Fluctuation contribution to spin Hall effect in superconductors

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We theoretically study the contribution of superconducting fluctuation to extrinsic spin Hall effects in two- and three-dimensional electron gas and intrinsic spin Hall effects in two-dimensional electron gas with Rashba-type spin-orbit interaction. The Aslamazov-Larkin, density of states, and Maki-Thompson terms have logarithmic divergence $\ln\epsilon$ in the limit $\epsilon = (T - T_c)/T_c \to +0$ in two-dimensional systems for both extrinsic and intrinsic spin Hall effects except the Maki-Thompson terms in extrinsic effect, which are proportional to $(\epsilon - \gamma_\varphi)^{-1}\ln\epsilon$ with a cutoff γ_φ in two-dimensional systems. We found that the fluctuation effects on the extrinsic spin Hall effect have an opposite sign to that in the normal state and thus suppress the spin Hall conductivity. In contrast, those on the intrinsic spin Hall effect have the same sign and enhance it.

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I. INTRODUCTION

Spin current has opened a venue to manipulate condensed matter systems. Spin transport experiments were conducted by Tedrow and Meservey [1], who demonstrated that current flow across a ferromagnet-superconductor interface was spin polarized. Subsequently, Aronov discussed that spin injection from ferromagnet to nonmagnetic metals could be used to amplify the electron spin resonance signals [2]. Johnson and Silsbee demonstrated that nonlocal response against the local charge current injection from a ferromagnetic metal to nonmagnetic metal could be utilized to measure the spin relaxation time [3]. In the nonlocal response, a major role is played by the propagation of nonconserved spin current over a mesoscopic scale termed a spin diffusion length. In addition to finding efficient ways for spin injection and the study of nonlocal response due to spin diffusion, spin-charge conversion (spin Hall effect [4–17] and spin galvanic effect [18,19]) is also an important issue in physics of spin transport. The spin Hall effect is categorized into two groups according to the origin, viz., extrinsic spin Hall effect [4–6] caused by the spin-orbit interaction in the disorder potential and intrinsic spin Hall effect [10,11] that occurs in a perfect crystal with the electric band structure split by the spin-orbit interaction.

Those issues in spin transport have been addressed not only in normal metals but in superconductors [20]. A theory of spin current injected into superconductors by taking account of charge imbalance and spin imbalance of quasiparticles was developed in 1995 [21]. In 2012, Hübler *et al.* reported spin transport in superconducting Al over distances of several microns, exceeding the normal-state spin-diffusion length and the charge-imbalance length [22]. Wakamura *et al.* observed spin-relaxation times in superconducting Nb, which is four times longer than that in the normal state [23]. Wakamura *et al.* [24] reported, in another paper, inverse spin Hall effect (ISHE), conversion from spin current to charge current in superconductors NbN. Recently, several efficient ways of injection of spin current into superconductors near the transition temperature T_c have been discussed theoretically in Refs. [25,26] in terms of spin-pumping, spin-Seebeck effect, and strong coupling between spin and energy in spin-splitting quasiparticles [27]. Jeon *et al.* reported that the conversion efficiency of magnon spin to quasiparticle charge in superconducting Nb via ISHE is enhanced compared with that in the normal state near T_c [28,29]. Enhancement of the ISHE signal was observed even at temperatures up to twice T_c [29]. Those experimental results imply the importance of superconducting fluctuation effects on spin transport near T_c. We note that earlier theoretical studies [7,30,31] but one [32] have focused on spin Hall effect in superconductors below T_c.

Fluctuation effects on transport properties above the superconducting transition temperature T_c were studied by Aslamazov and Larkin [33], Maki [34], and Thompson [35] on electric conductivity [36]. Dominant fluctuation processes contributing to electric conductivity are the charge transport by the fluctuating Cooper pairs [33], the reduction of the density of states (DOS) by the presence of the fluctuating Cooper pairs, and the scattering of electrons by the fluctuation of Cooper pairs. These three processes are represented by the Feynman diagrams, each of which is called the Aslamazov-Larkin(AL) terms, DOS terms, and Maki-Thompson (MT)

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terms, respectively. The MT terms for electric conductivity contain the anomalous part, which diverges for all temperatures above T_c in one- and two-dimensional systems. This anomalous part is cut off by the phase-breaking parameter with various origins such as the paramagnetic impurities, magnetic fields, inelastic phonon scattering, and nonlinear fluctuation effects (See Secs. 8.3.3 and 8.3.4 in Ref. [36]). Depending on the phase-breaking parameter, either the AL terms or the MT terms dominantly contribute to electric conductivity. The superconducting fluctuation effect on extrinsic anomalous Hall effects [37], which is closely related to the spin Hall effect, was studied by Li and Levchenko [38]. In Ref. [32], the spin Hall effect in the presence of a magnetic field was studied by consideration of the AL terms cooperating with the Hartree approximation.

In this paper, we discuss the fluctuation effects on the spin Hall effect in superconductors above T_c in the absence of magnetic fields with the lowest order processes of the fluctuation propagator. We study the extrinsic spin Hall effect in two and three-dimensional electron gas by incorporating the superconducting fluctuations in the model used by Tse and Das Sarma [9]. We also investigate the intrinsic spin Hall effect by taking account of the superconducting fluctuations in the model used by Sinova *et al.* [11], viz., the two-dimensional electron gas with the Rashba spin-orbit interaction.

The rest of the present paper is organized as follows. In Sec. II, we address the fluctuation effects on extrinsic spin Hall effects in the presence of side jump and skew scattering processes in two- and three-dimensional electron gas. In Sec. III, we discuss the fluctuation effects on intrinsic spin Hall effects in two-dimensional electron gas with Rashba spin-orbit interaction. In Sec. IV, we discuss singularity near T_c and magnitude of fluctuation contribution in AL, DOS, and MT terms. We also raise several issues to be addressed in the future. In Sec. V, we conclude the present paper. We defer the details of derivation in Secs. II and III to Supplemental Material [39]. We also list the symbols used in this paper in Sec. III in the Supplemental Material.

Throughout this paper, we set the Boltzmann constant to be unity (i.e., $k_B = 1$) and take the electric charge of the carriers to be negative ($-e < 0$).

II. FLUCTUATION EFFECTS ON EXTRINSIC SPIN HALL CONDUCTIVITY

Near and above the superconducting transition temperature T_c, we take account of the superconducting fluctuation via three types of the process: the AL term, the MT term, and the DOS term. Those are known to be the most diverging in the electric conductivity when $T \to T_c + 0$.

A. Model

We consider the extrinsic spin Hall effect in the system with the Hamiltonian

$$
\begin{aligned}
H &= H_0 + H_{\rm SO} + \mathcal{V} + H_{\rm int},\\
H_0 &= \sum_{\alpha\boldsymbol{k}} \psi^\dagger_{\boldsymbol{k}\alpha} \frac{\hbar^2 \boldsymbol{k}^2}{2m} \psi_{\boldsymbol{k}\alpha},\\
H_{\rm SO} &= -\frac{i\lambda_{\rm E}^2}{4V} \sum_{\alpha\beta} \sum_{\boldsymbol{k}\boldsymbol{k}'} \psi^\dagger_{\boldsymbol{k}\alpha} (\boldsymbol{k}\times\boldsymbol{k}')\cdot\boldsymbol{\sigma}_{\alpha\beta} \mathcal{V}_{\boldsymbol{k}-\boldsymbol{k}'} \psi_{\boldsymbol{k}'\beta},\\
\mathcal{V} &= \frac{1}{V}\sum_{\alpha}\sum_{\boldsymbol{k}\boldsymbol{k}'} \mathcal{V}_{\boldsymbol{k}-\boldsymbol{k}'} \psi^\dagger_{\boldsymbol{k}\alpha}\psi_{\boldsymbol{k}'\alpha},\\
H_{\rm int} &= -\frac{g}{V}\sum_{\boldsymbol{k}\boldsymbol{k}'\boldsymbol{q}} \psi^\dagger_{\boldsymbol{k}+\boldsymbol{q},\uparrow}\psi^\dagger_{-\boldsymbol{k},\downarrow}\psi_{-\boldsymbol{k}',\downarrow}\psi_{\boldsymbol{k}'+\boldsymbol{q},\uparrow}.
\end{aligned}
\tag{1}
$$

Here $\mathcal{V}$ and $H_{\rm SO}$ are, respectively, the potential and the spin-orbit interaction due to the impurities. m is mass of electron, V is volume of the system, $\boldsymbol{\sigma}$ is the Pauli matrices, $\lambda_{\rm E}$ is the coupling constant of spin-orbit interaction in the extrinsic spin Hall effect, and $\psi_{\boldsymbol{k}\alpha}$ is the Fourier transform of the annihilation operator of the electron with a z component of spin α. We denote by $H_{\rm int}$ the BCS-type two-body attraction. The random average of the impurity potential is given by

$$\langle \mathcal{V}_{\boldsymbol{k}}\mathcal{V}_{\boldsymbol{k}'}\rangle = n_{\rm i} v_0^2 V \delta_{\boldsymbol{k}+\boldsymbol{k}',0}$$

and

$$\langle \mathcal{V}_{\boldsymbol{k}}\mathcal{V}_{\boldsymbol{k}'}\mathcal{V}_{\boldsymbol{k}''}\rangle = n_{\rm i} v_0^3 V \delta_{\boldsymbol{k}+\boldsymbol{k}'+\boldsymbol{k}'',0},$$

where $n_{\rm i}$ is the density of impurities and v_0 is the uniform component of the Fourier transform of the potential of the single impurities.

