Probing Nieh-Yan anomaly through phonon dynamics in the Kramers-Weyl semimetals of chiral crystals

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Nieh-Yan anomaly describes the nonconservation of chiral charges induced by the coupling between Dirac fermions and torsion fields. Since the torsion field is beyond general relativity, this effect remains hypothetical and its relevance to our universe is unclear in the context of high-energy physics. In this work, we propose that the phonons can induce a torsion field for the Kramers-Weyl fermions through electron-phonon interaction in a nonmagnetic chiral crystal, thus leading to the occurrence of the Nieh-Yan anomaly. As a consequence, the Nieh-Yan term can strongly influence the phonon dynamics and lead to the helicity of acoustic phonons, namely, two transverse phonon modes mix with each other to form a circular polarization with a nonzero angular momentum and the phonon angular momentum forms a hedgehog texture in the momentum space. The phonon helicity can be probed through measuring the total phonon angular momentum driven by a temperature gradient.

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I. INTRODUCTION

Weyl fermion is a two-component relativistic fermion with a definite chirality and serves as the building block for fermions in quantum field theory [1]. The chirality of Weyl fermions can give rise to a variety of physical effects, including chiral anomaly [2,3], chiral magnetic effect [4], mixed axial-gravitational anomaly [5,6], chiral torsion effect [7–9], and Nieh-Yan (NY) anomaly [10–13]. Weyl fermions can also emerge as low-energy quasiparticles in condensed matter systems. These systems are dubbed "Weyl semimetals" [14–18], which have been demonstrated in a number of materials through observing the surface Fermi arc [19–23], a negative magnetoresistance [24,25], a negative magnetothermoelectric resistance [14–18,26].

In Weyl semimetals, position-dependent and time-dependent perturbations, such as magnetic fluctuations [27], strain [28–31], and structure inhomogeneity [32,33], can shift the position of Weyl nodes and thus act as an emergent gauge field, dubbed the "pseudogauge field" [34,35]. The chiral zero Landau levels induced by pseudogauge fields have been observed experimentally in photonic and acoustic Weyl metamaterials [32,33]. Furthermore, these perturbations can also give corrections to the Fermi velocity of Weyl fermions and thus play the role of the frame fields (also called tetrad or vierbein) [36,37], allowing to mimic Weyl fermions in the curved spacetime [26,38–42]. An exciting theoretical proposal is to realize the NY anomaly [10–13] due to the torsion field in Weyl semimetals [43–51], which may be probed through Hall viscosity [43], topological magnetotorsional effect [52], anomalous thermal Hall conductance [45], or the sound-wave-induced current oscillations in tilted Weyl semimetal interfaces under magnetic fields [53]. However, all these physical phenomena require the time reversal (TR) breaking, and thus cannot appear in Weyl semimetals with TR symmetry. However, the NY anomaly itself does not require to break TR. Therefore, one may wonder what anomalous response that reflects the NY anomaly can appear in TR-invariant Weyl semimetals and how it is different from TR-breaking Weyl semimetals.

Here we turn to the so-called "Kramers-Weyl" (KW) fermions in chiral crystals with TR symmetry and will show that this system provides an appealing platform to explore physics of Weyl fermions in the curved space. Chiral crystals are the crystal structures with a well-defined handedness and nonmagnetic chiral crystals can generally host the KW fermions when taking into account spin-orbit coupling (SOC) [54–58]. In this work, we consider acoustic phonons in the KW semimetal phase of chiral crystals and demonstrate that the phonons will induce a torsion field for the KW fermions. By integrating out the KW fermions, we prove that the phonon self-energy contains an off-diagonal term that originates from the NY anomaly. While this self-energy correction has no influence on the longitudinal phonon mode, it will mix two transverse modes and give rise to the phonon angular momentum (PAM) at a finite momentum. In particular, the induced PAM reverse its sign for opposite phonon momentum, as shown in Fig. 1(a), and form a hedgehog texture in the momentum space for one phonon branch [Fig. 1(b)], analogous to the helical spin texture in spin-orbit coupled electronic band structures. Thus, we term it as "phonon helicity" [59]. The phonon helicity can be probed through measuring the total PAM induced by a temperature gradient [60].

II. ELECTRON-PHONON COUPLING IN CHIRAL CRYSTALS

In nonmagnetic chiral crystals, all the energy bands are at least doubly degenerate at high symmetry momenta, labeled

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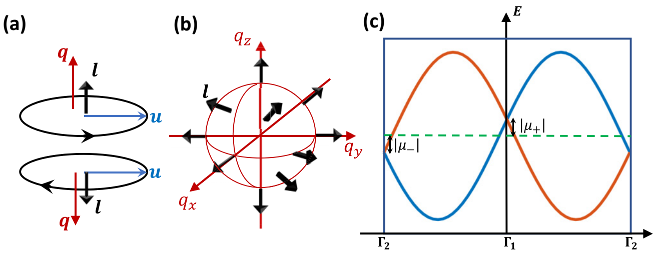

FIG. 1. (a) The PAM **l** is parallel with the phonon momentum **q** and its sign reverses for opposite **q** for one branch of phonon modes. This feature of transverse phonon modes is referred as "phonon helicity." Here **u** labels the displacement vector. (b) The hedgehog texture of the PAM **l** is formed in the **q**-space. (c) Energy dispersion between two time-reversal invariant momenta $\mathbf{\Gamma}_1$ and $\mathbf{\Gamma}_2$ for the KW semimetal phase of chiral crystals. The red and blue curves are for two Kramers' partner. The green dashed line depicts the Fermi energy, near which the linear term dominates. Here μ_+ and μ_- label the energy difference between the Fermi energy and the Weyl nodes at $\mathbf{\Gamma}_1$ and $\mathbf{\Gamma}_2$, respectively.

as $\mathbf{\Gamma}_i$ $(i = 1, 2 \ldots, 8)$, due to TR and SOC can lift this spin degeneracy for the momentum away from $\mathbf{\Gamma}_i$, thus giving rise to the KW fermions. As shown in the Secs. I and II in Appendices, the low-energy physics can be described by an isotropic Hamiltonian,

$$H_0 = \frac{\hbar^2 k^2}{2m_0^*} + \hbar v_f \mathbf{k}\cdot\sigma - \mu, \tag{1}$$

expanded around $\mathbf{\Gamma}_i$ up to the k^2 terms, where σ labels the Pauli matrix for spin, m_0^* is the effective mass, v_f is the Fermi velocity, and μ is the chemical potential. As discussed in Ref. [61], this isotropic Hamiltonian remains valid for the chiral point groups T and O, while for other chiral point groups, the Fermi velocity v_f becomes anisotropic and should be replaced by a tensor. We only focus on the isotropic Hamiltonian in this work. In chiral crystals, there are normally multiple Fermi surfaces, and here we only focus on the "KW semimetal phase" regime $\mu \ll \frac{\hbar^2\Lambda^2}{2m_0^*}$ with the momentum cutoff $\Lambda = \frac{2m_0^* v_f}{\hbar}$, where the linear-$k$ terms dominate over the k^2 terms at the Fermi surfaces around different $\mathbf{\Gamma}_i$, as depicted in Fig. 1(c).

The electron-phonon coupling in this system can be derived from the standard $\mathbf{k}\cdot\mathbf{p}$ theory up to the second-order perturbation terms with the help of Schrieffer-Wolf transformation (Sec. I C of Appendices). The TR symmetry $\hat{T} = i\sigma_y\mathcal{K}$, where $\mathcal{K}$ is the complex conjugate, gives a strong constraint on the form of the resulting effective Hamiltonian. In this work, we focus on the acoustic phonons, which couple to electrons through an internal strain, described by the strain tensor $u_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$ with the displacement vector u_i $(i, j = x, y, z)$. Since the strain tensor is even under $\hat{T}$, it cannot directly couple to either electron momentum **k** or spin σ, both of which are odd under $\hat{T}$. This implies that the strain tensor *cannot* play the role of the vector potential of pseudogauge field. The symmetry-allowed strain-electron coupling Hamiltonian is written as (Sec. II of Appendices)

$$H_{\rm ep} = (C_1 + g_0 k_j\sigma_j)u_{ii} + g_1 u_{ij} k_i \sigma_j, \tag{2}$$

up to the order of $k_i\sigma_j$, since both u_{ij} and $T_{ij} = k_i\sigma_j$ are rank-2 tensors. We have assumed the summation over the duplicated indices. C_1, g_0, and g_1 are three independent parameters. The C_1 term provides a correction to the chemical potential, while both g_0 and g_1 terms give the corrections to the Fermi velocity tensor. Microscopically, both g_0 and g_1 terms can be obtained from the second-order perturbations that combine the $\mathbf{k}\cdot\mathbf{p}$ term with the bare electron-phonon coupling term (Sec. I C of Appendices).

We notice that the strain tensor only couples to spin Pauli matrix σ through linear-k term, and the absence of k-independent term means that strain cannot play the role of the vector potential of pseudogauge field for KW fermions (but scalar potential is still possible). Consequently, the Weyl nodes are always pinned at TR invariant momenta for the KW semimetals. This is in sharp contrast to TR-breaking Weyl semimetals and other generic TR-preserving inversion-breaking Weyl semimetals, in which Weyl nodes exist at arbitrary momenta and strain can serve as the vector pseudogauge potential (see more discussions on this difference in Sec. II of Appendices).

With H_0 and H_{ep}, the effective action is then given by $S_{\rm eff} = \int d\tau d^3 r \mathcal{L}_{\rm eff}$ with (Sec. III of Appendices)

$$\mathcal{L}_{\rm eff} = \hat{\psi}^\dagger_{\mathbf{\Gamma_i}}\left(\frac{\partial}{\partial\tau} - \mu - A_0(\mathbf{r}) + \frac{\chi}{2}\left\{e_a^j, (-i\partial_j)\right\}\sigma^a\right)\hat{\psi}_{\mathbf{\Gamma_i}}, \tag{3}$$

where $\hat{\psi}_{\mathbf{\Gamma_i}}$ is the fermion field operator, $\{,\}$ labels the anti-commutation operation, τ is the imaginary time, $A_0 = C_1 u_{ii}$, $\chi = \text{sign}(\hbar v_f)$ and $e_a^j = \delta_a^j + \Delta_a^j$ with the $\mathbf{\Delta}$-field given by $\Delta_a^j = g_0 u_{ii}\delta_{ja} + g_1 u_{ja}$ $(i, j, a = x, y, z)$. The Fermi velocity $\hbar v_f$ is absorbed by rescaling the spatial coordinate, while we still keep track of the chirality χ (the sign of Fermi velocity). The e_a^j can be interpreted as the frame field [36,37] and it only involves nontrivial spatial part due to TR symmetry. The effective Lagrangian (3) describes the Weyl fermions in a space with nontrivial frame field but zero spin connection, which is known as the "Weitzenböck spacetime" studied in "teleparallel gravity theory" [62]. Zero spin connection means vanishing Riemann curvature while nontrivial frame field implies nonzero torsion field, defined as $T_{ij}^a = \partial_i \bar{e}_j^a - \partial_j \bar{e}_i^a$, where $\bar{e}_i^a$ is the coframe field $\bar{e}_i^a e_a^j = \delta_i^j$ (Sec. IV of Appendices). Thus, strain or acoustic phonon can induce a torsion field for the KW fermions in chiral crystals. We emphasize that interpreting e_a^j as a frame field is only for making the connection to the notation of torsion field and Nieh-Yan anomaly in high energy physics, and there is some subtle difference between the "effective" frame field defined here and that in the continuum field description of dislocations in elasticity theory [63]. The derivations below about stress-stress correlation functions and physical consequence of helical phonon modes are independent of such connection.

III. STRESS-STRESS CORRELATION FUNCTION AND NIEH-YAN ANOMALY

We next separate the effective Lagrangian into two parts, $\mathcal{L}_{\rm eff} = \mathcal{L}_0 + \mathcal{L}_1$, where $\mathcal{L}_0 = \hat{\psi}^\dagger_{\mathbf{\Gamma_i}}(\frac{\partial}{\partial\tau} - \mu_{\mathbf{\Gamma_i}} + \chi((-i\nabla)\cdot\sigma))\hat{\psi}_{\mathbf{\Gamma_i}}$ and $\mathcal{L}_1 = \hat{\psi}^\dagger_{\mathbf{\Gamma_i}}(-A_0 + \frac{\chi_{\mathbf{\Gamma_i}}}{2}\{\Delta_a^j, (-i\partial_j)\}\sigma^a)\hat{\psi}_{\mathbf{\Gamma_i}}$. Here both the A_0 and $\mathbf{\Delta}$ fields in $\mathcal{L}_1$ are related to the strain field induced by acoustic phonons and thus treated as perturbations. By integrating out the KW fermions in $\mathcal{L}_{\text{eff}}$, we find the effective action $W[A_0, \Delta]$ for the A_0 and $\mathbf{\Delta}$ fields (Eq. (70) in Sec. III of Appendices), which provide the corrections to the phonon effective action. Here we focus on the term W_{NY} for the $\mathbf{\Delta}$ field

$$W_{\text{NY}}[\mathbf{\Delta}] = \frac{1}{2}\sum_{\tilde{q}} \Delta_a^i(\tilde{q})\Delta_b^j(-\tilde{q})\Phi_{ij}^{ab}(\tilde{q}), \tag{4}$$

where $\tilde{q} = (\mathbf{q}, i\nu_m)$ and $\sum_{\tilde{q}} = \frac{1}{\beta V}\sum_{\mathbf{q}, i\nu_m}$. The stress-stress correlation function Φ_{ij}^{ab} is defined as $\Phi_{ij}^{ab}(\tilde{q}) = \sum_{\tilde{k}} \text{Tr}_\sigma[\mathcal{T}_i^a(\tilde{k}, \tilde{k}-\tilde{q})\mathcal{G}_0(\tilde{k}-\tilde{q})\mathcal{T}_j^b(\tilde{k}-\tilde{q}, \tilde{k})\mathcal{G}_0(\tilde{k})]$, where $\tilde{k} = (\mathbf{k}, i\omega_n)$, $\mathcal{G}_0 = [i\omega_n + \mu - \chi(\mathbf{k}\cdot\sigma)]^{-1}$ and the stress tensor $\mathcal{T}_i^a(\mathbf{k}, \mathbf{k}') = \frac{\chi}{2}(k_i + k_i')\sigma^a$ with $i, a = x, y, z$. Here we have dropped the $\mathbf{\Gamma_i}$ index and only consider the contribution from one KW fermion. It should be emphasized that the chemical potential μ here is always measured from the energy of the Weyl node.

Our next task is to show that Φ_{ij}^{ab} includes the contribution from the NY anomaly. To see that, we consider $i\nu_m = 0$ and treat both the momentum $\mathbf{q}$ and the chemical potential μ as perturbation. In this limit, we can first expand $\Phi_{ij}^{ab}(\mathbf{q}) \approx \Phi_{ij}^{ab}(0) + (\partial_{q_l}\Phi_{ij}^{ab})_{\mathbf{q}=\mathbf{0}}q_l$ up to the linear order in $\mathbf{q}$. Since $\mathbf{\Delta}$ is directly proportional to the strain tensor, the zero-order term $\Phi_{ij}^{ab}(0)$ just provides the corrections to the elastic moduli (Sec. VII of Appendices). The coefficient of linear-$\mathbf{q}$ term, denoted as $\Phi_{ij,l}^{ab} = (\partial_{q_l}\Phi_{ij}^{ab})_{\mathbf{q}=\mathbf{0}}$, can be further expanded as $\Phi_{ij,l}^{ab}(\mu) \approx \Phi_{ij,l}^{ab}(\mu = 0) + (\partial_\mu\Phi_{ij,l}^{ab})_{\mu=0}\mu$ up to the linear order in μ. Direct calculations in Sec. V of Appendices show that $\Phi_{ij,l}^{ab}(\mu = 0) = 0$ and $(\partial_\mu\Phi_{ij,l}^{ab})_{\mu=0} = i\epsilon^{alb}\delta_{ij}\chi\mathcal{F}_0$ with ϵ^{alb} the Levi-Civita symbol, $\mathcal{F}_0 = \frac{1}{3\pi^2\beta}\sum_{i\omega_n}\int_0^\Lambda k^4 dk \frac{1}{D^4}(1 - \frac{4(i\omega_n)^2}{D^2})$ with the momentum cutoff Λ and $D^2 = (i\omega_n)^2 - k^2$. The effective action reads

$$W_{\text{NY}}[\mathbf{\Delta}] = -\frac{\epsilon^{alb}\delta_{ij}\chi\mu\mathcal{F}_0}{2}\int d^3r d\tau\, \Delta_a^i(\partial_l\Delta_b^j) \tag{5}$$

after the Fourier transform into the real space.

Now we will show that W_{NY} is a manifestation of the NY anomaly. In literature [44,45,52], the effective action for the NY anomaly in a Weyl semimetal with minimal two Weyl nodes has been derived as $S_{\text{NY}} = \mathcal{F}\int d^4x\eta_{ab}\epsilon^{\mu\nu\lambda\rho}b_\mu\bar{e}_\nu^a\partial_\lambda\bar{e}_\rho^b$, where b_μ $(\mu = 0, x, y, z)$ is the separation between two Weyl nodes in the energy-momentum space. Now let us consider two KW fermions with opposite chiralities at two TR-invariant momenta, together forming one massless Dirac fermion. The effective action (5) can be applied to both KW fermions and the full action will be given by $W_{\text{NY}}[\mathbf{\Delta}] = -\mathcal{F}_0\int d^3r d\tau\epsilon^{alb}\delta_{ij}\langle\mu\rangle_\chi\Delta_a^i(\partial_l\Delta_b^j)$, where $\langle\mu\rangle_\chi = \frac{1}{2}(\mu_+ - \mu_-)$ is the chiral chemical potential with $\pm$ labels KW fermion chirality. Since $\mu_\pm$ is measured from the energy of each Weyl node, $\langle\mu\rangle_\chi$ just corresponds to the separation b_0 between two Weyl nodes in energy [Fig. 1(c)]. Therefore, W_{NY} is exactly the b_0 term in S_{NY}. We notice that the b_0 term is TR-even while the $b_{x,y,z}$ terms are TR-odd. Thus, W_{NY} does not include the $b_{x,y,z}$ terms that response for Hall viscosity [43], anomalous thermal Hall conductance [45], etc.

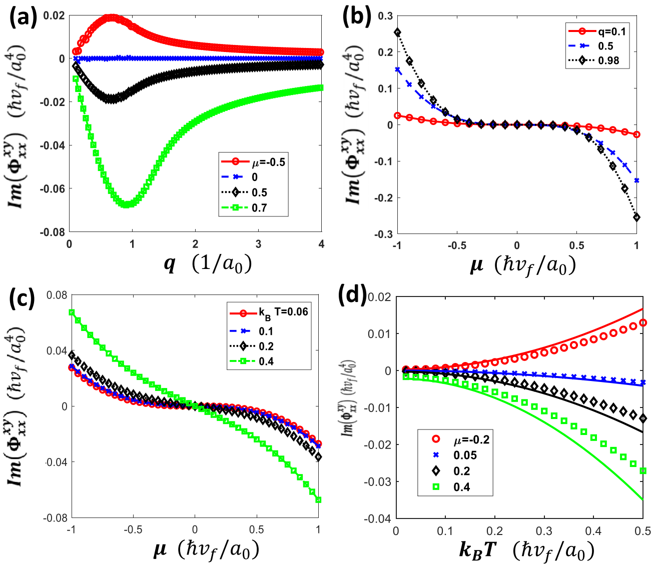

FIG. 2. (a) $\text{Im}(\Phi_{xx}^{xy})$ as a function of q for $\omega = 0$, $k_BT = 0.05$, and $\mu = -0.5, 0, 0.5, 0.7$; (b) $\text{Im}(\Phi_{xx}^{xy})$ as a function of μ for $\omega = 0$, $k_BT = 0.05$, and $q = 0.1, 0.5, 0.98$; (c) $\text{Im}(\Phi_{xx}^{xy})$ as a function of μ for $k_BT = 0.06, 0.1, 0.2, 0.4$, $\omega = 0$, and $q = 0.1$; (d) $\text{Im}(\Phi_{xx}^{xy})$ as a function of k_BT for $\omega = 0$, $q = 0.1$, and $\mu = -0.2, 0.05, 0.2, 0.4$. Here the lines are from the analytical expression (6). The length is in unit of characteristic length $a_0 = 1/\Lambda$. The momentum and energy is in unit of $1/a_0$ and $\hbar v_f/a_0$, and Φ_{xx}^{xy} is in unit of $\hbar v_f/a_0^4$.

This is the main difference in the effective action between TR-breaking and TR-preserving Weyl semimetals.

Next let us discuss the coefficient $\mathcal{F}_0$ in W_{NY}, which is found to take form of $\mathcal{F}_0 = F_0 + F_1(k_BT)^2$ for the finite temperature (Sec. V of Appendices). Dimension counting suggests the dimension of $\frac{1}{L^2}$ for $\mathcal{F}_0$ where L is the length dimension. Thus, F_0 is *not* universal and generally depends on the form of ultraviolet momentum cutoff, and our calculation gives $F_0 = 0$ due to the effective Lorentz invariance with an isotropic momentum cutoff [49] (see more discussions in Sec. IV of Appendices). However, for the temperature-dependent term, $(k_BT)^2$ exactly carries the dimension of $\frac{1}{L^2}$. Consequently, the parameter F_1 is dimensionless and universal [45]. Our calculations give $F_1 = -\frac{1}{12}$, consistent with the previous results from different approaches [45,46,48], and this T^2-dependent term is called "thermal NY anomaly."

The above discussion applies to the case of a single KW fermion or a single Dirac fermion in the limit $\mu \to 0$. For multiple KW fermions, one can introduce a single chiral chemical potential $\langle\mu\rangle_\chi = \frac{1}{2}\sum_{\mathbf{\Gamma}_i}\chi_{\mathbf{\Gamma}_i}\mu_{\mathbf{\Gamma}_i}$ for different $\mathbf{\Gamma}_i$ into Eq. (5), where $\chi_{\mathbf{\Gamma_i}}$ and $\mu_{\mathbf{\Gamma_i}}$ the chirality and chemical potential of the KW fermion at $\mathbf{\Gamma_i}$. We next consider numerically evaluate the expression of the correlation function $\Phi_{ij}^{ab}(\mathbf{q}, i\omega_n)$. Here we consider the momentum $\mathbf{q}$ along the z-direction ($\mathbf{q} = q\hat{e}_z$) and numerically compute the component Φ_{xx}^{xy} (Sec. VI of Appendices), which is shown in Fig. 2. Φ_{xx}^{xy} is found to be purely imaginary and $\text{Im}(\Phi_{xx}^{xy})$ as a function of q is shown in Fig. 2(a) for different μ. $\text{Im}(\Phi_{xx}^{xy})$ is linearly proportional to q for $q \ll 1$. Furthermore, $\text{Im}(\Phi_{xx}^{xy})$ vanishes at $\mu = 0$ and its sign reverses for opposite μ. Figure 2(b) shows the dependence of $\text{Im}(\Phi_{xx}^{xy})$ on μ for different q at a low temperature. $\text{Im}(\Phi_{xx}^{xy})$ depends on μ asymmetrically, and for μ close to zero, one can see that $\mathrm{Im}(\Phi_{xx}^{xy})$ is flat and almost zero. This suggests that the linear μ term in $\mathrm{Im}(\Phi_{xx}^{xy})$ should almost vanish, consistent with our analytical result $F_0 = 0$. Figure 2(c) shows $\mathrm{Im}(\Phi_{xx}^{xy})$ as a function of μ for different temperatures k_BT. With increasing temperature, the linear-μ term in $\mathrm{Im}(\Phi_{xx}^{xy})$ gradually appears. Finally, we plot $\mathrm{Im}(\Phi_{xx}^{xy})$ as a function of k_BT for different μ in Fig. 2(d), which shows a T^2-dependence, consistent with the thermal NY anomaly. At small μ and q, this temperature dependence can be well captured by the expression

$$\mathrm{Im}(\Phi_{xx}^{xy}) \approx -q\left[\frac{\mu^3}{12\pi^2} + \frac{\mu(k_BT)^2}{12}\right], \tag{6}$$

as shown by the blue curve in Fig. 2(d) for $\mu = 0.05$ Here the T^2 term (μ linear) reproduces our early derivation of Eq. (5) and we further expand it up to μ^3 terms (Sec. VI of Appendices). The derivation of Eq. (6) from numerics increases with μ (the red, black and green curves in Fig. 2(d) for $\mu = -0.2, 0.2, 0.4$, respectively). We also notice that our expression of $\mathrm{Im}(\Phi_{xx}^{xy})$ match those of the chiral vortical conductivity for energy flux [64,65].

IV. HELICITY OF ACOUSTIC PHONONS

Since the $\mathbf{\Delta}$-field is related to the strain field $\mathbf{u}$, it is natural to expect the NY anomaly term will influence the acoustic phonon dynamics (elastic wave). As derived in Sec. VII of Appendices, a new NY-related term can be induced in the equation of motion for phonons,

$$\frac{d^2}{dt^2}\mathbf{u} = c_t^2\nabla^2\mathbf{u} + \left(c_l^2 - c_t^2\right)\nabla(\nabla\cdot\mathbf{u}) + \frac{\xi_0}{2\rho_0}\nabla\times(\nabla^2\mathbf{u}), \tag{7}$$

where $\mathbf{u}$ is the displacement field, ρ_0 is the density, $c_{t(l)}$ is the velocity of transverse (longitudinal) modes, and $\xi_0 = -\frac{g_1^2}{(\hbar v_f)^4}[\frac{\langle\mu^3\rangle_\chi}{12\pi^2} + \frac{\langle\mu\rangle_\chi}{12}(k_BT)^2]$, with $\langle\mu^3\rangle_\chi = \frac{1}{2}\sum_{\mathbf{\Gamma_i}}\chi_{\mathbf{\Gamma_i}}\mu_{\mathbf{\Gamma_i}}^3$. The ξ_0 term does not affect longitudinal modes but will change the dispersion of transverse modes to $\omega_s^t = \sqrt{c_t^2q^2 + s\frac{|\xi_0|}{2\rho_0}q^3}$ for the s-mode ($s = \pm$). Here we always assume ξ_0 is small enough, so that we do not have any imaginary frequency within the momentum cutoff. Importantly, these two transverse modes carry angular momentum $\mathbf{l}_s(\mathbf{q})$, defined as $l_{s,i}(\mathbf{q}) = \hbar\mathbf{u}_{0,s}^\dagger(\mathbf{q})M_i\mathbf{u}_{0,s}(\mathbf{q})$ [60,66–69], where $\mathbf{u}_{0,s}(\mathbf{q})$ is the polarization vector for the mode $s = \pm$ and the $\mathbf{M}$ matrix is given by $(M_i)_{jk} = (-i)\epsilon_{ijk}$ with $i, j, k = x, y, z$. Direct calculation gives (Sec. VII of Appendices) $\mathbf{l}_s(\mathbf{q}) = s\hbar\frac{\xi_0}{|\xi_0|}\frac{\mathbf{q}}{q}$, which satisfies the relation $\mathbf{l}_s(\mathbf{q}) = -\mathbf{l}_s(-\mathbf{q})$ due to the TR symmetry (phonon helicity) [70] and forms a hedgehog texture in the momentum space, as shown in Fig. 1(b).

