Adiabatic continuity of the spinful quantum Hall states

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By using the extended Hubbard model of anyons, we numerically demonstrate the adiabatic deformation of the spinful quantum Hall (QH) states by transmutation of statistical fluxes. While the ground state is always spin-polarized in a series of $\nu = 1$ integer QH systems, the adiabatic continuity between the singlet QH states at $\nu = 2$ and $\nu = 2/5$ is confirmed. These results are consistent with the composite fermion theory with spin. The many-body Chern number of the ground-state multiplet works as an adiabatic invariant and also explains the discrete change of the topological degeneracy during the evolution. The generalized Středa formula of spinful systems is justified.

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I. INTRODUCTION

In these decades, topology has been coming to the fore in condensed matter physics. The integer quantum Hall (IQH) effect [1,2] is a prototypical example of the topologically nontrivial phase, where the topological nature of the Chern number is the origin of the quantization of the Hall conductance [3,4]. Topological invariants also work as order parameters beyond the Ginzburg-Landau theory based on the breaking symmetry, which demonstrates how topology brings further diversity to phases of matter. The electron-electron interaction gives even more enriched topological phenomena. The fractional quantum Hall (FQH) effect [5,6] is topologically ordered [7,8] and hosts fractionalized excitations carrying the fractional charge and fractional statistics [9–13]. Even though the origin of the energy gap is intrinsically different in the IQH and the FQH effects, the composite fermion theory [14,15] enables us to understand their underlying physics in a unified scheme: the FQH state at the filling factor $\nu = p/(2mp \pm 1)$ with p, m integers is interpreted as the $\nu = p$ IQH state of composite fermions carrying $2m$ fluxes. Their adiabatic continuity by trading the external fluxes for the statistical ones was demonstrated in various situations [16–22], which justifies the validity of the composite fermion picture. Intermediate states of this adiabatic transformation are anyonic. Although systems of anyons on a torus have some algebraic constraints [23,24] that come from the braid group [25], the gap remains open and the many-body Chern number [26] remains constant during the adiabatic evolution. This describes the discrete change of the topological degeneracy in a similar form of the Středa formula [27], which we call the generalized Středa formula [18].

The internal degree of freedom generates further diversity in the FQH effects. A typical system is the spinful FQH systems where small Zeeman splitting is neglected [28–33]. Multilayer systems [34–38] and (multilayer) graphene [39–55] also give exotic FQH states that cannot be observed in single component systems. They are not only fundamentally interesting in their own right, but may also provide a platform for topological quantum computation based on the non-Abelian braidings [56,57], which has attracted great interest over the past few decades. The composite fermion theory is remarkably useful in the multicomponent FQH systems as well [15]. For example, the spin structure of the FQH states in the limit of vanishing Zeeman energy depends strongly on the filling factor. This selection rule for the spin can be predicted by the corresponding IQH state of composite fermions [33]. This can be applied to other degrees of freedom, such as a layer index [37] and the valley degree [43]. The main goal of this work is to justify the validity of this picture in terms of the adiabatic continuity and to reveal the topological properties during the evolution of the flux-attachment on a torus.

Below we numerically analyze the extended Hubbard model of two-component anyons. They are spinful anyons but it can be applied to other degrees of freedom such as the layer index. We demonstrate that the spin-singlet IQH state at $\nu = 2$ is adiabatically connected to the $\nu = 2/5$ singlet FQH state [28] while the topological degeneracy changes discretely. The adiabatic continuity between the bosonic IQH state at $\nu = 2$ [58,59] and the singlet FQH state at $\nu = 2/3$ [33] is also confirmed. On the other hand, a series of $\nu = 1$ IQH system always gives maximally spin-polarized ground states. These results are consistent with the composite fermion theory with spin [33]. We also confirm that the many-body Chern number of the ground state remains constant during the adiabatic evolution and also describes the discrete change of the topological degeneracy. This justifies the validity of the generalized Středa formula [18] in spinful QH systems.

II. EXTENDED ANYON-HUBBARD MODEL

A. Spinful anyons

Let us consider a two-dimenional (2D) toroidal system of anyons on a square lattice under the magnetic field. Anyons with two components are labeled by spin with $S = 1/2$, but it can be applied to other degrees of freedom such as the layer index. Counterclockwise exchanges of particles with the same spin give the phase factor $e^{i\theta}$. Namely, particles with $\theta/\pi = 0 \pmod 2$ and $1 \pmod 2$ are bosons and fermions, respectively. Otherwise, they are anyons. Note that the statistical phases are not well defined for exchanges of opposite spins since the many-body configuration does not turn back to the original one. In other words, this operation does not form a closed loop in the configuration space, which implies that the change of the phase depends on a choice of basis. On the other hand, the two successive operations, where spin $\uparrow$ moves around spin $\downarrow$ or vice versa, form a closed loop in the configuration space. We assign this to the phase factor $e^{i2\theta}$. Namely, a local move around another always gives $e^{i2\theta}$ irrespective of their spins, which is a physically natural extension of the fractional statistics to two-component systems [60,61]. In the following, we consider the QH systems where the fundamental particles are two-component anyons. This should not be confused with the anyonic quasiparticles of the FQH effect.