The charge and spin currents are, respectively, given by

$$
\begin{aligned}
\boldsymbol{j}_{\rm c} &= \boldsymbol{j}_{\rm c}^{(1)} + \boldsymbol{j}_{\rm c}^{(2)}\\
&\equiv -e\sum_{\alpha\beta}\sum_{\boldsymbol{k}\boldsymbol{k}'} \psi^\dagger_{\boldsymbol{k}\alpha}\left[\frac{\hbar\boldsymbol{k}}{m}\delta_{\alpha\beta}\delta_{\boldsymbol{k}\boldsymbol{k}'} - \frac{i\lambda_{\rm E}^2}{4\hbar V}\boldsymbol{\sigma}_{\alpha\beta}\times(\boldsymbol{k}-\boldsymbol{k}')\mathcal{V}_{\boldsymbol{k}-\boldsymbol{k}'}\right]\psi_{\boldsymbol{k}'\beta},
\end{aligned}
\tag{2}
$$

$$
\begin{aligned}
\boldsymbol{j}_{\rm s} &= \boldsymbol{j}_{\rm s}^{(1)} + \boldsymbol{j}_{\rm s}^{(2)}\\
&\equiv -\frac{e}{2}\sum_{\alpha\beta}\sum_{\boldsymbol{k}\boldsymbol{k}'} \psi^\dagger_{\boldsymbol{k}\alpha}\left[\frac{\hbar\boldsymbol{k}}{m}\sigma^z_{\alpha\beta}\delta_{\boldsymbol{k}\boldsymbol{k}'} - \frac{i\lambda_{\rm E}^2}{4\hbar V}\hat{z}\delta_{\alpha\beta}\times(\boldsymbol{k}-\boldsymbol{k}')\mathcal{V}_{\boldsymbol{k}-\boldsymbol{k}'}\right]\psi_{\boldsymbol{k}'\beta}.
\end{aligned}
\tag{3}
$$

The spin Hall conductivity is defined as $j_{{\rm s}\mu} = \sum_\nu \sigma_{\mu\nu}E_\nu$ and it can be calculated by

$$
\begin{aligned}
\sigma_{\mu\nu}(\boldsymbol{q},\omega) &= \frac{\Phi_{\mu\nu}(\boldsymbol{q},\hbar\omega+i\delta) - \Phi_{\mu\nu}(\boldsymbol{q},i\delta)}{i(\omega+i\delta)},\\
\Phi_{\mu\nu}(\boldsymbol{q},i\omega_\nu) &= \frac{1}{V}\int_0^\beta \langle T_u[j_{{\rm s},\mu,\boldsymbol{q}}(u) j_{{\rm c},\nu,-\boldsymbol{q}}(0)]\rangle e^{i\omega_\nu u} du,
\end{aligned}
$$

where $\omega_\nu = 2\nu\pi T$ is the bosonic Matsubara frequency. From these expressions with omitting $H_{\rm int}$, the spin Hall conductivity in the normal state via the side-jump and the skew-scattering processes, respectively, were obtained as [9]

$$\sigma_{xy}^{\rm SJ(normal)} = \frac{e^2\hbar}{2Dm}\lambda_{\rm E}^2 k_{\rm F}^2 N(0), \tag{4}$$

$$\sigma_{xy}^{\rm SS(normal)} = \frac{\pi e^2\hbar^2}{2D^2m^2}\lambda_{\rm E}^2 v_0 k_{\rm F}^4 N(0)^2\tau. \tag{5}$$

Here $k_{\rm F}$ is the Fermi wave vector, $N(0)$ is DOS at the Fermi surface, and $D = 2, 3$ is spatial dimension. The symbol τ denotes the mean-free time $\tau = \frac{\hbar}{2\pi N(0) n_{\rm i} v_0^2}$. We take account

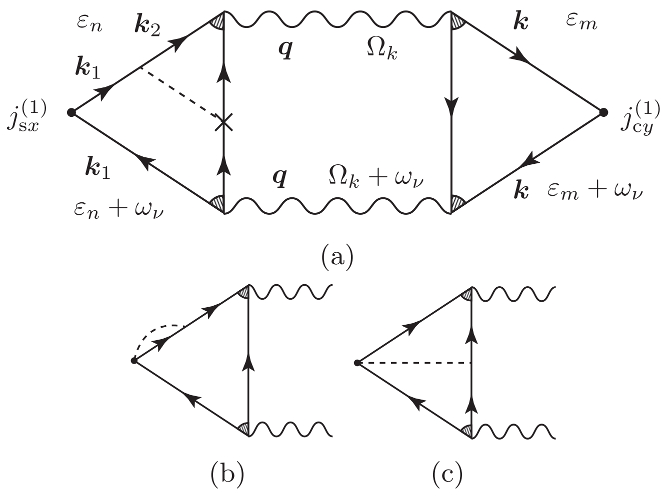

FIG. 1. The diagram for the AL term in the presence of side jump.

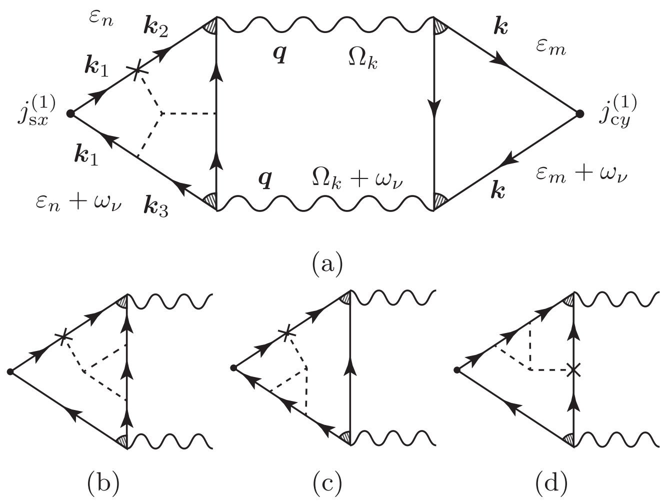

FIG. 2. The diagram for the AL term in the presence of skew scattering.

of superconducting fluctuations with the fluctuation propagator following Ref. [36]. The fluctuation propagator, which is the vertex part of the effective two-body interaction, can be calculated by taking the infinite sum of series of diagrams of $H_{\rm int}$ in the ladder approximation (see Secs. 6.2 and 6.4.2 in Refs. [36]). It is given by

$$L(\boldsymbol{q}, i\Omega_k) = -\frac{1}{VN(0)}\frac{1}{\epsilon + \psi\left(\frac{1}{2}+\frac{|\Omega_k|}{4\pi T}\right) - \psi\left(\frac{1}{2}\right) + \xi_{\rm SC}^2\boldsymbol{q}^2}, \tag{6}$$

where $\psi(x)$ is digamma function, $\Omega_k = 2k\pi T$ is the bosonic Matsubara frequency, and

$$\xi_{\rm SC}^2 = -\frac{v_{\rm F}^2\tau^2}{D}\left[\psi\left(\frac{1}{2}+\frac{\hbar}{4\pi T\tau}\right) - \psi\left(\frac{1}{2}\right) - \frac{\hbar}{4\pi T\tau}\psi^{(1)}\left(\frac{1}{2}\right)\right] \tag{7}$$

denotes the squared coherence length.

B. Aslamazov-Larkin terms

The Feynman diagrams [of the charge current-spin current correlation function $\Phi_{xy}^{\rm AL}(0, i\omega_\nu)$] for the AL terms in spin-charge current correlation function with side jump are shown in Fig. 1 and those for AL terms with skew scattering are shown in Fig. 2. In the figure, the shaded region represents the Cooperon (see Secs. 6.4.1 and Fig. 6.3 in Ref. [36]), which is given by

$$C(\boldsymbol{q}, \varepsilon_1, \varepsilon_2) = \frac{|\widetilde{\varepsilon}_1 - \widetilde{\varepsilon}_2|}{|\varepsilon_1 - \varepsilon_2| + \frac{\hbar}{\tau}\frac{(\hbar v_{\rm F} q)^2/D}{|\widetilde{\varepsilon}_1-\widetilde{\varepsilon}_2|^2}\theta(-\varepsilon_1\varepsilon_2)}.$$

The wavy lines, the dotted lines, and the cross marks in Fig. 1, respectively, represent the fluctuation propagator, the random average of the impurity potentials, and the spin-orbit interactions.