The total PAM is defined as $\mathbf{I}^{ph} = \sum_{s,\mathbf{q}}\mathbf{l}_s(\mathbf{q})[f(\omega_s^t) + \frac{1}{2}]$ [60,66–69], where $f(\omega_s^t)$ describes the distribution function for phonons. At the equilibrium state, the total PAM vanishes due to the TR symmetry. The phonon helicity $\mathbf{l}_s(\mathbf{q})$ can give rise to a total phonon angular momentum $\mathbf{I}^{\mathrm{ph}}$ by driving a thermal current with a temperature gradient [60], as shown in Fig. 3(a). This is in analog to the Edelstein effect of electrons due to spin-orbit coupling [71]. In the linear response regime, the response tensor α_{ij}, defined as $I_i^{ph} = \alpha_{ij}\frac{\partial T}{\partial x_j}$, can be evaluated from the formula [60] $\alpha_{ij} = -\frac{\tilde{\tau}}{V}\sum_{\mathbf{q},s}l_{s,i}v_{s,j}^{ph}\frac{\partial f_0(\omega_s^t)}{\partial T}$,

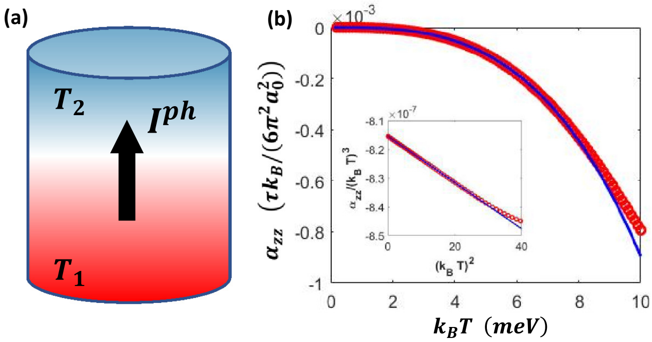

FIG. 3. (a) When there is a temperature gradient in the sample, $T_1 > T_2$, a total PAM $\mathbf{I}^{\mathrm{ph}}$ is generated. (b) α_{zz} as a function of k_BT. Here the red circles are for the full numerical calculations of Eq. (135) in Sec. VI of Appendices and the blue lines are from the analytical expression Eq. (8). Inset: $\alpha_{zz}/(k_BT)^3$ as a function of $(k_BT)^2$.

where $\tilde{\tau}$ is the phonon relaxation time, $\mathbf{v}_s^{ph} = \partial\omega_s^t/\partial\mathbf{q}$ is the group velocity of s-phonon mode and $f_0(\omega_s^t)$ is the Bose-Einstein distribution function. Since our model is isotropic, only the diagonal components $\alpha_{xx} = \alpha_{yy} = \alpha_{zz}$ are nonzero and we derive the approximated expression

$$\alpha_{zz} \approx \frac{2\pi^2}{45}\frac{\tilde{\tau}k_B}{a_0^2}\left(\frac{k_BTa_0}{\hbar c_t}\right)^3\left(\frac{\xi_0}{c_t^2\rho_0a_0}\right) \tag{8}$$

in the low temperature limit, where a_0 is lattice constant. Equation (8) fits well with the numerical calculations, as shown in Fig. 3(b). It is instructive to plot α_{zz}/T^3 as a function of T^2, which reveals a linear line dependence at the low temperature, as shown in the inset of Fig. 3(b), from which the intercept and the slope of the curve determine the contribution of the normal and thermal NY anomaly, respectively. For the estimate of the magnitude of α_{zz}, we choose $T \sim 300\mathrm{K}$, $c_t \sim 1500$ m/s, $\mu \sim 0.1$ eV, $v_f \sim 10^5$ m/s, $g_1 \sim 0.1$ eV Å and $\rho_0 \sim 10^3$ kg/m^3 and find the response coefficient $\alpha_{zz} \sim 5.5\times10^{-7}(\frac{\tau}{1s})\frac{Js}{Km^2}$, which is comparable to the α value for GaN, Te, and Se calculated from the first principles calculations in Ref. [60].

V. DISCUSSION AND CONCLUSION

To summarize, we have demonstrated that acoustic phonons can induce a torsion field for the KW fermions that gives rise to the NY anomaly. Strikingly, we find the phonon helicity can be induced by NY anomaly term and probed through measuring PAM driven by a temperature gradient. Phonon dynamics in Weyl/Dirac semimetals was previously studied in the context of chiral anomaly [72–78] and Kohn anomaly [79,80]. The concept of PAM has also been explored both theoretically [66,67,69,81–87] and experimentally [68,88]. The PAM due to temperature gradient can be measured through the rigid-body rotation or the phonon orbital magnetic moments, and the theoretical estimate of the magnitude of the induced PAM suggests these phenomena are detectable in experiments [60]. The Coulomb interaction is considered in Sec. VIII of Appendices and will not contribute to the screening of the electron-phonon interaction for the relevant phonon modes due to nonzero PAM. Similar physics can also occur in optical phonon dynamics (Sec. I.B of Appendices), and thus may allow for an optical detection of the NY anomaly. We also notice a similar phenomenon of helical electrodynamics which is induced by the anomalies in Weyl/Dirac semimetals [89,90], which implies more helical quasiparticles may exist and be related to the anomalies.

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APPENDIX A: $k \cdot p$ HAMILTONIAN AND ELECTRON-PHONON INTERACTION OF THE KRAMERS-WEYL FERMIONS

In this Appendix, we will describe the $k \cdot p$ theory for the KW fermions in chiral crystals. To be general, we start from the Schrödinger equation

$$\left[\frac{\hat{\mathbf{p}}^2}{2m_0} + \hat{V}(\mathbf{r})\right]\Psi(\mathbf{r}) = E\Psi(\mathbf{r}), \tag{A1}$$

where m_0 is the electron mass, $\hat{\mathbf{p}} = -i\hbar\nabla$ and the periodic potential $\hat{V}(\mathbf{r}+\mathbf{R_n}) = \hat{V}(\mathbf{r})$ including both the lattice potential and spin-orbit coupling. The Bloch theorem requires $\Psi(\mathbf{r}) = e^{i\mathbf{K}\cdot\mathbf{r}}\mathfrak{u}(\mathbf{r})$, where $\mathfrak{u}(\mathbf{r}+\mathbf{R_n}) = \mathfrak{u}(\mathbf{r})$ and $\mathbf{K}$ is the crystal momentum.

Let us consider the perturbation expansion around a time-reversal (TR) invariant momentum $\mathbf{\Gamma}_i$ ($i = 1, 2, \ldots$ labels different TR invariant momenta) and assume that the Schrödinger Eq. (A1) has been solved at $\mathbf{\Gamma}_i$ with the eigenenergy $E_{\mathbf{\Gamma}_i,n}$ and eigen-wave function $\psi_{\mathbf{\Gamma}_i,n}(\mathbf{r}) = e^{i\mathbf{\Gamma_i}\cdot\mathbf{r}}\mathfrak{u}_{\mathbf{\Gamma}_i,n}(\mathbf{r})$. We are interested in the momentum $\mathbf{K} = \mathbf{\Gamma_i} + \mathbf{k}$ where $\mathbf{k}$ is considered as a perturbation, and expand the wave function $\mathfrak{u}_{\mathbf{K}}(\mathbf{r}) = \sum_n C_n \mathfrak{u}_{\mathbf{\Gamma_i}}(\mathbf{r})$. Thus, $\psi_{\mathbf{K}}(\mathbf{r}) = e^{i\mathbf{K}\cdot\mathbf{r}}\mathfrak{u}_{\mathbf{K}}(\mathbf{r}) = e^{i(\mathbf{\Gamma_i}+\mathbf{k})\cdot\mathbf{r}}\sum_n C_n\mathfrak{u}_{\mathbf{\Gamma_i}}(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}}\sum_n C_n\psi_{\mathbf{\Gamma_i},\mathbf{n}}(\mathbf{r})$. The Schrödinger equation at $\mathbf{K} = \mathbf{\Gamma_i} + \mathbf{k}$ can be expanded as

$$\left(E_{\mathbf{\Gamma_i},m} + \frac{\hbar^2k^2}{2m_0}\right)\delta_{nm}C_m + \sum_n \mathcal{P}_{mn}\cdot\mathbf{k}C_n = EC_m \tag{A2}$$

and $\mathcal{P}_{mn} = \frac{\hbar}{m_0}\langle\psi_{\mathbf{\Gamma_i},m}|\hat{\mathbf{p}}|\psi_{\mathbf{\Gamma_i},n}\rangle$, and the corresponding Hamiltonian can be written as

$$H = H_0 + H_{\mathbf{k}\cdot\mathbf{p}}, \tag{A3}$$

where

$$(H_0)_{mn} = \left(E_{\mathbf{\Gamma_i},m} + \frac{\hbar^2k^2}{2m_0}\right)\delta_{nm} \tag{A4}$$

and

$$(H_{\mathbf{k}\cdot\mathbf{p}})_{mn} = \mathcal{P}_{mn}\cdot\mathbf{k}. \tag{A5}$$

Since we focus on TR symmetry, we denote the quantum number $m = (\alpha, s)$, where s labels different spin states, and α labels other band indices. With the TR operator $\hat{T}$, we define the bands of a Kramers' pair as $\hat{T}\psi_{\mathbf{\Gamma}_i,\alpha,s} = s\psi_{\mathbf{\Gamma}_i,\alpha,\bar{s}}$ and $E_{\mathbf{\Gamma}_i,\alpha,s} = E_{\mathbf{\Gamma}_i,\alpha,\bar{s}}$, where $\bar{s} = -s$. We may assume around $\mathbf{\Gamma}_i$, only one Kramers' pair of bands, denoted as $\psi_{\mathbf{\Gamma}_i,0,s}$ ($\alpha = 0$), contribute to the low-energy physics and thus the low-energy effective Hamiltonian can be written as

$$(H_{\text{eff}})_{ss'} = \left(E_{\mathbf{\Gamma}_i,0} + \frac{\hbar^2k^2}{2m_0^*}\right)\delta_{ss'} + \mathcal{P}_{0,ss'}\cdot\mathbf{k}, \tag{A6}$$

with $s, s' = \pm$. Here we only keep the correction to the electron mass ($m_0 \to m_0^*$) for the terms of the second order in $\mathbf{k}$. In general, the effective mass m_0^* should be a tensor, but we only consider the quadratic term to provide a cutoff for the valid regime of the KW fermion physics and thus simply treat m_0^* as a scalar. We can decompose the parameter matrix $\mathcal{P}_0$ as $\mathcal{P}^j_{0,ss'} = \sum_a \mathcal{P}^j_{0,a}(\sigma^a)_{ss'}$ where $\sigma^{a=x,y,z}$ are three Pauli matrices. Here TR symmetry requires that the linear $\mathbf{k}$ term cannot couples to identity matrix. With $\text{Tr}(\sigma^a\sigma_b) = 2\delta^a_b$, we have $\mathcal{P}^j_{0,a} = \frac{1}{2}\text{Tr}(\mathcal{P}^j_0\sigma_a)$. Correspondingly, the effective Hamiltonian can be written as

$$H_{\text{eff}} = E_{\mathbf{\Gamma}_i,0} + \frac{\hbar^2k^2}{2m_0^*} + \sum_{a,j}\mathcal{P}^j_{0,a}\sigma^a k_j. \tag{A7}$$

The parameters $\mathcal{P}^j_{0,a}$ will be constrained by the crystal symmetry of the system. Since the chiral crystals only involve rotation symmetry, we consider the full rotation symmetry here. As discussed in Appendix B, one can show that up to a unitary transformation, the effective Hamiltonian can be written as

$$H_{\text{eff},0}(\mathbf{k}) = E_{\mathbf{\Gamma}_i,0} + \frac{\hbar^2k^2}{2m_0^*} + \hbar v_f\mathbf{k}\cdot\sigma, \tag{A8}$$

where the Fermi velocity is given by $\hbar v_f = \frac{\hbar}{2m_0}\text{Tr}(\mathcal{P}^x_0\sigma_x) = \frac{\hbar}{2m_0}\text{Tr}(\mathcal{P}^y_0\sigma_y) = \frac{\hbar}{2m_0}\text{Tr}(\mathcal{P}^z_0\sigma_z)$ (with full rotation symmetry) and its sign determines the chirality of the KW fermions. It should be mentioned that the above Hamiltonian is also valid for the crystals with the O and T chiral point groups [61]. In the second quantization language, we can write down $H_{\text{eff},0} = \frac{1}{V}\sum_{\mathbf{k}}\hat{c}^\dagger_{0,\mathbf{k}}H_{\text{eff},0}(\mathbf{k})\hat{c}_{0,\mathbf{k}}$. In the real space, the Hamiltonian is written as

$$H_{\text{eff},0} = \int d^3r\hat{\psi}^\dagger_{\Gamma_i}(\mathbf{r})\left[E_{\mathbf{\Gamma}_i,0} - \frac{\hbar^2\nabla^2}{2m_0^*} + \hbar v_f(-i\nabla)\cdot\sigma\right]\hat{\psi}_{\Gamma_i}(\mathbf{r}). \tag{A9}$$

The Fourier transform is given by

$$\hat{\psi}(\mathbf{r}) = \tfrac{1}{V}\textstyle\sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}}\hat{c}_{\mathbf{k}}, \tag{A10}$$

$$\hat{c}_{\mathbf{k}} = \textstyle\int d^3re^{-i\mathbf{k}\cdot\mathbf{r}}\hat{\psi}(\mathbf{r}), \tag{A11}$$

with V the volume of the system, where we have suppressed the lower indices.

1. Deformation potential for electron-acoustic phonon interaction

The electron-phonon coupling for acoustic phonons should vanish in the long-wave length limit with the phonon momentum $\mathbf{q} \to 0$, while there is no such constraint for optical phonons. Thus, our starting deformation Hamiltonians for acoustic phonons and optical phonons are slightly different. For the acoustic phonons, we will start from the continuous model with the electron field described by $\hat{\psi}_{\Gamma_i}(\mathbf{r})$ while the acoustic phonon modes can be described by the displacement field $\mathbf{u}(\mathbf{r})$. In the case of more than one atoms in one unit cell, the acoustic phonon modes correspond to the polarization vector with the same amount of shift in the same direction for all the atoms in one unit cell and thus we can still use one vector field. The displacement field $\mathbf{u}$ can give rise to the polarization $\mathbf{P}$ which couples to electron density through its divergence $\nabla \cdot \mathbf{P}$. Let us assume this coupling is local as the lowest-order perturbation. Thus, our starting point will be the volume deformation Hamiltonian

$$H_{\mathrm{ep}} = g_a \int d^3 r \hat{\psi}^\dagger_{\Gamma_i}(\mathbf{r})(\mathbf{r})(\nabla\cdot\mathbf{u})\hat{\psi}_{\Gamma_i}(\mathbf{r}), \tag{A12}$$

with the coupling constant g_a. Here the spin index s does not appear because volume deformation potential does not involve spin. $\nabla \cdot \mathbf{u} = \sum_i u_{ii}$ is the trace part of the strain tensor (volume dilation), which is defined as $\bar{u}$ below. In the momentum space, we have

$$H_{\mathrm{ep}} = \frac{g_a}{V^2}\sum_{\mathbf{k},\mathbf{q}} \hat{c}^\dagger_{0,\mathbf{k}}\bar{u}_{\mathbf{q}}\hat{c}_{0,\mathbf{k}-\mathbf{q}}, \tag{A13}$$

where $\bar{u}_{\mathbf{q}} = \int d^3 r \bar{u}(\mathbf{u}) e^{-i\mathbf{q}\cdot\mathbf{r}}$. Here we only consider the lowest-order bands, and thus only the volume dilation enters into the effective Hamiltonian. If we consider all the other bands, then other components of the strain tensor can also appear to couple different bands and the most general form of the electron-acoustic phonon coupling is given by

$$H_{\mathrm{ep}} = \sum_{\alpha\beta,ij}\frac{g^\lambda_{\alpha\beta}}{V^2}\sum_{\mathbf{k},\mathbf{q}} \hat{c}^\dagger_{\alpha,\mathbf{k}} u_\lambda(\mathbf{q})\hat{c}_{\beta,\mathbf{k}-\mathbf{q}}, \tag{A14}$$

where we use a single parameter λ to label different components of strain tensor ($u_\lambda = u_{ij}$ for different $i, j = x, y, z$). The form of the coupling parameter $g^\lambda_{\alpha\beta}$ can be constrained by the crystal symmetry, which will be discussed in Appendix B. The electron-optical phonon interaction is derived in Appendix I.

2. Second-order perturbation terms from Schrieffer-Wolf transformation

We next consider the second-order perturbation contribution to the electron-phonon interaction through the Schrieffer-Wolf transformation. We consider the full $\mathbf{k}\cdot\mathbf{p}$ Hamiltonian including the electron-phonon interaction around $\boldsymbol{\Gamma}_i$ with the form $H_{\boldsymbol{\Gamma}_\mathbf{i}} = H_0 + H_1$ with the perturbation Hamiltonian $H_1 = H_{\mathbf{k}\cdot\mathbf{p}} + H_{\mathrm{ep}}$, where $H_{\mathbf{k}\cdot\mathbf{p}}$ is given by Eq. (A5) with $m = (\alpha, s)$, $n = (\beta, t)$ and H_{ep} is given by the Eq. (A14) for acoustic phonons and the Eq. (I3) for optical phonons, in which $\mathbf{K}$ should be replaced by $\mathbf{k}$ since all the momenta are expanded around $\boldsymbol{\Gamma}_\mathbf{i}$.

To get the second-order perturbation terms, we consider the Schrieffer-Wolf transformation

$$H_{\mathrm{eff}} = e^{-S} H_{\boldsymbol{\Gamma}_\mathbf{i}} e^S \approx H_0 + H_1 - [S, H_0] - [S, H_1] + \tfrac{1}{2}[S,[S,H_0]] + \cdots, \tag{A15}$$

and we require the first-order term to vanish, $H_1 - [S, H_0] = 0$. The corresponding second-order term is given by

$$H_{\mathrm{eff}} \approx H_0 - \tfrac{1}{2}[S, H_1]. \tag{A16}$$

Since $H_1 = H_{\mathbf{k}\cdot\mathbf{p}} + H_{\mathrm{ep}}$, we can also decompose $S = S_1 + S_2$ with $[S_1, H_0] = H_{\mathbf{k}\cdot\mathbf{p}}$ and $[S_2, H_0] = H_{\mathrm{ep}}$. One can show that

$$S_1 = \sum_{\mathbf{k},\alpha\neq\beta,st}\frac{1}{\Delta_{\beta\alpha}}\hat{c}^\dagger_{\alpha s,\mathbf{k}}(\mathcal{P}_{\alpha s,\beta t}\cdot\mathbf{k})\hat{c}_{\beta t,\mathbf{k}} \tag{A17}$$

and

$$S_2 = \frac{1}{V}\sum_{\mathbf{k},\mathbf{q},\alpha\neq\beta,st,\lambda}\frac{1}{\Delta_{\beta\alpha}} g^\lambda_{\alpha s,\beta t} u_\lambda(\mathbf{q})\hat{c}^\dagger_{\alpha s,\mathbf{k}}\hat{c}_{\beta t,\mathbf{k}-\mathbf{q}}, \tag{A18}$$

where $\Delta_{\beta\alpha} = E_\beta - E_\alpha$. Here we choose optical phonons as the example and keep the lowest-order terms, so we can drop the index $\mathbf{K}$, $\mathbf{q}$ for the coupling constant $g^\lambda_{\alpha s,\beta t}$. With the decomposition of S, the effective Hamiltonian includes

$$H_{\mathrm{eff}} \approx H_0 - \tfrac{1}{2}([S_1, H_{\mathbf{k}\cdot\mathbf{p}}] + [S_2, H_{\mathbf{k}\cdot\mathbf{p}}] + [S_1, H_{\mathrm{ep}}] + [S_2, H_{\mathrm{ep}}]). \tag{A19}$$

The first perturbation term is given by

$$-\tfrac{1}{2}[S_1, H_{\mathbf{k}\cdot\mathbf{p}}] = \frac{1}{2V}\sum_{\mathbf{k},\mu\nu,\alpha\beta\gamma}\left(\frac{1}{\Delta_{\alpha\gamma}}+\frac{1}{\Delta_{\beta\gamma}}\right)\left(\mathcal{P}^\mu_{\alpha\gamma}\cdot\mathbf{k}\right) \times \left(\mathcal{P}^\nu_{\gamma\beta}\cdot\mathbf{k}\right)\hat{c}^\dagger_{\alpha,\mathbf{k}}\sigma_\mu\sigma_\nu\hat{c}_{\beta,\mathbf{k}}, \tag{A20}$$

which provides a correction to the effective mass. The second and the third term together give

$$\begin{aligned} -\tfrac{1}{2}&([S_2, H_{\mathbf{k}\cdot\mathbf{p}}] + [S_1, H_{\mathrm{ep}}]) \\ &= \frac{1}{2V^2}\sum_{\mathbf{k},\mathbf{q},\mu\nu,\alpha\beta\gamma,\lambda}\left(\frac{1}{\Delta_{\alpha\gamma}}+\frac{1}{\Delta_{\beta\gamma}}\right)\Big(g^{\lambda,\mu}_{\alpha\gamma}\left[\mathcal{P}^\nu_{\gamma\beta}\cdot(\mathbf{k}-\mathbf{q})\right] \\ &\quad + \left(\mathcal{P}^\mu_{\alpha\gamma}\cdot\mathbf{k}\right)g^{\lambda,\nu}_{\gamma\beta}\Big)u_\lambda(\mathbf{q})\hat{c}^\dagger_{\alpha,\mathbf{k}}\sigma_\mu\sigma_\nu\hat{c}_{\beta,\mathbf{k}-\mathbf{q}}, \end{aligned} \tag{A21}$$

where $g^\lambda_{\alpha s,\beta t} = \sum_\mu g^{\lambda,\mu}_{\alpha\beta}(\sigma_\mu)_{st}$. The last term $-\frac{1}{2}[S_2, H_{\mathrm{ep}}]$ is of order g^2 and we neglect this term.

Now we apply the above second-order perturbation formalism to the Kramers-Weyl fermions to project into the lowest-energy subspace with $\alpha = \beta = 0$, and obtain the effective Hamiltonian

$$H_{\mathrm{eff},\boldsymbol{\Gamma}_i} = H_{\mathrm{eff},0} + H_{\mathrm{eff,ep}}, \tag{A22}$$

where

$$H_{\mathrm{eff},0} = \frac{1}{V}\sum_{\mathbf{k}}\hat{c}^\dagger_{0,\mathbf{k}}\left(E_{\boldsymbol{\Gamma}_i,0} + \frac{\hbar^2 k^2}{2m_0^*} + \hbar v_f(\mathbf{k}\cdot\sigma)\right)\hat{c}_{0,\mathbf{k}} \tag{A23}$$

and

$$H_{\mathrm{eff,ep}} = H^{(0)}_{\mathrm{eff,ep}} + H^{(2)}_{\mathrm{eff,ep}} = \frac{1}{V^2}\sum_{\mathbf{k},\mathbf{q},\lambda} g^\lambda_o u_\lambda(\mathbf{q})\hat{c}^\dagger_{0,\mathbf{k}}\hat{c}_{0,\mathbf{k}-\mathbf{q}} \tag{A24}$$

$$+\frac{1}{V^2}\sum_{\mathbf{k},\mathbf{q},\mu\nu,\gamma,\lambda}\frac{1}{\Delta_{0\gamma}}\left(g^{\lambda,\mu}_{0\gamma}\left[\mathcal{P}^\nu_{\gamma 0}\cdot(\mathbf{k}-\mathbf{q})\right] + \left(\mathcal{P}^\mu_{0\gamma}\cdot\mathbf{k}\right)g^{\lambda,\nu}_{\gamma 0}\right) \times u_\lambda(\mathbf{q})\hat{c}^\dagger_{0,\mathbf{k}}\sigma_\mu\sigma_\nu\hat{c}_{0,\mathbf{k}-\mathbf{q}}. \tag{A25}$$

Here we have made the approximation of isotropic effective mass

$$\frac{\hbar^2}{2m_0^*} = \frac{\hbar^2}{2m_0} + \sum_{\mu,\gamma\neq 0} \frac{1}{\Delta_{0\gamma}} \langle (\mathcal{P}^\mu_{0\gamma}\cdot \mathbf{k})(\mathcal{P}^\mu_{\gamma 0}\cdot\mathbf{k})\rangle_{\hat{k}}, \tag{A26}$$

where $\langle ...\rangle_{\hat{k}}$ means the average over all different directions of the momentum $\mathbf{k}$.

Since $(\sigma_\mu)^2 = 1, \sigma_0\sigma_a = \sigma_a\sigma_0 = \sigma_a, \sigma_a\sigma_b = \delta_{ab} + i\epsilon_{abc}\sigma_c$ with $a, b, c = x, y, z$, we may define

$$\mathcal{C}_0^{\lambda,j} = \sum_{\mu,\gamma\neq 0} \frac{1}{\Delta_{0\gamma}} g^{\lambda\mu}_{0\gamma}\mathcal{P}^{\mu,j}_{\gamma 0} \tag{A27}$$

and

$$\mathcal{C}_a^{\lambda,j} = \sum_{\gamma\neq 0} \frac{1}{\Delta_{0\gamma}} \left(g^{\lambda 0}_{0\gamma}\mathcal{P}^{a,j}_{\gamma 0} + g^{\lambda a}_{0\gamma}\mathcal{P}^{0,j}_{\gamma 0} + i\sum_{bc}\epsilon_{abc} g^{\lambda b}_{0\gamma}\mathcal{P}^{c,j}_{\gamma 0}\right), \tag{A28}$$

and $H^{(2)}_{\text{eff,ep}}$ can be rewritten as

$$H^{(2)}_{\text{eff,ep}} = \frac{1}{V^2}\sum_{\mathbf{k},\mathbf{q},\mu,\lambda} \left(\mathcal{C}^{\lambda,j}_\mu u_\lambda(\mathbf{q})\hat{c}^\dagger_{0,\mathbf{k}}\sigma^\mu (k_j - q_j)\hat{c}_{0,\mathbf{k}-\mathbf{q}} + \text{H.c.}\right). \tag{A29}$$

Transforming back to the real space, we find that $H_{\text{eff},0}$ is given by Eq. (A9), $H^{(0)}_{\text{eff,ep}}$ is given by Eq. (I9) and

$$H^{(2)}_{\text{eff,ep}} = \sum_{\mu,\lambda,j}\int d^3r u_\lambda(\mathbf{r})\big(\mathcal{C}^{\lambda,j}_\mu \hat{\psi}^\dagger_{\Gamma_i}(\mathbf{r})\sigma^\mu[-i\partial_j\hat{\psi}_{\Gamma_i}(\mathbf{r})] + \left(\mathcal{C}^{\lambda,j}_\mu\right)^*\left[i\partial_j\hat{\psi}^\dagger_{\Gamma_i}(\mathbf{r})\right]\sigma^\mu\hat{\psi}_{\Gamma_i}(\mathbf{r})\big). \tag{A30}$$

Now let us impose the TR symmetry on the above expression and we find $\mathcal{C}^{\lambda,j}_0 = -(\mathcal{C}^{\lambda,j}_0)^*$ so that $\mathcal{C}^{\lambda,j}_0$ should be pure imaginary while $\mathcal{C}^{\lambda,j}_a = (\mathcal{C}^{\lambda,j}_a)^*$ $(a = x, y, z)$ so that $\mathcal{C}^{\lambda,j}_a$ should be real. This means that for the $\mathcal{C}^{\lambda,j}_0$ term, the Hamiltonian will only depend on $\partial_j u_\lambda(\mathbf{r})$, while the $\mathcal{C}^{\lambda,j}_a$ terms $(a = x, y, z)$ depend on $u_\lambda(\mathbf{r})$. In the long wavelength limit, the $\mathcal{C}^{\lambda,j}_0$ term will be much smaller than the $\mathcal{C}^{\lambda,j}_a$ terms. Thus, we only consider the $\mathcal{C}^{\lambda,j}_a$ terms below.