B. Hamiltonian

Modeling the anyons as fermions with the statistical fluxes, we define the Hamiltonian $H = H_{\text{kin}} + H_{\text{int}}$ with

$$H_{\text{kin}} = -t \sum_{\alpha,\langle ij\rangle} c^\dagger_{i\alpha} e^{i\phi_{ij}} e^{i\theta_{ij}} c_{j\alpha}, \tag{1}$$

$$H_{\text{int}} = U \sum_i n_{i\uparrow} n_{i\downarrow} + V \sum_{\langle ij\rangle} n_i n_j, \tag{2}$$

where $c^\dagger_{i\alpha}$ is the creation operator for a fermion [62] with spin $\alpha = \uparrow, \downarrow$ on site i, $n_{i\alpha} \equiv c^\dagger_{i\alpha} c_{i\alpha}$, $n_i \equiv n_{i\uparrow} + n_{i\downarrow}$, $\langle ij\rangle$ indicates the summation over the nearest-neighbor pairs of sites, and $e^{i\phi_{ij}}$ describes the external magnetic field [63]. The statistical fluxes are introduced by $e^{i\theta_{ij}}$, see below for details. As shown in Eq. (4) below, θ_{ij} is an operator. The notation of the gauge field θ_{ij} should not be confused with the statistical phase θ. When putting $\theta_{ij} = 0$, the Hamiltonian describes a fermionic system with $\theta/\pi = 1 \pmod 2$. Unless $e^{i2\theta} = 1$, i.e., bosons or fermions, particles carry fractional fluxes, which implies that θ_{ij} is ill defined if two or more particle coordinates coincide. To avoid the singularities, we set $U = +\infty$ that results in the hard-core constraint $c^\dagger_{i\alpha} c^\dagger_{i\beta} = 0$ for any spin α, β. The other parameters are set as $t = 1$ and $V \geqslant 0$. We use the Lanczos method to diagonalize the Hamiltonian.

Our system preserves SU(2) spin-rotational symmetry since the Hamiltonian is expressed as

$$H = \sum_{ij} (t_{ij} \boldsymbol{c}^\dagger_i \boldsymbol{c}_j + V_{ij} n_i n_j), \tag{3}$$

where $\boldsymbol{c}^\dagger_i = (c^\dagger_{i\uparrow}, c^\dagger_{i\downarrow})$, t_{ij} is a function of the operators n_k's, and V_{ij} is a constant. This is obviously invariant under the transformation $\boldsymbol{c}^\dagger_i \to \boldsymbol{c}^\dagger_i u$ with $u \in \text{SU}(2)$.

C. Statistical fluxes with spin

The gauge field θ_{ij} in Eq. (1) is assigned so that the phase accumulated by a particle exchange by hoppings is $e^{i\theta}$. Following the method in Refs. [18,24,64,65], we first construct the hopping Hamiltonian for spinless anyons under with the statistical phase θ under the magnetic field as $H'_{\text{kin}} = -t \sum_{\langle ij\rangle} c^\dagger_i e^{i\phi_{ij}} e^{i\theta'_{ij}} c_j$, where $\theta'_{ij} = \sum_{k\neq i,j} A_{ijk} c^\dagger_k c_k$ with A_{ijk} real. The boundary conditions are modified to ensure the braid group on a torus as described in the next paragraph. Within this framework, we then define the Hamiltonian in Eq. (1) with

$$\theta_{ij} = \sum_{k\neq i,j} A_{ijk} (c^\dagger_{k\uparrow} c_{k\uparrow} + c^\dagger_{k\downarrow} c_{k\downarrow}) = \sum_{k\neq i,j} A_{ijk} n_k \tag{4}$$

under the same boundary conditions. Hoppings in H'_{kin} properly give the phase factors $e^{i\theta}$ and $e^{i2\theta}$ for particle exchanges and for moves of particles around another, respectively. In the same manner, the hoppings in H_{kin} give $e^{i\theta}$ and $e^{i2\theta}$ for particle exchanges within the same spin and for moves around another particle irrespective of their spins as well.

On a torus, states of spinless anyons are not completely determined by the positions of particles [23,24,64,65]. This is derived from algebraic constraints of the braid group on a torus. Accordingly, the Hilbert space of anyons with $\theta/\pi = n/m$ (n, m: coprimes) is spanned by the basis $|\{\boldsymbol{r}_k\}; w\rangle$, where $\{\boldsymbol{r}_k\}$ is the particle configuration and $w = 1, \dots, m$ is the additional label associated with the boundaries: when a particle crosses the boundary in the x (y) direction, the label is shifted from w to $w - 1$ (the phase factor $e^{iw\theta}$ is given). This realizes the nonlocal nature of anyons, which we employ in our spinful system in the same way. This condition does not break the SU(2) symmetry. The dimension of the Hilbert space with $\theta/\pi = n/m$ is given by $\dim H = m\binom{N_x N_y}{N_p}\binom{N_p}{N_\uparrow}$, where $N_x \times N_y$ is the lattice number and $N_p = N_\uparrow + N_\downarrow$ is the total particle number.