We denote $\mathcal{B}_{\rm c}(\boldsymbol{q}, \omega, \Omega)$ by the triangular part containing the charge current vertex (i.e., renormalized charge current vertex) and $\mathcal{B}_{\rm s}(\boldsymbol{q}, \omega, \Omega)$ by the renormalized spin current vertex, respectively. The response function is then given by

$$\Phi_{xy}(0, i\omega_\nu) = \frac{1}{V}\sum_{\boldsymbol{q}} T\sum_{\Omega_k} L(\boldsymbol{q}, i\Omega_k)L(\boldsymbol{q}, i\Omega_k + i\omega_\nu) \times \mathcal{B}_{\rm s}(\boldsymbol{q}, i\Omega_k, i\omega_\nu)\mathcal{B}_{\rm c}(\boldsymbol{q}, i\Omega_k, i\omega_\nu). \tag{8}$$

The $O(\omega)$ contributions in Eq. (8) yield the spin Hall conductivity. In the electric conductivity, we can deduce the main contribution in the AL term by setting $\omega = 0$ and $\Omega = 0$ in the charge current vertices $\mathcal{B}_{\rm c}(\boldsymbol{q}, \omega, \Omega)$ and retaining the $O(\omega)$ part in the fluctuation propagators. In contrast, in the spin Hall conductivity, the contribution with $\mathcal{B}_{\rm s}(\boldsymbol{q}, 0, 0)$ vanishes, and thus we have to maintain the frequencies Ω, ω to be finite. Accordingly, the procedure to calculate the spin Hall conductivity, which is given below, is slightly different from that for the AL term in electric conductivity,

(1) List the relevant diagrams (Fig. 1 for AL+ side jump and Fig. 2 for AL + skew scattering) and write the expressions for the charge current-spin current correlation function $\Phi_{xy}^{\rm AL}(0, i\omega_\nu)$.

(2) Expand $\mathcal{B}_{\rm c}(\boldsymbol{q}, \omega, \Omega)$ and $\mathcal{B}_{\rm s}(\boldsymbol{q}, \omega, \Omega)$ with respect to $\boldsymbol{q}$ up to the first order as $\mathcal{B}_{\rm s}(\boldsymbol{q}, \omega, \Omega) \approx -i\zeta_{\rm s}Vq_yB_{\rm s}(\omega, \Omega)$ and $\mathcal{B}_{\rm c}(\boldsymbol{q}, \omega, \Omega) \approx \zeta_{\rm c}q_yVB_{\rm c}(\omega, \Omega)$ with coefficients $\zeta_{\rm s}$ and $\zeta_{\rm c}$.

(3) Perform integrals in the expressions for $B_{\rm c}$ and $B_{\rm s}$ with respect to internal wave vectors $\boldsymbol{k}_1, \boldsymbol{k}_2 \cdots$.

(4) Transform the sum with Ω_k to contour integral.

(5) Expand the resultant expression with respect to ω after analytic continuation $i\omega_\nu \to \hbar\omega$.

(6) Retain the most singular terms in the limit of $\epsilon \to 0$.

(7) Perform summation in $B_{\rm c}(\boldsymbol{q}, \omega, \Omega)$, $B_{\rm s}(\boldsymbol{q}, \omega, \Omega)$ with respect to internal frequencies ε_n and ε_m.

(8) Integrate the resultant expression with respect to $\boldsymbol{q}$.

The details of the calculation along these procedures are given in Secs. I A and I B in the Supplemental Material [39]. See also the derivation of Eq. (16) in Ref. [40] concerning step 6.

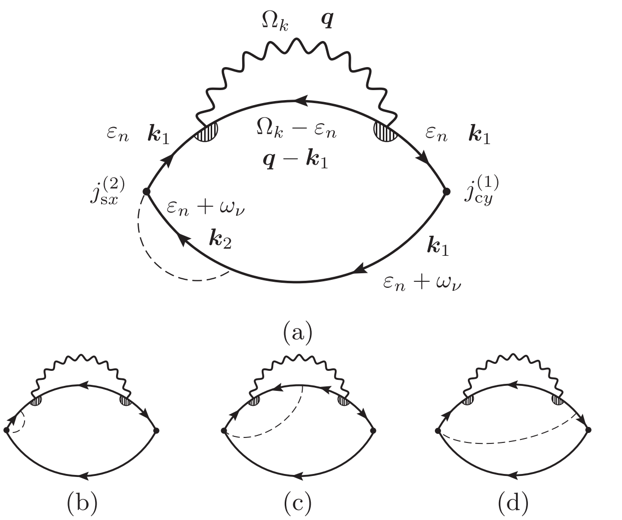

FIG. 3. Diagrams for DOS term in the presence of side jump process.

The resultant expressions for the AL term of the spin Hall conductivity in the side jump and skew scattering process are given by

$$\frac{\sigma_{xy}^{\mathrm{AL-SJ}}}{\sigma_{xy}^{\mathrm{SJ(normal)}}} = 2\frac{T}{\varepsilon_{\mathrm{F}}}\begin{cases}\ln\frac{1}{\epsilon} & (D=2)\\ \frac{2}{\pi\xi_{\mathrm{SC}}} & (D=3),\end{cases}$$

$$\frac{\sigma_{xy}^{\mathrm{AL-SS}}}{\sigma_{xy}^{\mathrm{SS(normal)}}} = \frac{2T}{\pi DN(0)\xi_{\mathrm{SC}}^{D}}\left(\frac{\tau}{\hbar}\right)^2\left[2\frac{\hbar}{4\pi T\tau}\psi^{(1)}\left(\frac{1}{2}\right) + \frac{\hbar}{4\pi T\tau}\psi^{(1)}\left(\frac{1}{2}+\frac{\hbar}{4\pi T\tau}\right) + 3\psi\left(\frac{1}{2}\right) - 3\psi\left(\frac{1}{2}+\frac{\hbar}{4\pi T\tau}\right)\right] \times \begin{cases}\ln\frac{1}{\epsilon} & (D=2)\\ \frac{2}{\pi} & (D=3),\end{cases}$$

where we normalize the results by dividing them by Eqs. (4) and (5).

C. DOS terms

The DOS terms for the spin Hall conductivity are calculated in a way similar to those for electric conductivity. The procedure to calculate the spin Hall conductivity is given below.

(1) List the relevant diagrams (Fig. 3 for DOS + side jump, and Fig. 4 for DOS + skew scattering) and write the expressions for the charge current-spin current correlation function $\Phi_{xy}^{\mathrm{DOS}}(0, i\omega_\nu)$.

(2) Put $\Omega_k = 0$ in all quantities and $\boldsymbol{q} = 0$ in all but $L(\boldsymbol{q}, i\Omega_k)$.

(3) Perform integration with respect to internal wave vectors $\boldsymbol{k}_1, \boldsymbol{k}_2 \cdots$ with the use of residue theorem.

(4) Reduce the sum with ε_n in the polygamma functions.

(5) Expand the resultant expression with respect to ω after analytic continuation $i\omega_\nu \to \hbar\omega$.

(6) Perform integration in $L(\boldsymbol{q}, 0)$ with respect to $\boldsymbol{q}$.

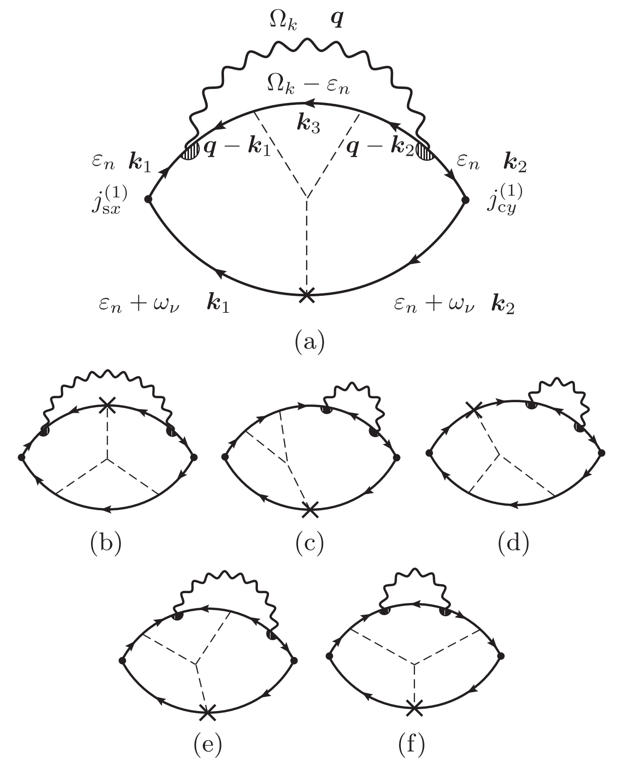

FIG. 4. Diagrams of DOS terms in the presence of skew scattering.