The Hamiltonians (A9), (I9), and (A30) together form the starting point for our study on the electron-phonon coupling in the Kramers-Weyl semimetals, and we can rewrite them in a more compact form

$$H_{\text{eff}} = \int d^3r\Bigg[\hat{\psi}^\dagger_{\Gamma_i}(\mathbf{r})\left(-\frac{\hbar^2}{2m_0^*}\nabla^2 - \mu - A_0(\mathbf{r})\right)\hat{\psi}_{\Gamma_i}(\mathbf{r}) + \frac{\hbar v_f}{2}\sum_{a,j}\big(\hat{\psi}^\dagger_{\Gamma_i}(\mathbf{r})e^j_a\sigma^a[-i\partial_j\hat{\psi}_{\Gamma_i}(\mathbf{r})] + \left[i\partial_j\hat{\psi}^\dagger_{\Gamma_i}(\mathbf{r})\right]e^j_a\sigma^a\hat{\psi}_{\Gamma_i}(\mathbf{r})\big)\Bigg], \tag{A31}$$

where the chemical potential μ has included the energy $E_{\Gamma_i,0}$, $A_0 = g^\lambda_o u_\lambda(\mathbf{r})$ is from the zero-order term of electron-phonon coupling and behaves as a chemical potential fluctuation (or scalar potential), and the frame field $e^j_a = \delta^j_a + \Delta^j_a(\mathbf{r})$ with $\Delta^j_a(\mathbf{r}) = \frac{2}{\hbar v_f}\sum_\lambda \mathcal{C}^{\lambda,j}_a u_\lambda(\mathbf{r})$ comes from the second-order perturbation involving electron-phonon coupling. In deriving the above Hamiltonian, we only used the TR symmetry and the spatial rotation symmetry will give additional constraints on the parameters $\mathcal{C}^{\lambda,j}_a$ and reduce the number of independent parameters, as discussed in the Appendix B. From the above form of the Hamiltonian, it is clear that the phonon provides a background frame field of the curved space for the KW fermions.

We emphasize that here we consider the strain-induced background field felt by the KW fermions through electron-phonon interaction, and this background field is not necessary to be the same as that in the continuum field description of dislocations in elasticity theory [63]. The major difference is related to the difference between distortion tensor and strain tensor. In the continuum field description of dislocations, the coframe field $\bar{e}^a_\mu$ is directly given by the unsymmetrized distortion tensor, defined as $w_{\mu a} = \partial_\mu u_a$, which is a gradient term. Thus, as a curl of $\bar{e}^a_\mu$, one can immediately show that the torsion $T^a_{\mu\nu} = \partial_\mu\bar{e}^a_\nu - \partial_\nu\bar{e}^a_\mu = \partial_\mu\partial_\nu u_a - \partial_\nu\partial_\mu u_a = 0$. Thus, in the continuum field description of dislocations [63], the nonzero torsion can only be related to dislocation which breaks the integrability condition. However, in this work, we consider the nontrivial spacetime geometry that the KW fermions see through the electron-phonon interaction, which is induced by the strain field $u_{\mu a}$, instead of the distortion field $w_{\mu a}$. From the above derivation in this Appendix and also Eq. (2) in the main text, the "effective" frame/coframe fields that the KW fermions see in this case should be proportional to the symmetrized strain field $u_{\mu a} = \partial_\mu u^a + \partial_a u_\mu$, and its curl will no longer vanish $T^a_{\mu\nu} = \partial_\mu\bar{e}^a_\nu - \partial_\nu\bar{e}^a_\mu \propto \partial_\mu(\partial_\nu u_a + \partial_a u_\nu) - \partial_\nu(\partial_\mu u_a + \partial_a u_\mu) = \partial_\mu\partial_a u_\nu - \partial_\nu\partial_a u_\mu \neq 0$. We emphasize again that the "effective spacetime geometry" that the Weyl fermions see is not necessary to be the same as the spacetime geometry from the continuum field description of elasticity since it originates from the microscopic electron-phonon interaction. In other words, one can start from the electron-phonon interaction Hamiltonian [Eq. (2) in the main text] and perform the whole calculation of integrating out the electrons without introducing the whole notation of frame or coframe fields, and the whole results remain valid. The notation of frame/coframe fields is only useful to make the analog with the torsion and the Nieh-Yan anomaly in high-energy physics. One may ask if the strain tensor or distortion tensor should enter into the form of electron-phonon interactions (or electron-elastic wave interaction) in the effective theory for Weyl fermions. As we know, the distortion tensor can be decomposed into a symmetric strain tensor and an antisymmetric rotation tensor, $w_{\mu a} = u_{\mu a} + \omega_{\mu a}$, where $\omega_{\mu a} = \partial_\mu u_a - \partial_a u_\mu$. Normally, we expect only the strain tensor part $u_{\mu a}$ should be responsible for the electron-phonon interaction, while the electron energy should not be influenced by the rotation part $\omega_{\mu a}$. Indeed, there are a large number of papers in literature to consider the the coupling between deformation potential from lattice distortion and electrons from the microscopic models, such as tight-binding model, and only the strain tensor enters the coupling form in these papers [30,42,74,91]. However, we also notice that people consider distortion tensor coupled to electron momentum in some papers [43,92], but these works are more motivated by the analog to the high-energy physics. In Ref. [93], they use distortion tensor for the continuous model and the strain tensor for the tight-binding model, and make the comparison between these two cases. In our work, we developed a $k \cdot p$ theory to describe the effective theory of Weyl fermions from the microscopic electron-phonon interaction (deformation potential theory). Reference [42] considered a tight-binding model for Weyl semimetals and derived both the effective spacetime geometry and pseudogauge field for low-energy Weyl fermions. Their effective model with the frame/coframe fields of Weyl fermions from the tight-binding model is complementary to our results from the $k \cdot p$ theory and one can see that the "effective torsion" for the Weyl fermions can be induced by a strain field. Based on our above argument, we next consider the relation between transverse elastic waves (or acoustic phonons) and the torsion spacetime. For example, we can consider a transverse elastic wave $u_x(z) = u_{x0}\cos(\omega t - qz)$ propagating along the z axis, which can induce a shear strain $u_{xz}(z) \propto \partial_x u_z + \partial_z u_x = qu_{x0}\sin(\omega t - qz)$. With the definition of the effect coframe field $\hat{e}^z_x(z) \propto u_{xz}(z)$, a nonzero dynamical torsion field $T^z_{zx}(z) = \partial_z \hat{e}^z_x(z) - \partial_x \hat{e}^z_z(z) \propto \partial_z u_{xz}(z) = -q^2 u_{x0}\cos(\omega t - qz)$ can be induced. This gives a simple example of how elastic wave can induce dynamical torsion field in Weyl semimetals.

APPENDIX B: SYMMETRY CONSTRUCTION OF THE ELECTRON-STRAIN INTERACTION HAMILTONIAN FOR THE KW FERMIONS

In this Appendix, we will consider the construction of the effective Hamiltonian for the KW fermions from the symmetry principle [94,95]. Here we will focus on the acoustic phonons, which couple to electrons through the strain tensor u_{ij}.

The chiral crystals only involve rotation symmetries and to simplify the problem, let us first consider the isotropic systems with the full rotation symmetry [$SO(3)$] and TR symmetry. In this case, we can classify all the physical operators in terms of their angular momentum. The Hamiltonian should be invariant under the full rotation and thus should carry the angular momentum 0. Both the momentum $\mathbf{k}$ and the spin σ carry the angular momentum 1, and thus allow to construct two terms k^2 and $\mathbf{k}\cdot\sigma$. As discussed above, the acoustic phonons couple to electrons through the strain tensor u_{ij}, which is a rank-2 tensor and symmetric with respect to i and j ($u_{ij} = u_{ji}$). For the momentum $\mathbf{k}$ and spin σ, one can define another rank-2 tensor $T_{ij} = k_i\sigma_j$, which is actually the stress tensor of Weyl fermions. Now we need to construct the invariant terms based on two rank-2 tensors u_{ij} and T_{ij} and there are two ways: (1) $\sum_{ij} u_{ii}T_{jj}$ and $\sum_{ij} u_{ij}T_{ij}$. Thus, we can write down the Hamiltonian as

$$H_{\text{eff}} = C_0 + C_1\bar{u} + C_3 k^2 + (C_2 + g_0\bar{u})(\mathbf{k}\cdot\sigma) + g_1\sum_{ij} u_{ij}T_{ij}, \tag{B1}$$

up to the order of k^2 and $u_{ij}k_l$, where $\bar{u} = \sum_i u_{ii}$ is the trace of strain tensor. For TR symmetry $\hat{\mathcal{T}}$, $\mathbf{k}$ and σ are TR-odd while u_{ij} and T_{ij} is TR-even. Therefore, all the terms above are allowed for a TR-invariant Hamiltonian. We may further consider the strain as a field with spatial and temporal variations and thus require to change $u_{ij}T_{ij} = u_{ij}k_i\sigma_j$ to $\frac{1}{2}\{u_{ij}, (-i\partial_i)\}\sigma_j$ and $\bar{u}(\mathbf{k}\cdot\sigma)$ to $\frac{1}{2}\{\bar{u}, (-\mathbf{i}\nabla\cdot\sigma)\}$. The corresponding Hamiltonian can be written as

$$H_{\text{eff}} = \int d^3r\hat{\psi}^\dagger(\mathbf{r})\bigg(C_0 + C_1\bar{u} - C_3\nabla^2 + C_2(-i\nabla\cdot\sigma) + \frac{g_0}{2}\{\bar{u}, (-i\nabla\cdot\sigma)\} + \frac{g_1}{2}\{u_{ij}, (-i\partial_i)\}\sigma_j\bigg)\hat{\psi}(\mathbf{r}). \tag{B2}$$

As compared with Eq. (A31), one finds $\mu = -C_0$, $A_0 = C_1\bar{u}$, $C_3 = \frac{\hbar^2}{2m_0^*}$, $C_2 = \hbar v_f$, and $\Delta^j_a = \frac{1}{\hbar v_f}(g_0\bar{u}\delta_{ja} + g_1 u_{ja})$. Since we only focus on the spatial component, the upper and lower indices do not have specific meaning as in general relativity.

Here we notice that there are two independent parameters that characterize the coupling between strain tensor u_{ij} and the stress tensor T_{ij}. This conclusion can also be obtained by classifying the rank-2 tensors according to their angular momentum. The strain tensor can be decomposed into two parts, $u_{J=0;M=0} = \sum_i u_{ii} = \bar{u}$ with the angular momentum 0 and $u_{J=2;M=\pm2,\pm1,0}$ with the angular momentum 2. The explicit form of $u_{J=2;M}$ is given by

$$u_{2,2} = \tfrac{1}{2}[u_{xx} - u_{yy} + i(u_{xy} + u_{yx})], \tag{B3}$$

$$u_{2,1} = \left(-\tfrac{1}{2}\right)[u_{xz} + u_{zx} + i(u_{yz} + u_{zy})], \tag{B4}$$

$$u_{2,0} = \sqrt{\frac{1}{6}}(2u_{zz} - u_{xx} - u_{yy}), \tag{B5}$$

$$u_{2,-1} = \tfrac{1}{2}[u_{xz} + u_{zx} - i(u_{yz} + u_{zy})], \tag{B6}$$

$$u_{2,-2} = \tfrac{1}{2}[u_{xx} - u_{yy} - i(u_{xy} + u_{yx})]. \tag{B7}$$

Similarly, the stress tensor component with the angular momentum 2 can also be written as

$$T_{2,2} = \tfrac{1}{2}[T_{xx} - T_{yy} + i(T_{xy} + T_{yx})] \tag{B8}$$

$$T_{2,1} = \left(-\tfrac{1}{2}\right)[T_{xz} + T_{zx} + i(T_{yz} + T_{zy})] \tag{B9}$$

$$T_{2,0} = \sqrt{\frac{1}{6}}(2T_{zz} - T_{xx} - T_{yy}) \tag{B10}$$

$$T_{2,-1} = \tfrac{1}{2}[T_{xz} + T_{zx} - i(T_{yz} + T_{zy})] \tag{B11}$$

$$T_{2,-2} = \tfrac{1}{2}[T_{xx} - T_{yy} - i(T_{xy} + T_{yx})]. \tag{B12}$$

The term $\sum_m u_{2,m}T_{2,-m}$ is invariant under the rotation. Collecting all the invariant terms, we obtain the form of the Hamiltonian

$$H_{\text{eff}} = C_1 u_{0,0} + C_3 k^2 + (C_2 + \tilde{g}_0 u_{0,0})(\mathbf{k}\cdot\sigma) + \tilde{g}_1\sum_{m=\pm2,\pm1,0} u_{2,m}T_{2,-m}. \tag{B13}$$

One can easily check that this Hamiltonian is the same as Eq. (B1).

When reducing the symmetry group from $SO(3)$ to chiral point group symmetry, the number of independent parameters will increase. Here we take the O group as an example and other chiral symmetry groups can

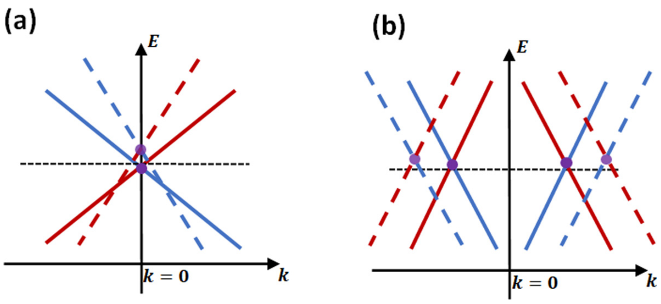

FIG. 4. (a) The Weyl point is pinned at $\mathbf{k}=0$ (or other TR-invariant momenta) by TR symmetry for the Kramers-Weyl semimetals and the strain can only shift Weyl nodes in energy but *not* its momentum ($\mathbf{k}=0$). (b) The Weyl points for a generic inversion-breaking TR-preserving Weyl semimetals, in which strain can also induce the momentum shift of Weyl nodes. Here the dashed lines show the dispersion after adding the strain and the purple spots depict the location of Weyl nodes.

be worked out in a similar manner. According to the character table of O group [95], there are five irreducible representations (IRRs), labeled by A_1, A_2, E, T_1, and T_2. The Hamiltonian should be invariant and thus belongs to the A_1 IRR. Both $\mathbf{k}$ and σ belong to the T_1 and since $T_1 \otimes T_1 = A_1 + E + T_1 + T_2$, we have one term $\mathbf{k}\cdot\sigma$ can be constructed to be invariant. For the strain tensor, $u_{0,0} = \sum_i u_{ii}$ belongs to the A_1 IRR, $(u_{xx}-u_{yy}, 2u_{zz}-u_{xx}-u_{yy})$ belongs to the E IRR, and (u_{xy}, u_{yz}, u_{zx}) belongs to the T_2 IRR. This allows us to construct the following invariant terms: (1) $u_{0,0}\mathbf{k}\cdot\sigma$ and $u_{0,0}k^2$; (2) $(2u_{zz}-u_{xx}-u_yy)k_z\sigma_z + (2u_{xx}-u_{yy}-u_{zz})k_x\sigma_x + (2u_{yy}-u_{xx}-u_{zz})k_y\sigma_y$; (3) $u_{xy}(k_x\sigma_y + k_y\sigma_x) + u_{yz}(k_y\sigma_z + k_z\sigma_y) + u_{zx}(k_z\sigma_x + k_x\sigma_z)$. Therefore, the first four terms in the Hamiltonian (B1) remain the same while the last term is changed to

$$g_1[(2u_{zz}-u_{xx}-u_yy)k_z\sigma_z + (2u_{xx}-u_{yy}-u_{zz})k_x\sigma_x + (2u_{yy}-u_{xx}-u_{zz})k_y\sigma_y] + \tag{B14}$$

$$g_2[u_{xy}(k_x\sigma_y + k_y\sigma_x) + u_{yz}(k_y\sigma_z + k_z\sigma_y) + u_{zx}(k_z\sigma_x + k_x\sigma_z)], \tag{B15}$$

which possesses two independent parameters g_1 and g_2.

Based on the above derived effective Hamiltonian, we below discuss the major difference between the Kramers-Weyl semimetals and conventional TR-preserving inversion-breaking Weyl semimetals. Figure 4(a) schematically shows the Weyl nodes location for the Kramers-Weyl semimetals without and with the strain (shown by the solid and dashed lines, separately). Due to TR symmetry, the Kramers-Weyl nodes are always pinned at $k=0$, and this general property remains in the presence of strain that respects TR. This fact can also be understood from the effective model [Eqs. (1) and (2) in the main text]. For the Weyl nodes $\mathbf{k}\cdot\sigma$, TR $\hat{T} = i\sigma_y K$ directly acts on spin Pauli matrix σ and the strain *cannot* induce any constant (k-independent) coupling to σ since σ is TR-odd while any function of strain tensor must be TR-even. Therefore, the strain tensor *cannot* contributes to the vector potential part $\vec{A}$ of the pseudogauge field while it can only contribute to the scalar potential A_0. Without the vector potential $\vec{A}$, the scalar part A_0 of strain-induced pseudogauge field itself cannot give rise to the axial anomaly. This situation is changed for generic TR-preserving inversion-breaking Weyl semimetals, as shown in Fig. 4(b). The TR symmetry now relates one Weyl point at the momentum $\mathbf{k}$ to another Weyl point at the momentum $-\mathbf{k}$, so the strain can play the role of both vector potential $\vec{A}$ (shift in the momentum space) and the scalar potential A_0 (shift in energy) for the pseudogauge field, as shown in Fig. 4(b). In this situation, the pseudogauge field due to strain itself can in principle give rise to the contribution to the axial anomaly equation through the form of $\epsilon^{\mu\nu\rho\lambda}\partial_\mu A_\nu \partial_\rho A_\lambda$, where A_ν and A_λ both can come from pseudogauge fields. Therefore, there is a substantial difference for the strain effect between the Kramers-Weyl semimetals and generic inversion-breaking Weyl semimetals: for Kramers-Weyl semimetals, the strain can only induce A_0 and the torsion field (higher-order-k correction), and as discussed below, this fact means that only the NY anomaly is possible, while for generic inversion-breaking Weyl semimetals, the strain can both $\vec{A}$ and A_0, as well as the more general gravitational field, and thus can lead to all different types of anomaly, including axial anomaly, mixed axial-gravitational anomaly and NY anomaly. It should be noted that the axial anomaly due to the strain-induced pseudogauge field can also affect the thermal transport in Weyl semimetals. For example, it is shown that axial anomaly induced by pseudogauge fields in strained Weyl semimetals can give rise to a directional asymmetry of the heat transfer and thus strongly modify thermal conductivity in this system [96]. The pseudomagnetic magnetic fields due to torsional mechanical strain can also influence thermoelectric transport in Weyl semimetals under additional magnetic fields [97].

APPENDIX C: EFFECTIVE ACTION AND CORRELATION FUNCTION

In this Appendix, we will show the derivation of the formalism for the effective action for the phonon dynamics and the corresponding correlation functions.

From the Hamiltonian (A31), the effective action can be written as

$$\begin{aligned} S_{\text{eff}} &= \int_0^\beta d\tau \int d^3r \left[\hat{\psi}^\dagger_{\Gamma_i}\frac{\partial}{\partial\tau}\hat{\psi}_{\Gamma_i} + \mathcal{H}_{\text{eff}}\right] \\ &= \int_0^\beta d\tau \int d^3r \left[\hat{\psi}^\dagger_{\Gamma_i}\left(\frac{\partial}{\partial\tau} - \mu - A_0(\mathbf{r}) - \frac{\hbar^2}{2m_0^*}\nabla^2\right)\hat{\psi}_{\Gamma_i}\right. \\ &\quad \left. + \frac{\hbar v_f}{2}\sum_{a,j}\left(\hat{\psi}^\dagger_{\Gamma_i} e^j_a \sigma^a(-i\partial_j\hat{\psi}_{\Gamma_i}) + \left(i\partial_j\hat{\psi}^\dagger_{\Gamma_i}\right)e^j_a\sigma^a\hat{\psi}_{\Gamma_i}\right)\right], \end{aligned} \tag{C1}$$

where we have used the imaginary time $\tau = it$ and $e^j_a = \delta^j_a + \Delta^j_a$.

To implement the perturbation calculations, we may separate the full action into two parts $S_{\text{eff}} = S_0 + S_1$, where

$$S_0 = \int_0^\beta d\tau \int d^3r \hat{\psi}^\dagger_{\Gamma_i}\left(\frac{\partial}{\partial\tau} - \mu + \hbar v_f(-i\nabla\cdot\sigma)\right)\hat{\psi}_{\Gamma_i} \tag{C2}$$

and

$$S_1=\int_0^\beta d\tau\int d^3r\left[-\hat{\psi}^\dagger_{\Gamma_i}A_0(\mathbf{r})\hat{\psi}_{\Gamma_i}+\frac{\hbar v_f}{2}\sum_{a,j}\left(\hat{\psi}^\dagger_{\Gamma_i}\Delta^j_a\sigma^a(-i\partial_j\hat{\psi}_{\Gamma_i})+\left(i\partial_j\hat{\psi}^\dagger_{\Gamma_i}\right)\Delta^j_a\sigma^a\hat{\psi}_{\Gamma_i}\right)\right]. \tag{C3}$$

Now let us define the Fourier transform as

$$\hat{\psi}_{\Gamma_i}(\mathbf{r},\tau)=\frac{1}{\beta V}\sum_{i\omega_n,\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{r}-i\omega_n\tau}\hat{\psi}_{\Gamma_i}(\mathbf{k},i\omega_n), \tag{C4}$$

$$A_0(\mathbf{r},\tau)=\frac{1}{\beta V}\sum_{i\nu_m,\mathbf{q}}e^{i\mathbf{q}\cdot\mathbf{r}-i\nu_m\tau}A_0(\mathbf{q},i\nu_m), \tag{C5}$$

$$\Delta^j_a(\mathbf{r},\tau)=\frac{1}{\beta V}\sum_{i\nu_m,\mathbf{q}}e^{i\mathbf{q}\cdot\mathbf{r}-i\nu_m\tau}\Delta^j_a(\mathbf{q},i\nu_m). \tag{C6}$$

Then, we have

$$S_0=\frac{1}{\beta V}\sum_{i\omega_n,\mathbf{k}}\hat{\psi}^\dagger(\mathbf{k},i\omega_n)[-i\omega_n-\mu+\hbar v_f(\mathbf{k}\cdot\sigma)]\hat{\psi}(\mathbf{k},i\omega_n) \tag{C7}$$

and

$$S_1=\frac{1}{(\beta V)^2}\sum_{i\omega_n,i\nu_m,\mathbf{k},\mathbf{q}}\Bigg[-\hat{\psi}^\dagger_{\Gamma_i}(\mathbf{k},i\omega_n)A_0(\mathbf{q},i\nu_m)\hat{\psi}_{\Gamma_i}(\mathbf{k}-\mathbf{q},i\omega_n-i\nu_m)$$
$$+\frac{\hbar v_f}{2}\sum_{a,j}\hat{\psi}^\dagger_{\Gamma_i}(\mathbf{k},i\omega_n)\Delta^j_a(\mathbf{q},i\nu_m)\sigma^a(2k_j-q_j)\hat{\psi}_{\Gamma_i}(\mathbf{k}-\mathbf{q},i\omega_n-i\nu_m)\Bigg]. \tag{C8}$$

Here $i\omega_n=i\frac{(2n+1)\pi}{\beta}$ and $i\nu_m=i\frac{2m\pi}{\beta}$. We have dropped the quadratic term $\frac{\hbar^2k^2}{2m_0^*}$ in S_0 and instead, we limit the momentum summation within the range set by the cutoff $\Lambda=\frac{2m_0^*v_f}{\hbar}$, where the quadratic term $\frac{\hbar^2\Lambda^2}{2m_0^*}$ is at the same order as the linear term $\hbar v_f\Lambda$.

Below, we introduce the notation $\tilde{k}=(i\omega_n,\mathbf{k})$ for short so that the summation over both the frequency and the momentum can be simplified as $\sum_{\tilde{k}}=\frac{1}{\beta V}\sum_{i\omega_n,\mathbf{k}}$. One should keep in mind the value of $i\omega_n$ is for boson or for fermion operators. The full action can be written as

$$S=S_0+S_1=\sum_{\tilde{k}}\hat{\psi}^\dagger_{\tilde{k}}\left(-\mathcal{G}_0^{-1}\right)\hat{\psi}_{\tilde{k}}+\sum_{\tilde{k},\tilde{k}'}\hat{\psi}^\dagger_{\tilde{k}}\mathfrak{X}(\tilde{k},\tilde{k}')\hat{\psi}_{\tilde{k}'}, \tag{C9}$$

where

$$\mathcal{G}_0=[i\omega_n+\mu-\hbar v_f(\mathbf{k}\cdot\sigma)]^{-1} \tag{C10}$$

and

$$\mathfrak{X}(\tilde{k},\tilde{k}')=-A_0(\tilde{q}=\tilde{k}-\tilde{k}')+\sum_{a,i}\mathcal{T}^a_i(\tilde{k},\tilde{k}')\Delta^i_a(\tilde{q}=\tilde{k}-\tilde{k}'), \tag{C11}$$

with the stress tensor operator $\mathcal{T}^a_i(\tilde{k},\tilde{k}')=\frac{\hbar v_f}{2}(k_i+k'_i)\sigma^a$ for Weyl fermions.

Since the full action is quadratic in fermion operators, one can directly integrate out the Weyl fermions to get the effective action for the A_0 and Δ fields. Let us consider the partition function

$$Z=\int\mathcal{D}(\hat{\psi}^\dagger\hat{\psi})\exp(-S)=\mathrm{Det}(-\mathcal{G}_0^{-1}+\mathfrak{X}) \tag{C12}$$

and the zero-order partition function

$$Z_0=\int\mathcal{D}(\hat{\psi}^\dagger\hat{\psi})\exp(-S_0)=\mathrm{Det}\left(-\mathcal{G}_0^{-1}\right), \tag{C13}$$

where Det is the determinant. The effective action $W[A_0,\Delta]$ is defined as $Z=Z_0e^{-W}$ and thus

$$\begin{aligned}W[A_0,\Delta]&=-\ln[\mathrm{Det}(-\mathcal{G}_0^{-1}+\mathfrak{X})]+\ln\left[\mathrm{Det}\left(-\mathcal{G}_0^{-1}\right)\right]\\&=-\mathrm{Tr}\left[\ln\left(-\mathcal{G}_0^{-1}+\mathfrak{X}\right)-\ln\left(-\mathcal{G}_0^{-1}\right)\right]\\&=-\mathrm{Tr}[\ln(1-\mathfrak{X}\mathcal{G}_0)]=\sum_n\frac{1}{n}\mathrm{Tr}[(\mathfrak{X}\mathcal{G}_0)^n],\end{aligned} \tag{C14}$$

which gives us the perturbation expansion. The first-order term is given by $\mathrm{Tr}(\mathfrak{X}\mathcal{G}_0)=\sum_{\tilde{k}}\mathfrak{X}(\tilde{k},\tilde{k})\mathcal{G}_0(\tilde{k})$. This involve only the terms $A_0(\tilde{q}=0)$ and $\Delta^i_a(\tilde{q}=0)$ in $\mathfrak{X}$. Since we only consider the fluctuation for $\mathfrak{X}$ and thus can choose $A_0(\tilde{q}=0)=0$ and $\Delta^i_a(\tilde{q}=0)=0$ when defining S_1. Thus, the first-order term will be zero.