The existence of the label w implies that $\dim H$ with $\theta/\pi = n/m$ is m times larger than that with fermions or bosons even for the same particle and the same site numbers, meaning that H changes discretely as θ is changed continuously. Nevertheless, as shown below, the energy gaps of the QH states behave smoothly in the evolution of the flux-attachment although the ground-state degeneracy is discretely changed. Using this smoothness found in a dense set of the energy gaps, we define "adiabatic continuity."

III. ADIABATIC CONTINUITY

By the above setup, we investigate the adiabatic continuity of the spinful QH states under the flux-attachment transformation [14,16,33]. This transformation trades the external magnetic fluxes for the statistical fluxes while keeping their total number constant, i.e.,

$$N_\phi + N_p \frac{\theta}{\pi} = \text{const.}, \tag{5}$$

where N_ϕ is the number of external fluxes and $N_p = N_\uparrow + N_\downarrow$ is the particle number. This implies that a fermionic system at $\nu \equiv N_p/N_\phi = p$ is transformed to systems of anyons with the statistical angle θ at

$$\nu = \frac{p}{p(1-\theta/\pi)+1}. \tag{6}$$

We call such a set of transformed systems the family of the $\nu = p$ IQH system. Figure 1 plots $1/\nu$ as a function of θ/π

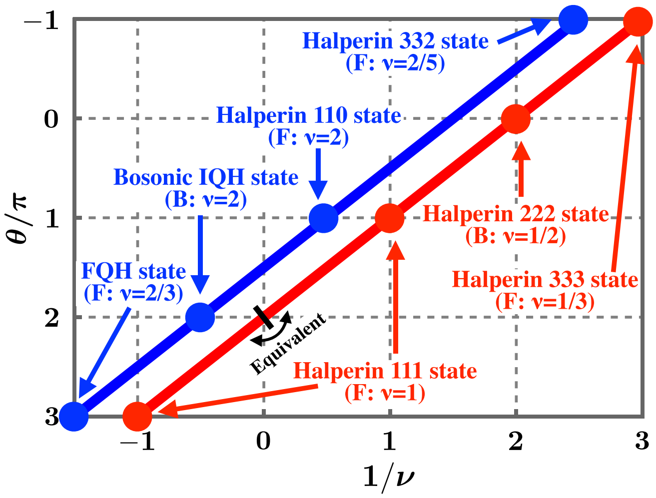

FIG. 1. Each line represents the family of the $\nu = 1$ IQH system (red) and the family of $\nu = 2$ IQH system (blue). These paths are given by Eq. (6) with $p = 1$ and 2, respectively. B and F in the parentheses stand for bosons and fermions, respectively. The red line contains the lattice analog of the Halperin lll state [28] at $\nu = 1/l$ with $l = 1, 2, 3$ (systems at $\nu < 0$ in this family are trivially mapped to that at $\nu > 0$). The blue line contains the lattice analogues of the Halperin 110 state, the Halperin 332 state, $\nu = -2$ (equivalently $\nu = 2$) bosonic IQH state [58,59], and the singlet $\nu = -2/3$ (equivalently $\nu = 2/3$) FQH state [33].

using Eq. (6) with $p = 1$ and 2. This figure also shows various states that each path contains as discussed below.

A. Family of the $\nu = 1$ IQH system

1. Energy gap

Let us first consider a family of the $\nu = 1$ IQH system. We numerically demonstrate that the lattice analog of the Halperin lll state [28] emerges at $\nu = 1/l$ with $l = 1, 2, 3$ and the intermediate systems of anyons also give maximally polarized ground state, see Fig. 1. This result justifies the composite fermion theory with spin [33].

In Fig. 2(a), we plot the energy gap as functions of $1/\nu$, setting $N_p = 4$, $V = 0$, and the system size as $N_x \times N_y = 9 \times 9$. Here, $1/\nu$ (namely N_ϕ) and θ/π change under the constraint in Eq. (5) with, implying that the filling factor is given by Eq. (6) with $p = 1$. The dimension of the Hilbert space at $S_z^{\rm tot} = 0$ with $\theta/\pi = n/7$ is given by $\dim H = 69877080$. The SU(2) spin-rotational symmetry allows us to label the eigenstates with total spin $S_{\rm tot}$ [66]. At $\nu = 1$, we obtain the maximally polarized IQH state with $S_{\rm tot} = S_{\rm tot}^{\rm max} = 2$, which is the lattice analog of the Halperin 111 state. Even though ν is an integer, the Hubbard interaction is crucial here since the lowest Landau level (LLL) is partially filled at $\nu = 1$ (Note that the filling factor is defined as $\nu = N_p/N_\phi$. While the LLL has $2N_\phi$ single-particle states, where 2 comes from the spin degree of freedom, the particle number is $N_p = N_\phi$ at $\nu = 1$). Following the argument of the flat band ferromagnetism [67–70], one expects the spin-polarized ground state. The first excited state gives $S = S_{\rm tot}^{\rm max} - 1$, which is consistent with the spin wave of the polarized QH states. This means that the obtained finite gap is a finite-size effect, but it survives as

