$\bar{g}_1 = -\frac{g_1}{\pi v_F} = \frac{\kappa\gamma}{\pi^2}, \tag{102}$$

and we find that the beta function for $\bar{g}_1$ is of the form of Eq. (72) with:

$$\beta_0 = -\frac{\kappa}{2}, \quad \beta_1 = \frac{\kappa}{4}. \tag{103}$$

By using these values and Eq. (102), we find that the RG invariant scale in Eq. (73) agrees precisely with Eq. (62).

As noted in Ref. [3], the beta function β_1 coincides with the one of the chiral Gross-Neveu model [36]. In fact, it can be shown explicitly that the Hamiltonian $H_0 + H_I$, where only the couplings g_1, g_2 are considered, is a particular case of the chiral Gross-Neveu model. This explains the relationship between the beta functions. We give some details of this equivalence in Appendix B.

VI. CONCLUSIONS

In this paper, we have extended and deepened the connection found in Refs. [1–3] between the energy gap, the large order behavior of perturbation theory, and renormalons in 1D models of many-body fermions with an attractive interaction. We have seen that the weak-coupling behavior of the energy gap in the multicomponent Gaudin-Yang model can be predicted from the large order behavior of the perturbative series for the ground state energy. This series diverges factorially due to renormalon diagrams. When the number of components is large, the leading renormalon diagrams (which turn out to be ring diagrams) correctly reproduce the exponential term in the energy gap. Moreover, as in asymptotically free theories in two dimensions (2D), the leading and subleading terms in the large order behavior can be obtained from the beta function of the theory, as computed in the relativistic approximation near the Fermi points. This also implies a connection between the gap and the beta function, noted long ago in Ref. [16]. To establish these relationships, we have performed a detailed calculation of the energy gap directly from the BA solution in the multicomponent case, generalizing in this way the results of Ref. [15] for $\kappa = 2$.

Although the integrability of the model makes it possible to test our ideas in detail, the connection we have found should be valid more generally. For example, the results of this paper, combined with the ones in Ref. [3], suggest that the coupling dependency of the energy gap in the multicomponent Hubbard model (which is not integrable) is given by

$$\Delta \approx u^{1/\kappa}\exp\left[-\frac{2\pi}{\kappa u}\sin\left(\frac{\pi n}{\kappa}\right)\right], \quad u \to 0, \tag{104}$$

where u is the coupling constant, and n is the density (see Ref. [3] for more details and clarifications on the notation). There are additional coupling-independent factors that depend on n [37] and κ but which we cannot yet fully ascertain. The exponent appearing in this expression can be interpreted as due to the contribution of renormalon diagrams dominating in the large κ limit. In quantum chromodynamics (QCD), renormalons have been instrumental in determining nonperturbative scales [10], and it is gratifying that the same principles shed light on the energy gap of many-fermion systems.

There are various avenues opened by this investigation. One important issue would be to systematically understand the corrections to the results presented in this paper. As we have mentioned, the minimal scale in Eq. (58) is the leading approximation to a fully-fledged transseries, and it multiplies a power series in the coupling constant. As emphasized in this paper, these subleading corrections can in principle be computed by following any of the three roads we have considered. We could, for example, use the BA equations; we could determine them from the subleading contributions to the large order behavior; and we could try to understand them from the beta function, by including higher loops and higher modes. It would also be interesting to connect these corrections to the behavior of diagrams. In fact, this should be done already to reproduce the prefactor $\gamma^{2/\kappa}$ in Eq. (71). It might be possible to do this by considering diagrams which are subleading in the large κ expansion.

Another interesting avenue is to find a description of the model in the $1/\kappa$ expansion, along the lines of what was done for the principal chiral field in Refs. [38–40]. This might require us to study a regime of the model in which different bound states are present in a prescribed way, as in Refs. [38,39]. We have found encouraging indications that the BA equations for the multicomponent Gaudin-Yang model might simplify in an appropriate large κ regime, but more work is needed.