Next we consider the second-order term, given by

$$W[A_0,\Delta]=\frac{1}{2}\mathrm{Tr}(\mathfrak{X}\mathcal{G}_0\mathfrak{X}\mathcal{G}_0)=\frac{1}{2}\sum_{\tilde{k}_1,\tilde{k}_2}\mathrm{Tr}_\sigma[\mathfrak{X}(\tilde{k}_1,\tilde{k}_2)\mathcal{G}_0(\tilde{k}_2)\mathfrak{X}(\tilde{k}_2,\tilde{k}_1)\mathcal{G}_0(\tilde{k}_1)]$$
$$=\frac{1}{2}\sum_{\tilde{q}}\left[A_0(\tilde{q})A_0(-\tilde{q})\Pi_0(\tilde{q})+\sum_{ij,ab}\Delta_a^i(\tilde{q})\Delta_b^j(-\tilde{q})\Phi_{ij}^{ab}(\tilde{q})-\sum_{i,a}A_0(\tilde{q})\Delta_a^i(-\tilde{q})\Theta_i^a(\tilde{q})-\sum_{i,a}\Delta_a^i(\tilde{q})A_0(-\tilde{q})\Theta_i^a(-\tilde{q})\right], \quad \text{(C15)}$$

where three types of correlation functions are defined as

$$\Pi_0(\tilde{q})=\sum_{\tilde{k}}\mathrm{Tr}_\sigma[\mathcal{G}_0(\tilde{k}-\tilde{q})\mathcal{G}_0(\tilde{k})], \quad \text{(C16)}$$

$$\Phi_{ij}^{ab}(\tilde{q})=\sum_{\tilde{k}}\mathrm{Tr}_\sigma\left[\mathcal{T}_i^a(\tilde{k},\tilde{k}-\tilde{q})\mathcal{G}_0(\tilde{k}-\tilde{q})\mathcal{T}_j^b(\tilde{k}-\tilde{q},\tilde{k})\mathcal{G}_0(\tilde{k})\right], \quad \text{(C17)}$$

$$\Theta_i^a(\tilde{q})=\sum_{\tilde{k}}\mathrm{Tr}_\sigma[\mathcal{G}_0(\tilde{k}-\tilde{q})\mathcal{T}_i^a(\tilde{k}-\tilde{q},\tilde{k})\mathcal{G}_0(\tilde{k})]. \quad \text{(C18)}$$

Here Π_0 is the density-density correlation function, Φ_{ij}^{ab} is the stress-stress correlation function, while Θ_i^a is the stress-density correlation function.

APPENDIX D: A BRIEF REVIEW OF NIEH-YAN ANOMALY

In this Appendix, we will first review the Weyl/Dirac fermions in the curved space and the Nieh-Yan anomaly. Then we will discuss the connection of the stress-stress correlation function Φ_{ij}^{ab} and Nieh-Yan anomaly in our formalism. In a curved space, the action for Dirac fermions can be written as [36,37]

$$S_D=\int d^4x\bar{\psi}\left(\frac{i}{2}\gamma^a\{e_a^\mu,\nabla_\mu\}-m\right)\psi, \quad \text{(D1)}$$

where γ^as are the standard γ-matrices with $\{\gamma^a,\gamma^b\}=2\eta^{ab}$ and η^{ab} is the metric in the Minkowski space, and e_a^μ is the frame field. Here the Einstein summation rule has been assumed. The frame fields satisfy $\eta^{ab}e_a^\mu e_b^\nu=g^{\mu\nu}$ with the metric $g^{\mu\nu}$ in the curved space, so that $\{\gamma^\mu,\gamma^\nu\}=2g^{\mu\nu}$ where $\gamma^\mu=\gamma^a e_a^\mu$. $\nabla_\mu=\partial_\mu+\Gamma_\mu$ is the covariant derivative with the spin connection $\Gamma_\mu=\frac{i}{2}\eta_{ac}\Gamma^c_{b\mu}J^{ab}$ and the generator $J^{ab}=-\frac{i}{4}[\gamma^a,\gamma^b]$ of the Lorentz group.

It is convenient to introduce the so-called spin connection 1-form $\mathbf{\Gamma}_b^c=\Gamma^c_{b\mu}d\mathbf{x}^\mu$ and the coframe field 1-form $\mathbf{e}^a=\bar{e}^a_\mu d\mathbf{x}^\mu$, where the coframe field is the inverse of the frame field, $e_a^\nu\bar{e}^a_\mu=\delta^\nu_\mu$. The Cartan's structure equations of the Riemann-Cartan spacetime [98] define the curvature 2-form $\mathbf{R}_b^a=d\mathbf{\Gamma}_b^a+\mathbf{\Gamma}_c^a\wedge\mathbf{\Gamma}_b^c$ and the torsion 2-form $\mathbf{T}^a=d\mathbf{e}^a+\mathbf{\Gamma}_c^a\wedge\mathbf{e}^c$. In the standard general relativity, the torsion is assumed to be zero (zero torsion constraint), leading to the relation between the coframe field and the spin connection ($d\mathbf{e}^a+\mathbf{\Gamma}_c^a\wedge\mathbf{e}^c=0$). However, in the Einstein-Cartan theory [98,99], the torsion can be nonzero and consequently, the frame field (or coframe field) and the spin connection should be treated as two independent fields.

By comparing the effective action (A31) for our system with the action of Dirac field in the curved space, it is clear that the electron-phonon coupling can create a nontrivial coframe field $\mathbf{e}^a$ but the spin connection $\mathbf{\Gamma}_b^c$ is still zero. Consequently, the Weyl fermions in our system will feel nonzero torsion, given by $\mathbf{T}^a=d\mathbf{e}^a$, but zero curvature $\mathbf{R}_b^a=0$. This is in sharp contrast to the standard general relativity with nonzero curvature but zero torsion, and is known as the "Weitzenböck spacetime," which was studied in the "teleparallel gravity theory" [62].

When the Dirac fermion is coupled to an electromagnetic field, quantum correction can give rise to the anomalous nonconservation of the chiral charge, known as the "chiral anomaly" [1], given by

$$\partial_\mu\langle j^{5\mu}\rangle=\frac{1}{16\pi^2}\epsilon^{\mu\nu\lambda\rho}F_{\mu\nu}F_{\lambda\rho}, \quad \text{(D2)}$$

where $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$ is the strength of the gauge field. The chiral anomaly effect plays an essential role in predicting and understanding a number of physical phenomena in Weyl semimetals. In our system, the phonon will induce the torsion field for the KW fermions, instead of the pseudogauge field. It turns out that the torsion field can also contribute to the nonconservation of the chiral charge, which is known as the Nieh-Yan anomaly [10–13] and given by

$$\partial_\mu\langle j^{5\mu}\rangle=\frac{\mathcal{F}}{4}\eta_{ab}\epsilon^{\mu\nu\lambda\rho}T^a_{\mu\nu}T^b_{\lambda\rho}, \quad \text{(D3)}$$

where $\mathcal{F}$ is some nonuniversal coefficient. Since the spin connection vanishes, the torsion field $\mathbf{T}^a=d\mathbf{e}^a$ gives the field strength of the coframe field $\mathbf{e}^a$ and its components are given by $T^a_{\mu\nu}=\partial_\mu\bar{e}^a_\nu-\partial_\nu\bar{e}^a_\mu$ and the Nieh-Yan anomaly equation is also written as

$$\partial_\mu\langle j^{5\mu}\rangle=\mathcal{F}\eta_{ab}\epsilon^{\mu\nu\lambda\rho}\partial_\mu\bar{e}^a_\nu\partial_\lambda\bar{e}^b_\rho. \quad \text{(D4)}$$

Unlike the chiral anomaly with the universal dimensionless coefficient $\frac{1}{16\pi^2}$ (Here we follow the convention of high energy physics and set $e=\hbar=1$), the coefficient $\mathcal{F}$ in the Nieh-Yan anomaly has the dimension $[1/L]^2$, where $[L]$ labels the dimension of length. This can be easily obtained from the dimension counting as follows. In the Dirac action, the fermion field has the dimension $[1/L]^{3/2}$ and the chiral current $\langle j^{5\mu}\rangle=\langle\bar{\psi}\gamma^5\psi\rangle$ has the dimension $[1/L]^3$. Thus, the left-hand side of the anomaly equation ($\partial_\mu\langle j^{5\mu}\rangle$) has the dimension $[1/L]^4$. However, the coframe field $\bar{e}^a_\nu$ is dimensionless and thus $\partial_\mu\bar{e}^a_\nu\partial_\lambda\bar{e}^b_\rho$ has the dimension $[1/L]^2$. This means that the coefficient $\mathcal{F}$ has the dimension $[1/L]^2$. It was shown in literature that $\mathcal{F}$ is proportional to the ultraviolet (UV) momentum cutoff Λ^2 [12,12,45]. Due to the nonuniversal nature of this coefficient, the relevance of Nieh-Yan (NY) term to the anomaly for relativistic fermions has been debated for long in high energy physics. Chandia and Zanelli first proposed that NY term can appear in anomaly equation [12]. Later, Kreimer and Mielke suggested that NY term should be cancelled by the counter term in the renormalization group argument [100]. A series of subsequent papers were devoted to clarifying this issue [101–106]. While the status of the NY anomaly in relativistic systems remains controversial, the nonrelativistic condensed matter systems have an explicit UV cutoff and thus its appearance in condensed matter physics is plausible. A number of recent papers in the field of Weyl/Dirac semimetals and topological insulators discuss this possibility [43–51]. It was suggested by Nissinen that the presence or absence of NY anomaly in condensed matter systems relies on how the UV cutoff is chosen [47,49]. Particularly, it was shown that the momentum separation between two Weyl nodes can give rise to an anisotropic cutoff along different momentum directions, which determines the magnitude of the anomaly. However, for relativistic Weyl fermions with Lorentz invariance up to arbitrary scales, the anomaly will be cancelled. Besides the temperature independent term, it was recently proposed [45,46,48] that at a finite temperature, the coefficient takes the form $\mathcal{F} = F_0 + F_1(k_BT)^2$ with $F_0 = a_0\Lambda^2$ and $F_1 = -\frac{1}{12}$. The temperature-dependent term is proportional to $(k_BT)^2$, which absorbs the dimension $[1/L]^2$, and thus, the coefficient F_1 becomes dimensionless and universal in the sense that it is only proportional to the central charge of $(1+1)$-dimensional Dirac fermion, which was proposed in Ref. [45]. Thus, the anomaly induced by this temperature-dependent term is dubbed "thermal Nieh-Yan anomaly."

To compare with our results, it is convenient to derive the effective action that corresponds to the Nieh-Yan anomaly. Since the right-hand side of Eq. (D4) is actually a total derivative, the chiral current can be given by $\langle j^{5\mu}\rangle = \mathcal{F}\eta_{ab}\epsilon^{\mu\nu\lambda\rho}\bar{e}^a_\nu\partial_\lambda\bar{e}^b_\rho$. With $\langle j^{5\mu}\rangle = \frac{\delta S_{\text{eff}}}{\delta A_{5\mu}}$, we have

$$S_{\text{NY}} = \mathcal{F}\int d^4x\eta_{ab}\epsilon^{\mu\nu\lambda\rho}A_{5\mu}\bar{e}^a_\nu\partial_\lambda\bar{e}^b_\rho, \tag{D5}$$

where $A_{5\mu}$ is the chiral gauge potential. Since the phonon can only induce chiral chemical potential in our case, we only keep the A_{50} term in the above action and thus obtain

$$S_{\text{NY}} = \mathcal{F}\int d^4x\eta_{ab}\epsilon^{\nu\lambda\rho}A_{50}\bar{e}^a_\nu\partial_\lambda\bar{e}^b_\rho, \tag{D6}$$

where $\nu, \lambda, \rho = x, y, z$.

In our effective action [Eq. (3)] of the main text, the frame field is given by $e^j_a = \delta^j_a + \Delta^j_a$ and the corresponding coframe field is $\bar{e}^a_j = \delta^a_j - \Delta^j_a$ with $a, j = x, y, z$. Since we only concern the spatial coordinate here, the upper and lower indices do not have much meaning. Furthermore, Δ^j_a is proportional to the strain tensor u_{ja} which is symmetric with respect to j and a. We next focus on the stress-stress correlation function term (C17) in the effective action (C15). We may expand $\Phi^{ab}_{ij}(\tilde{q})$ as a function of $\tilde{q}$, $\Phi^{ab}_{ij}(\tilde{q}) = \Phi^{ab}_{ij}(0) + (\partial_{\mathbf{q}}\Phi^{ab}_{ij})_{\tilde{q}=0}\cdot\mathbf{q} + (\partial_{\omega_n}\Phi^{ab}_{ij})_{\tilde{q}=0}\omega_n + \cdots$. Here we only focus on the term that is linearly proportional to $\mathbf{q}$ and denote $\Phi^{ab}_{ij,l} = (\partial_{q_l}\Phi^{ab}_{ij})_{\tilde{q}=0}$. Then, the corresponding term in the action is given by

$$W_{\text{NY}}[\Delta] = \frac{1}{2}\sum_{\tilde{q},ijl,ab}\Phi^{ab}_{ij,l}\big[\Delta^i_a(\tilde{q})\Delta^j_b(-\tilde{q})q_l\big],$$
$$= \frac{1}{2}\sum_{ijl,ab}\int d^3rd\tau\Phi^{ab}_{ij,l}\Delta^i_a(\mathbf{r},\tau)\big[i\partial_l\Delta^j_b(\mathbf{r},\tau)\big]. \tag{D7}$$

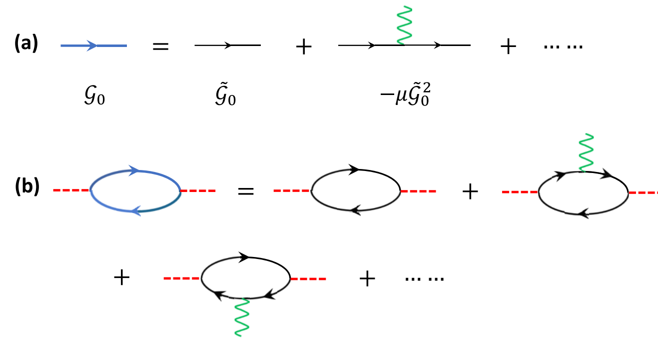

FIG. 5. (a) Feynman Diagram for the expansion of the Green's function $\mathcal{G}_0$ up to the linear order in μ. (b) Feynman diagram for the expansion of the stress-stress correlation function Φ^{ab}_{ij}. Here the blue lines are for the Green's function $\mathcal{G}_0$, the black lines are for the Green's function $\tilde{\mathcal{G}}_0$ at $\mu = 0$, the green wiggly line is for the vertex coupled to μ and the red dashed lines are for the stress tensor $\mathcal{T}^a_i$.

As demonstrated in Appendix E, the coefficient $\Phi^{ab}_{ij,l}$ is purely imaginary and proportional to $\delta_{ij}\epsilon^{abl}$, where ϵ is the Levi-civita symbol. Moreover, this term is linearly proportional to the chemical potential μ for a small μ. Therefore, by replacing $\bar{e}^a_\nu$ with Δ^i_a, W_{NY} takes the exact form as S_{NY} for the Nieh-Yan anomaly, and we call this term as the Nieh-Yan term below. Actually, the symmetry analysis of the possible terms for the strain tensors, given in the Appendix G, suggests that the Nieh-Yan term is the only term that is allowed by symmetry at this order of $\mathbf{q}$ in a uniform isotropic system.

It is well-known that the quantum anomaly normally comes from the triangle diagram in the perturbation expansion. Below we will show that by expanding the stress-stress correlation function Φ^{ab}_{ij} up to the linear order in μ, we actually evaluate the triangle diagram and thus, our method here is equivalent to the evaluation of the triangle diagram. Since the μ-dependence of the stress-stress correlation function Φ^{ab}_{ij} in Eq. (C17) comes from the Green's function $\mathcal{G}_0$, we first expand $\mathcal{G}_0$ in Eq. (C10) as

$$\mathcal{G}_0 = \tilde{\mathcal{G}}_0 + \frac{\partial\mathcal{G}_0}{\partial\mu}\bigg|_{\mu=0}\mu + \ldots, \tag{D8}$$

where

$$\tilde{\mathcal{G}}_0 = \mathcal{G}_0(\mu = 0) = [i\omega_n - \hbar v_f(\mathbf{k}\cdot\sigma)]^{-1} \tag{D9}$$

and

$$\frac{\partial\mathcal{G}_0}{\partial\mu}\bigg|_{\mu=0} = -\tilde{\mathcal{G}}_0^2. \tag{D10}$$

This expansion of the Green's function is shown in Fig. 5(a). Next we consider Φ_{ij}^{ab}, which can be expanded as

$$\begin{aligned}\Phi_{ij}^{ab}(\tilde{q}) &= \sum_{\tilde{k}} \mathrm{Tr}_\sigma\big(\mathcal{T}_i^a(\tilde{k},\tilde{k}-\tilde{q})\mathcal{G}_0(\tilde{k}-\tilde{q})\mathcal{T}_j^b(\tilde{k}-\tilde{q},\tilde{k})\mathcal{G}_0(\tilde{k})\big)\\ &\approx \sum_{\tilde{k}} \mathrm{Tr}_\sigma\big(\mathcal{T}_i^a(\tilde{k},\tilde{k}-\tilde{q})(\tilde{\mathcal{G}}_0(\tilde{k}-\tilde{q})-\mu\tilde{\mathcal{G}}_0^2(\tilde{k}-\tilde{q}))\mathcal{T}_j^b(\tilde{k}-\tilde{q},\tilde{k})(\tilde{\mathcal{G}}_0(\tilde{k})-\mu\tilde{\mathcal{G}}_0^2(\tilde{k}))\big)\\ &\approx \tilde{\Phi}_{ij}^{ab}(\tilde{q})-\mu\sum_{\tilde{k}}\big[\mathrm{Tr}_\sigma\big(\mathcal{T}_i^a(\tilde{k},\tilde{k}-\tilde{q})\tilde{\mathcal{G}}_0(\tilde{k}-\tilde{q})\mathcal{T}_j^b(\tilde{k}-\tilde{q},\tilde{k})\tilde{\mathcal{G}}_0^2(\tilde{k})\big)+\mathrm{Tr}_\sigma\big(\mathcal{T}_i^a(\tilde{k},\tilde{k}-\tilde{q})\tilde{\mathcal{G}}_0^2(\tilde{k}-\tilde{q})\mathcal{T}_j^b(\tilde{k}-\tilde{q},\tilde{k})\tilde{\mathcal{G}}_0(\tilde{k})\big)\big].\end{aligned} \tag{D11}$$

Here $\tilde{\Phi}_{ij}^{ab}(\tilde{q}) = \sum_{\tilde{k}} \mathrm{Tr}_\sigma(\mathcal{T}_i^a(\tilde{k},\tilde{k}-\tilde{q})\tilde{\mathcal{G}}_0(\tilde{k}-\tilde{q})\mathcal{T}_j^b(\tilde{k}-\tilde{q},\tilde{k})\tilde{\mathcal{G}}_0(\tilde{k}))$ is the stress-stress correlation function at $\mu = 0$ and the next two terms in the last step is linear in μ. From the diagram in Fig. 5(b), it is clear that these two terms are for the triangle diagrams that response for the quantum anomaly.

APPENDIX E: ANALYTICAL EVALUATION OF THE NIEH-YAN TERM

In this Appendix, we will evaluate the coefficient $\Phi_{ij,l}^{ab}$ analytically. With Eq. (C17), we have

$$\begin{aligned}\Phi_{ij,l}^{ab} = \left(\frac{\partial\Phi_{ij}^{ab}}{\partial q_l}\right)_{\tilde{q}=0} &= \sum_{\tilde{k}} \mathrm{Tr}_\sigma\Big(\big(\partial_{q_l}\mathcal{T}_i^a(\tilde{k},\tilde{k}-\tilde{q})\big)_{\tilde{q}=0}\mathcal{G}_0(\tilde{k})\mathcal{T}_j^b(\tilde{k},\tilde{k})\mathcal{G}_0(\tilde{k})+\mathcal{T}_i^a(\tilde{k},\tilde{k})(\partial_{q_l}\mathcal{G}_0(\tilde{k}-\tilde{q}))_{\tilde{q}=0}\mathcal{T}_j^b(\tilde{k},\tilde{k})\mathcal{G}_0(\tilde{k})\\ &\quad+\mathcal{T}_i^a(\tilde{k},\tilde{k})\mathcal{G}_0(\tilde{k})\big(\partial_{q_l}\mathcal{T}_j^b(\tilde{k}-\tilde{q},\tilde{k})\big)_{\tilde{q}=0}\mathcal{G}_0(\tilde{k})\Big)\\ &= \sum_{\tilde{k}} \mathrm{Tr}_\sigma\bigg(\Big(-\frac{\hbar v_f}{2}\Big)(\delta_{il}\sigma^a\mathcal{G}_0(\tilde{k})\mathcal{T}_j^b(\tilde{k},\tilde{k})\mathcal{G}_0(\tilde{k})+\mathcal{T}_i^a(\tilde{k},\tilde{k})\mathcal{G}_0(\tilde{k})\delta_{jl}\sigma^b\mathcal{G}_0(\tilde{k}))\\ &\quad+\mathcal{T}_i^a(\tilde{k},\tilde{k})(\partial_{k_l}\mathcal{G}_0(\tilde{k}))\mathcal{T}_j^b(\tilde{k},\tilde{k})\mathcal{G}_0(\tilde{k})\bigg).\end{aligned} \tag{E1}$$

The Green's function $\mathcal{G}_0$ is given in Eq. (C10) and here we focus on the situation with a small chemical potential μ, so we expand $\mathcal{G}_0$ up to the linear order in μ as $\mathcal{G}_0(\mu) = \mathcal{G}_0(\mu=0) + (\partial_\mu\mathcal{G}_0)_{\mu=0}\mu$ with

$$\mathcal{G}_0(\mu=0) = (i\omega_n - \hbar v_f(\mathbf{k}\cdot\sigma))^{-1} = \frac{i\omega_n + \hbar v_f(\mathbf{k}\cdot\sigma)}{(i\omega_n)^2 - (\hbar v_f k)^2} \tag{E2}$$

and

$$(\partial_\mu\mathcal{G}_0)_{\mu=0} = -[i\omega_n - \hbar v_f(\mathbf{k}\cdot\sigma)]^{-2} = -[\mathcal{G}_0(\mu=0)]^2. \tag{E3}$$

Below we absorb $\hbar v_f$ into the definition of the momentum $\mathbf{k}$, but keep track on the chirality of the Weyl fermions. Thus, we define the chirality as $\chi = \mathrm{sign}(\hbar v_f)$, and obtain

$$\mathcal{G}_0(\mu=0) = \frac{i\omega_n + \chi(\mathbf{k}\cdot\sigma)}{D^2}, \tag{E4}$$

where $D^2 = (i\omega_n)^2 - k^2$ and

$$(\partial_\mu\mathcal{G}_0)_{\mu=0} = -[\mathcal{G}_0(\mu=0)]^2 = -\frac{[i\omega_n + \chi(\mathbf{k}\cdot\sigma)]^2}{D^4} = -\frac{(i\omega_n)^2 + k^2 + 2i\omega_n\chi\mathbf{k}\cdot\sigma}{D^4}. \tag{E5}$$

Thus, the Green's function is given by

$$\mathcal{G}_0(\mathbf{k}, i\omega_n) = \frac{i\omega_n + \chi(\mathbf{k}\cdot\sigma)}{D^2} - \frac{[(i\omega_n)^2 + k^2 + 2i\omega_n\chi\mathbf{k}\cdot\sigma]\mu}{D^4}. \tag{E6}$$

Let us first look at the first term in Eq. (E1) and the direct calculation gives

$$\begin{aligned}\sum_{\tilde{k}} \mathrm{Tr}_\sigma(\delta_{il}\sigma^a\mathcal{G}_0(\tilde{k})\mathcal{T}_j^b(\tilde{k},\tilde{k})\mathcal{G}_0(\tilde{k})) &= \sum_{\tilde{k}} \frac{\delta_{il}k_j}{D^4}\left(i\omega_n + \mu - \frac{2(i\omega_n)^2\mu}{D^2}\right)\left(1 - \frac{2i\omega_n\mu}{D^2}\right)(\sigma^a\sigma^b\chi(\mathbf{k}\cdot\sigma) + \sigma^a\chi(\mathbf{k}\cdot\sigma)\sigma^b)\\ &= \sum_{\tilde{k}} \frac{\chi\delta_{il}k_j^2}{D^4}\left(i\omega_n + \mu - \frac{2(i\omega_n)^2\mu}{D^2}\right)\left(1 - \frac{2i\omega_n\mu}{D^2}\right)\mathrm{Tr}_\sigma(\sigma^a\sigma^b\sigma^j + \sigma^a\sigma^j\sigma^b)\\ &= \sum_{\tilde{k}} \frac{\chi\delta_{il}k_j^2}{D^4}\left(i\omega_n + \mu - \frac{2(i\omega_n)^2\mu}{D^2}\right)\left(1 - \frac{2i\omega_n\mu}{D^2}\right)\mathrm{Tr}_\sigma(\sigma^a 2\delta^{bj}) = 0.\end{aligned} \tag{E7}$$

In the second step of the above derivation, we collect all the terms that have even order in the momentum $\mathbf{k}$ since these are the nonzero term under the integral of the momentum angle. Similar calculation also shows the second term in Eq. (E1) is also zero. Thus, only the last term is left nonzero. To evaluate this term, we first need to calculate $\partial_{k_l}\mathcal{G}_0(\tilde{k})$, which is given by

$$\begin{aligned}
\partial_{k_l}\mathcal{G}_0(\tilde{k}) &= \frac{\chi\sigma^l}{D^2} + (i\omega_n + \chi(\mathbf{k}\cdot\sigma))\frac{\partial(1/D^2)}{\partial k_l} - \mu\frac{2k^l + 2i\omega_n\chi\sigma^l}{D^4} - \mu((i\omega_n)^2 + k^2 + 2i\omega_n\chi\mathbf{k}\cdot\sigma)\frac{\partial(1/D^4)}{\partial k_l} \\
&= \frac{\chi\sigma^l}{D^2} + (i\omega_n + \chi(\mathbf{k}\cdot\sigma))\frac{2k^l}{D^4} - \mu\frac{2k^l + 2i\omega_n\chi\sigma^l}{D^4} - \mu((i\omega_n)^2 + k^2 + 2i\omega_n\chi\mathbf{k}\cdot\sigma)\frac{4k^l}{D^6} \\
&= \frac{\chi\sigma^l}{D^2} + (i\omega_n + \chi(\mathbf{k}\cdot\sigma))\frac{2k^l}{D^4} - \frac{\mu}{D^6}\left((2k^l + 2i\omega_n\chi\sigma^l)((i\omega_n)^2 - k^2) + 4k^l((i\omega_n)^2 + k^2 + 2i\omega_n\chi\mathbf{k}\cdot\sigma)\right) \\
&= \frac{\chi\sigma^l}{D^2} + (i\omega_n + \chi(\mathbf{k}\cdot\sigma))\frac{2k^l}{D^4} - \frac{2\mu i\omega_n\chi\sigma^l}{D^4} - \frac{\mu}{D^6}(6k^l(i\omega_n)^2 + 2k^2k^l + 8k^l(i\omega_n)\chi\mathbf{k}\cdot\sigma).
\end{aligned} \tag{E8}$$

By substituting these terms into the last term in Eq. (E1), we find

$$\begin{aligned}
\Phi^{ab}_{ij,l} &= \sum_{\tilde{k}} \mathrm{Tr}_\sigma\left(\mathcal{T}^a_i(\tilde{k},\tilde{k})(\partial_{k_l}\mathcal{G}_0(\tilde{k}))\mathcal{T}^b_j(\tilde{k},\tilde{k})\mathcal{G}_0(\tilde{k})\right) \\
&= \sum_{\tilde{k}} \mathrm{Tr}_\sigma\Bigg(\sigma^a k_i\left(\frac{\chi\sigma^l}{D^2} + (i\omega_n + \chi(\mathbf{k}\cdot\sigma))\frac{2k^l}{D^4} - \frac{2\mu i\omega_n\chi\sigma^l}{D^4} - \frac{\mu}{D^6}(6k^l(i\omega_n)^2 + 2k^2k^l + 8k^l(i\omega_n)\chi\mathbf{k}\cdot\sigma)\right) \\
&\qquad \times \sigma^b k_j\left(\frac{i\omega_n + \chi(\mathbf{k}\cdot\sigma)}{D^2} - \frac{\mu}{D^4}((i\omega_n)^2 + k^2 + 2i\omega_n\chi\mathbf{k}\cdot\sigma)\right)\Bigg) \\
&= \sum_{\tilde{k}} \mathrm{Tr}_\sigma\Bigg(\sigma^a k_i\left(\frac{\chi\sigma^l}{D^2} + \chi(\mathbf{k}\cdot\sigma)\frac{2k^l}{D^4} - \frac{2\mu i\omega_n\chi\sigma^l}{D^4} - \frac{\mu}{D^6}8k^l(i\omega_n)\chi\mathbf{k}\cdot\sigma\right)\sigma^b k_j\left(\frac{i\omega_n}{D^2} - \frac{((i\omega_n)^2 + k^2)\mu}{D^4}\right) \\
&\qquad + \sigma^a k_i\left(i\omega_n\frac{2k^l}{D^4} - \frac{\mu}{D^6}(6k^l(i\omega_n)^2 + 2k^2k^l)\right)\sigma^b k_j\left(\frac{\chi(\mathbf{k}\cdot\sigma)}{D^2} - \frac{(2i\omega_n\chi\mathbf{k}\cdot\sigma)\mu}{D^4}\right)\Bigg).
\end{aligned} \tag{E9}$$