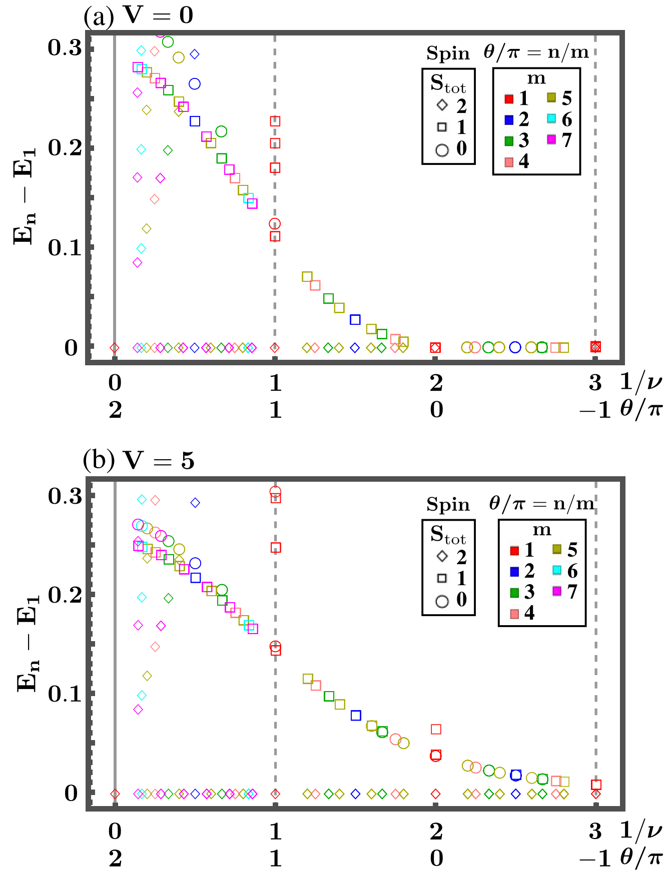

FIG. 2. Energy gaps as functions of $1/\nu$ for (a) $V = 0$ and (b) $V = 5$. We set $N_p = 4$ and $N_x \times N_y = 9 \times 9$. We plot the lowest 15 energies within the $S_z^{\rm tot} = 0$ sector at each $1/\nu$. The number of plots looks less than 15 because of the topological degeneracy. The ground-state degeneracy for each $1/\nu$ is shown in Fig. 3(a). Since its number increases as $1/\nu$ increases, only the first excited energy is plotted in the region $1 < 1/\nu$. The vertical dashed lines represent fermionic systems.

$1/\nu$ increases and then closes at $\nu = 1/2$. This suggests that the spin-polarization at the fractional fillings is understood by the maximally polarized IQH state [33].

The gap closing at $\nu = 1/2$ is explained by the composite fermion theory [14,33]. Noting $U = \infty$ and $V = 0$ on a lattice, let us consider an interaction $\sum_{ij} \delta^2(z_i - w_j)$ in the continuum system in the disk geometry, where $z_j = x_j - iy_j$ and $w_j = x_j - iy_j$ are the positions of particles with $\alpha =\uparrow$ and $\downarrow$, respectively. Within the LLL, this interaction gives the zero-energy degenerate eigenfunctions of bosons at $\nu = 1/2$:

$$\Psi^S_{\nu=\frac{1}{2}} = \prod_{i<j}(z_i - z_j)\prod_{i<j}(w_i - w_j)\prod_{i,j}(z_i - w_j)\Phi^S_{\nu=1}. \quad (7)$$

Here, $\Phi^S_{\nu=1}$ is a LLL projected state at $\nu = 1$ with total spin S, which is macroscopically degenerate for general S as the LLL is partially filled while $\Phi^{S=S_{\rm max}}_{\nu=1}$ is unique. The gap closing at $\nu = 1/2$ in Fig. 2(a) is consistent with this fact.

The discussion implies that the adiabatic continuity in a wider range of ν is established by turning on the nearest-neighbor interaction V. Figure 2(b) is the same as Fig. 2(a), but for $V = 5$. The gap at $\nu = 1/2$ becomes finite and the