The details of the calculations are given in Secs. I C and I D in the Supplemental Material [39]. By following the above steps, we arrive at the expressions for the DOS terms for extrinsic spin Hall conductivity with the side jump and skew-scattering process,

$$\frac{\sigma_{xy}^{\mathrm{DOS-SJ}}}{\sigma_{xy}^{\mathrm{SJ(normal)}}} = -\frac{T}{2\pi N(0)\xi_{\mathrm{SC}}^{D}}\left(\frac{\tau}{\hbar}\right)^2 \times \left[2\psi\left(\frac{1}{2}\right) - 2\psi\left(\frac{1}{2}+\frac{\hbar}{4\pi T\tau}\right) + 3\frac{\hbar}{4\pi T\tau} \times \psi^{(1)}\left(\frac{1}{2}\right) - \left(\frac{\hbar}{4\pi T\tau}\right)^2\psi^{(2)}\left(\frac{1}{2}\right)\right] \times \begin{cases}\ln\frac{1}{\epsilon} & (D=2)\\ \frac{2}{\pi} & (D=3),\end{cases}$$

$$\frac{\sigma_{xy}^{\mathrm{DOS-SS}}}{\sigma_{xy}^{\mathrm{SS(normal)}}} = -\frac{T}{\pi N(0)\xi_{\mathrm{SC}}^{D}}\left(\frac{\tau}{\hbar}\right)^2 \times \left\{-3\left[\psi\left(\frac{1}{2}+\frac{\hbar}{4\pi T\tau}\right) - \psi\left(\frac{1}{2}\right)\right] + \frac{\hbar}{4\pi T\tau}\left[\psi^{(1)}\left(\frac{1}{2}+\frac{\hbar}{4\pi T\tau}\right) + 3\psi^{(1)}\left(\frac{1}{2}\right)\right] - \left(\frac{\hbar}{4\pi T\tau}\right)^2\psi^{(2)}\left(\frac{1}{2}\right)\right\}\begin{cases}\ln\frac{1}{\epsilon} & (D=2)\\ \frac{2}{\pi} & (D=3).\end{cases}$$

The opposite sign between the DOS terms and the spin Hall conductivity in the normal state is consistent with the suppression of the DOS by the superconducting fluctuation.

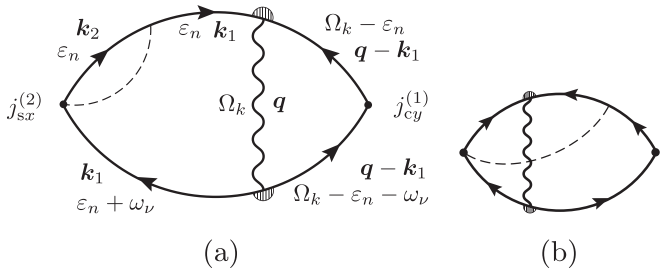

FIG. 5. Diagrams of MT terms in the presence of side jump.

D. Maki-Thompson terms

The MT terms are calculated similarly to those for DOS terms. The procedure to calculate the MT terms is given below. We note that the MT terms with the side-jump process turn out to vanish in a way similar to that for the MT terms in the extrinsic anomalous Hall effect [38].

(1) List the relevant diagrams (Fig. 5 for MT + side jump and Fig. 6 for MT + skew scattering) and write the expressions for the charge current-spin current correlation function $\Phi_{xy}^{\mathrm{MT}}(0, i\omega_\nu)$.

(2) Put $i\Omega_k = 0$ in all quantities.

(3) Perform integration with respect to internal wave vectors $\boldsymbol{k}_1, \boldsymbol{k}_2 \cdots$ with the use of the residue theorem.

(4) Perform summation over ε_n.

(5) Expand the resultant expression with respect to ω after analytic continuation $i\omega_\nu \to \hbar\omega$.

(6) Separate the *regular part* and *anomalous part*. All factors but $L(\boldsymbol{q}, 0)$ in the former are regular in the limit of $\boldsymbol{q} \to 0$ while the anomalous part contains a singular factor in addition to $L(\boldsymbol{q}, 0)$.

(7) Integrate the regular part with $\boldsymbol{q}$ after setting $\boldsymbol{q} \to 0$ in all quantities but $L(\boldsymbol{q}, 0)$.

(8) Integrate the anomalous part with $\boldsymbol{q}$ after introducing a phase-breaking relaxation time τ_φ to cutoff IR divergence.

The diagrams of the MT terms with skew scattering are shown in Fig. 6.

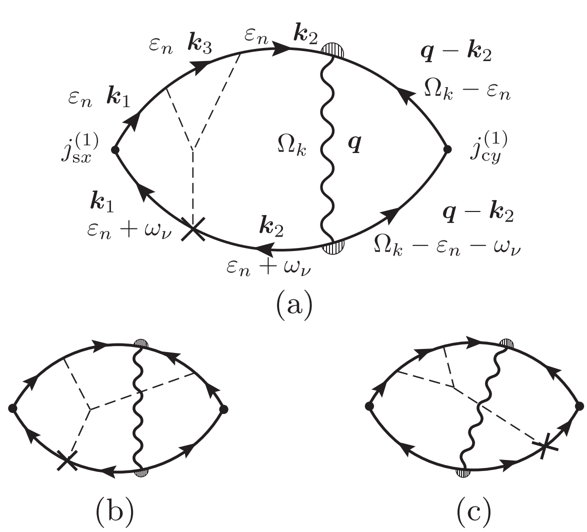

FIG. 6. Diagrams of MT terms in the presence of skew scattering.

As for the regular part, we can proceed in a way similar to that in the DOS terms and obtain

$$\frac{\sigma_{xy}^{\mathrm{MT-SS(reg)}}}{\sigma_{xy}^{\mathrm{SS(normal)}}} = \frac{7\zeta(3)}{16\pi^3 N(0)\xi_{\mathrm{SC}}^D T} \begin{cases} \ln\frac{1}{\epsilon} & (D=2) \\ \frac{2}{\pi} & (D=3). \end{cases}$$

As for the anomalous part, on the other hand, we introduce the phase-breaking time τ_φ to cutoff the IR divergence as in the case of electric conductivity [35]. We then obtain

$$\frac{\sigma_{xy}^{\mathrm{MT-SS(an)}}}{\sigma_{xy}^{\mathrm{SS(normal)}}} = -\frac{\pi}{128 N(0)\xi_{\mathrm{SC}}^D T} \begin{cases} \frac{1}{\epsilon-\gamma_\varphi}\ln\frac{\epsilon}{\gamma_\varphi} & (D=2) \\ \frac{1}{\sqrt{\epsilon}+\sqrt{\gamma_\varphi}} & (D=3), \end{cases}$$

with the dimensionless cutoff $\gamma_\varphi = \pi/8T\tau_\varphi$. The details of the calculation are available in Secs. I E and I F in the Supplemental Material [39].

III. FLUCTUATION EFFECTS ON INTRINSIC SPIN HALL CONDUCTIVITY IN TWO-DIMENSIONAL SYSTEMS WITH RASHBA-TYPE SPIN-ORBIT INTERACTION

A. Model

We consider the intrinsic spin Hall effect in the system, where the Hamiltonian in the normal state is given by

$$H = \frac{\boldsymbol{p}^2}{2m} I - \frac{\lambda_{\mathrm{I}}}{\hbar} \boldsymbol{\sigma} \cdot (\hat{z} \times \boldsymbol{p}), \tag{9}$$

where I and $\boldsymbol{\sigma}$ are the 2 by 2 unit matrix and the Pauli matrices in the spin space. The spin Hall conductivity is given by [11]

$$\sigma_{xy}^{(\mathrm{normal})} = \frac{e^2}{8\pi\hbar} \tag{10}$$

with $\lambda_{\mathrm{I}} \to 0$. In the original paper [11], the authors derived the spin Hall conductivity mainly using the Bloch equation for the spinor in the momentum space. We present a detailed derivation using the Green's function in Sec. II A in the Supplemental Material [39] as a basis of calculation of the fluctuation contribution to the intrinsic spin Hall effect, which will be presented in Sec. III C.

Exceptional simplicity of this model, viz., the combination of parabolic band dispersion and linear momentum dependence of the spin-orbit interaction, makes the spin Hall conductivity vanish in the presence of spin-conserving impurities, even in the limit of weak scatterers [16,41–47]. However, this model is of importance in addressing the two-body interaction effect on the spin Hall effect [45,48]. We will thus adopt the Rashba model and add the BCS-type attractive short-range interaction Eqs. (5) to consider the superconducting fluctuation contribution to the spin Hall effect. The superconducting property of this model below the transition temperature was discussed in Refs. [49,50].