In the above, since only the terms with even number of the momentum $\mathbf{k}$ can be nonzero, we thus obtain two terms, and let us calculate them separately. For the first term, we have

$$\begin{aligned}
&\sum_{\tilde{k}} \mathrm{Tr}_\sigma\left(\sigma^a k_i\left(\frac{\chi\sigma^l}{D^2}\left(1 - \frac{2i\omega_n\mu}{D^2}\right) + \chi(\mathbf{k}\cdot\sigma)\frac{2k^l}{D^4}\left(1 - \frac{4i\omega_n\mu}{D^2}\right)\right)\sigma^b k_j\left(\frac{i\omega_n}{D^2} - \frac{((i\omega_n)^2 + k^2)\mu}{D^4}\right)\right) \\
&= \sum_{\tilde{k}} \chi k_i k_j\left(\frac{i\omega_n}{D^2} - \frac{((i\omega_n)^2 + k^2)\mu}{D^4}\right)\left(\frac{1}{D^2}\left(1 - \frac{2i\omega_n\mu}{D^2}\right)\mathrm{Tr}_\sigma(\sigma^a\sigma^l\sigma^b) + \frac{2k^l}{D^4}\left(1 - \frac{4i\omega_n\mu}{D^2}\right)\mathrm{Tr}_\sigma(\sigma^a(\mathbf{k}\cdot\sigma)\sigma^b)\right) \\
&= \sum_{\tilde{k}} \chi k_i k_j\left(\frac{i\omega_n}{D^2} - \frac{((i\omega_n)^2 + k^2)\mu}{D^4}\right)\left(\frac{1}{D^2}\left(1 - \frac{2i\omega_n\mu}{D^2}\right)2i\epsilon^{alb} + \frac{2k^l k_m}{D^4}\left(1 - \frac{4i\omega_n\mu}{D^2}\right)2i\epsilon^{amb}\right) \\
&= -\sum_{\tilde{k}} \frac{2\chi\mu k_i k_j}{D^6}\left(2(i\omega_n)^2\left(i\epsilon^{alb} + 4i\epsilon^{amb}\frac{k^l k_m}{D^2}\right) + ((i\omega_n)^2 + k^2)\left(i\epsilon^{alb} + 2i\epsilon^{amb}\frac{k^l k_m}{D^2}\right)\right).
\end{aligned} \tag{E10}$$

In the last step above, we have dropped all the terms with odd number of $i\omega_n$ since the frequency summation will make them vanishing. For the second term, we have

$$\begin{aligned}
&\sum_{\tilde{k}} \mathrm{Tr}_\sigma\left(\sigma^a k_i\left(i\omega_n\frac{2k^l}{D^4} - \frac{\mu}{D^6}(6k^l(i\omega_n)^2 + 2k^2k^l)\right)\sigma^b k_j\left(\frac{\chi(\mathbf{k}\cdot\sigma)}{D^2} - \frac{(2i\omega_n\chi\mathbf{k}\cdot\sigma)\mu}{D^4}\right)\right) \\
&= \sum_{\tilde{k}} \chi k_i k_j\left(i\omega_n\frac{2k^l}{D^4} - \frac{\mu}{D^6}(6k^l(i\omega_n)^2 + 2k^2k^l)\right)\left(\frac{1}{D^2} - \frac{2i\omega_n\mu}{D^4}\right)\mathrm{Tr}_\sigma(\sigma^a\sigma^b(\mathbf{k}\cdot\sigma)) \\
&= \sum_{\tilde{k}} \frac{2\chi k_i k_j k^l}{D^6}\left(i\omega_n - \frac{\mu}{D^2}(3(i\omega_n)^2 + k^2)\right)\left(1 - \frac{2i\omega_n\mu}{D^2}\right)\mathrm{Tr}_\sigma(\sigma^a\sigma^b(\mathbf{k}\cdot\sigma)) \\
&= -\sum_{\tilde{k}} \frac{2\chi\mu k_i k_j k^l}{D^8}(2(i\omega_n)^2 + (3(i\omega_n)^2 + k^2))\mathrm{Tr}_\sigma(\sigma^a\sigma^b(\mathbf{k}\cdot\sigma)) = -\sum_{\tilde{k}} \frac{4\chi\mu k_i k_j k^l k_m}{D^8}(5(i\omega_n)^2 + k^2)i\epsilon^{abm}.
\end{aligned} \tag{E11}$$

Again here we only pick up the terms with even order of $i\omega_n$. Finally, we can add these two terms together and obtain

$$\begin{aligned}\left(\frac{\partial \Phi_{ij}^{ab}}{\partial q_l}\right)_{\tilde{q}=0} &= -\sum_{\tilde{k}} \frac{2\chi\mu k_i k_j}{D^6}\left((3(i\omega_n)^2+k^2)i\epsilon^{alb}+(5(i\omega_n)^2+k^2)2i\epsilon^{amb}\frac{k^l k_m}{D^2}\right)-\sum_{\tilde{k}}\frac{4\chi\mu k_i k_j k^l k_m}{D^8}(5(i\omega_n)^2+k^2)i\epsilon^{abm}\\ &= -\sum_{\tilde{k}} \frac{2\chi\mu k_i k_j}{D^6}\left((3(i\omega_n)^2+k^2)i\epsilon^{alb}+(10(i\omega_n)^2+2k^2)i\epsilon^{amb}\frac{k^l k_m}{D^2}+(10(i\omega_n)^2+2k^2)i\epsilon^{abm}\frac{k^l k_m}{D^2}\right)\\ &= -\sum_{\tilde{k}} \frac{2\chi\mu k_i k_j}{D^6}(3(i\omega_n)^2+k^2)i\epsilon^{alb} = 2i\epsilon^{alb}\chi\mu\sum_{\tilde{k}}\frac{k_i k_j}{D^4}\left(1-\frac{4(i\omega_n)^2}{D^2}\right). \end{aligned} \tag{E12}$$

In the above expression, the integral over the momentum $\mathbf{k}$ can only be nonzero when $i = j$. Furthermore, the integral over the polar and azimuthal angles of the momentum gives rise to

$$\int d(\cos\theta)d\varphi k_x^2 = \int d(\cos\theta)d\varphi k_y^2 = \int d(\cos\theta)d\varphi k_z^2 = \frac{4\pi}{3}k^2 \tag{E13}$$

with $\mathbf{k} = (k, \theta, \varphi)$ in the spherical coordinate. Thus, let us define the function

$$\mathcal{F}_0 = \frac{2}{\beta V}\sum_{\mathbf{k},i\omega_n}\frac{k_x^2}{D^4}\left(1-\frac{4(i\omega_n)^2}{D^2}\right) = \frac{1}{3\pi^2\beta}\sum_{i\omega_n}\int_0^\Lambda k^4 dk\frac{1}{D^4}\left(1-\frac{4(i\omega_n)^2}{D^2}\right), \tag{E14}$$

where a cutoff Λ in the momentum space has been assumed, and

$$\Phi_{ij,l}^{ab} = \left(\frac{\partial \Phi_{ij}^{ab}}{\partial q_l}\right)_{\tilde{q}=0} = i\epsilon^{alb}\delta_{ij}\chi\mu\mathcal{F}_0, \tag{E15}$$

from which one can see that $\Phi_{ij,l}^{ab}$ is indeed pure imaginary ($\mathcal{F}_0$ is real, as shown below), and proportional to $\epsilon^{alb}\delta_{ij}$. Thus, we demonstrate the effective action (H9) together with Eq. (E15) indeed takes the form of Nieh-Yan term.

Next we will evaluate the coefficient $\mathcal{F}_0$ in Eq. (E14) analytically. Let us separate $\mathcal{F}_0$ into $\mathcal{F}_{01}$ and $\mathcal{F}_{02}$ with

$$\mathcal{F}_{01} = \frac{1}{3\pi^2\beta}\sum_{i\omega_n}\int_0^\Lambda dk\frac{k^4}{((i\omega_n)^2-k^2)^2} \tag{E16}$$

and

$$\mathcal{F}_{02} = \frac{4}{3\pi^2\beta}\sum_{i\omega_n}\int_0^\Lambda dk\frac{k^4(i\omega_n)^2}{((i\omega_n)^2-k^2)^3}, \tag{E17}$$

so that

$$\mathcal{F}_0 = \mathcal{F}_{01} - \mathcal{F}_{02}. \tag{E18}$$

Here we have substitute the expression for D^2.

We perform the frequency summation in $\mathcal{F}_{01}$ and consider the contour integral

$$I = \oint_R \frac{dz}{2\pi i}f(z)n_F(z) \tag{E19}$$

with

$$f(z) = \frac{1}{(z^2-k^2)^2} \tag{E20}$$

in the complex-z plane with the radius R of z. In the limit $R \to \infty$, since the integrand $f(z)n_F(z)$ decays to zero fast enough, this integral should be zero. However, this integral can re-expressed as the integral around the poles of the integrand in the complex-z plane, and thus is determined by the residuals of these poles according to the residual theorem. We next need to figure out all the poles and the corresponding residuals for the integrand $f(z)n_F(z)$.

For the Fermi function $n_F(z) = \frac{1}{e^{\beta z}+1}$, the poles are $z_n = i\omega_n = i\frac{(2n+1)\pi}{\beta}$ and the corresponding residuals for $f(z)n_F(z)$ are $\mathrm{Res}[f(z)n_F(z), z = z_n] = -\frac{1}{\beta}f(i\omega_n)$.

There are another two poles $z_\pm = \pm k$ from the function $f(z)$. Since they are not simple poles, we need a bit more work to extract the residuals. For $z = z_+ = k$, let us denote $f(z)n_F(z) = \frac{g(z)}{(z-k)^2}$ with $g(z) = \frac{n_F(z)}{(z+k)^2}$. We need to expand $g(z)$ around $z_+ = k$ and pick up the first-order term $(\frac{\partial g(z)}{\partial z})_{z=k}(z-k)$ since the coefficient in this term will give the residual of $f(z)n_F(z)$. Direct calculations give rise to

$$\frac{\partial g(z)}{\partial z}=-\frac{2}{(z+k)^3}n_F(z)+\frac{1}{(z+k)^2}\partial_z n_F(z)\partial_z n_F(z)=-\frac{\beta e^{\beta z}}{(e^{\beta z}+1)^2}=-\frac{\beta}{4(\cosh(\beta z/2))^2}, \tag{E21}$$

and thus the corresponding residual is

$$\mathrm{Res}[f(z)n_F(z),z=z_+]=\left(\frac{\partial g(z)}{\partial z}\right)_{z=k}=-\frac{2}{(2k)^3}n_F(k)-\frac{1}{(2k)^2}\frac{\beta}{4(\cosh(\beta k/2))^2}=-\frac{1}{4k^3}n_F(k)-\frac{1}{16k^2}\frac{\beta}{(\cosh(\beta k/2))^2}. \tag{E22}$$

For $z=z_-=-k$, we denote $f(z)n_F(z)=\frac{g(z)}{(z+k)^2}$ with $g(z)=\frac{n_F(z)}{(z-k)^2}$. With

$$\frac{\partial g(z)}{\partial z}=-\frac{2}{(z-k)^3}n_F(z)+\frac{1}{(z-k)^2}\partial_z n_F(z), \tag{E23}$$

we obtain

$$\begin{aligned}\mathrm{Res}[f(z)n_F(z),z=z_-]&=\left(\frac{\partial g(z)}{\partial z}\right)_{z=-k}=-\frac{2}{(-2k)^3}n_F(-k)-\frac{1}{(-2k)^2}\frac{\beta}{4[\cosh(\beta k/2)]^2}\\&=\frac{1}{4k^3}n_F(-k)-\frac{1}{16k^2}\frac{\beta}{[\cosh(\beta k/2)]^2}.\end{aligned} \tag{E24}$$

Putting all the results together, we have

$$\begin{aligned}I&=-\frac{1}{\beta}\sum_{i\omega_n}f(i\omega_n)+\mathrm{Res}[f(z)n_F(z),z=z_+]+\mathrm{Res}[f(z)n_F(z),z=z_-]=0\\&\to\frac{1}{\beta}\sum_{i\omega_n}f(i\omega_n)=\mathrm{Res}[f(z)n_F(z),z=z_+]+\mathrm{Res}[f(z)n_F(z),z=z_-]\\&=\frac{1}{4k^3}(n_F(-k)-n_F(k))-\frac{1}{8k^2}\frac{\beta}{[\cosh(\beta k/2)]^2}=\frac{1}{4k^3}\tanh(\beta k/2)-\frac{1}{8k^2}\frac{\beta}{[\cosh(\beta k/2)]^2},\end{aligned} \tag{E25}$$

and

$$\begin{aligned}\mathcal{F}_{01}&=\frac{1}{3\pi^2\beta}\sum_{i\omega_n}\int_0^\Lambda dk\frac{k^4}{[(i\omega_n)^2-k^2]^2}=\frac{1}{3\pi^2}\int_0^\Lambda dk k^4\left(\frac{1}{4k^3}\tanh(\beta k/2)-\frac{1}{8k^2}\frac{\beta}{[\cosh(\beta k/2)]^2}\right)\\&=\frac{1}{3\pi^2}\int_0^\Lambda dk\left(\frac{k}{4}\tanh(\beta k/2)-\frac{k^2}{8}\frac{\beta}{[\cosh(\beta k/2)]^2}\right).\end{aligned} \tag{E26}$$

We may consider the coefficient $\mathcal{F}_{01}$ at the zero temperature and its correction from the finite temperature, $\mathcal{F}_{01}=\mathcal{F}_{01}(T=0)+\delta\mathcal{F}_{01}(T)$. At zero temperature ($\beta\to\infty$), the first term in the above integral diverges as Λ^2 since $\tanh(\beta\Lambda/2)\to 1$ while the second term converges (actually vanishes). Thus, we have

$$\mathcal{F}_{01}(T=0)=\frac{1}{3\pi^2}\int_0^\Lambda dk\frac{k}{4}=\frac{\Lambda^2}{24\pi^2} \tag{E27}$$

and the finite temperature correction is given by

$$\begin{aligned}\delta\mathcal{F}_{01}(T)&=\frac{1}{3\pi^2}\int_0^\Lambda dk\left(\frac{k}{4}[\tanh(\beta k/2)-1]-\frac{k^2}{8}\frac{\beta}{[\cosh(\beta k/2)]^2}\right)\\&=\frac{1}{3\pi^2}\int_0^\infty d(2x/\beta)\left(\frac{2x}{4\beta}[\tanh(x)-1]-\frac{(2x)^2}{8\beta^2}\frac{\beta}{[\cosh(x)]^2}\right)\\&=\frac{1}{3\pi^2\beta^2}\int_0^\infty dx\left(x[\tanh(x)-1]-\frac{x^2}{[\cosh(x)]^2}\right),=-\frac{1}{24\beta^2}=-\frac{(k_BT)^2}{24}\end{aligned} \tag{E28}$$

where we have used $x=\beta k/2$ and $\Lambda\to\infty$.

For $\mathcal{F}_{02}$, we should choose

$$f(z)=\frac{z^2}{(z^2-k^2)^3} \tag{E29}$$

in Eq. (E19). The poles and residuals for $n_F(z)$ remain the same. For $f(z)$, the poles are still at $z_\pm=\pm k$, but the residuals are changed. For $z=z_+=k$, we denote $f(z)n_F(z)=\frac{g(z)}{(z-k)^3}$ with $g(z)=\frac{z^2}{(z+k)^3}n_F(z)$. Since this is a pole of third order, we need to evaluate $(\frac{1}{2}\frac{\partial^2 g(z)}{\partial z^2})_{z=k}$. Direct calculations give rise to

$$\frac{\partial g(z)}{\partial z} = \left(\frac{2z}{(z+k)^3} - \frac{3z^2}{(z+k)^4}\right)n_F(z) + \frac{z^2}{(z+k)^3}\partial_z n_F(z),$$
$$\frac{\partial^2 g(z)}{\partial z^2} = \left(\frac{2}{(z+k)^3} - \frac{12z}{(z+k)^4} + \frac{12z^2}{(z+k)^5}\right)n_F(z) + 2\left(\frac{2z}{(z+k)^3} - \frac{3z^2}{(z+k)^4}\right)\partial_z n_F(z) + \frac{z^2}{(z+k)^3}\partial_z^2 n_F(z),$$
$$\partial_z n_F(z) = -\frac{\beta e^{\beta z}}{(e^{\beta z}+1)^2} = -\frac{\beta}{4[\cosh(\beta z/2)]^2},$$
$$\partial_z^2 n_F(z) = -\frac{\beta^2 e^{\beta z}}{(e^{\beta z}+1)^2} + \frac{2\beta^2 e^{2\beta z}}{(e^{\beta z}+1)^3} = \beta^2\frac{e^{2\beta z}-e^{\beta z}}{(e^{\beta z}+1)^3} = \beta^2\frac{e^{\beta z}}{(e^{\beta z}+1)^2}\frac{e^{\beta z}-1}{e^{\beta z}+1} = \frac{\beta^2}{4[\cosh(\beta z/2)]^2}\tanh(\beta z/2) \quad \text{(E30)}$$

and the residual is

$$\begin{aligned}\mathrm{Res}[f(z)n_F(z), z=z_+] = \frac{1}{2}\left(\frac{\partial^2 g(z)}{\partial z^2}\right)_{z=k} &= \left(\frac{1}{8k^3} - \frac{3}{8k^3} + \frac{3}{16k^3}\right)n_F(k) + \left(\frac{2}{8k^2} - \frac{3}{16k^2}\right)\partial_z n_F(k) + \frac{1}{16k}\partial_z^2 n_F(k),\\ &= -\frac{1}{16k^3}n_F(k) - \frac{1}{16k^2}\frac{\beta}{4[\cosh(\beta k/2)]^2} + \frac{1}{16k}\frac{\beta^2}{4[\cosh(\beta k/2)]^2}\tanh(\beta k/2).\end{aligned} \quad \text{(E31)}$$

Similarly for For $z = z_- = -k$, we denote $f(z)n_F(z) = \frac{g(z)}{(z+k)^3}$ with $g(z) = \frac{z^2}{(z-k)^3}n_F(z)$. Direct calculations give

$$\frac{\partial g(z)}{\partial z} = \left(\frac{2z}{(z-k)^3} - \frac{3z^2}{(z-k)^4}\right)n_F(z) + \frac{z^2}{(z-k)^3}\partial_z n_F(z),$$
$$\frac{\partial^2 g(z)}{\partial z^2} = \left(\frac{2}{(z-k)^3} - \frac{12z}{(z-k)^4} + \frac{12z^2}{(z-k)^5}\right)n_F(z) + 2\left(\frac{2z}{(z-k)^3} - \frac{3z^2}{(z-k)^4}\right)\partial_z n_F(z) + \frac{z^2}{(z-k)^3}\partial_z^2 n_F(z), \quad \text{(E32)}$$

and the residual is

$$\begin{aligned}\mathrm{Res}[f(z)n_F(z), z=z_-] = \frac{1}{2}\left(\frac{\partial^2 g(z)}{\partial z^2}\right)_{z=-k} &= \left(-\frac{1}{8k^3} + \frac{3}{8k^3} - \frac{3}{16k^3}\right)n_F(-k) + \left(\frac{2}{8k^2} - \frac{3}{16k^2}\right)\partial_z n_F(-k) - \frac{1}{16k}\partial_z^2 n_F(-k),\\ &= \frac{1}{16k^3}n_F(-k) - \frac{1}{16k^2}\frac{\beta}{4[\cosh(\beta k/2)]^2} + \frac{1}{16k}\frac{\beta^2}{4[\cosh(\beta k/2)]^2}\tanh(\beta k/2).\end{aligned} \quad \text{(E33)}$$

Putting all the results together, we have

$$\begin{aligned}\frac{1}{\beta}\sum_{i\omega_n} f(i\omega_n) &= \mathrm{Res}[f(z)n_F(z), z=z_+] + \mathrm{Res}[f(z)n_F(z), z=z_-]\\ &= \frac{1}{16k^3}(n_F(-k) - n_F(k)) - \frac{1}{8k^2}\frac{\beta}{4[\cosh(\beta k/2)]^2} + \frac{1}{8k}\frac{\beta^2}{4[\cosh(\beta k/2)]^2}\tanh(\beta k/2)\\ &= \frac{1}{16k^3}\tanh(\beta k/2) - \frac{1}{8k^2}\frac{\beta}{4[\cosh(\beta k/2)]^2} + \frac{1}{8k}\frac{\beta^2}{4[\cosh(\beta k/2)]^2}\tanh(\beta k/2),\end{aligned} \quad \text{(E34)}$$

and

$$\begin{aligned}\mathcal{F}_{02} &= \frac{4}{3\pi^2}\int_0^\Lambda dk k^4 \frac{1}{\beta}\sum_{i\omega_n}\frac{(i\omega_n)^2}{[(i\omega_n)^2 - k^2]^3}\\ &= \frac{4}{3\pi^2}\int_0^\Lambda dk k^4\left(\frac{1}{16k^3}\tanh(\beta k/2) - \frac{1}{8k^2}\frac{\beta}{4[\cosh(\beta k/2)]^2} + \frac{1}{8k}\frac{\beta^2}{4[\cosh(\beta k/2)]^2}\tanh(\beta k/2)\right)\\ &= \frac{1}{24\pi^2}\int_0^\Lambda dk\left(2k\tanh(\beta k/2) - \frac{\beta k^2}{[\cosh(\beta k/2)]^2} + \frac{\beta^2 k^3}{[\cosh(\beta k/2)]^2}\tanh(\beta k/2)\right).\end{aligned} \quad \text{(E35)}$$

We can decompose $\mathcal{F}_{02}$ into $\mathcal{F}_{02} = \mathcal{F}_{02}(T=0) + \delta\mathcal{F}_{02}(T)$ and find

$$\mathcal{F}_{02}(T=0) = \frac{1}{12\pi^2}\int_0^\Lambda dk k = \frac{\Lambda^2}{24\pi^2} \quad \text{(E36)}$$

and

$$\delta\mathcal{F}_{02}(T) = \frac{1}{24\pi^2}\int_0^\Lambda dk\left(2k[\tanh(\beta k/2)-1] - \frac{\beta k^2}{[\cosh(\beta k/2)]^2} + \frac{\beta^2 k^3}{[\cosh(\beta k/2)]^2}\tanh(\beta k/2)\right)$$
$$= \frac{1}{24\pi^2\beta^2}\int_0^\infty d(2x)\left(4x[\tanh(x)-1] - \frac{(2x)^2}{[\cosh(x)]^2} + \frac{(2x)^3}{[\cosh(x)]^2}\tanh(x)\right)$$
$$= \frac{1}{3\pi^2\beta^2}\int_0^\infty dx\left(x[\tanh(x)-1] - \frac{x^2}{[\cosh(x)]^2} + \frac{2x^3}{[\cosh(x)]^2}\tanh(x)\right) = \frac{1}{24\beta^2} = \frac{(k_BT)^2}{24}. \tag{E37}$$

where we have used $x=\beta k/2$ and $\Lambda\to\infty$.

Now we can see that from $\mathcal{F}_0 = \mathcal{F}_{01} - \mathcal{F}_{02}$, we have

$$\mathcal{F}_0(T=0) = \mathcal{F}_{01}(T=0) - \mathcal{F}_{02}(T=0) = 0, \tag{E38}$$

and

$$\delta\mathcal{F}_0(T) = \delta\mathcal{F}_{01}(T) - \delta\mathcal{F}_{02}(T) = -\frac{(k_BT)^2}{24} - \frac{(k_BT)^2}{24} = -\frac{(k_BT)^2}{12}, \tag{E39}$$

It is interesting to notice that the cutoff-dependent term cancels between $\mathcal{F}_{01}$ and $\mathcal{F}_{02}$, while the temperature-dependent term remains. Thus, $\mathcal{F}_0 = F_0 + F_1(k_BT)^2$ with $F_0 = 0$ and $F_1 = -\frac{1}{12}$. It should be emphasized that F_0 depends on the form of UV momentum cutoff, and our model here has the effective Lorentz invariance with an isotropic momentum cutoff, and thus $F=0$ is expected [49]. In contrast, our result $F_1 = -\frac{1}{12}$ recovers the value derived in literature [45,46,48], implying the universal property of the thermal Nieh-Yan term.

APPENDIX F: NUMERICAL METHODS FOR EVALUATING STRESS-STRESS CORRELATION FUNCTION

In this Appendix, we will describe our numerical methods to evaluate the stress-stress correlation functions. Here we only focus on the stress-stress correlation function $\Phi^{ab}_{ij}(\mathbf{q}, i\nu_m)$ in Eq. (C17). One can rewrite the Green's function in Eq. (C10) as

$$\mathcal{G}_0 = \sum_{s=\pm}\frac{P_s(\mathbf{k})}{i\omega_n - \xi_\mathbf{k}}, \tag{F1}$$

where $\xi_{s,\mathbf{k}} = s\hbar v_f k - \mu$ and the projection operator $P_s(\mathbf{k}) = \frac{1}{2}(1 + s\frac{\mathbf{k}\cdot\sigma}{k})$. Then after the Matsubara frequency summation, Eq. (C17) can be simplified as

$$\Phi^{ab}_{ij}(\mathbf{q}, i\nu_m) = \frac{1}{V}\sum_{\mathbf{k},ss'}\frac{n_F(\xi_{s',\mathbf{k}-\mathbf{q}}) - n_F(\xi_{s,\mathbf{k}})}{i\nu_m + \xi_{s',\mathbf{k}-\mathbf{q}} - \xi_{s,\mathbf{k}}}G^{ab}_{ij}(s\mathbf{k}, s'\mathbf{k}-\mathbf{q}), \tag{F2}$$

with

$$G^{ab}_{ij}(s\mathbf{k}, s'\mathbf{k}-\mathbf{q}) = \mathrm{Tr}_\sigma\left[\mathcal{T}^a_i(\mathbf{k},\mathbf{k}-\mathbf{q})P_{s'}(\mathbf{k}-\mathbf{q})\mathcal{T}^b_j(\mathbf{k}-\mathbf{q},\mathbf{k})P_s(\mathbf{k})\right]. \tag{F3}$$

Direct evaluation of G^{ab}_{ij} gives

$$G^{ab}_{ij}(s\mathbf{k}, s'\mathbf{k}-\mathbf{q}) = \frac{(\hbar v_f)^2}{8}(2k_i - q_i)(2k_j - q_j)\left(\delta^{ab} + i\epsilon^{abl}s\frac{k_l}{k} + i\epsilon^{alb}s'\frac{k_l - q_l}{|\mathbf{k}-\mathbf{q}|} + \frac{ss'}{k|\mathbf{k}-\mathbf{q}|}\left[k_b(k_a - q_a) + k_a(k_b - q_b) - \delta^{ab}\mathbf{k}\cdot(\mathbf{k}-\mathbf{q})\right]\right). \tag{F4}$$

Up to now, the whole formalism is general and can be calculated numerically in principles. As our purpose here is to extract the term that is related to the Nieh-Yan anomaly, we want to look for the terms $\Phi^{ab}_{ij}(\mathbf{q}, i\nu_m) \propto \delta_{ij}$. Furthermore, let us choose the momentum along the z-direction, $\mathbf{q} = (0, 0, q_z)$, and then we should expect that the relevant component should be $\Phi^{xy}_{ii}(q_z, i\nu_m)$. Thus, our numerical simulations focus on the component $\Phi^{xy}_{xx}(q_z, i\nu_m)$ below ($\Phi^{xy}_{xx}(q_z, i\nu_m) = \Phi^{xy}_{yy}(q_z, i\nu_m)$). For this component, we have

$$G^{xy}_{xx}(s\mathbf{k}, s'\mathbf{k}-\mathbf{q}) = \frac{(\hbar v_f)^2}{2}k_x^2\left(is\frac{k_z}{k} - is'\frac{k_z - q_z}{|\mathbf{k}-\mathbf{q}|} + \frac{ss'}{k|\mathbf{k}-\mathbf{q}|}2k_xk_y\right). \tag{F5}$$

Now let us consider the momentum integral in Eq. (F2), and choose the spherical coordinate for the momentum, $\mathbf{k} = (k\sin\theta_k\cos\varphi_k, k\sin\theta_k\sin\varphi_k, k\cos\theta_k)$. Furthermore, we denote $\mathbf{k}' = \mathbf{k}-\mathbf{q} = (k'\sin\theta'_k\cos\varphi'_k, k'\sin\theta'_k\sin\varphi'_k, k'\cos\theta'_k) = (k\sin\theta_k\cos\varphi_k, k\sin\theta_k\sin\varphi_k, k\cos\theta_k - q_z)$. Thus, we have $\varphi'_k = \varphi_k$, $k'\sin\theta'_k = k\sin\theta_k$ and $k'\cos\theta'_k = k\cos\theta_k - q_z$. The latter two equalities give rise to $k' = \sqrt{k^2 + q_z^2 - 2kq_z\cos\theta_k}$ and $\cos\theta'_k = \frac{k\cos\theta_k - q_z}{k'}$.