As mentioned in our previous papers [1–3], a fundamental issue is to find a first-principles procedure to calculate the energy gap from the path integral, by some generalization of perturbation theory that considers renormalon physics. In QCD, such a procedure is provided, for some observables, by the operator product expansion, combined with the existence of nontrivial vacuum condensates. It would be fascinating to extend these methods to nonrelativistic models like the one studied in this paper.

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APPENDIX A: NONPERTURBATIVE SCALE FROM RING DIAGRAMS

As we have already argued in Ref. [1], the large order behavior of the perturbative series in Eq. (68) is due to renormalon diagrams. To study these diagrams in a systematic way, it is useful to consider a $1/\kappa$ expansion (in quantum electrodynamics or QCD, one would consider an expansion in $1/N_f$, where N_f is the number of flavors [10]). In the multicomponent Gaudin-Yang model, the dominant diagrams in the large κ limit are the so-called *ring diagrams* (see, e.g., Refs. [41,42]). It was shown in Ref. [1] that these diagrams lead to the correct exponential term in Eq. (71). However, as we will briefly show here, they do not lead to the correct prefactor, which is subleading in the $1/\kappa$ expansion.

The ground state energy for the Gaudin-Yang model with κ spin components in Eq. (64) has a $1/\kappa$ expansion of the form:

$$e(\lambda;\kappa) = e_0(\lambda) + \frac{1}{\kappa} e_1(\lambda) + \cdots . \tag{A1}$$

In this equation, λ is the 't Hooft parameter in Eq. (63), $e_0(\lambda)$ is the free gas result plus the Hartree term, while $e_1(\lambda)$ is given by a resummation of ring diagrams [1]:

$$e_1(\lambda) = \lambda - \frac{\pi}{4}\int_0^\infty dy\, y \int_0^\infty d\nu \times \left\{ \log\left[1 - \frac{\kappa^2\gamma}{\pi^2}F(y,\nu)\right] + \frac{\kappa^2\gamma}{\pi^2}F(y,\nu)\right\}, \tag{A2}$$

where

$$F(y,\nu) = \frac{1}{2y}\log\left[\frac{\left(\frac{y}{2}+1\right)^2+\nu^2}{\left(\frac{y}{2}-1\right)^2+\nu^2}\right]. \tag{A3}$$

As noted in Refs. [3,43], in similar situations, the integral in Eq. (A2) has an exponentially small imaginary piece which must be canceled by nonperturbative effects not captured by the diagrammatic expansion. We can then reconstruct these nonperturbative effects by properly expanding the imaginary part of the integral, which occurs when the argument of the log becomes negative. The condition:

$$1 - \frac{\kappa^2\gamma}{\pi^2}F(y,\nu,\gamma) < 0, \tag{A4}$$

defines a compact region $\mathcal{R}$ in the first quadrant of the (y,ν) plane. The region is delimited by the curve defined by the equation:

$$\nu^2 = \frac{-(y-2)^2 + \exp\left(-\frac{2\pi^2 y}{\kappa^2\gamma}\right)(y+2)^2}{4\left[1-\exp\left(-\frac{2\pi^2 y}{\kappa^2\gamma}\right)\right]}. \tag{A5}$$

Since $\nu \geqslant 0$, we must find the limits of integration $y = y_\pm$ where the boundary line crosses the real axis. Let us define the nonperturbative parameter:

$$\alpha = \exp\left(-\frac{2\pi^2}{\kappa^2\gamma}\right), \tag{A6}$$

and let us change variables from y to u, where

$$y = 2 + 4\alpha u. \tag{A7}$$

The equation for the endpoints $u_\pm$ is

$$\exp(-2\alpha\log\alpha u_\pm) \mp \left(\frac{1}{u_\pm} + \alpha\right) = 0, \tag{A8}$$

which can be easily solved in a power series expansion in the two variables α, $\log\alpha$ (this is a simple example of a transseries, see Refs. [6–8]). For the first few orders, we find

$$u_\pm = \pm 1 + \alpha[2\log(\alpha) + 1] \pm \alpha^2[6\log^2(\alpha) + 6\log(\alpha) + 1] + \cdots . \tag{A9}$$