B. Fluctuation propagator

In this subsection, we derive the fluctuation propagator in the presence of the Rashba-type spin-orbit interaction. We start with the Hamiltonian, which is the sum of the Rashba model [Eq. (9)] and BCS-type two-body attractive interaction [Eq. (1)]. Figure 7 shows the diagram of the fluctuation propagator.

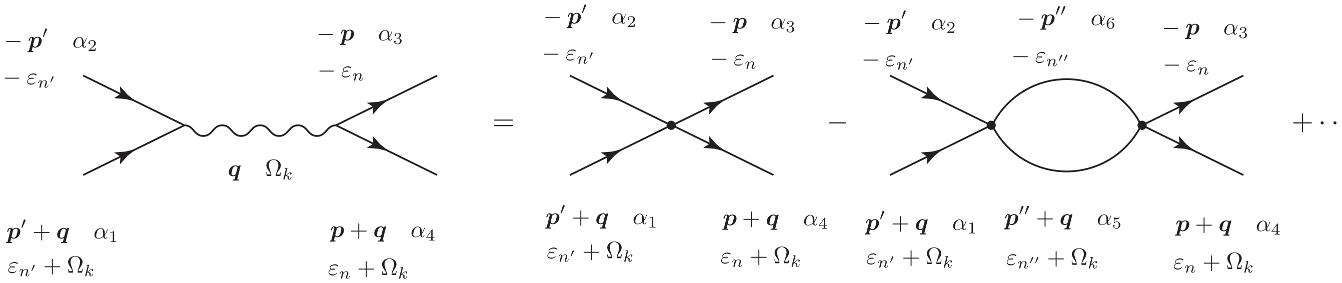

FIG. 7. Feynman diagram for fluctuation propagator.

We first rewrite the two-body interaction with the use of the creation operator $c^\dagger_{\boldsymbol{k}\pm}$ and annihilation operator $c_{\boldsymbol{k}\pm}$ of the eigenstate $|\chi_\pm(\boldsymbol{k})\rangle \equiv (\pm(k_y+ik_x)/k, 1)/\sqrt{2}$ of Eq. (9),

$$H_{\rm int} = -\frac{g}{4V}\sum_{\boldsymbol{p}\boldsymbol{p}'\boldsymbol{q}}\sum_{\alpha_1\sim\alpha_4} \times\ \alpha_2\alpha_3 e^{i(\theta_p-\theta_{p'})} c^\dagger_{\boldsymbol{p}+\boldsymbol{q},\alpha_1} c^\dagger_{-\boldsymbol{p},\alpha_2} c_{-\boldsymbol{p}',\alpha_3} c_{\boldsymbol{p}'+\boldsymbol{q},\alpha_4},$$

with

$$e^{i\theta_p} = \frac{p_x+ip_y}{p}.$$

The spin and wave-vector dependent coefficients that appear in Eq. (7) cancel out but at the left and right ends.

Consequently, the summation over the series of diagrams can be carried out in a way similar to that for superconductors without spin-orbit interaction, i.e.,

$$L_{\alpha_2\alpha_3}(\boldsymbol{q},\boldsymbol{p},\boldsymbol{p}',i\Omega_k) = -\frac{1}{4V}\alpha_2\alpha_3 e^{i(\theta_p-\theta_{p'})}[g^{-1}-\Pi(\boldsymbol{q},\Omega_k)]^{-1},$$

where we introduce the notations

$$\Pi(\boldsymbol{q},\Omega_k) \equiv \frac{1}{4}\sum_{\alpha\beta}\frac{T}{V}\sum_{\varepsilon_n,\boldsymbol{p}}\mathcal{G}_\alpha(\boldsymbol{p}+\boldsymbol{q},i\varepsilon_{n+k})\mathcal{G}_\beta(-\boldsymbol{p},-i\varepsilon_n)$$
$$\approx N(0)\left[\psi\left(\frac{1}{2}+\frac{|\Omega_k|}{4\pi T}+\frac{\hbar\omega_{\rm D}}{2\pi T}\right) - \psi\left(\frac{1}{2}+\frac{|\Omega_k|}{4\pi T}\right)+\frac{A(\boldsymbol{q})}{2(4\pi T)^2}\psi^{(2)}\left(\frac{1}{2}+\frac{|\Omega_k|}{4\pi T}\right)\right],$$

where

$$A(\boldsymbol{q}) \equiv 2(\lambda_{\rm I}k_{\rm F})^2 + \left[\left(\frac{\hbar^2}{m}\right)^2+\left(\frac{\lambda_{\rm I}}{k_{\rm F}}\right)^2\right]\frac{(k_{\rm F}q)^2}{D}.$$

The transition temperature $T_{\rm c}$ is determined by the condition that $L(\boldsymbol{q}=0, i\Omega_k=0)$ diverges at $T\to T_{\rm c}$, i.e., $g^{-1}-\Pi(0,0)=0$. With the use of this condition, we arrive at the expression of the fluctuation propagator,

$$L_{\alpha_1\alpha_2\alpha_3\alpha_4}(\boldsymbol{q},\boldsymbol{p},\boldsymbol{p}',i\Omega_k) = \left(\frac{\alpha_2}{2}e^{-i\theta_{p'}}\right)\left(\frac{\alpha_3}{2}e^{i\theta_p}\right)L(\boldsymbol{q},i\Omega_k),$$

where $L(\boldsymbol{q},i\Omega_k)$ is the fluctuation propagator without the spin-orbit interaction, which coincides with Eqs. (6) and (7) in the clean limit.

C. Intrinsic spin Hall conductivity

The diagrams for the AL terms and DOS terms are shown in Figs. 8 and 9. The outline of the procedure to calculate the AL terms and DOS terms for the intrinsic spin Hall effect are the same as that for the extrinsic spin Hall effect. The details of the calculation along these procedures are given in Secs. II C and II D in the Supplemental Material [39]. The resultant expression for the AL term of the intrinsic spin Hall conductivity normalized by Eq. (10) is given by

$$\frac{\sigma^{\rm AL}_{xy}}{\sigma^{\rm (normal)}_{xy}} = \frac{\pi^2}{98\zeta(3)^2}\frac{T^3}{\varepsilon_{\rm F}(\lambda_{\rm I}k_{\rm F})^2}\left[{\rm Im}\psi^{(1)}\left(\frac{1}{2}+\frac{i\lambda_{\rm I}k_{\rm F}}{2\pi T}\right)\right]^2\ln\frac{1}{\epsilon},$$

which reduces, when $\lambda_{\rm I}k_{\rm F}\ll T$, to

$$\frac{\sigma^{\rm AL}_{xy}}{\sigma^{\rm (normal)}_{xy}} = \frac{1}{2}\frac{T}{\varepsilon_{\rm F}}\ln\frac{1}{\epsilon}.$$

The DOS term for intrinsic spin Hall conductivity is given by

$$\frac{\sigma^{\rm DOS}_{xy}}{\sigma^{\rm (normal)}_{xy}} = -\frac{\pi}{28\zeta(3)}\frac{T^2}{\varepsilon_{\rm F}\lambda_{\rm I}k_{\rm F}}\left[-{\rm Im}\psi^{(1)}\left(\frac{1}{2}+\frac{i\lambda_{\rm I}k_{\rm F}}{2\pi T}\right)\right]\ln\frac{1}{\epsilon}.$$

For $\lambda_{\rm I}k_{\rm F}\ll T$, we obtain

$$\frac{\sigma^{\rm DOS}_{xy}}{\sigma^{\rm (normal)}_{xy}} = -\frac{1}{4}\frac{T}{\varepsilon_{\rm F}}\ln\frac{1}{\epsilon}.$$

The MT terms in intrinsic spin Hall effects are calculated in a way similar to those in extrinsic spin Hall effect, but there are no anomalous terms in the MT terms for intrinsic spin Hall effect, and thus the cutoff is not necessary to be introduced.

The diagram for the MT term in the intrinsic spin Hall effect is shown in Fig. 10. The MT term for intrinsic spin Hall

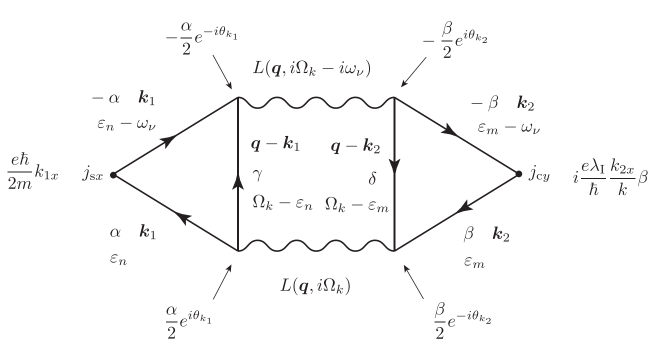

FIG. 8. Feynman diagram for the AL term for intrinsic spin Hall effect.