By choosing $\mathbf{q}$ along the z direction, $\xi_{s,\mathbf{k}-\mathbf{q}}$ only involves k and θ_k and does not depend on φ_k. Thus, we can define

$$\begin{aligned}\bar{G}_{xx}^{xy}(ss';k,\cos\theta_k,q_z) &= \int \frac{d\varphi_k}{2\pi} G_{xx}^{xy}(s\mathbf{k}, s'\mathbf{k}-\mathbf{q})\\ &= \frac{(\hbar v_f)^2}{4\pi}\int_0^{2\pi} d\varphi_k k^2 \sin^2\theta_k\cos^2\varphi_k\left(is\cos\theta_k - is'\cos\theta_k' + \frac{ss'}{kk'}2k^2\sin^2\theta_k\cos\varphi_k\sin\varphi_k\right)\\ &= \frac{(\hbar v_f)^2}{4}k^2\sin^2\theta_k(is\cos\theta_k - is'\cos\theta_k'),\end{aligned} \tag{F6}$$

and then

$$\Phi_{xx}^{xy}(q_z, i\nu_m) = \sum_{ss'}\int \frac{k^2 dk d\cos\theta_k}{(2\pi)^2}\frac{n_F(\xi_{s',\mathbf{k}-\mathbf{q}}) - n_F(\xi_{s,\mathbf{k}})}{i\nu_m + \xi_{s',\mathbf{k}-\mathbf{q}} - \xi_{s,\mathbf{k}}}\bar{G}_{xx}^{xy}(ss';k,\cos\theta_k,q_z). \tag{F7}$$

We may decompose $\Phi_{xx}^{xy}(q_z, i\nu_m)$ into several different components $\Phi_{xx}^{xy}(q_z, i\nu_m) = \Phi_{xx}^{xy,-}(q_z, i\nu_m) + \Phi_{xx}^{xy,+}(q_z, i\nu_m)$, where $\pm$ labels the contribution from different s bands. We further decompose $\Phi_{xx}^{xy,\pm}(q_z, i\nu_m) = \Phi_{xx,1}^{xy,\pm}(q_z, i\nu_m) + \Phi_{xx,2}^{xy,\pm}(q_z, i\nu_m)$, where 1 and 2 here are for the intraband and interband contribution. Explicitly,

$$\begin{aligned}\Phi_{xx,1}^{xy,+}(q_z, i\nu_m) &= \int \frac{k^2 dk d\cos\theta_k}{(2\pi)^2}\frac{n_F(\xi_{+,\mathbf{k}-\mathbf{q}}) - n_F(\xi_{+,\mathbf{k}})}{i\nu_m + \xi_{+,\mathbf{k}-\mathbf{q}} - \xi_{+,\mathbf{k}}}\bar{G}_{xx}^{xy}(++;k,\cos\theta_k,q_z)\\ \Phi_{xx,2}^{xy,+}(q_z, i\nu_m) &= \int \frac{k^2 dk d\cos\theta_k}{(2\pi)^2}\left(\frac{n_F(\xi_{+,\mathbf{k}-\mathbf{q}})}{i\nu_m + \xi_{+,\mathbf{k}-\mathbf{q}} - \xi_{-,\mathbf{k}}}\bar{G}_{xx}^{xy}(-+;k,\cos\theta_k,q_z) - \frac{n_F(\xi_{+,\mathbf{k}})}{i\nu_m + \xi_{-,\mathbf{k}-\mathbf{q}} - \xi_{+,\mathbf{k}}}\bar{G}_{xx}^{xy}(+-;k,\cos\theta_k,q_z)\right)\\ \Phi_{xx,1}^{xy,-}(q_z, i\nu_m) &= \int \frac{k^2 dk d\cos\theta_k}{(2\pi)^2}\frac{n_F(\xi_{-,\mathbf{k}-\mathbf{q}}) - n_F(\xi_{-,\mathbf{k}})}{i\nu_m + \xi_{-,\mathbf{k}-\mathbf{q}} - \xi_{-,\mathbf{k}}}\bar{G}_{xx}^{xy}(--;k,\cos\theta_k,q_z)\\ \Phi_{xx,2}^{xy,-}(q_z, i\nu_m) &= \int \frac{k^2 dk d\cos\theta_k}{(2\pi)^2}\left(\frac{n_F(\xi_{-,\mathbf{k}-\mathbf{q}})}{i\nu_m + \xi_{-,\mathbf{k}-\mathbf{q}} - \xi_{+,\mathbf{k}}}\bar{G}_{xx}^{xy}(+-;k,\cos\theta_k,q_z) - \frac{n_F(\xi_{-,\mathbf{k}})}{i\nu_m + \xi_{+,\mathbf{k}-\mathbf{q}} - \xi_{-,\mathbf{k}}}\bar{G}_{xx}^{xy}(-+;k,\cos\theta_k,q_z)\right).\end{aligned} \tag{F8}$$

Since the above integral involves the singular points, we need to treat it as the Cauchy principal value integral. To numerically evaluate the integral, we need some more simplification. First, the integral over θ_k can be replaced by the integral over k'. All the above integrals takes the form of

$$J_1 = \int \frac{k^2 dk d\cos\theta_k}{(2\pi)^2}\mathcal{G}(k,\cos\theta_k,q_z). \tag{F9}$$

With $\int d^3k'\delta(\mathbf{k}'-\mathbf{k}+\mathbf{q}) = \int d^3k'\delta(k' - \sqrt{k^2+q_z^2-2kq_z\cos\theta_k}) = 1$, we have [107]

$$\begin{aligned}J_1 &= \int \frac{k^2 dk d\cos\theta_k}{(2\pi)^2}\int d^3k'\delta\left(k' - \sqrt{k^2+q_z^2-2kq_z\cos\theta_k}\right)\mathcal{G}(k,\cos\theta_k,q_z)\\ &= \frac{1}{(2\pi)^2}\int_0^\Lambda dk\int_{|k-q|}^{k+q} dk'\frac{kk'}{q_z}\mathcal{G}\left(k, \cos\theta_k = \frac{k^2+q_z^2-k'^2}{2kq_z}, q_z\right).\end{aligned} \tag{F10}$$

Here the integral range of k' is determined by requiring $|\cos\theta_k| \leqslant 1$. Next one may absorb the velocity $\hbar v_f$ into the definition of the momentum, $\tilde{k} = \hbar v_f k, \tilde{k}' = \hbar v_f k', q = \hbar v_f q_z$. Thus, one can see that $\mathcal{G}$ does not give any additional factors while the integral J_1 should contain an additional prefactor $\frac{1}{(\hbar v_f)^3}$. Now $\mathcal{G}$ should be a function of $\tilde{k}, \tilde{k}', q$, denoted as $\mathcal{G}(\tilde{k}, \tilde{k}', q)$. Finally, one can take the transformation $\tilde{k} = \frac{1}{2}(x+y)$ and $\tilde{k}' = \frac{1}{2}(x-y)$ and the integral can be transformed into

$$J_1 = \frac{1}{(2\pi)^2(\hbar v_f)^3}\int_q^\Lambda dx\int_{-q}^q dy\frac{x^2-y^2}{8q}\mathcal{G}\big(\tilde{k} = \tfrac{1}{2}(x+y), \tilde{k}' = \tfrac{1}{2}(x-y), q\big). \tag{F11}$$

Using the equation, we can rewrite the integrals as

$$\begin{aligned}\Phi_{xx,1}^{xy,+}(q,\omega+i\eta) &= \frac{1}{(2\pi)^2(\hbar v_f)^3}\int_q^\Lambda dx\int_{-q}^q dy\frac{x^2-y^2}{8q}\frac{n_F\left(\frac{x-y}{2}-\mu\right) - n_F\left(\frac{x+y}{2}-\mu\right)}{\omega+i\eta-y}\bar{G}_{xx}^{xy}(++;x,y,q)\\ \Phi_{xx,2}^{xy,+}(q,\omega+i\eta) &= \frac{1}{(2\pi)^2(\hbar v_f)^3}\int_q^\Lambda dx\int_{-q}^q dy\frac{x^2-y^2}{8q}\left(\frac{n_F\left(\frac{x-y}{2}-\mu\right)}{\omega+i\eta_m+x}\bar{G}_{xx}^{xy}(-+;x,y,q) - \frac{n_F\left(\frac{x+y}{2}-\mu\right)}{\omega+i\eta-x}\bar{G}_{xx}^{xy}(+-;x,y,q)\right)\end{aligned}$$

$$\Phi_{xx,1}^{xy,-}(q,\omega+i\eta)=\frac{1}{(2\pi)^2(\hbar v_f)^3}\int_q^\Lambda dx\int_{-q}^q dy\frac{x^2-y^2}{8q}\frac{n_F\left(-\frac{x-y}{2}-\mu\right)-n_F\left(-\frac{x+y}{2}-\mu\right)}{\omega+i\eta+y}\bar{G}_{xx}^{xy}(--;x,y,q)$$

$$\Phi_{xx,2}^{xy,-}(q,\omega+i\eta)=\frac{1}{(2\pi)^2(\hbar v_f)^3}\int_q^\Lambda dx\int_{-q}^q dy\frac{x^2-y^2}{8q}$$
$$\times\left(\frac{n_F\left(-\frac{x-y}{2}-\mu\right)}{\omega+i\eta-x}\bar{G}_{xx}^{xy}(+-;x,y,q)-\frac{n_F\left(-\frac{x+y}{2}-\mu\right)}{\omega+i\eta+x}\bar{G}_{xx}^{xy}(-+;x,y,q)\right), \tag{F12}$$

where

$$\bar{G}_{xx}^{xy}(ss';x,y,q)=\frac{(x+y)^2}{16}(1-\cos^2\theta_k)(is\cos\theta_k-is'\cos\theta_k'), \tag{F13}$$

$$\cos\theta_k=\frac{k^2+q_z^2-k'^2}{2kq_z}=\frac{xy+q^2}{(x+y)q}, \tag{F14}$$

$$\cos\theta_k'=\frac{k\cos\theta_k-q_z}{k'}=\frac{xy-q^2}{(x-y)q}, \tag{F15}$$

and we have replaced $i\nu_m$ by $\omega+i\eta$. Now one can see that the poles of the integral are determined by $x=\pm\omega$ or $y=\pm\omega$, and the corresponding Cauchy integral is easy to deal with numerically. These expressions form the basis for our numerical calculations.

To recover the unit, one notices that q,x,y in the above expression are in the unit of energy. $\hbar v_f$ is in the unit of $[E\cdot L]$, and thus we need to further choose certain length unit of our system, which is denoted as a_0. A natural choice will be the inverse of the momentum cutoff $a_0=1/\Lambda$. Then, the unit of energy can be chosen as $\hbar v_f/a_0$. From the expression of Φ_{xx}^{xy}, one can easily find that Φ_{xx}^{xy} is in the unit of energy density or equivalently $\hbar v_f/a_0^4$. For the numerical calculations, we simply choose $\hbar v_f=1$ and $a_0=1$, and this is also convenient to compare with analytical results in Appendix E. The main results of our numerical calculations are summarized in Fig. 2 in the main text.

Below we provide some more understanding on the analytical aspects of the formalism. We consider the small $\mathbf{q}$ expansion of Eq. (F7) at $i\nu_m=0$, and perform the perturbation expansion. The calculation here is equivalent to the calculations in Appendix E, except that we below keep the chemical potential up to μ^3. For a small $\mathbf{q}=q_z\hat{e}_z$, we have the perturbation expansion

$$\xi_{s',\mathbf{k}-\mathbf{q}}\approx s'\hbar v_f k-\mu-s'\hbar v_f q\cos\theta_k=\xi_{s',\mathbf{k}}-s'\hbar v_f q_z\cos\theta_k, \tag{F16}$$

$$n_F(\xi_{s',\mathbf{k}-\mathbf{q}})\approx n_F(\xi_{s',\mathbf{k}})-\frac{\partial n_F}{\partial\xi_{s',\mathbf{k}}}s'\hbar v_f q_z\cos\theta_k, \tag{F17}$$

$$\cos\theta_{k'}\approx\cos\theta_k-\frac{q_z}{k}\sin^2\theta_k, \tag{F18}$$

up to the first order in q_z. Now let us denote $\Phi_{xx}^{xy}=\sum_{s,s'}\Phi_{xx,ss'}^{xy}$ and evaluate $\Phi_{xx,ss'}^{xy}$ separately. First, let us consider $s=s'$ and we have

$$\bar{G}_{xx}^{xy}(ss;k,\cos\theta_k,q_z)\approx\frac{is}{4}(\hbar v_f)^2\sin^4\theta_k kq_z, \tag{F19}$$

which only possesses the linear-q_z term. Thus,

$$\Phi_{xx,ss}^{xy}=\int\frac{k^2dkd\cos\theta_k}{(2\pi)^2}\frac{\partial n_F}{\partial\xi_{s,\mathbf{k}}}\bar{G}_{xx}^{xy}(ss;k,\cos\theta_k,q_z)=\frac{is}{4}(\hbar v_f)^2q_z\int_{-1}^1 d\cos\theta_k\sin^4\theta_k\int_0^\Lambda k^3dk\frac{\partial n_F}{\partial\xi_{s,\mathbf{k}}}. \tag{F20}$$

With $\frac{\partial n_F}{\partial\xi_{s,\mathbf{k}}}=-\frac{\beta e^{\beta\xi_{s,\mathbf{k}}}}{(1+e^{\beta\xi_{s,\mathbf{k}}})^2}$ and $x=\beta\xi_{s,\mathbf{k}}$, we find

$$\Phi_{xx,ss}^{xy}=-i\frac{4q_z}{15(2\pi)^2}\frac{(k_BT)^3}{(\hbar v_f)^2}\left(\int_{-\beta\mu}^\infty dx(x+\beta\mu)^3\frac{e^x}{(1+e^x)^2}+\int_{\beta\mu}^\infty dx(-x+\beta\mu)^3\frac{e^x}{(1+e^x)^2}\right). \tag{F21}$$

With

$$\int_{\beta\mu}^\infty dxx^2\frac{e^x}{(1+e^x)^2}\approx\frac{\pi^2}{6}-\frac{(\beta\mu)^3}{12}, \tag{F22}$$

$$\int_{\beta\mu}^\infty dx\frac{e^x}{(1+e^x)^2}\approx\frac{1}{2}-\frac{\beta\mu}{4}+\frac{(\beta\mu)^3}{48}, \tag{F23}$$

$$\int_{-\beta\mu}^{\beta\mu}dx(x+\beta\mu)^3\frac{e^x}{(1+e^x)^2}\approx(\beta\mu)^4, \tag{F24}$$

we have

$$\sum_s \Phi^{xy}_{xx,ss} = -i\frac{q_z}{15\pi^2(\hbar v_f)^2}[\mu^3 + \mu\pi^2(k_BT)^2]. \tag{F25}$$

For $s' = -s$,

$$\bar{G}^{xy}_{xx}(s, -s; k, \cos\theta_k, q_z) \approx \frac{is}{4}(\hbar v_f)^2 \sin^2\theta_k(2k^2\cos\theta_k - kq_z\sin^2\theta_k), \tag{F26}$$

which contains both q_z-independent and linear-q_z terms. In addition,

$$\begin{aligned}\frac{n_F(\xi_{-s,\mathbf{k}-\mathbf{q}}) - n_F(\xi_{s,\mathbf{k}})}{\xi_{-s,\mathbf{k}-\mathbf{q}} - \xi_{s,\mathbf{k}}} &\approx \frac{n_F(\xi_{-s,\mathbf{k}}) - n_F(\xi_{s,\mathbf{k}}) + \frac{\partial n_F}{\partial\xi_{-s,\mathbf{k}}}s\hbar v_f q\cos\theta_k}{-2s\hbar v_f k + s\hbar v_f q\cos\theta_k}\\
&= -\frac{1}{2s\hbar v_f k}\left(n_F(\xi_{-s,\mathbf{k}}) - n_F(\xi_{s,\mathbf{k}}) + q_z\cos\theta_k\left(\frac{\partial n_F}{\partial\xi_{-s,\mathbf{k}}}s\hbar v_f + \frac{1}{2k}(n_F(\xi_{-s,\mathbf{k}}) - n_F(\xi_{s,\mathbf{k}}))\right)\right)\end{aligned} \tag{F27}$$

up to the linear order in q_z. Putting all these results together, we have

$$\begin{aligned}\Phi^{xy}_{xx,s,-s} = &\int \frac{k^2dkd\cos\theta_k}{(2\pi)^2}\frac{(-i\hbar v_f q_z)}{8}\bigg(-(n_F(\xi_{-s,\mathbf{k}}) - n_F(\xi_{s,\mathbf{k}}))\sin^4\theta_k + 2k\cos^2\theta_k\sin^2\theta_k\\
&\times\left(\frac{\partial n_F}{\partial\xi_{-s,\mathbf{k}}}s\hbar v_f + \frac{1}{2k}(n_F(\xi_{-s,\mathbf{k}}) - n_F(\xi_{s,\mathbf{k}}))\right)\bigg)\\
= &\int \frac{k^2dk}{(2\pi)^2}\frac{(-i\hbar v_f q_z)}{15}\left(-\frac{3}{2}(n_F(\xi_{-s,\mathbf{k}}) - n_F(\xi_{s,\mathbf{k}})) + \frac{\partial n_F}{\partial\xi_{-s,\mathbf{k}}}s\hbar v_f k\right),\end{aligned} \tag{F28}$$

where we have used

$$\int_{-1}^{1} d\cos\theta_k \sin^4\theta_k = \frac{16}{15}, \tag{F29}$$

$$\int_{-1}^{1} d\cos\theta_k \sin^2\theta_k\cos^2\theta_k = \frac{4}{15}. \tag{F30}$$

In the above expression, since $\sum_s(n_F(\xi_{-s,\mathbf{k}}) - n_F(\xi_{s,\mathbf{k}})) = 0$, the first term will vanish after the summation over s. Thus,

$$\sum_s \Phi^{xy}_{xx,s,-s} = \sum_s \frac{(-is(\hbar v_f)^2 q_z)}{15}\int \frac{dk}{(2\pi)^2}k^3\frac{\partial n_F}{\partial\xi_{-s,\mathbf{k}}} = \sum_s \frac{is(\hbar v_f)^2 q_z}{15}\int\frac{dk}{(2\pi)^2}k^3\frac{\beta e^{\beta\xi_{-s,\mathbf{k}}}}{(e^{\beta\xi_{-s,\mathbf{k}}}+1)^2}. \tag{F31}$$

Now let us define $x = \beta\xi_{-s,\mathbf{k}}$ and find

$$\begin{aligned}\sum_s \Phi^{xy}_{xx,s,-s} &= \frac{-iq_z}{15\beta^3(2\pi\hbar v_f)^2}\left(\int_{\beta\mu}^{\infty}dx(-x+\beta\mu)^3\frac{e^x}{(1+e^x)^2} + \int_{-\beta\mu}^{\infty}dx(x+\beta\mu)^3\frac{e^x}{(1+e^x)^2}\right)\\
&\approx \frac{(-i)q_z}{15(2\pi\hbar v_f)^2}(\mu^3 + \mu\pi^2(k_BT)^2).\end{aligned} \tag{F32}$$

Putting all the results together, we have

$$\begin{aligned}\Phi^{xy}_{xx} &= \sum_{ss'}\Phi^{xy}_{xx,ss'} = -i\frac{q_z}{12\pi^2(\hbar v_f)^2}(\mu^3 + \mu\pi^2(k_BT)^2) = -i\frac{q_z\mu^3}{12\pi^2(\hbar v_f)^2} - i\frac{q_z\mu}{12(\hbar v_f)^2}(k_BT)^2\\
&= \frac{iq_z}{(\hbar v_f)^2}\mu\left(-\frac{\mu^2}{12\pi^2} - \frac{(k_BT)^2}{12}\right).\end{aligned} \tag{F33}$$

The second term reproduces the result of Eqs. (E15) and (E39) for the thermal Nieh-Yan anomaly (choosing $\hbar v_f = 1$), while the first term is of order μ^3 and thus not included in the calculations of Appendix E, which only keep the terms up to linear μ. Therefore, this analytical result is consistent with the results given in Appendix E. Equation (F33) can fit well with our numerical results at small μ, as shown in Fig. 2(d) in the main text.

APPENDIX G: ACOUSTIC PHONON DYNAMICS IN THE KRAMERS-WEYL SEMIMETALS

In this Appendix, we will analyze the influence on the effective action (D7) on the phonon dynamics. We focus on the acoustic phonons here, and as discussed in Appendix B, the A_0 and Δ fields are related to the strain tensor as $A_0(\tilde{q}) = C_1\bar{u}(\tilde{q})$ and $\Delta_a^j(\tilde{q}) = \frac{1}{\hbar v_f}[g_0\bar{u}(\tilde{q})\delta_{ja} + g_1 u_{ja}(\tilde{q})]$, where the strain tensor u_{ij} is treated as a fluctuating field for acoustic phonons and $\bar{u} = \sum_i u_{ii}$. By substituting the forms of A_0 and Δ fields, we find

$$W = \frac{1}{2}\sum_{\tilde{q}}\left(\bar{u}(\tilde{q})\Lambda_0(\tilde{q})\bar{u}(-\tilde{q}) + \sum_{i,a}\bar{u}(\tilde{q})\Lambda_{2,i}^a(\tilde{q})u_{ia}(-\tilde{q}) + \sum_{ij,ab}u_{ia}(\tilde{q})\Lambda_{3,ij}^{ab}(\tilde{q})u_{jb}(-\tilde{q})\right), \tag{G1}$$

where

$$\Lambda_0(\tilde{q}) = C_1^2\Pi_0(\tilde{q}) + \sum_{i,j}\frac{g_0^2}{(\hbar v_f)^2}\Phi_{ij}^{ij}(\tilde{q}) - \sum_i \frac{C_1 g_0}{\hbar v_f}\left[\Theta_i^i(\tilde{q}) + \Theta_i^i(-\tilde{q})\right], \tag{G2}$$

$$\Lambda_{2,i}^a(\tilde{q}) = \sum_j \frac{g_0 g_1}{(\hbar v_f)^2}\left[\Phi_{ji}^{ja}(\tilde{q}) + \Phi_{ij}^{aj}(-\tilde{q})\right] - \frac{2C_1 g_1}{\hbar v_f}\Theta_i^a(\tilde{q}), \tag{G3}$$

$$\Lambda_{3,ij}^{ab}(\tilde{q}) = \frac{g_1^2}{(\hbar v_f)^2}\Phi_{ij}^{ab}(\tilde{q}). \tag{G4}$$

We can expand $\Lambda_0(\tilde{q})$, $\Lambda_{2,i}^a(\tilde{q})$, and $\Lambda_{3,ij}^{ab}(\tilde{q})$ as a function of $\tilde{q}$, $\Lambda(\tilde{q}) = \Lambda(0) + (\partial_{\mathbf{q}}\Lambda)_{\tilde{q}=0}\cdot\mathbf{q} + (\partial_{\omega_n}\Lambda)_{\tilde{q}=0}\omega_n + \ldots$. At the zero-order term, we have

$$\begin{aligned} W^{(0)} &= \frac{1}{2}\sum_{\tilde{q}}\left(\bar{u}(\tilde{q})\Lambda_0(0)\bar{u}(-\tilde{q}) + \sum_{i,a}\bar{u}(\tilde{q})\Lambda_{2,i}^a(0)u_{ia}(-\tilde{q}) + \sum_{ij,ab}u_{ia}(\tilde{q})\Lambda_{3,ij}^{ab}(0)u_{jb}(-\tilde{q})\right) \\ &= \frac{1}{2}\int_0^\beta d\tau\int d^3r\left(\bar{u}(\mathbf{r},\tau)\Lambda_0(0)\bar{u}(\mathbf{r},\tau) + \sum_{i,a}\bar{u}(\mathbf{r},\tau)\Lambda_{2,i}^a(0)u_{ia}(\mathbf{r},\tau) + \sum_{ij,ab}u_{ia}(\mathbf{r},\tau)\Lambda_{3,ij}^{ab}(0)u_{jb}(\mathbf{r},\tau)\right). \end{aligned} \tag{G5}$$

Here all the terms are quadratic in the strain tensor u_{ij} and let us consider the general form of the effective action for acoustic phonons (elastic wave), which is defined as [108,109]

$$S_{ph,0} = \frac{1}{2}\int dt d^3r\left(\rho\sum_j \partial_t u_j\partial_t u_j - \sum_{ijkl}\lambda_{ijkl}\partial_i u_j\partial_k u_l\right), \tag{G6}$$

where the rank-4 tensor λ_{ijkl} is the elastic modulus. It is clear that $W^{(0)}$ just provides the correction of the elastic modulus λ_{ijkl}.