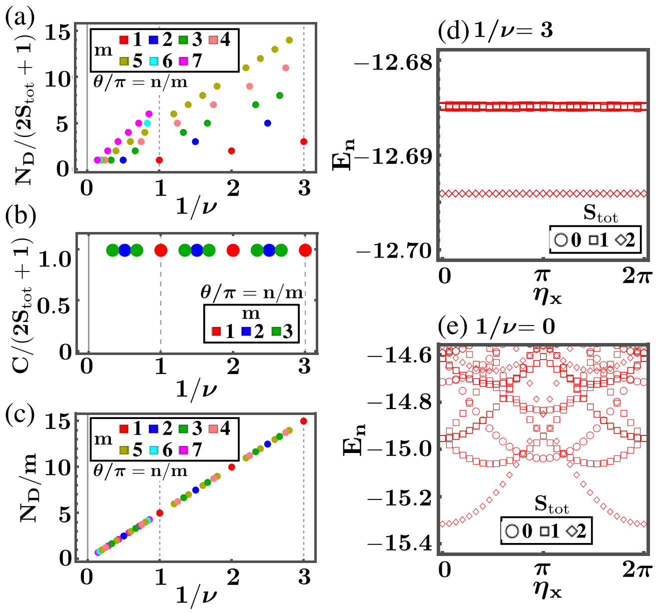

FIG. 3. (a) Ground-state degeneracy N_D divided by the spin degeneracy $2S_{\rm tot}+1$ ($=5$ in this case) as a function of $1/\nu$. (b) Many-body Chern number. (c) Ground-state degeneracy N_D divided by the denominator of θ/π. (d,e) Spectral flows at (d) $1/\nu = 3$ of fermions and (e) $1/\nu = 0$ of bosons. In the all panels, the same setting as Fig. 2(b) is used.

$\nu = 1$ IQH state is adiabatically connected to the lattice analog of the Laughlin state (Halperin 333 state) at $\nu = 1/3$.

2. Topological degeneracy and Chern number

Because of the spin polarization, topological properties of the obtained ground state are identical to that of spinless systems [18] except for the $(2S_{\rm tot}^{\rm max}+1)$-fold spin degeneracy. Assuming that the states are degenerate if their energy difference is less than 0.001 in Fig. 2(b), we plot in Fig. 3(a) the ground-state degeneracy N_D. It changes discretely even though the energy gap behaves smoothly. In Fig. 3(b), imposing the twisted boundary conditions, we compute the many-body Chern number [26]

$$C = \frac{1}{2\pi i}\int_{T^2} d^2\eta F, \tag{8}$$

where $\vec{\eta} = (\eta_x, \eta_y)$ is the twisted angles, $T^2 = [0, 2\pi]\times[0, 2\pi]$, $F = (\partial A_y/\partial \eta_x) - (\partial A_x/\partial \eta_y)$, $A_{x(y)} = {\rm Tr}\,[\Phi^\dagger(\partial\Phi/\partial\eta_{x(y)})]$, and $\Phi(\vec{\eta}) = [|G_1(\vec{\eta})\rangle, \dots, |G_{N_D}(\vec{\eta})\rangle]$ is the ground-state multiplet. As shown in Fig. 3(b), the Chern number C works as an adiabatic invariant numerically.

Extending the generalized Středa formula for spinless anyons [18], the discrete change of the degeneracy is described by the many-body Chern number as

$$\frac{\Delta N_D}{\Delta(m/\nu)} = C, \tag{9}$$

where m is the denominator of θ/π and Δ represents the difference for two possible cases in a family. This works even for polarized states with the spin degeneracy because the additional factors appear in both sides of Eq. (9) as $N_D \to (2S_{\rm tot}^{\rm max}+1)N_D$ and $C \to (2S_{\rm tot}^{\rm max}+1)C$. In Fig. 3(c), we plot N_D/m as a function of $1/\nu$. The slope is surely identical to C ($=5$ in this case), which is consistent with Eq. (9).

Let us mention the twisted boundary conditions. In our model, they are defined as follows: when an anyon hops across the boundary in x (y) direction, the phase factor $e^{i\eta_x\delta_{w1}}$ ($e^{i\eta_y}$) is given to the basis $|\{\boldsymbol{r}_{k\uparrow}\}, \{\boldsymbol{r}_{k\downarrow}\}; w\rangle$. In Figs. 3(d) and 3(e), we plot the energies at $1/\nu = 3$ and $1/\nu = 0$ of Fig. 2(b) as functions of η_x fixing $\eta_y = 0$. Note that $1/\nu = N_\phi/N_p = 0$ means the absence of the external magnetic field. While the FQH state at $\nu = 1/3$ is insensitive to the boundary condition, the spectral flow at $1/\nu = 0$ gives the strong η_x-dependence and the gap is closed. This is consistent with the emergence of Nambu-Goldston modes of the bosonic superconductor [71].

B. Family of the $\nu = 2$ IQH system

1. Energy gap

Let us next consider a family of the $\nu = 2$ IQH system. We demonstrate the adiabatic continuity between the singlet FQH states at $\nu = 2$ and $\nu = 2/5$, which correspond to the Halperin 110 state and Halperin 332 state, respectively. It is also shown that the $\nu = -2$ bosonic IQH state [58,59], a symmetry-protected topological phase of bosons discussed in Ref. [72–74], is adiabatically connected to the singlet FQH state at $\nu = -2/3$ [33], see Fig. 1. These results are consistent with the composite fermion theory with spin [33].