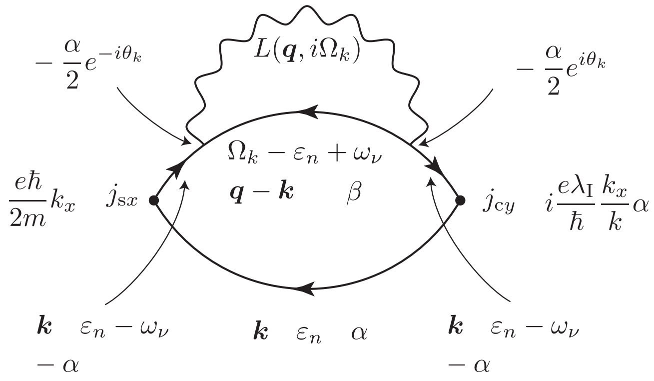

FIG. 9. Feynman diagram of DOS term in intrinsic spin Hall conductivity.

conductivity is given by

$$\frac{\sigma_{xy}^{\mathrm{MT}}}{\sigma_{xy}^{(\mathrm{normal})}} = \frac{2\pi^2}{7\zeta(3)}\frac{T^3}{\varepsilon_{\mathrm{F}}(\lambda_{\mathrm{I}}k_{\mathrm{F}})^2}\left[\mathrm{Re}\psi\left(\frac{1}{2}+\frac{i\lambda_{\mathrm{I}}k_{\mathrm{F}}}{2\pi T}\right) - \psi\left(\frac{1}{2}\right)\right]\ln\frac{1}{\epsilon},$$

which reduces, when $\lambda_{\mathrm{I}}k_{\mathrm{F}} \ll T$, to

$$\frac{\sigma_{xy}^{\mathrm{MT}}}{\sigma_{xy}^{(\mathrm{normal})}} = \frac{1}{2}\frac{T}{\varepsilon_{\mathrm{F}}}\ln\frac{1}{\epsilon}.$$

See Sec. II E in the Supplemental Material [39] for details of the calculation.

IV. DISCUSSION

A. Summary of the resuls for D = 2

We summarize the results for the $D = 2$ case, where the spin Hall conductivity diverges in the limit $\epsilon \to +0$. In Tables I (extrinsic effects in the dirty limit), II (extrinsic effects in the clean limit), and III (intrinsic effects), we note two properties common in extrinsic and intrinsic effects. One is that the singularity in the AL terms is $\ln(1/\epsilon)$, which is weaker than the power-law singularity in the AL terms in electric conductivity. The power-counting argument is given in Sec. IV B. As another point, we notice that all contributions contain the factor $1/\varepsilon_{\mathrm{F}}$, which weakens the fluctuation effect. The origin of this factor is discussed in Sec. IV C.

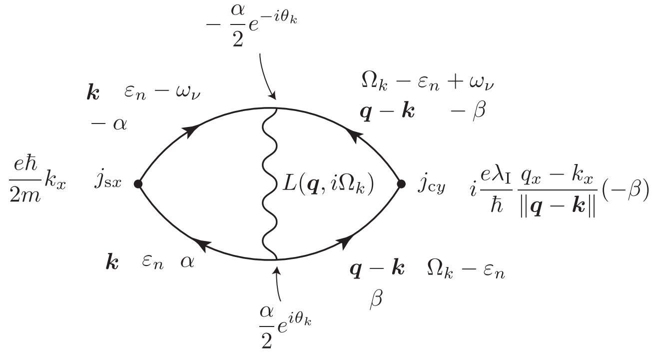

FIG. 10. Feynman diagram of MT term in intrinsic Hall conductivity.

TABLE I. Extrinsic spin Hall conductivity for $D = 2$ in the dirty limit. The results are normalized by the spin Hall conductivity in the normal state, $\sigma_{xy}^{\mathrm{SJ(normal)}}$ or $\sigma_{xy}^{\mathrm{SS(normal)}}$.

	Side jump	Skew scattering
AL/normal	$2\frac{T}{\varepsilon_{\mathrm{F}}}\ln\frac{1}{\epsilon}$	$4\frac{T}{\varepsilon_{\mathrm{F}}}\ln\frac{1}{\epsilon}$
DOS/normal	$-\frac{7\zeta(3)}{\pi^3}\frac{\hbar/\tau}{\varepsilon_{\mathrm{F}}}\ln\frac{1}{\epsilon}$	$-\frac{14\zeta(3)}{\pi^3}\frac{\hbar/\tau}{\varepsilon_{\mathrm{F}}}\ln\frac{1}{\epsilon}$
MT (reg)/normal	0	$\frac{7\zeta(3)}{\pi^3}\frac{\hbar/\tau}{\varepsilon_{\mathrm{F}}}\ln\frac{1}{\epsilon}$
MT (an)/normal	0	$-\frac{\pi}{8}\frac{\hbar/\tau}{\varepsilon_{\mathrm{F}}}\frac{1}{\epsilon-\gamma_\varphi}\ln\frac{\epsilon}{\gamma_\varphi}$

First, we discuss the extrinsic case. We summarize the dominant contribution in Table IV for the dirty limit and Table V for the clean limit. In the dirty limit, the DOS terms with the side jump process are dominant when $\sigma_{xy}^{\mathrm{SJ(normal)}} \gg \sigma_{xy}^{\mathrm{SS(normal)}}$. When $\sigma_{xy}^{\mathrm{SJ(normal)}} \ll \sigma_{xy}^{\mathrm{SS(normal)}}$, either anomalous MT terms or the sum of the DOS term and the regular part of the MT terms is dominant, depending on the magnitude of γ_φ. In the clean limit, the DOS terms with the side-jump process is dominant when $\sigma_{xy}^{\mathrm{SJ(normal)}} \gg \sigma_{xy}^{\mathrm{SS(normal)}}$. When $\sigma_{xy}^{\mathrm{SJ(normal)}} \ll \sigma_{xy}^{\mathrm{SS(normal)}}$, either the anomalous part of the MT term or the DOS term is dominant, depending on the relative magnitude of $T\tau/\hbar$ and $1/(\epsilon - \gamma_\varphi)$. Both in the dirty and clean limits, dominant contributions have signs opposite to that in the normal state. We give an estimate of the dominant contribution in the fluctuation effect based on the parameters for Nb and clean Al when $\gamma_\varphi \ll 1$ in Figs. 11 and 12. We have assumed in these estimations γ_φ independent of temperature but, in reality, importance of temperature dependence in $\gamma_\varphi \ll 1$ has been pointed out [51,52].

Next, we discuss the intrinsic case. All terms are independent of $\lambda_{\mathrm{I}}k_{\mathrm{F}}$ as in the normal state when $\lambda_{\mathrm{I}}k_{\mathrm{F}} \ll T$, and thus a tiny Rashba-type spin-orbit interaction makes the contribution of fluctuations finite.

Fluctuation effects on intrinsic spin Hall conductivity except for the DOS term has the same sign as that in the normal state, in contrast to the extrinsic case. In Ref. [45], the lowest order correction due to two-body *repulsive* interactions was found to suppress the intrinsic spin Hall conductivity in the two-dimensional Rashba model. The enhancement of spin Hall conductivity due to two-body attraction in the present paper and the suppression due to repulsion in Ref. [45] seem consistent with each other.

TABLE II. Extrinsic spin Hall conductivity for $D = 2$ in the clean limit. The results are normalized by the spin Hall conductivity in the normal state, $\sigma_{xy}^{\mathrm{SJ(normal)}}$ or $\sigma_{xy}^{\mathrm{SS(normal)}}$.

	Side jump	Skew scattering
AL/normal	$2\frac{T}{\varepsilon_{\mathrm{F}}}\ln\frac{1}{\epsilon}$	$2\frac{T}{\varepsilon_{\mathrm{F}}}\ln\frac{1}{\epsilon}$
DOS/normal	$-\frac{2\pi^3}{7\zeta(3)}\frac{T}{\varepsilon_{\mathrm{F}}}\frac{T\tau}{\hbar}\ln\frac{1}{\epsilon}$	$-\frac{4\pi^3}{7\zeta(3)}\frac{T}{\varepsilon_{\mathrm{F}}}\frac{T\tau}{\hbar}\ln\frac{1}{\epsilon}$
MT (reg)/normal	0	$2\frac{T}{\varepsilon_{\mathrm{F}}}\ln\frac{1}{\epsilon}$
MT (an)/normal	0	$-\frac{\pi^4}{28\zeta(3)}\frac{T}{\varepsilon_{\mathrm{F}}}\frac{1}{\epsilon-\gamma_\varphi}\ln\frac{\epsilon}{\gamma_\varphi}$

TABLE III. Intrinsic spin Hall conductivity via Rashba-type spin-orbit interaction for $\lambda_{\rm I} k_{\rm F} \ll T$. The results are normalized by that in the normal state, $\sigma_{xy}^{(\rm normal)} = e^2/8\pi\hbar$ [Eq. (10)].