Next we focus on the first-order term with the linear $\mathbf{q}$ dependence, which takes the form

$$\begin{aligned} W^{(1)} &= \frac{1}{2}\sum_{\tilde{q}}\left(\sum_i(\partial_i\Lambda_0)\bar{u}(\tilde{q})q_i\bar{u}(-\tilde{q}) + \sum_{ij,a}(\partial_j\Lambda_{2,i}^a)\bar{u}(\tilde{q})q_j u_{ia}(-\tilde{q}) + \sum_{ijl,ab}(\partial_l\Lambda_{3,ij}^{ab})u_{ia}(\tilde{q})q_l u_{jb}(-\tilde{q})\right) \\ &= \frac{1}{2}\int_0^\beta d\tau\int d^3r\left(\sum_j(\partial_j\Lambda_0)\bar{u}(\mathbf{r},\tau)\left(i\frac{\partial}{\partial r_j}\bar{u}(\mathbf{r},\tau)\right) + \sum_{ij,a}(\partial_j\Lambda_{2,i}^a)\bar{u}(\mathbf{r},\tau)\left(i\frac{\partial}{\partial r_j}u_{ia}(\mathbf{r},\tau)\right)\right. \\ &\qquad\left. + \sum_{ijk,ab}(\partial_k\Lambda_{3,ij}^{ab})u_{ia}(\mathbf{r},\tau)\left(i\frac{\partial}{\partial r_k}u_{jb}(\mathbf{r},\tau)\right)\right), \end{aligned} \tag{G7}$$

where $(\partial_j\Lambda) = (\frac{\partial\Lambda}{\partial q_j})_{\tilde{q}=0}$ is just a number. From Eq. (G4), one can see that the last term is just the Nieh-Yan term. Although the above expression seems quite complex, the symmetry gives a strong constraint on the form of the allowed effective action. It is clear that the above action contributes to the following general form of the effective action of acoustic phonons

$$S_{ph,1} = \int dt d^3r\sum_{ijklm}\xi_{ijklm}(\partial_i u_{jk})u_{lm}, \tag{G8}$$

where $\partial_i = \frac{\partial}{\partial r_i}$. Now the question is what is the most general form of the coefficient ξ_{ijklm}, a rank-5 tensor, for a uniform isotropic system. Since the strain tensor u_{ij} is symmetric with respect to the indices i and j, the coefficient ξ_{ijklm} should satisfy the relations $\xi_{ijklm} = \xi_{ikjlm} = \xi_{ijkml}$. Furthermore, it also satisfies the anti-symmetric relation $\xi_{ijklm} = -\xi_{ilmjk}$. These relations together with the full rotation symmetry will fix the form of ξ_{ijklm}.

It is also equivalent to ask how to construct an invariant term with the form of $(\partial_i u_{jk})u_{lm}$. We analyze this problem based on the irreducible representation of the angular momentum. As discussed in Appendix B, the symmetric rank-2 tensor u_{ij} can be decomposed to two parts, $\bar{u}$ with the angular momentum $\mathbf{0}$ and $u_{J=2,M}$ with the angular momentum $\mathbf{2}$. Since the expression involves two strain tensors, the corresponding angular momentum decomposition can be given by

$$\mathbf{2} \otimes \mathbf{2} = \mathbf{4} \oplus \mathbf{3} \oplus \mathbf{2} \oplus \mathbf{1} \oplus \mathbf{0}, \tag{G9}$$

$$\mathbf{2} \otimes \mathbf{0} = \mathbf{2}, \tag{G10}$$

$$\mathbf{0} \otimes \mathbf{0} = \mathbf{0}, \tag{G11}$$

where the bold number represents the total angular momentum. Furthermore, the expression $(\partial_i u_{jk})u_{lm}$ involves one ∂_i in addition to two strain tensors, and ∂_i possesses the angular momentum $\mathbf{1}$. To make the term $(\partial_i u_{jk})u_{lm}$ invariant (with the angular momentum $\mathbf{0}$), only the irreducible representation with the angular momentum $\mathbf{1}$ in $\mathbf{2} \otimes \mathbf{2}$ can lead to $\mathbf{1} \otimes \mathbf{1} = \mathbf{2} \oplus \mathbf{1} \oplus \mathbf{0}$. From such angular momentum combination, only a single term is allowed for ξ_{ijklm} in a uniform isotropic system. This term can be explicitly constructed as

$$S_{ph,1} = \xi_0 \int dtd^3r \sum_{ijklm} \delta_{jl}\epsilon_{ikm}(\partial_i u_{jk})u_{lm}, \tag{G12}$$

so that the parameter ξ_{ijklm} is given by

$$\xi_{ijklm} = \frac{\xi_0}{4}(\delta_{jl}\epsilon_{ikm} + \delta_{kl}\epsilon_{ijm} + \delta_{jm}\epsilon_{ikl} + \delta_{km}\epsilon_{ijl}). \tag{G13}$$

We can apply the above analysis to the action $W^{(1)}$, and it is clear that it is impossible to construct an invariant term from the first two terms in Eq. (G7), and thus the coefficient $\partial_j \Lambda_0$ and $\partial_j \Lambda^a_{2,i}$ must vanish. The effective action then takes the form

$$W^{(1)} = \frac{1}{2}\int_0^\beta d\tau \int d^3r \left(\sum_{ijk,ab} (i\partial_k \Lambda^{ab}_{3,ij}) u_{ia}(\mathbf{r},\tau) \times \left(\frac{\partial}{\partial r_k} u_{jb}(\mathbf{r},\tau) \right) \right), \tag{G14}$$

with the coefficient

$$\begin{aligned}(i\partial_k \Lambda^{ab}_{3,ij}) &= i\frac{g_1^2}{(\hbar v_f)^2}\left(\partial_k \Phi^{ab}_{ij}(\tilde{q})\right)_{\tilde{q}=0} = i\frac{g_1^2}{(\hbar v_f)^2}\Phi^{ab}_{ij,k} \\ &= \frac{g_1^2}{(\hbar v_f)^4}\epsilon^{abk}\delta_{ij}\chi\mu\mathcal{F}_0. \end{aligned} \tag{G15}$$

Here we have used Eq. (E15) and restored the coefficient $(\hbar v_f)^2$ in $\Phi^{ab}_{ij,k}$. Substituting the coefficient and also changing from imaginary time to real time, we have

$$\begin{aligned} W^{(1)} &= -\frac{g_1^2 \chi\mu\mathcal{F}_0}{2(\hbar v_f)^4} \sum_{ijk,ab} \int dtd^3r \epsilon^{abk}\delta_{ij} u_{ia}(\mathbf{r},\tau) \\ &\quad \times \left(\frac{\partial}{\partial r_k} u_{jb}(\mathbf{r},\tau) \right) \\ &= \frac{g_1^2 \chi\mu\mathcal{F}_0}{2(\hbar v_f)^4} \sum_{ijk,ab} \int dtd^3r \epsilon^{kab}\delta_{ij} \\ &\quad \times \left(\frac{\partial}{\partial r_k} u_{ia}(\mathbf{r},\tau) \right) u_{jb}(\mathbf{r},\tau). \end{aligned} \tag{G16}$$

Comparing with $S_{ph,1}$ in Eq. (G12), one can see that the Nieh-Yan term can contribute to a new term in the phonon effective action and the corresponding coefficient is given by

$$\xi_0 = \frac{g_1^2 \chi\mu\mathcal{F}_0}{2(\hbar v_f)^4}. \tag{G17}$$

We may further use the derivation of Eq. (F33) to get a more complete result, up to the order of μ^3. We also include multiple KW fermions and obtain

$$\xi_0 = -\frac{g_1^2}{(\hbar v_f)^4}\left(\frac{\langle \mu^3 \rangle_\chi}{12\pi^2} + \frac{\langle \mu \rangle_\chi}{12}(k_B T)^2 \right), \tag{G18}$$

where $\langle \mu \rangle_\chi = \frac{1}{2}\sum_{\mathbf{\Gamma_i}} \chi_{\mathbf{\Gamma_i}} \mu_{\mathbf{\Gamma_i}}$ and $\langle \mu^3 \rangle_\chi = \frac{1}{2}\sum_{\mathbf{\Gamma_i}} \chi_{\mathbf{\Gamma_i}} \mu^3_{\mathbf{\Gamma_i}}$ represent the average of μ and μ^3 terms over all the $\mathbf{\Gamma}_i$ momenta, respectively. Here $\chi_{\mathbf{\Gamma_i}}$ is the chirality and $\mu_{\mathbf{\Gamma_i}}$ is the chemical potential with respect to the Weyl node for the KW fermion at $\mathbf{\Gamma_i}$.

In the above discussion, there is another term with the form of $u_{ij}\omega_n u_{kl}$ at the same order and in the time-domain, this term should have the form of $u_{ij}(\partial_\tau u_{kl})$. In the elastic theory, this term is nothing but the viscosity, generally defined as [108–110]

$$S_{ph,2} = \int dtd^3r \sum_{ijkl} \eta_{ijkl}(\partial_t u_{jk})u_{lm}. \tag{G19}$$

However, the existence of the viscosity requires the breaking of TR symmetry, while TR symmetry exists in our system. Thus, the viscosity coefficient η_{ijkl} extracted from our model must vanish.

Now we hope to solve the acoustic phonon dispersion (or elastic wave) in our system based on the actions (G6) and (G12). For the uniform isotropic system, there are two independent parameters for the elastic moduli λ_{ijkl} and the corresponding equation of motion for the displacement field $\mathbf{u}$ is given by

$$\frac{d^2}{dt^2}\mathbf{u} = c_t^2\nabla^2\mathbf{u} + \left(c_l^2 - c_t^2\right)\nabla(\nabla\cdot\mathbf{u}), \tag{G20}$$

where c_l and c_t are the velocities of longitudinal and transverse waves. The Nieh-Yan term $S_{ph,1}$ can add an additional term into the above elastic wave equation, as given by

$$\begin{aligned}\frac{\delta S_{ph,1}}{\delta u_n} &= \frac{\xi_0}{2}\epsilon_{ikm}(\delta_{kn}\partial_i\partial_j u_{jm} + \delta_{jn}\partial_i\partial_k u_{jm} - \delta_{mn}\partial_i\partial_j u_{jk} - \delta_{jn}\partial_i\partial_m u_{jk})\\ &= \frac{\xi_0}{2}(\epsilon_{inm}\partial_i\partial_j u_{jm} + \epsilon_{ikm}\partial_i\partial_k u_{nm} - \epsilon_{ikn}\partial_i\partial_j u_{jk} - \epsilon_{ikm}\partial_i\partial_m u_{nk})\\ &= \frac{\xi_0}{2}(\epsilon_{inm}\partial_i\partial_j u_{jm} - \epsilon_{ikn}\partial_i\partial_j u_{jk}) = \frac{\xi_0}{2}\epsilon_{inm}\partial_i\partial_j(\partial_j u_m + \partial_m u_j) = \frac{\xi_0}{2}\epsilon_{inm}\partial_i(\nabla^2 u_m) = -\frac{\xi_0}{2}(\nabla\times(\nabla^2\mathbf{u}))_n. \end{aligned} \tag{G21}$$

By including the Nieh-Yan term, the equation of motion for the displacement field is now written as

$$\frac{d^2}{dt^2}\mathbf{u} = c_t^2\nabla^2\mathbf{u} + (c_l^2 - c_t^2)\nabla(\nabla\cdot\mathbf{u}) + \frac{\xi_0}{2\rho_0}\nabla\times(\nabla^2\mathbf{u}), \tag{G22}$$

where ρ_0 is the mass density of the elastic media. We can decompose the displacement field into the longitudinal and transverse components, $\mathbf{u} = \mathbf{u_l} + \mathbf{u_t}$. One can show that the longitudinal and transverse components are still decoupled from each other. The longitudinal part does not have any correction from the Nieh-Yan term, while the equation of motion for the transverse part is revised as

$$\frac{d^2}{dt^2}\mathbf{u}_t = c_t^2\nabla^2\mathbf{u}_t + \frac{\xi_0}{2\rho_0}\nabla\times(\nabla^2\mathbf{u}_t). \tag{G23}$$

Let us take $\mathbf{u}_t = \mathbf{u}_0 e^{i\mathbf{q}\cdot\mathbf{r}-i\omega t}$ and the equation of motion is changed to

$$\omega^2\mathbf{u}_0 = c_t^2 q^2\mathbf{u}_0 + \frac{\xi_0 q^2}{2\rho_0} i\mathbf{q}\times\mathbf{u}_0. \tag{G24}$$

Now let us choose $\mathbf{q} = q\hat{\mathbf{e}}_\mathbf{z}$ and $\mathbf{u}_0 = (u_{0x}, u_{0y}, 0)^T$ (the superscript T here represents transpose) and then the equation of motion (G24) can be written as

$$\omega^2\begin{pmatrix}u_{0x}\\ u_{0y}\end{pmatrix} = \begin{pmatrix}c_t^2 q^2 & -i\frac{\xi_0}{2\rho_0}q^3\\ i\frac{\xi_0}{2\rho_0}q^3 & c_t^2 q^2\end{pmatrix}\begin{pmatrix}u_{0x}\\ u_{0y}\end{pmatrix}. \tag{G25}$$

This eigen equation can be easily solved. Two branches of dispersion relations for the transverse modes are given by

$$\omega_s^t = \sqrt{c_t^2 q^2 + s\frac{|\xi_0|}{2\rho_0}q^3} = q\sqrt{c_t^2 + s\frac{|\xi_0|}{2\rho_0}q}, \tag{G26}$$

from which one can see that the Nieh-Yan term mainly corrects the phonon velocity and gives rise to the different velocities for two transverse phonon modes. The corresponding normalized eigen vectors are given by

$$\mathbf{u}_{0,s} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ is\frac{\xi_0}{|\xi_0|}\\ 0\end{pmatrix}, \tag{G27}$$

so that $\mathbf{u}_{0,s}^\dagger(\mathbf{k})\mathbf{u}_{0,s}(\mathbf{k}) = 1$. The angular momentum of the phonon modes is defined as [60,66–69]

$$l_{s,i}(\mathbf{q}) = \hbar\mathbf{u}_{0,s}^\dagger(\mathbf{q})M_i\mathbf{u}_{0,s}(\mathbf{q}), \tag{G28}$$

where $s = \pm$, $i = x, y, z$, and $(M_i)_{jk} = (-i)\epsilon_{ijk}$. One can easily show that only $l_{s,z}$ is nonzero and $l_{\pm,z} = \pm\hbar$ for the momentum $\mathbf{q} = q\hat{\mathbf{e}}_\mathbf{z}$. Due to the full rotation symmetry of our model, the general form of angular momentum for the s-mode is given by

$$\mathbf{l}_s = s\hbar\hat{\mathbf{q}}\frac{\xi_0}{|\xi_0|}, \tag{G29}$$

with $\hat{\mathbf{q}} = \frac{\mathbf{q}}{q}$.

Next let us consider the phonon total angular momentum induced by a temperature gradient. The total phonon angular momentum per volume can be related to the temperature gradient by

$$I_i^{ph} = \alpha_{ij}\frac{\partial T}{\partial r_j}, \tag{G30}$$

where $i, j = x, y, z$ and $\mathbf{r}$ is the spatial coordinate. From Ref. [60], the response coefficient α_{ij} is derived as

$$\alpha_{ij} = -\frac{\tau}{V}\sum_{\mathbf{q},s} l_{s,i}v_{s,j}^{ph}\frac{\partial f_0(\omega_s^t)}{\partial T} \tag{G31}$$

in the linear response regime, where τ is the phonon relaxation time, $\mathbf{v}_s^{ph}$ is the group velocity of s-phonon mode and $f_0(\omega_s^t)$ is the Bose distribution function for phonons. The group velocity of the transverse mode should be given by

$$\mathbf{v}_s^{ph} = \frac{\partial\omega_s^t}{\partial\mathbf{q}} = \frac{\partial\omega_s^t}{\partial q}\frac{\partial q}{\partial\mathbf{q}} = \frac{c_t^2 + \frac{3s}{4\rho_0}|\xi_0|q}{\sqrt{c_t^2 + \frac{s}{2\rho_0}|\xi_0|q}}\hat{\mathbf{q}}. \tag{G32}$$

For our isotropic system, there is only a single nonzero independent component $\alpha_{xx} = \alpha_{yy} = \alpha_{zz}$, so let us evaluate α_{zz}, which is given by

$$\begin{aligned}\alpha_{zz} &= -\tau\hbar\int\frac{d^3q}{(2\pi)^3}\sum_s\frac{sq_z^2}{q^2}\frac{\xi_0}{|\xi_0|}\frac{c_t^2 + \frac{3s}{4\rho_0}|\xi_0|q}{\sqrt{c_t^2 + \frac{s}{2\rho_0}|\xi_0|q}}\frac{\partial f_0}{\partial T}\\ &= -\frac{\tau\hbar}{6\pi^2}\int dq\sum_s sq^2\frac{\xi_0}{|\xi_0|}\frac{c_t^2 + \frac{3s}{4\rho_0}|\xi_0|q}{\sqrt{c_t^2 + \frac{s}{2\rho_0}|\xi_0|q}}\frac{\partial f_0}{\partial T},\end{aligned} \tag{G33}$$

where

$$\frac{\partial f_0(\omega)}{\partial T} = k_B\beta^2\hbar\omega\frac{e^{\beta\hbar\omega}}{(e^{\beta\hbar\omega}-1)^2}. \tag{G34}$$

Here we have recovered $\hbar$ to get the correct unit and neglected the temperature dependence in ξ_0. This expression can be evaluated numerically. The angular part of the phonon momentum in the above expression can be integrated out since $\frac{\partial f_0(\omega)}{\partial T}$ only depends on the magnitude of $\mathbf{q}$. To get some analytical understanding, we can make the further approximations. According to the Bose distribution $f_0(\omega)$, we know that the phonons are mainly excited in the energy range determined by the temperature T, $\hbar\omega \ll k_B T$. This means the contribution in the above expression mainly comes from the momentum range $q < \frac{k_B T}{\hbar c_t}$. Now let us consider the momentum range, in which the Nieh-Yan term contribution to the velocity $|\xi_0|q/\rho_0$ is much smaller than c_t^2. This is naturally satisfied when the temperature $k_B T \ll \frac{\hbar c_t^3 \rho_0}{|\xi_0|}$ and in this temperature range, we can treat $|\xi_0|q$ as a perturbation and expand the integrand in Eq. (G33). First, we have

$$\frac{c_t^2 + \frac{3s}{4\rho_0}|\xi_0|q}{\sqrt{c_t^2 + \frac{s}{2\rho_0}|\xi_0|q}} \approx c_t\left(1 + \frac{3s}{4\rho_0 c_t^2}|\xi_0|q\right)\left(1 - \frac{s}{4\rho_0 c_t^2}|\xi_0|q\right) \approx c_t\left(1 + \frac{s}{2\rho_0 c_t^2}|\xi_0|q\right) \tag{G35}$$

and

$$\frac{\partial f_0}{\partial T} = \frac{\partial f_0(\omega_0)}{\partial T} + \frac{\partial^2 f_0}{\partial T \partial \omega}\delta\omega_s, \tag{G36}$$

where $\omega_0 = c_t q$ and $\delta\omega_s = \frac{s|\xi_0|}{4\rho_0 c_t}q^2$. Let us define $x = \beta\hbar\omega_0 = \frac{\hbar c_t q}{k_B T}$ and then we obtain

$$\frac{\partial f_0(\omega_0)}{\partial T} = k_B\beta^2\hbar\omega_0\frac{e^{\beta\hbar\omega_0}}{(e^{\beta\hbar\omega_0}-1)^2} = \frac{x}{T}\frac{e^x}{(e^x-1)^2}, \tag{G37}$$

and

$$\begin{aligned}\frac{\partial^2 f_0(\omega_0)}{\partial T \partial \omega} &= k_B\beta^2\hbar\frac{e^{\beta\hbar\omega_0}}{(e^{\beta\hbar\omega_0}-1)^3} \\ &\quad\times[e^{\beta\hbar\omega_0}(1-\beta\hbar\omega_0) - \beta\hbar\omega_0 - 1] \\ &= \frac{\hbar}{k_B T^2}\frac{e^x}{(e^x-1)^3}[e^x(1-x) - x - 1].\end{aligned} \tag{G38}$$

Thus, we have

$$\begin{aligned}\alpha_{zz} &\approx -\frac{\tau\hbar}{6\pi^2}\int q^2 dq \sum_s s c_t\frac{\xi_0}{|\xi_0|}\left(1 + \frac{s}{2\rho_0 c_t^2}|\xi_0|q\right)\left(\frac{\partial f_0(\omega_0)}{\partial T} + \frac{\partial^2 f_0}{\partial T\partial\omega}\frac{s|\xi_0|}{4\rho_0 c_t}q^2\right) \\ &= -\frac{\tau\hbar}{6\pi^2}\int q^2 dq c_t\left(\frac{\partial f_0(\omega_0)}{\partial T}\frac{\xi_0}{\rho_0 c_t^2}q + \frac{\partial^2 f_0}{\partial T\partial\omega}\frac{\xi_0}{2\rho_0 c_t}q^2\right) \\ &= -\frac{\tau\hbar}{6\pi^2}\left(\frac{k_B T}{c_t}\right)^3\frac{\xi_0}{\rho_0 c_t^2\hbar^3}\int x^2 dx\left(\frac{\partial f_0(\omega_0)}{\partial T}\frac{x k_B T}{\hbar} + \frac{\partial^2 f_0}{\partial T\partial\omega}\frac{x^2}{2\hbar^2}(k_B T)^2\right) \\ &= -\frac{\tau\xi_0 k_B}{6\pi^2\rho_0 c_t^5\hbar^3}(k_B T)^3\int x^4 dx\left(\frac{e^x}{(e^x-1)^2} + \frac{1}{2}\frac{e^x}{(e^x-1)^3}[e^x(1-x) - x - 1]\right)\end{aligned} \tag{G39}$$

In the above expression, the integral of x is convergent, and thus just gives a number. Therefore, all the temperature dependence is given by the coefficient before the x-integral. With the expression (G18) for ξ_0, we expect α_{zz} has T^3 dependence for the standard Nieh-Yan term and T^5 dependence for the thermal Nieh-Yan term. Let us denote $c_0 = \int x^4 dx(\frac{e^x}{(e^x-1)^2} + \frac{1}{2}\frac{e^x}{(e^x-1)^3}[e^x(1-x) - x - 1]) = -4\pi^4/15$, and then we have

$$\begin{aligned}\alpha_{zz} &\approx -\frac{\tau c_0 \xi_0 k_B}{6\pi^2\rho_0 c_t^5\hbar^3}(k_B T)^3 \\ &= \frac{\tau c_0 k_B g_1^2}{6\pi^2\rho_0(\hbar v_f)^4 c_t^5\hbar^3}\left(\frac{\langle\mu^3\rangle_\chi}{12\pi^2}(k_B T)^3 + \frac{\langle\mu\rangle_\chi}{12}(k_B T)^5\right).\end{aligned} \tag{G40}$$

To give a reasonable estimate of the magnitude of the effect, we rewrite the above equation as

$$\alpha_{zz} = -\frac{c_0}{6\pi^2}\frac{\hbar}{T a_0^2}\left(\frac{\tau k_B T}{\hbar}\right)\left(\frac{k_B T a_0}{\hbar c_t}\right)^3\left(\frac{\xi_0}{c_t^2\rho_0 a_0}\right), \tag{G41}$$

where a_0 is some characteristic length scale and we can choose it as the lattice constant. The overall unit of α_{zz} is given by $\frac{\hbar}{T a_0^2}$ and the remaining parts in the above expression are dimensionless, making it convenient for us to estimate the magnitude. Let us next estimate each term separately. We consider the temperature $T \sim 300$ K (around room temperature). $\hbar c_t/a_0$ is the acoustic phonon energy at the Brillouin zone boundary, which is estimated as $\hbar c_t/a_0 \sim 10$ meV. This corresponds to the speed of sound wave around $c_t \sim 1500$ m/s for a lattice constant $a_0 \sim 1$Å, which is typical in a metal. With these assumptions, the dimensionless factor $(\frac{k_B T a_0}{\hbar c_t})^3 \sim 17.3$. $\hbar\sqrt{\frac{\xi_0}{\rho_0 a_0^3}}$ is the splitting of two transverse phonon modes at the Brillouin zone boundary induced by Nieh-Yan term. We first give an estimate of ξ_0 based on the expression (G18). We choose $\mu \sim 0.1$ eV, $v_f \sim 10^5$ m/s (or $\hbar v_f \sim 0.6$ eV Å) and $T \sim 300$ K. The strain-electron coupling constant g_1 share the same unit as $\hbar v_f$ and we choose $g_1 \sim 0.1$ eVÅ. With these choices of parameters, we have $\xi_0 \sim 7.6\times10^{-7}$ eV/Å^2. With $\rho_0 \sim 10^3$ kg/m^3, we have $\hbar\sqrt{\frac{\xi_0}{\rho_0 a_0^3}} \sim 0.07$ meV. Thus, the dimensionless factor $(\frac{\xi_0}{c_t^2\rho_0 a_0}) \sim 5\times10^{-5}$. With the above estimates, the response coefficient $\alpha_{zz} \sim 5.5\times10^{-7}(\frac{\tau}{1s})\frac{Js}{Km^2}$.

APPENDIX H: ELECTRON-ELECTRON INTERACTION

Since the electron-phonon interaction may be influenced by electron-electron interaction, one may wonder how electron-electron interaction affects the proposed phenomena here. In this Appendix, we will demonstrate that electron-electron interaction is actually decoupled from the phonon modes discussed in this work and thus all the results remain valid even taking into account electron-electron interaction.

To see that, let us start from the effective action $S_{\rm eff}$ in Eq. (C1) with an additional electron-electron Coulomb interaction. We consider the Stratonovich-Hubbard transformation by introducing an additional φ field with the effective action

$$S_C=\int d\tau d^3r\big(\tfrac{1}{2}[\nabla\varphi(\mathbf{r},\tau)]^2+ie\varphi(\mathbf{r},\tau)\hat{\psi}^\dagger_{\mathbf{\Gamma_i}}(\mathbf{r},\tau)\hat{\psi}_{\mathbf{\Gamma_i}}(\mathbf{r},\tau)\big). \tag{H1}$$

With the Fourier transform

$$\hat{\psi}_{\mathbf{\Gamma_i}}(\mathbf{r},\tau)=\sum_{\tilde{k}}e^{i\mathbf{k}\cdot\mathbf{r}-i\omega_n\tau}\hat{\psi}_{\mathbf{\Gamma_i}}(\tilde{k}), \tag{H2}$$

$$\varphi(\mathbf{r},\tau)=\sum_{\tilde{q}}e^{i\mathbf{q}\cdot\mathbf{r}-i\nu_m\tau}\varphi(\tilde{q}), \tag{H3}$$

we have

$$S_C=\sum_{\tilde{q}}\frac{q^2}{2}\varphi(\tilde{q})\varphi(-\tilde{q})+ie\sum_{\tilde{k},\tilde{q}}\varphi(\tilde{q})\hat{\psi}^\dagger_{\mathbf{\Gamma_i}}(\tilde{k})\hat{\psi}_{\mathbf{\Gamma_i}}(\tilde{k}-\tilde{q}), \tag{H4}$$

where $\sum_{\tilde{k}}=\frac{1}{\beta V}\sum_{\mathbf{k},i\omega_n}$. Now let us perform the transformation

$$\tilde{\varphi}(-\tilde{q})=\varphi(-\tilde{q})+i\frac{e}{q^2}\sum_{\tilde{k}}\hat{\psi}^\dagger_{\mathbf{\Gamma_i}}(\tilde{k})\hat{\psi}_{\mathbf{\Gamma_i}}(\tilde{k}-\tilde{q}) \tag{H5}$$

$$\tilde{\varphi}(\tilde{q})=\varphi(\tilde{q})+i\frac{e}{q^2}\sum_{\tilde{k}}\hat{\psi}^\dagger_{\mathbf{\Gamma_i}}(\tilde{k})\hat{\psi}_{\mathbf{\Gamma_i}}(\tilde{k}+\tilde{q}) \tag{H6}$$

and

$$\begin{aligned}\sum_{\tilde{q}}\frac{q^2}{2}\tilde{\varphi}(\tilde{q})\tilde{\varphi}(-\tilde{q})&=\sum_{\tilde{q}}\frac{q^2}{2}\left(\varphi(\tilde{q})+i\frac{e}{q^2}\sum_{\tilde{k}}\hat{\psi}^\dagger_{\mathbf{\Gamma_i}}(\tilde{k})\hat{\psi}_{\mathbf{\Gamma_i}}(\tilde{k}+\tilde{q})\right)\left(\varphi(-\tilde{q})+i\frac{e}{q^2}\sum_{\tilde{k}}\hat{\psi}^\dagger_{\mathbf{\Gamma_i}}(\tilde{k})\hat{\psi}_{\mathbf{\Gamma_i}}(\tilde{k}-\tilde{q})\right)\\&=\sum_{\tilde{q}}\left(\frac{q^2}{2}\varphi(\tilde{q})\varphi(-\tilde{q})+ie\varphi(\tilde{q})\sum_{\tilde{k}}\hat{\psi}^\dagger_{\mathbf{\Gamma_i}}(\tilde{k})\hat{\psi}_{\mathbf{\Gamma_i}}(\tilde{k}-\tilde{q})-\frac{e^2}{2q^2}\sum_{\tilde{k}\tilde{k}'}\hat{\psi}^\dagger_{\mathbf{\Gamma_i}}(\tilde{k})\hat{\psi}_{\mathbf{\Gamma_i}}(\tilde{k}+\tilde{q})\hat{\psi}^\dagger_{\mathbf{\Gamma_i}}(\tilde{k}')\hat{\psi}_{\mathbf{\Gamma_i}}(\tilde{k}'-\tilde{q})\right).\end{aligned} \tag{H7}$$

From the above quality, we have

$$S_C=\sum_{\tilde{q}}\frac{q^2}{2}\tilde{\varphi}(\tilde{q})\tilde{\varphi}(-\tilde{q})+\frac{e^2}{2q^2}\sum_{\tilde{q}\tilde{k}\tilde{k}'}\hat{\psi}^\dagger_{\mathbf{\Gamma_i}}(\tilde{k})\hat{\psi}_{\mathbf{\Gamma_i}}(\tilde{k}+\tilde{q})\hat{\psi}^\dagger_{\mathbf{\Gamma_i}}(\tilde{k}')\hat{\psi}_{\mathbf{\Gamma_i}}(\tilde{k}'-\tilde{q}), \tag{H8}$$

the latter term of which is indeed four-fermion Coulomb interaction with the coefficient $V_q=\frac{e^2}{q^2}$ the Fourier transform of Coulomb interaction in the momentum space (we have set $\epsilon_0=1$).