To determine the imaginary part of $e_1(\lambda)$, we must calculate

$$\int_{u_-}^{u_+} 4\alpha(2+4\alpha u)\nu(u)du. \tag{A10}$$

This can be done by expanding the integrand into factors of $(u-u_+)^m(u-u_-)^k$ and $u(u-u_+)^m(u-u_-)^k$ at each order in α before performing the integration and then resuming at each order in α the resulting polynomials in u_+ and u_-. When all this is done, we obtain the following expansion for the imaginary part of $e_1(\lambda)$:

$$\begin{aligned}\operatorname{Im} e_1(\lambda) = {}& 2\pi^2\exp\left(-\frac{\pi^2}{\lambda}\right) + \frac{8\pi^2\exp\left(-\frac{2\pi^2}{\lambda}\right)}{\lambda^2} \\ &\times\left(\lambda^2 - \frac{3\pi^2}{2}\lambda + \frac{\pi^4}{2}\right) \\ &+ \frac{6\pi^2\exp\left(-\frac{3\pi^2}{\lambda}\right)}{\lambda^4} \\ &\times\left(3\lambda^4 - 14\pi^2\lambda^3 + 21\pi^4\lambda^2 - 12\pi^6\lambda + \frac{9\pi^8}{4}\right) \\ &+ \cdots .\end{aligned} \tag{A11}$$

The leading, exponentially small effect has the correct exponent to match Eq. (71) but not the correct prefactor. A similar phenomenon was found in the Hubbard model in Ref. [3]. This is due to the fact that ring diagrams capture the diagrammatic structure at the first nontrivial order in the $1/\kappa$ expansion, while the prefactor $\gamma^{1/\kappa}$ is subleading in $1/\kappa$. By considering renormalon diagrams of order $1/\kappa^2$, one might be able to reproduce this prefactor [44].

An interesting application of the above calculation is a precise formula for the large order behavior of the coefficients

$c_\ell^{(1)}$ in the perturbative expansion of $e_1(\lambda)$:

$$e_1(\lambda) = \sum_{\ell\geqslant 0} c_\ell^{(1)} \lambda^\ell. \tag{A12}$$

These coefficients appear in the $1/\kappa$ expansion of the coefficients $c_\ell(\kappa)$ of Eq. (65):

$$c_\ell(\kappa) = c_\ell^{(0)} + \frac{1}{\kappa} c_\ell^{(1)} + \cdots . \tag{A13}$$

If we write

$$\mathrm{Im}\, e_1(\lambda) = \sum_{j\geqslant 1} \sum_{i=0}^{2j-2} a_{j,i} \lambda^{-i} \exp\left(-\frac{j\pi^2}{\lambda}\right), \tag{A14}$$

we find

$$c_\ell^{(1)} \sim -\sum_{j\geqslant 1} \sum_{i=0}^{2j-2} \frac{\Gamma(\ell+i)}{(\pi^2 j)^{\ell+i}} a_{j,i}. \tag{A15}$$

By appropriately truncating the sum over j, we can obtain from Eq. (A15) very accurate values for the perturbative coefficients $c_\ell^{(1)}$.

APPENDIX B: RELATION TO THE CHIRAL GROSS-NEVEU MODEL

The relativistic model we have used in our RG analysis turns out to be closely related to the Thirring model and, more precisely, to the chiral Gross-Neveu model (similar relations have been pointed out in Refs. [45,46]). To see this, we consider the general form of the Thirring Lagrangian, given by

$$\mathcal{L} = i\bar{\Psi}\partial\!\!\!/\Psi - \frac{1}{2} g J_\mu^\alpha J^{\mu\alpha}, \quad J_\mu^\alpha = \sum_j \bar{\Psi}_j \gamma_\mu T^\alpha \Psi_j, \tag{B1}$$

where $\Psi_{\alpha,j}$ ($j = 1, \ldots, N_f$ and $\alpha = 1, \ldots, N_c$) is a N_f dimensional vector of Dirac spinors in a N_c dimensional representation of a compact Lie group G, and T^α, $\alpha = 1, \ldots, \dim(G)$ are a basis for the representations of its Lie algebra such that $\mathrm{Tr}[T^\alpha T^\beta] = \frac{1}{2}\delta^{\alpha\beta}$.