We plot the energy gap as functions of $1/\nu$ in Fig. 4(a), setting $N_p = 4$, $V = 0$, and $N_x \times N_y = 8 \times 8$. The dimension of the Hilbert space at $S_z^{\rm tot} = 0$ with $\theta/\pi = n/7$ is given by $\dim H = 26685792$. At $\nu = 2$, we obtain the spin-singlet IQH state with $S_{\rm tot} = S_{\rm tot}^{\rm min} \equiv 0$. If the Hubbard U vanishes, the ground state is a completely occupied LLL (Halperin 110 state). The obtained ground state is expected to be topologically equivalent to that even for the infinite Hubbard U since the density per site is very small. The energy gap at $\nu = 2$ survives as $1/\nu$ increases. It closes at $\nu = 2/5$ since maximally polarized states at least can be ground states in the fermionic system with $U = \infty$. Figure 4(b) is the same as Fig. 4(a), but for $V = 5$. This indicates that the nearest-neighbor interaction brings the singlet ground state at $\nu = 2/5$, and consequently the adiabatic continuity between the $\nu = 2$ and $\nu = 2/5$ is established. As mentioned below, the ground state at $\nu = 2/5$ is five-fold degenerate and its many-body Chern number C is 2, which is consistent with the Halperin 332 state. (Detailed discussions about the topological degeneracy and the many-body Chern number C are given below.) Although whether the gap survives in the thermodynamic limit is an open question, our demonstration for the fixed system size includes important scientific information.

In a family of $\nu = 2$ IQH system, two systems at $+\nu$ and $-\nu$ are not identical since they are not mapped to each other by simply reversing the magnetic and the statistical fluxes. In Fig. 4(b), a singlet ground state is obtained at $\nu = -2$ with $\theta/\pi = 2$. This is unique and gives $C = -2$, which is consistent with the bosonic IQH state [58,59]. Figure 4(b) suggests that this is adiabatically connected to the singlet FQH state of fermions at $\nu = -2/3$ [33].

Let us now focus on the vicinity of $1/\nu = 0$ with $\theta/\pi = 3/2$ (semions) in Figs. 4. The energy gap looks symmetric

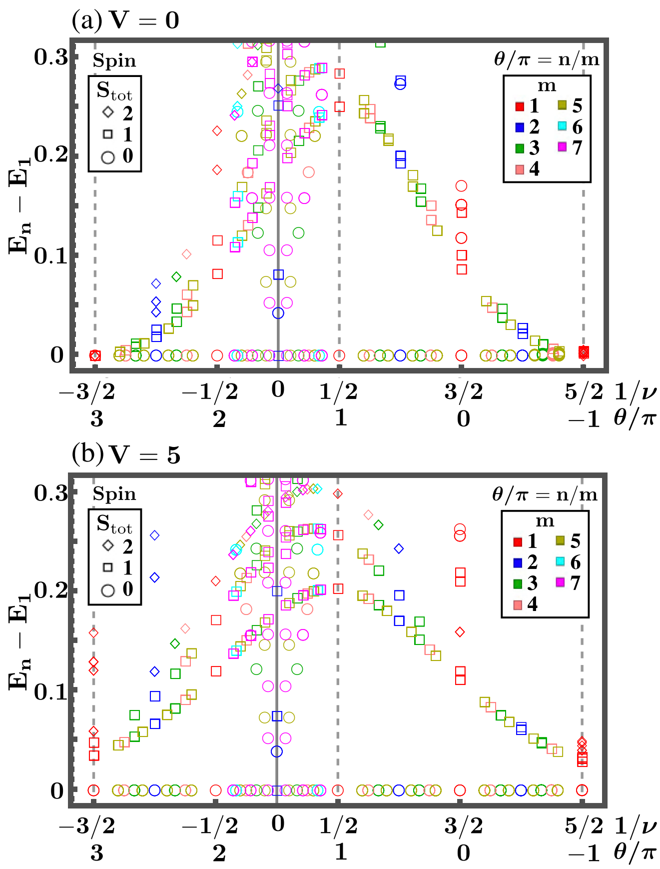

FIG. 4. Energy gaps as functions of $1/\nu$ for (a) $V = 0$ and (b) $V = 5$. We set $N_p = 4$ and $N_x \times N_y = 8 \times 8$. We plot the lowest 25 energies within the $S_z^{tot} = 0$ sector at each $1/\nu$. The number of plots looks less than 25 because of the topological degeneracy. The ground-state degeneracy for each $1/\nu$ is shown in Fig. 6(a). The vertical dashed lines represents fermionic systems.

around $1/\nu = 0$ and closes at $1/\nu = 0$. In fact, this symmetry is exact since the energy with $S_{tot} = 0$ should be an even function of $1/\nu$ when $N_p = 4$ (see the Appendix). We then plot the energy gap but for $N_p = 6$ in Fig. 5. The di-

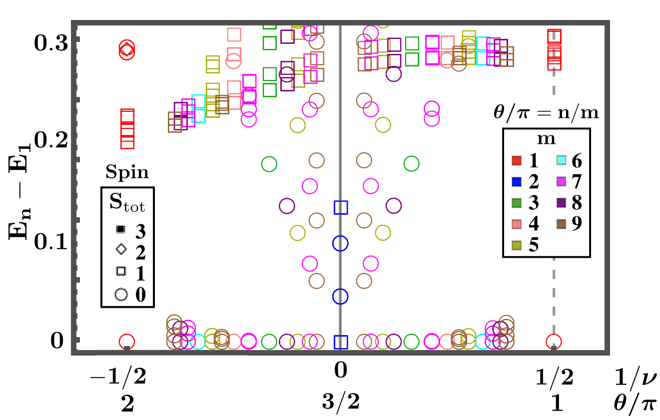

FIG. 5. Energy gaps as functions of $1/\nu$ for $V = 5$. We set $N_p = 6$ and $N_x \times N_y = 6 \times 5$. We plot the lowest 15 energies within the $S_z^{tot} = 0$ sector at each at each $1/\nu$. The vertical dashed lines represent fermionic systems.