Now let us compare the second term in S_C [Eq. (H1)] with the first term in S_1 [Eq. (C3)], and one can see that the A_0 field is identical to the $-ie\varphi$ field. Similar to the derivation for the effective action $W[A_0,\Delta]$ in Eq. (C15), we consider the effective action for the φ and $\mathbf{\Delta}$ fields (we drop the A_0 field here) by integrating out the fermion operators, and the resulting effective action is given by

$$\begin{aligned}W_1[\varphi,\mathbf{\Delta}]&=\frac{1}{2}\sum_{\tilde{q}}\big[q^2\varphi(\tilde{q})\varphi(-\tilde{q})-e^2\Pi_0(\tilde{q})\varphi(\tilde{q})\varphi(-\tilde{q})+\Phi^{ab}_{ij}(\tilde{q})\Delta^i_a(\tilde{q})\Delta^j_b(-\tilde{q})+ie\Theta^a_i(\tilde{q})\varphi(\tilde{q})\Delta^i_a(-\tilde{q})\\&\qquad+ie\Theta^a_i(-\tilde{q})\Delta^i_a(\tilde{q})\varphi(-\tilde{q})\big]\\&=\frac{1}{2}\sum_{\tilde{q}}\big[D^{-1}(\tilde{q})\varphi(\tilde{q})\varphi(-\tilde{q})+\Phi^{ab}_{ij}(\tilde{q})\Delta^i_a(\tilde{q})\Delta^j_b(-\tilde{q})+ie\Theta^a_i(\tilde{q})\varphi(\tilde{q})\Delta^i_a(-\tilde{q})+ie\Theta^a_i(-\tilde{q})\Delta^i_a(\tilde{q})\varphi(-\tilde{q})\big],\end{aligned} \tag{H9}$$

where the duplicated indices should be summed over. Here $D^{-1}(\tilde{q})=q^2-e^2\Pi_0(\tilde{q})=D_0^{-1}-e^2\Pi_0(\tilde{q})$ defines the full Green's function for the φ field.

Next let us consider the path integral

$$Z_1[\mathbf{\Delta}]=\int\mathcal{D}\varphi e^{-W_1[\varphi,\mathbf{\Delta}]} \tag{H10}$$

and we can perform the transformation

$$\tilde{\varphi}(\tilde{q})=\varphi(\tilde{q})+ie\Theta^a_i(-\tilde{q})\Delta^i_a(\tilde{q})D(\tilde{q}) \tag{H11}$$

$$\tilde{\varphi}(-\tilde{q})=\varphi(-\tilde{q})+ie\Theta^a_i(\tilde{q})\Delta^i_a(-\tilde{q})D(-\tilde{q}). \tag{H12}$$

Correspondingly,

$$\frac{1}{2}\sum_{\tilde{q}} D^{-1}(\tilde{q})\tilde{\varphi}(\tilde{q})\tilde{\varphi}(-\tilde{q}) = \frac{1}{2}\sum_{\tilde{q}} D^{-1}(\tilde{q})[\varphi(\tilde{q}) + ie\Theta_i^a(-\tilde{q})\Delta_a^i(\tilde{q})D(\tilde{q})][\varphi(-\tilde{q}) + ie\Theta_j^b(\tilde{q})\Delta_b^j(-\tilde{q})D(-\tilde{q})]$$
$$= \frac{1}{2}\sum_{\tilde{q}} \big[D^{-1}(\tilde{q})\varphi(\tilde{q})\varphi(-\tilde{q}) + ie\varphi(\tilde{q})\Theta_j^b(\tilde{q})\Delta_b^j(-\tilde{q}) + ie\Theta_i^a(-\tilde{q})\Delta_a^i(\tilde{q})\varphi(-\tilde{q})$$
$$- e^2\Theta_i^a(-\tilde{q})\Delta_a^i(\tilde{q})D(\tilde{q})\Theta_j^b(\tilde{q})\Delta_b^j(-\tilde{q})\big] \tag{H13}$$

and

$$W_1[\varphi, \mathbf{\Delta}] = \frac{1}{2}\sum_{\tilde{q}} \big[D^{-1}(\tilde{q})\tilde{\varphi}(\tilde{q})\tilde{\varphi}(-\tilde{q}) + \big(\Phi_{ij}^{ab}(\tilde{q}) + e^2\Theta_i^a(-\tilde{q})D(\tilde{q})\Theta_j^b(\tilde{q})\big)\Delta_a^i(\tilde{q})\Delta_b^j(-\tilde{q})\big]. \tag{H14}$$

Now one can see that the $\tilde{\varphi}$ and $\mathbf{\Delta}$ fields are decoupled from each other. Let us write $Z_1[\mathbf{\Delta}] \sim e^{-W_2[\mathbf{\Delta}]}$ and then

$$W_2[\mathbf{\Delta}] = \frac{1}{2}\sum_{\tilde{q}} \big[\Phi_{ij}^{ab}(\tilde{q}) + e^2\Theta_i^a(-\tilde{q})D(\tilde{q})\Theta_j^b(\tilde{q})\big]\Delta_a^i(\tilde{q})\Delta_b^j(-\tilde{q}). \tag{H15}$$

From this expression, we conclude that the full stress-stress correlation function, denoted as $\tilde{\Phi}_{ij}^{ab}(\tilde{q})$, should generally acquire a correction from the electron-electron interaction through

$$\tilde{\Phi}_{ij}^{ab}(\tilde{q}) = \Phi_{ij}^{ab}(\tilde{q}) + e^2\Theta_i^a(-\tilde{q})D(\tilde{q})\Theta_j^b(\tilde{q}), \tag{H16}$$

where the stress-density correlation function $\Theta_i^a(\tilde{q})$ is defined in Eq. (C18).

One can see that the $\Theta_i^a(\tilde{q})$ function couples electrons to acoustic phonons and this function involves one density vertex and one stress tensor vertex from Eq. (C18). Next we will show that for the phonon modes that we are interested in, the $\Theta_i^a(\tilde{q})$ function is always zero if the system has full rotation symmetry. In the above expressions, $a, b, i, j = x, y, z$ which is not convenient for the symmetry analysis. It is more convenient to relabel the strain and stress tensor with the angular momentum, as shown in Eqs. (B3)–(B12). The rotationally symmetric form of the Hamiltonian (B13) is useful for our symmetry analysis below.

To make the problem more concrete, let us consider the phonon momentum along the z direction, $\mathbf{q} = q_z\hat{e}_z$, and we first figure out which strain tensors, as well as the stress tensors, are involved in the Nieh-Yan term. To see that, let us write down explicitly the terms in Eq. (G12) along the $q_z\hat{e}_z$ direction, which involves

$$\delta_{jl}\epsilon_{zkm}(\partial_z u_{jk})u_{lm} = \epsilon_{zxy}(\partial_z u_{jx})u_{jy} + \epsilon_{zyx}(\partial_z u_{jy})u_{jx}$$
$$= [(\partial_z u_{xx})u_{xy} + (\partial_z u_{yx})u_{yy} + (\partial_z u_{zx})u_{zy}] - [(\partial_z u_{xy})u_{xx} + (\partial_z u_{yy})u_{yx} + (\partial_z u_{zy})u_{zx}]$$
$$= 2c\partial_z(u_{xx} - u_{yy}) + 2(\partial_z u_{zx})u_{zy}, \tag{H17}$$

in which all the total derivative term has been dropped since they will not contribute to the effective action. Now we notice that only the components $u_{xx} - u_{yy}$, u_{xy}, u_{zx}, and u_{zy} of the strain tensor are involved for the acoustic phonons along the z direction that we are interested in. From Eqs. (B3)–(B7), one can see that only $u_{J=2,M}$ with $M = \pm 1, \pm 2$ are involved, which means the the z-direction angular momentum M only takes ± 1 and ± 2, but not 0. However, the electron density operator carries z-direction angular momentum 0. Since the z-direction rotation symmetry is preserved for the momentum $\mathbf{q}$ along the z direction, we expect elections cannot directly couple to the phonon modes with higher angular momentum.

To make this argument more explicitly, let us define $\mathcal{T}_M^J$ as

$$\mathcal{T}_2^2 = \tfrac{1}{2}\big[\mathcal{T}_x^x - \mathcal{T}_y^y + i\big(\mathcal{T}_x^y + \mathcal{T}_y^x\big)\big], \tag{H18}$$

$$\mathcal{T}_1^2 = \big(-\tfrac{1}{2}\big)\big[\mathcal{T}_x^z + \mathcal{T}_z^x + i\big(\mathcal{T}_y^z + \mathcal{T}_z^y\big)\big], \tag{H19}$$

$$\mathcal{T}_0^2 = \sqrt{\frac{1}{6}}\big(2\mathcal{T}_z^z - \mathcal{T}_x^x - \mathcal{T}_y^y\big), \tag{H20}$$

$$\mathcal{T}_{-1}^2 = \tfrac{1}{2}\big[\mathcal{T}_x^z + \mathcal{T}_z^x - i\big(\mathcal{T}_y^z + \mathcal{T}_z^y\big)\big], \tag{H21}$$

$$\mathcal{T}_{-2}^2 = \tfrac{1}{2}\big[\mathcal{T}_x^x - \mathcal{T}_y^y - i\big(\mathcal{T}_x^y + \mathcal{T}_y^x\big)\big], \tag{H22}$$

and the stress-density correlation function can also be reconstructed as

$$\Theta_M^J(\mathbf{q}, i\nu_m) = \frac{1}{\beta V}\sum_{\mathbf{k}, i\omega_n} \mathrm{Tr}_\sigma\big[\mathcal{G}_0(\mathbf{k}-\mathbf{q}, i\omega_n - i\nu_m) \times \mathcal{T}_M^J(\mathbf{k}-\mathbf{q}, \mathbf{k})\mathcal{G}_0(\mathbf{k}, i\omega_n)\big] \tag{H23}$$

for the angular momentum J, M of the stress operator. Here we also add the density operator n which is an identity. Let us consider a symmetry operator $\mathcal{R}$, which transforms the stress tensor as $\mathcal{R}\mathcal{T}_M^J(\mathbf{k}-\mathbf{q}, \mathbf{k})\mathcal{R}^{-1} = \sum_{M'} D_{MM'}^J(\mathcal{R})\mathcal{T}_{M'}^J[\mathcal{R}^{-1}(\mathbf{k}-\mathbf{q}), \mathcal{R}^{-1}\mathbf{k}]$. The Green function should be invariant under the symmetry operation $\mathcal{R}\mathcal{G}_0(\mathbf{k}, i\omega_n)\mathcal{R}^{-1} = \mathcal{G}_0(\mathcal{R}^{-1}\mathbf{k}, i\omega_n)$. Thus, one can insert symmetry operation into the definition of stress-density correlation function and find

$$\begin{aligned}\Theta^J_M(\mathbf{q}, i\nu_m) &= \frac{1}{\beta V}\sum_{\mathbf{k},i\omega_n}\mathrm{Tr}_\sigma\left(\mathcal{R}\mathcal{G}_0(\mathbf{k}-\mathbf{q}, i\omega_n - i\nu_m)\mathcal{R}^{-1}\mathcal{R}\mathcal{T}^J_M(\mathbf{k}-\mathbf{q},\mathbf{k})\mathcal{R}^{-1}\mathcal{R}\mathcal{G}_0(\mathbf{k}, i\omega_n)\mathcal{R}^{-1}\right)\\ &= \frac{1}{\beta V}\sum_{\mathbf{k},i\omega_n}\mathrm{Tr}_\sigma\left(\mathcal{G}_0(\mathcal{R}^{-1}(\mathbf{k}-\mathbf{q}), i\omega_n - i\nu_m)\sum_{M'}D^J_{MM'}(\mathcal{R})\mathcal{T}^J_{M'}(\mathcal{R}^{-1}(\mathbf{k}-\mathbf{q}),\mathbf{k})\mathcal{G}_0(\mathcal{R}^{-1}\mathbf{k}, i\omega_n)\right)\\ &= \frac{1}{\beta V}\sum_{\mathbf{k}',i\omega_n,M'}D^J_{MM'}(\mathcal{R})\mathrm{Tr}_\sigma\left(\mathcal{G}_0(\mathbf{k}'-\mathcal{R}^{-1}\mathbf{q}), i\omega_n - i\nu_m)\mathcal{T}^J_{M'}(\mathbf{k}'-\mathcal{R}^{-1}\mathbf{q},\mathbf{k}')\mathcal{G}_0(\mathbf{k}', i\omega_n)\right)\\ &= \sum_{M'}D^J_{MM'}(\mathcal{R})\Theta^J_{M'}(\mathcal{R}^{-1}\mathbf{q}, i\nu_m). \end{aligned} \tag{H24}$$

In the above derivation, we have used $\mathbf{k}' = \mathcal{R}^{-1}\mathbf{k}$. The above equation gives rise to the symmetry constraint on the form of correlation functions. Along $\mathbf{q} = q_z\hat{e}_z$, we have $\mathcal{R}^{-1}\mathbf{q} = \mathbf{q}$ and

$$\Theta^J_M(q_z, i\nu_m) = \sum_{M'}D^J_{MM'}(\mathcal{R})\Theta^J_{M'}(q_z, i\nu_m). \tag{H25}$$

Now let us consider the orthonormality relation for irreducible representations of a symmetry group, given by $\sum_{\mathcal{R}} D^{\Gamma_j}_{MM'}(\mathcal{R})[D^{\Gamma'_j}_{NN'}(\mathcal{R})]^* = \frac{h}{l_j}\delta_{jj'}\delta_{MN}\delta_{M'N'}$. We should consider the symmetry group formed by the rotation along the z directions and thus only the z-directional angular momentum M is a good quantum number that characterizes the rotation group, while the total angular momentum J is not. Consequently, we expect the orthonormality relation is then given by $\sum_{\mathcal{R}} D^J_{MM'}(\mathcal{R})[D^{J'}_{NN'}(\mathcal{R})]^* \propto \delta_{MN}\delta_{M'N'}$. Now let us choose $D^{J'}_{NN'}$ to be $J'=0$ (identity representation), so $N = N' = 0$ and $D^0_{00} = 1$ for any $\mathcal{R}$. Then the orthonormality relation takes the form $\sum_{\mathcal{R}} D^J_{MM'}(\mathcal{R}) \propto \delta_{M0}\delta_{M'0}$, which leads to

$$\sum_{\mathcal{R}}\Theta^J_M(q_z, i\nu_m) = \sum_{M'}\sum_{\mathcal{R}}D^J_{MM'}(\mathcal{R})\Theta^J_{M'}(q_z, i\nu_m) \propto \delta_{M0} \rightarrow \Theta^J_M(q_z, i\nu_m) \propto \delta_{M0}. \tag{H26}$$

So the correlation function Θ^J_M can only be nonzero for the z-directional angular momentum $M = 0$. However, as we have shown above, the Nieh-Yan anomaly term only involves the strain tensor with $M = \pm 1, \pm 2$, and this concludes that the electron-electron Coulomb interaction will not contribute to the acoustic phonon modes, whose dynamics is influenced by the Nieh-Yan anomaly.

APPENDIX I: DEFORMATION POTENTIAL FOR ELECTRON-OPTICAL PHONON INTERACTION

The optical phonons involve the relative motions of atoms within one unit cell and thus we cannot directly start from the continuous model. Let us next consider the deformation potential

$$H_{\mathrm{ep}} = \int d^3r\hat{\Psi}^\dagger(\mathbf{r})\left(\sum_{\mathbf{n},\tau}\mathbf{u}_{\mathbf{n}\tau}\cdot\frac{\partial U(\mathbf{r}-\mathbf{R}_{\mathbf{n}\tau})}{\partial\mathbf{R}_{\mathbf{n}\tau}}\right)\hat{\Psi}(\mathbf{r}), \tag{I1}$$

where $\mathbf{R}_{\mathbf{n}\tau} = \mathbf{R}_\mathbf{n} + \mathfrak{r}_\tau$ with the lattice vector $\mathbf{R}_\mathbf{n}$ and the position $\mathfrak{r}_\tau$ for the atom τ in one unit cell. Here the electron field operator can be expanded as $\hat{\Psi}(\mathbf{r}) = \frac{1}{V}\sum_{\alpha s\mathbf{K}} e^{i\mathbf{K}\cdot\mathbf{r}}\mathfrak{u}_{\alpha s\mathbf{K}}(\mathbf{r})\hat{c}_{\alpha s\mathbf{K}}$, where s labels spin and α labels other band index, while $\mathbf{u}_{\mathbf{n}\tau}$ is the displacement field, which can be expanded as $\mathbf{u}_{\mathbf{n}\tau} = \frac{1}{\sqrt{NM_\tau}}\sum_{\lambda\mathbf{q}} Q_{\lambda,\mathbf{q}}\mathfrak{e}^\lambda_\tau e^{i\mathbf{q}\cdot\mathbf{R}_\mathbf{n}}$. Here M_τ is the atom mass, $Q_{\lambda,\mathbf{q}}$ labels the normal mode of the λ-phonon, and the polarization vector $\mathfrak{e}^\lambda_\tau$ satisfies the equation of motion

$$\omega^2_\lambda\mathfrak{e}^\lambda_{\tau,i} = \sum_{\xi j}D_\mathbf{k}(\tau i;\xi j)\mathfrak{e}^\lambda_{\xi,j}, \tag{I2}$$

where $i, j = x, y, z$ and $D_\mathbf{k}(\tau i;\xi j)$ is the dynamical matrix for phonons in the momentum space. Since the displacement $\mathbf{u}_{\mathbf{n}\tau}$ is real, $Q_{\lambda,\mathbf{q}} = Q^\dagger_{\lambda,-\mathbf{q}}$. By substituting the expansion of $\hat{\Psi}(\mathbf{r})$ and $\mathbf{u}_{\mathbf{n}\tau}$ into Eq. (I1) and after some straightforward simplifications, we obtain

$$H_{\mathrm{ep}} = \frac{1}{V^2}\sum_{\mathbf{K},\mathbf{q},\alpha s,\beta t,\lambda} g^\lambda_{\alpha s,\beta t}(\mathbf{K},\mathbf{q})Q_\lambda(\mathbf{q})\hat{c}^\dagger_{\alpha s,\mathbf{K}}\hat{c}_{\beta t,\mathbf{K}-\mathbf{q}}, \tag{I3}$$

where

$$g^\lambda_{\alpha s,\beta t}(\mathbf{K},\mathbf{q}) = \sum_\tau\sqrt{\frac{N}{M_\tau}}\mathfrak{e}^\lambda_\tau\cdot\langle\mathfrak{u}_{\mathbf{K},\alpha s}|e^{-i\mathbf{q}\cdot\mathbf{r}}\frac{\partial U(\mathbf{r}-\mathfrak{r}_\tau)}{\partial\mathfrak{r}_\tau}|\mathfrak{u}_{\mathbf{K}-\mathbf{q},\beta t}\rangle. \tag{I4}$$

We can again expand $g^\lambda_{\alpha s,\beta t}(\mathbf{K},\mathbf{q})$ in terms of the 2 by 2 matrices as $g^\lambda_{\alpha s,\beta t}(\mathbf{K},\mathbf{q}) = \sum_\mu g^\lambda_{\alpha\beta,\mu}(\mathbf{K},\mathbf{q})(\sigma^\mu)_{st}$, where $\mu = 0, x, y, z$. The electron-phonon coupling Hamiltonian is written as

$$H_{\mathrm{ep}} = \frac{1}{V^2}\sum_{\mathbf{K},\mathbf{q},\alpha\beta,\mu,\lambda} g^\lambda_{\alpha\beta,\mu}(\mathbf{K},\mathbf{q})Q_\lambda(\mathbf{q})\hat{c}^\dagger_{\alpha,\mathbf{K}}\sigma^\mu\hat{c}_{\beta,\mathbf{K}-\mathbf{q}}. \tag{I5}$$

The hermitian property of the Hamiltonian $H^\dagger_{\mathrm{ep}} = H_{\mathrm{ep}}$ requires $[g^\lambda_{\alpha\beta,\mu}(\mathbf{K},\mathbf{q})]^* = g^\lambda_{\beta\alpha,\mu}(\mathbf{K},-\mathbf{q})$.

The time-reversal symmetry provides a constraint on the form of $g^\lambda_{\alpha s,\beta t}(\mathbf{K},\mathbf{q})$. With $\hat{T}Q_\lambda(\mathbf{q})\hat{T}^{-1} = Q_\lambda(-\mathbf{q})$ and $\hat{T}\hat{c}^\dagger_\alpha(\mathbf{K})\hat{T}^{-1} = \hat{c}^\dagger_\alpha(-\mathbf{K})(i\sigma_y\mathcal{K})$, we have

$$\hat{T}H_{\mathrm{ep}}\hat{T}^{-1} = \sum_{\mathbf{K},\mathbf{q},\alpha\beta,\mu,\lambda}\left(g^\lambda_{\alpha\beta,\mu}(\mathbf{K},\mathbf{q})\right)^*Q_\lambda(-\mathbf{q})\hat{c}^\dagger_{\alpha,-\mathbf{K}}(i\sigma_y\mathcal{K})\sigma^\mu \times(i\sigma_y\mathcal{K})\hat{c}_{\beta,\mathbf{q}-\mathbf{K}} \tag{I6}$$

$$= \sum_{\mathbf{K},\mathbf{q},\alpha\beta,\mu,\lambda}\left(g^\lambda_{\alpha\beta,\mu}(-\mathbf{K},-\mathbf{q})\right)^*Q_\lambda(\mathbf{q})\hat{c}^\dagger_{\alpha,\mathbf{K}}(i\sigma_y\mathcal{K})\sigma^\mu \times(-i\sigma_y\mathcal{K})\hat{c}_{\beta,\mathbf{K}-\mathbf{q}}. \tag{I7}$$

Since $(i\sigma_y\mathcal{K})\sigma^0(-i\sigma_y\mathcal{K}) = \sigma^0$ and $(i\sigma_y\mathcal{K})\sigma^a(-i\sigma_y\mathcal{K}) = -\sigma^a$ $(a = x, y, z)$, the time-reversal symmetry $\hat{T}H_{\text{ep}}\hat{T}^{-1} = H_{\text{ep}}$ gives rise to $[g^\lambda_{\alpha\beta,0}(-\mathbf{K}, -\mathbf{q})]^* = g^\lambda_{\alpha\beta,0}(-\mathbf{K}, -\mathbf{q})$ and $[g^\lambda_{\alpha\beta,a}(-\mathbf{K}, -\mathbf{q})]^* = -g^\lambda_{\alpha\beta,a}(\mathbf{K}, \mathbf{q})$.

In the spirit of the $\mathbf{k}\cdot\mathbf{p}$ theory, we consider the electron-phonon coupling around time-reversal invariant momenta $\mathbf{\Gamma}_i = -\mathbf{\Gamma}_i$ and thus take the approximation $\mathbf{K} = \mathbf{\Gamma_i}$ for the parameter $g^\lambda_{\alpha\beta,\mu}(\mathbf{K}, \mathbf{q})$, so that $[g^\lambda_{\alpha\beta,0}(\mathbf{\Gamma_i}, -\mathbf{q})]^* = g^\lambda_{\alpha\beta,0}(\mathbf{\Gamma_i}, -\mathbf{q})$ and $[g^\lambda_{\alpha\beta,a}(\mathbf{\Gamma_i}, -\mathbf{q})]^* = -g^\lambda_{\alpha\beta,a}(\mathbf{\Gamma_i}, -\mathbf{q})$. Below we will drop the index $\mathbf{\Gamma}_i$ in the coupling constant g. Furthermore, we can assume the phonon momentum $\mathbf{q}$ is a small number and expand g as $g^\lambda_{\alpha\beta,\mu}(\mathbf{q}) = g^\lambda_{\alpha\beta,\mu}(\mathbf{q} = \mathbf{0}) + (\partial_\mathbf{q} g^\lambda_{\alpha\beta,\mu})_{\mathbf{q}=\mathbf{0}}\cdot\mathbf{q} + \dots$, and for optical phonons, we may only focus on the lowest-order term $g^\lambda_{\alpha\beta,\mu} = g^\lambda_{\alpha\beta,\mu}(\mathbf{q} = \mathbf{0})$. Thus, we should have $(g^\lambda_{\alpha\beta,0})^* = g^\lambda_{\alpha\beta,0}$ and $(g^\lambda_{\alpha\beta,a})^* = -g^\lambda_{\alpha\beta,a}$ $(a = x, y, z)$ and thus $g^\lambda_{\alpha\beta,0}$ is real while $g^\lambda_{\alpha\beta,a}$ is pure imaginary. The hermitian condition requires $(g^\lambda_{\alpha\beta,\mu})^* = g^\lambda_{\beta\alpha,\mu}$. Therefore, if we choose $\alpha = \beta$, then only $g^\lambda_{\alpha\alpha,0}$ does not vanish while the pure imaginary $g^\lambda_{\alpha\beta,a}$ only couples different bands $\alpha \neq \beta$.

Now let us look at the effective electron-phonon coupling for the band $\psi_{\mathbf{\Gamma}_i,0,s}$. Since now we choose $\alpha = \beta = 0$, it is clear that only $g^\lambda_{\alpha\alpha,0}$ is nonvanishing for the lowest-order terms. Therefore, at the lowest order, the electron-phonon coupling takes the form

$$H_{\text{ep},\mathbf{\Gamma}_i} = \frac{1}{V^2}\sum_{\mathbf{k},\mathbf{q},\lambda} g^\lambda_o Q_\lambda(\mathbf{q})\hat{c}^\dagger_{0,\mathbf{k}}\hat{c}_{0,\mathbf{k}-\mathbf{q}} \tag{I8}$$

for optical phonons, where g^λ_o is a real number. In the real space, we have

$$H_{\text{ep},\mathbf{\Gamma}_i} = \sum_\lambda g^\lambda_o \int d^3r Q_\lambda(\mathbf{r})\hat{\psi}^\dagger_{\Gamma_i}(\mathbf{r})\hat{\psi}_{\Gamma_i}(\mathbf{r}) \tag{I9}$$

for optical phonons, where we have defined $Q_\lambda(\mathbf{q}) = \int d^3r Q_\lambda(\mathbf{r})e^{-i\mathbf{q}\cdot\mathbf{r}}$.

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