AL/normal	$\frac{1}{2}\frac{T}{\varepsilon_{\rm F}}\ln\frac{1}{\epsilon}$
DOS/normal	$-\frac{1}{4}\frac{T}{\varepsilon_{\rm F}}\ln\frac{1}{\epsilon}$
MT/normal	$\frac{1}{2}\frac{T}{\varepsilon_{\rm F}}\ln\frac{1}{\epsilon}$

B. Power-counting of singularity in the limit $\epsilon \to 0$

We first consider the origin of the singularity near $\epsilon \to +0$ by a power counting argument. After that, we discuss the physical implication of this result.

Before considering the singularity of AL terms in the spin Hall conductivity, we first review the origin of the singularity in those terms in electric conductivity, which is given by

$$\sigma_{xx}^{\rm AL} \propto \frac{1}{i\omega}\left[\sum_{\boldsymbol{q}} \boldsymbol{q}^2 T \sum_{\Omega_k} L(\boldsymbol{q}, i\Omega_k) L(\boldsymbol{q}, i\Omega_k + i\omega_\nu) \times B_{\rm c}(i\Omega_k, i\omega_\nu)^2\right]_{i\omega_\nu \to \hbar\omega \approx 0}.$$

The charge current vertex in the zero frequency limit $B_{\rm c}(0,0)$ is nonzero. We can thus replace $B_{\rm c}(i\Omega_k, i\omega_\nu)$ by $B_{\rm c}(0,0)$ and obtain

$$\sigma_{xx}^{\rm AL} \propto \frac{1}{i\omega} B_{\rm c}(0,0)^2 \left[\sum_{\boldsymbol{q}} \boldsymbol{q}^2 T \sum_{\Omega_k} L(\boldsymbol{q}, i\Omega_k) \times L(\boldsymbol{q}, i\Omega_k + i\omega_\nu)\right]_{i\omega_\nu \to \hbar\omega \approx 0}$$
$$\propto B_{\rm c}(0,0)^2 \underbrace{\sum_{\boldsymbol{q}} \boldsymbol{q}^2}_{\epsilon^{(D+2)/2}} \int \underbrace{d\Omega}_{\epsilon} \underbrace{\coth\left(\frac{\Omega}{2T}\right)}_{\epsilon^{-1}} \underbrace{\frac{d}{d\Omega}}_{\epsilon^{-1}} [{\rm Im}\, \underbrace{L^R(\boldsymbol{q}, \Omega)}_{\epsilon^{-1}}]^2 \quad (11)$$

to extract the most diverging contribution $\epsilon \to 0$. Note that the ω-linear term comes from the fluctuation part and the Ω derivative of the fluctuation part appears in the last line.

We count the power of ϵ in Eq. (11). From the form of L, each quantity scales as $\Omega \propto \epsilon$ and $q \propto \epsilon^{1/2}$ and, accordingly, $\frac{d}{d\Omega} \propto \epsilon^{-1}$, $L \propto \epsilon^{-1}$, $dq\, q^{D+1} \propto \epsilon^{(D+2)/2}$. The fluctuation propagator L is appreciable when $\Omega/T \ll 1$, where we can replace $\coth(\Omega/2T) \approx (\Omega/2T)^{-1}$ thus $\coth(\Omega/2T)$ yields a factor of ϵ^{-1}. Consequently, we see that $\sigma_{xx}^{\rm AL} \propto \epsilon^{D/2-2}$.

TABLE IV. Dominant contribution in extrinsic spin Hall conductivity in the dirty limit.

	$\sigma_{xy}^{\rm SJ(normal)} \gg \sigma_{xy}^{\rm SS(normal)}$	$\sigma_{xy}^{\rm SJ(normal)} \ll \sigma_{xy}^{\rm SS(normal)}$
$\gamma_\varphi \ll 1$	DOS-SJ	MT(an)-SS
$\gamma_\varphi \gg 1$	DOS-SJ	DOS-SS + MT(reg)-SS

TABLE V. Dominant contribution in extrinsic spin Hall conductivity in the clean limit.

	$\sigma_{xy}^{\rm SJ(normal)} \gg \sigma_{xy}^{\rm SS(normal)}$	$\sigma_{xy}^{\rm SJ(normal)} \ll \sigma_{xy}^{\rm SS(normal)}$
$\frac{T\tau}{\hbar} \ll \frac{1}{\epsilon-\gamma_\varphi}$	DOS-SJ	MT(an)-SS
$\frac{T\tau}{\hbar} \gg \frac{1}{\epsilon-\gamma_\varphi}$	DOS-SJ	DOS-SS

We turn to the AL terms in spin Hall conductivity. In the zero frequency limit, $B_{\rm s}(0,0) = 0$ and the dominant contribution comes from the ω linear in $B_{\rm s}(i\Omega_k, i\omega_\nu)$ and we obtain

$$\sigma_{xy}^{\rm AL} \propto \underbrace{\sum_{\boldsymbol{q}} \boldsymbol{q}^2}_{\epsilon^{(D+2)/2}} \underbrace{L(\boldsymbol{q}, 0)^2}_{\epsilon^{-2}} B_{\rm c2}(0,0) \times \frac{d}{dx}[B_{\rm s2}(-x, x) + B_{\rm s2}(0, x)]_{x=0},$$

where the derivative of L does not appear but that of $B_{\rm s}$ does. $B_{\rm s}$ is regular in ω and thus the derivative of $B_{\rm s}$ does not yield any power of ϵ^{-1}. As a result, $\sigma_{xy}^{\rm AL} \propto \epsilon^0$. The power-counting argument does not distinguish $\ln\epsilon$ from ϵ^0 and thus this argument correctly accounts for singularity of $\sigma_{xy}^{\rm AL}$.

Next, we discuss the power of ϵ in the DOS terms and the MT terms. It suffices to consider the contribution from $\Omega_k = 0$ in electric conductivity and spin Hall conductivity. Consequently, the singularities in both quantities are the same.

The power of ϵ comes only from $\sum_{\boldsymbol{q}} L(\boldsymbol{q}, 0)$ in the DOS terms (for extrinsic and intrinsic cases) and the MT terms for the intrinsic case and the regular part of the MT terms for extrinsic case. $L \propto \epsilon^{-1}$, $\sum_{\boldsymbol{q}} \propto \int dq q^{D-1} \propto \epsilon^{D/2}$ and thus $\sigma_{xy}^{\rm DOS}, \sigma_{xy}^{\rm MT(reg)} \propto \epsilon^0$, which is consistent with $\sigma_{xy}^{\rm DOS}, \sigma_{xy}^{\rm MT(reg)} \propto \ln(1/\epsilon)$.

The power of ϵ comes only from $\sum_{\boldsymbol{q}} L(\boldsymbol{q}, 0)(\gamma_\varphi - \xi_{\rm SC}^2 \boldsymbol{q}^2)^{-1}$ in the anomalous part of the MT terms for the extrinsic case. $L \propto \epsilon^{-1}$, $\sum_{\boldsymbol{q}} \propto \int dq q^{D-1} \propto \epsilon^{D/2}$, $\xi_{\rm SC}^2 q^2 \propto \epsilon$ and thus $\sigma_{xy}^{\rm MT(an)} \propto (\gamma_\varphi - \epsilon)^{-1}$, which is consistent with $\sigma_{xy}^{\rm MT(an)} \propto (\gamma_\varphi - \epsilon)^{-1} \ln\epsilon$.

We have seen that the AL terms in spin Hall conductivity have weaker singularity than the AL terms in electric conductivity. The AL terms in electric conductivity represent

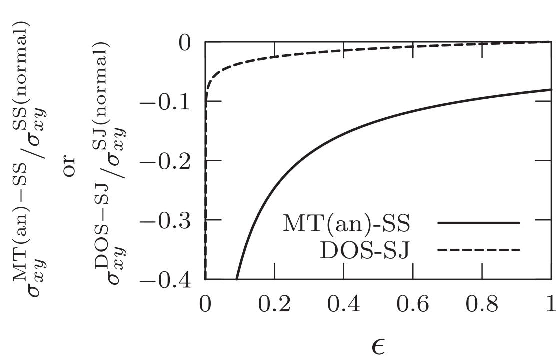

FIG. 11. Estimate of fluctuation effect based on the parameters for Nb.