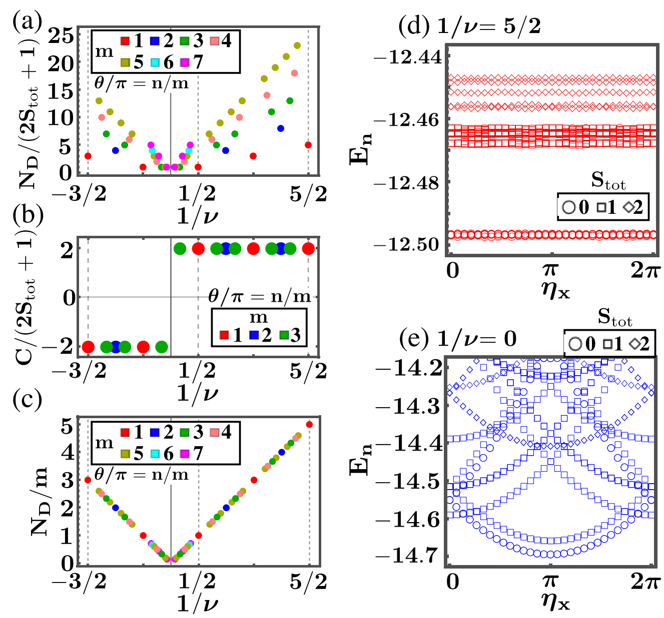

FIG. 6. (a) Ground-state degeneracy N_D divided by the spin degeneracy $2S_{tot} + 1$ (= 1 in this case) as a function of $1/\nu$. (b) Many-body Chern number. (c) Ground-state degeneracy N_D divided by the denominator of θ/π. (d,e) Spectral flows at (d) $1/\nu = 5/2$ of fermions and (e) $1/\nu = 0$ with $\theta/\pi = 3/2$. In the all panels, the same setting as Fig. 4(b) is used.

mension of the Hilbert space at $S_z^{tot} = 0$ with $\theta/\pi = n/9$ is given by $\dim H = 106\,879\,500$. Even though the systems no longer have the emergent symmetry, the energy gaps with $S_{tot} = 0$ at $\pm\nu$ are almost symmetric [e.g., the gap of the first excited states $\Delta E(1/\nu)$ at $1/\nu = \pm 1/6$ gives $\Delta E(1/6) - \Delta E(-1/6) \approx 0.0008$]. This symmetric behavior is consistent with the pairing of two semions [60,61,75–79] since the sign of fluctuations of $1/\nu$ has no influence for bosons. The gap closing at $1/\nu = 0$ is also consistent with the emergence of the Nambu-Goldston modes of the spinful anyon superconductor as in the case of the gap closing in Fig. 2. In Fig. 4(b), the larger gaps look symmetric around $1/\nu \approx 1/2$, which is still an open question.

2. *Topological degeneracy and Chern number*

Let us discuss the topological properties of the ground states. Assuming that states are degenerate if their energy difference is less than 0.001 in Fig. 4(b), we plot the ground-state degeneracy N_D and their many-body Chern number C in Figs. 6(a) and 6(b), respectively. Even though N_D discretely changes as ν is changed, C remains constant and its sign changes at $1/\nu = 0$. This suggests that each gap is characterized by the many-body Chern number C.

The validity of the generalized Středa formula in Eq. (9) is nontrivial for the singlet ground states, unlike in the spin-polarized case. In Fig. 6(c), we plot N_D/m as a function of $1/\nu$. The slope is identical to C (= ±2 in this case), which is actually consistent with Eq. (9). In fact, Eq. (9) generally holds in the spinful systems by assuming some conditions as derived below.

For simplicity, we consider a translationally invariant system as discussed in Ref. [18]. We define the magnetic translational operators $\tau_{i\alpha}$ and $\rho_{i\alpha}$ for ith fermions (with statistical fluxes) with spin $\alpha =\uparrow, \downarrow$ along noncontractible loops on the torus in the x and y directions, respectively. They satisfy

$$\rho_{i\alpha}^{-1}\tau_{j\beta}\rho_{i\alpha}\tau_{j\beta}^{-1} = e^{i2\theta}, \tag{10}$$

for any α, β since the left-hand side is transformed to a local move of the particle i around the particle j [23–25]. This implies

$$\left[\tau_{i\alpha}^{m}, \rho_{j\beta}\right] = 0, \tag{11}$$

for $\theta/\pi = n/m$, meaning that the Hamiltonian specified by twisted boundary conditions should commute with $\tau_{i\alpha}^m$ and $\rho_{j\beta}$. Then defining the translation operators of center of mass [80] in the same way of Ref. [18], we obtain at least $pm/|\nu|$-fold degeneracy at ν with $\theta/\pi = n/m$ in a family of $\nu = p$ IQH system. This reduces to

$$N_D = Cm/\nu, \tag{12}$$

by assuming that the ground state does not have any other degeneracy and gives the Chern number as $C = \text{sgn}\{\nu\} \times p$ as shown in Fig. 6(b). Taking its difference, we obtain Eq. (9).