We are interested in the case of $N_f = 1$ and $G = SU(\kappa)$, with the Dirac spinor in the fundamental representation ($N_c = \kappa$). This is the matter content of the chiral Gross-Neveu model (see, e.g., Ref. [47]). We work in $1+1$ dimensions with signature $(-,+)$. Explicitly, we use the following Dirac spinor conventions, where the chiral matrix is labeled γ_3:

$$\Psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}, \quad \bar{\Psi} = i\Psi^\dagger \gamma_0,$$
$$\gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$
$$\gamma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma_3 = \gamma_0\gamma_1, \tag{B2}$$

and the $SU(\kappa)$ index is implicit for the spinors. The kinetic term is easily expanded in the above convention as

$$i\bar{\Psi}\,\partial\!\!\!/\Psi = -\psi_+^*(\partial_0 - \partial_1)\psi_+ - \psi_-^*(\partial_0 + \partial_1)\psi_-, \tag{B3}$$

and $\psi_\pm$ match the κ-component left-/right-moving modes from the two Fermi points considered in Sec. V (up to a rescaling of the spatial direction by v_{F}).

To compare with the interactions in Eq. (78), we need to expand the vertex. It is useful to use two different Fierz identities. The first one is the Fierz identity in the Clifford algebra of two space-time dimensions:

$$(\gamma_\mu)_\alpha^\beta (\gamma^\mu)_\gamma^\delta = (\mathbb{I})_\alpha^\delta (\mathbb{I})_\gamma^\beta - (\gamma_3)_\alpha^\delta (\gamma_3)_\gamma^\beta, \tag{B4}$$

and the second one is the Fierz identity in the $SU(\kappa)$ Lie algebra:

$$(T^\alpha)_{ab}(T^\alpha)_{cd} = \frac{1}{2}\left(\delta_{ad}\delta_{cb} - \frac{1}{N}\delta_{ab}\delta_{cd}\right). \tag{B5}$$

Using these two together, we get, after some simple algebra,

$$\begin{aligned} -\frac{1}{2} g J_\mu^\alpha J^{\mu\alpha} &= \frac{g}{4}\left[(\bar{\Psi}\cdot\Psi)^2 - (\bar{\Psi}\cdot\gamma_3\Psi)^2 + \frac{1}{N}(\bar{\Psi}\cdot\gamma^\mu\Psi)(\bar{\Psi}\cdot\gamma_\mu\Psi)\right] \\ &= -g(\psi_+^*\cdot\psi_-)(\psi_-^*\cdot\psi_+) - \frac{g}{N}(\psi_+^*\cdot\psi_+)(\psi_-^*\cdot\psi_-), \end{aligned} \tag{B6}$$

where the inner products make explicit the sum over components. We can already identify the two vertices $g_{1,2}$ in Sec. V as $g_1 \propto -g$ and $g_2 \propto -g/\kappa$. Note that, according to our results in Sec. V, $g_1 - \kappa g_2$ is RG invariant. Here, we find an additional perspective on this fact: This combination is forced to be zero due to Lorentz and $SU(\kappa)$ invariance of the relativistic Lagrangian. Equation (B6) is also the interaction term for the chiral Gross-Neveu model in Ref. [47] with $g' = 0$. By using the results in Ref. [48], it is also possible to show in detail that the calculation of the beta function in Sec. V is identical to the one for the coupling g in the chiral Gross-Neveu model.

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