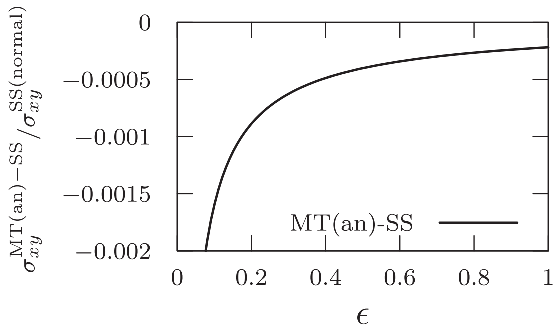

FIG. 12. Estimate of fluctuation effect based on the parameters for clean Al, where skew scattering is considered to be dominant.

the effect of transport carried by the dynamically fluctuating Cooper pairs [36]; the effects of those terms can also be described by the time-dependent-Ginzburg-Landau theory, which is the effective theory for bosons (Cooper pairs) obtained by integrating out the fermionic degrees of freedom. In the case of s-wave superconductors, however, the Cooper pairs carry electric charges but do not spin. Accordingly, the AL terms in the spin Hall conductivity represent a different physical process from that in electric conductivity. The ω-linear term in the response function comes from the fluctuation propagator in the case of the AL terms in electric conductivity, while the ω-linear term comes from the spin current vertex in the spin Hall conductivity. We could thus say that the AL terms in electric conductivity describe the dynamical effect of Cooper pairs. In contrast, the AL terms in spin Hall conductivity come from the dynamical part of the spin current vertex with the static effect of fluctuating Cooper pairs. This kind of dynamical aspect of the spin current vertex in the spin Hall effect has been pointed out in an earlier study [32], where the vortex spin Hall effect in the presence of magnetic field and spin accumulation is discussed.

Singularity in the DOS terms in spin Hall conductivity is the same as that in electric conductivity. This can be understood by recalling that the DOS terms represent the quasiparticle contribution. In this process, spin/charge is carried by quasiparticles. The presence of the fluctuating Cooper-pairs suppresses the DOS of the quasiparticles above the transition temperature [53,54]. Thereby, electric conductivity is suppressed. The quasiparticles carry spin as well as charge and thus those terms for the spin Hall conductivity diverge in a way similar to electric conductivity.

C. Magnitude of spin Hall conductivity; ϵ-independent factors

In Tables I–III, we notice the factor $\varepsilon_{\mathrm{F}}^{-1}$ in all cases. This factor reduces the effects of fluctuation on spin Hall conductivity. You can see that the integral $\sum_{\|\boldsymbol{q}\|<\xi_{\mathrm{SC}}^{-1}} \boldsymbol{q}^2 L^2(\boldsymbol{q},0)$ or $\sum_{\|\boldsymbol{q}\|<\xi_{\mathrm{SC}}^{-1}} L(\boldsymbol{q},0)$ yields a factor $1/\xi_{\mathrm{SC}}$ and reduces the magnitude of the spin Hall conductivity by counting the power of k_{F} in the expression. The small phase volume of $\boldsymbol{q}$ restricted by the condition $\|\boldsymbol{q}\| < \xi_{\mathrm{SC}}^{-1}$ or the *support* of $L(\boldsymbol{q},0)$ implies that a limited number of electrons can contribute to the fluctuation part of the spin Hall conductivity. This fact reflects an additional factor $\varepsilon_{\mathrm{F}}^{-1}$ in fluctuation spin Hall conductivity, compared to spin Hall conductivity in the normal state.

D. Relation to anomalous Hall effect

As mentioned in Sec. I, it is known that there is a connection between the extrinsic spin Hall effect and the extrinsic anomalous Hall effect [9,17,37]. In this subsection, we discuss the relation to Ref. [38], where the superconducting fluctuation on anomalous Hall effect was addressed.

The uniform component of the charge and spin current density operator can be written as Eqs. (10) and (11), respectively. These equations are identical except that (i) $\boldsymbol{j}_{\mathrm{s}}$ contains the factor $1/2$ (ii) $\sigma^z_{\alpha\beta}$ and $\delta_{\alpha\beta}$ are swapped. Therefore, the Feynman diagrams of the spin Hall effect and anomalous Hall effect become very similar. One of the differences is that diagrams of anomalous Hall effect contain an odd number of $\sigma^z_{\alpha\beta}$. In a ferromagnetic metal, physical quantities (such as DOS) for electrons with spin up and down have different values. Thus, we can incorporate the difference of the quantities into the coupling constant of spin-orbit interaction α_{so} by taking an average of spin direction (see Eq. (2.7) in Ref. [38]).

From the above discussion, we can rewrite the results in this paper to the results in Ref. [38] by replacing the strength of spin-orbit interaction λ_{E}^2 to $8\alpha_{\mathrm{so}}/p_{\mathrm{F}}^2$. However, the procedures in this paper for extracting the most diverging term slightly differ from that in Ref. [38]. Because of this, the results in Ref. [38] are different from our results by a numerical factor.

Besides, diagrams containing more fluctuation propagators have more factor of $\varepsilon_{\mathrm{F}}^{-1}$ as mentioned in the last of Sec. IV C. Li and Levchenko calculated the diagrams that contain more fluctuation propagators than diagrams in this paper and showed that these contributions have the factor of $\varepsilon_{\mathrm{F}}^{-2}$ (see Table 1 in Ref. [38]). The nonlinear fluctuation effects are more singular in the limit of $\epsilon \to 0$ than the lowest order contributions of the fluctuation effects and they are dominant when $\hbar/(\varepsilon_{\mathrm{F}}\tau) \ll \epsilon \ll \sqrt{\hbar/(\varepsilon_{\mathrm{F}}\tau)}$ in dirty 2D superconductors [38]. For simplicity, we restrict in this paper the lowest order contributions of the fluctuation effects. This treatment is valid when $\sqrt{\hbar/(\varepsilon_{\mathrm{F}}\tau)} \ll \epsilon \ll 1$ in 2D dirty superconductors.

E. Future issues

As we are motivated in the present paper, the experiments by Jeon *et al.* [29] imply important roles of superconducting fluctuations in spin injection into superconductors or spin-charge conversion in superconductors above T_{c}. As future issues, fluctuation effects on spin-pumping, spin-Seebeck, and charge-imbalance related to spin injection into superconductors are worthwhile to address.

Spin injection from magnets to metals can be driven by electromagnetic field (spin pumping) or thermal gradient at the interface (spin-Seebeck effect). While both subjects for superconductors have been addressed within the mean-field theory [25,26], fluctuation effects on these effects have yet to be considered. As developed in Refs. [25,26], the spin currents injected via spin-pumping and the spin-Seebeck effect depend on local magnetic susceptibility $\chi^R_{\mathrm{loc}}(\omega)$. In the limit $\omega \to 0$, the AL process vanishes as it occurs in spin Hall conductivity. When the dephasing is weak or moderate, the MT term becomes dominant [55].

Consideration of fluctuation effects on $\chi^R_{\mathrm{loc}}(\omega)$ for finite $\omega \neq 0$ will reveal fluctuation effects on spin-Seebeck and spin-pumping effects.

Charge imbalance is another issue to be addressed.

The inverse spin Hall voltage measured in experiments in Ref. [29] is considered to be a consequence of this charge imbalance caused by a spin-charge conversion of quasiparticles in the superconductor. Charge-imbalance has been discussed theoretically in the Boltzmann-type transport theory. It is appropriate to deal with the charge imbalance within the Green function formalism, to incorporate superconducting fluctuations.

The spin Hall effect in the normal state in Nb has been attributed to the intrinsic effect [56,57] based on a semiquantitative model reflecting the multiorbital electronic band structure. For a quantitative account of the experiments by Jeon *et al.* [29], a theoretical study on the fluctuation effects based on a realistic model is desirable. In future research developed in this direction, the fluctuation effects on spin transport in the simple models used in the present paper will serve as a basis for understanding the results of realistic models and experiments.

V. CONCLUSION

In this paper, we theoretically studied the effects of superconducting fluctuations on extrinsic spin Hall effects in two- and three-dimensional electron gas and intrinsic spin Hall effects in the two-dimensional Rashba model. The AL, DOS, and MT terms have logarithmic divergence $\ln \epsilon$ in the limit $\epsilon = (T - T_c)/T_c \to +0$ in two-dimensional systems for both extrinsic and intrinsic spin Hall effects except the MT terms in extrinsic effect, which are proportional to $(\epsilon - \gamma_\varphi)^{-1} \ln \epsilon$ with a cutoff γ_φ in two-dimensional systems. The fluctuation correction to the extrinsic spin Hall effect has an opposite sign to that in the normal state and suppresses the spin Hall effect. The correction to the intrinsic spin Hall effect has the same sign as that in the normal state and thus enhances the spin Hall effect. The study of fluctuation effects on spin injection to superconductors based on more realistic models as well as simple models is an important issue for the future.

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