In Figs. 6(d) and 6(e), we plot the energies as functions of η_x with $\eta_y = 0$ at $1/\nu = 5/2$ and $1/\nu = 0$. Here the systems in Fig. 4(b) are used. While the FQH state at $\nu = 2/5$ is nearly independent of the boundary condition, the spectral flow at $1/\nu = 0$ has the strong η_x-dependence and the gap is closed. This is consistent with the emergence of the Nambu-Goldston modes of the spinful anyon superconductor.

IV. CONCLUSION

In this paper, the extended Hubbard model of anyons is numerically analyzed. In a family of $\nu = 1$ IQH system, we confirm the maximally polarized ground states during the evolution of the flux-attachment. In a family of $\nu = 2$ IQH system, we show that the singlet IQH state at $\nu = 2$ is adiabatically connected to the $\nu = 2/5$ singlet FQH state. The adiabatic continuity between the bosonic IQH state at $\nu = 2$ and the singlet FQH at $\nu = 2/3$ is also confirmed. These results are consistent with the composite fermion theory with spin [33]. The many-body Chern number not only works as an adiabatic invariant, but also describes the discrete change of the topological degeneracy during the evolution.

The energy gap behaves continuously even though the degeneracy is discretely changed. Furthermore, the Chern number is invariant during the adiabatic evolution. The pattern of the degeneracy is rigorously determined by the Chern number as well as the filling factor and the statistical phase [18]. We expect that in adiabatic evolution, what should be deformed continuously is not the states themselves, but a gap between the sets of ground-state multiplets.

An extension of this argument to systems without SU(2) spin-rotational symmetry (e.g., the bilayer FQH effect) is an interesting issue as a future direction. According to the composite fermion theory for bilayer systems [37], one can differentiate between fluxes seen by other particles with the same spin and that seen by particles with the opposite spin. We expect that this diversity may give even more nontrivial adiabatic continuity beyond what the generalized Středa formula predicts.

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APPENDIX: EMERGENT SYMMETRY OF FOUR-PARTICLE SYSTEM

Let us consider the system with the statistical phase θ at the filling factor ν. Since this system is mapped to that with $(-\nu, -\theta)$ by reversing the magnetic and the statistical fluxes, the energy satisfies

$$E(\nu, \theta) = E(-\nu, -\theta). \tag{A1}$$

In the four-particle system, there is another constraint of $E(\nu, \theta)$. The eigenstates of $\boldsymbol{S}_{\text{tot}}^2$ and S_z^{tot} with $S_{\text{tot}} = S_z^{\text{tot}} = 0$ are doubly degenerate [81]:

$$|\pm\rangle = [(12) + e^{\pm i\phi}(13) + e^{\mp i\phi}(14)]/\sqrt{3}, \tag{A2}$$

with $(ij) \equiv (S_i^- S_j^- |\uparrow\uparrow\uparrow\uparrow\rangle + S_i^+ S_j^+ |\downarrow\downarrow\downarrow\downarrow\rangle)/\sqrt{2}$ and $\phi = 2\pi/3$. They satisfy

$$P_{34}|\pm\rangle = |\mp\rangle, \tag{A3}$$

where P_{34} is the exchange operator between the spins 3 and 4. This implies the following equivalence:

$$\Phi^\dagger P_{34}\Phi \simeq \text{diag}\{1, -1\}, \tag{A4}$$

where $\Phi = (|+\rangle, |-\rangle)$. Equations (A3) and (A4) give

$$E_{S_{\text{tot}}=0}(\nu, \theta) = E_{S_{\text{tot}}=0}(\nu, \theta + s\pi), \tag{A5}$$

with s an integer.

According to Eq. (6), a family of $\nu = p$ IQH system gives the following constraint between ν and θ:

$$\theta = \theta(\nu) \equiv \pi\left(\frac{p+1}{p} - \frac{1}{\nu}\right). \tag{A6}$$

This satisfies

$$\theta(-\nu) = -\theta(\nu) + \frac{2(p+1)}{p}\pi. \tag{A7}$$

With $p = 2$, we have

$$E_{S_{\text{tot}}=0}[-\nu, \theta(-\nu)] = E_{S_{\text{tot}}=0}[\nu, \theta(\nu)], \tag{A8}$$

where Eqs. (A1) and (A5) are used. This implies that the energy with $S_{\text{tot}} = 0$ is an even function of $1/\nu$.

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