Dipolar Bose-Hubbard model

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We study a simple model of interacting bosons on a d-dimensional cubic lattice whose dynamics conserves both total boson number and total boson dipole moment. This model provides a simple framework in which several remarkable consequences of dipole conservation can be explored. As a function of chemical potential and hopping strength, the model can be tuned between gapped Mott insulating phases and various types of gapless condensates. The condensed phase realized at large hopping strengths, which we dub a *Bose-Einstein insulator*, is particularly interesting: despite having a Bose condensate, it is insulating, and despite being an insulator, it is compressible.

I. INTRODUCTION

A growing body of work has demonstrated that in systems with a conserved charge, interesting phenomena can arise if the system's dynamics conserves higher mulitpolar moments of the charge, such as dipole or quadrupole moments. Systems with this type of dynamics have constrained kinematics, with the conservation laws restricting the manner in which charge is able to move. These systems have been shown to exhibit close connections with fractonic phases of matter [1–11], offer ways to realize robust ergodicity breaking [12–16] and anomalously slow diffusion [17–22], and are relevant for describing experiments in systems where ultracold atoms are prepared in strongly tilted optical lattices [13,23–25].

Our aim in this work is to develop a better understanding of the physical consequences of multipolar conservation laws, and in particular to examine how such conservation laws influence the competition between kinetic energy and interactions which is at the heart of much of modern condensed matter physics. To this end, we explore a simple model that we dub the *dipolar Bose-Hubbard model* (DBHM), which describes interacting bosons hopping on a d-dimensional cubic lattice in a manner that conserves both total boson number and total boson dipole moment. The Hamiltonian of the DBHM is

$$H_{\text{DBHM}} = H_{\text{hop}} + H_{\text{onsite}}$$
$$H_{\text{hop}} = -t \sum_{i,a} b_{i-a}^\dagger b_i^2 b_{i+a}^\dagger - t' \sum_{i,a} \sum_{b \neq a} b_i^\dagger b_{i+a} b_{i+a+b}^\dagger b_{i+b}$$
$$\quad + \text{H.c.}$$
$$H_{\text{onsite}} = -\mu \sum_i n_i + \frac{U}{2} \sum_i n_i(n_i - 1), \tag{1}$$

where t, t', μ, U are all positive coefficients, $n_i = b_i^\dagger b_i$ is the boson number operator on site i, and where the sums over a, b run over spatial unit vectors. The hopping terms proportional

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to t and t' capture the simplest boson hopping processes compatible with dipole conservation, and are illustrated in Fig. 1. The goal in this work is to understand the competition between H_{hop} and H_{onsite}, and by doing so to map out the quantum phase phase diagram of H_{DBHM}.

The conventional Bose-Hubbard model (BHM) [26], whose Hamiltonian is given by

$$H_{\text{BHM}} = -t_{\text{sp}} \sum_{i,a} b_i^\dagger b_{i+a} + \text{H.c.} + H_{\text{onsite}}, \tag{2}$$

provides a simple model of a transition between an interaction-driven Mott insulator at small single-particle hopping t_{sp}, and a kinetic-energy-driven superfluid at large t_{sp}. This model has been extremely well-studied, and is by now textbook material [27]. Despite the fact that the DBHM differs fundamentally from the conventional BHM only by the imposition of a single conservation law, we will see that the phase diagrams of the two models are markedly different. The large t, t' phase of H_{DBHM} is particularly interesting: it contains a Bose condensate and is compressible, and yet at the same time it is *insulating*, and displays no Meissner effect.

An outline of this paper is as follows. In Sec. II we discuss the mean-field phase diagram of the DBHM, which is summarized in the bottom panel of Fig. 2. In Sec. III we explore the rather remarkable phenomenology of the condensed phase realized at large t, t'. Section IV is devoted to an analysis of the nature of the phase transitions identified in Sec. II, and in Sec. VI we conclude with a short discussion that briefly touches on issues relevant to realizing the DBHM in experiment.

II. MEAN-FIELD THEORY AND PHASE DIAGRAMS

In this section we employ a simple mean-field analysis to sketch out the quantum phase diagram of the DBHM model (1) as a function of the hopping strength and chemical potential. Because we are fixing μ, we will be working in the grand canonical ensemble for boson number. We will however fix
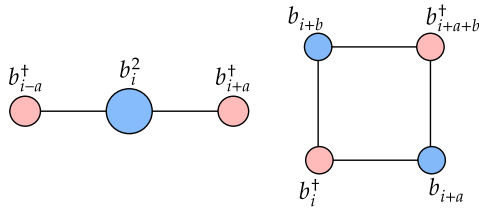
FIG. 1. Illustration of the simplest dipole-conserving hopping terms on the square lattice. Left: an operator that creates two dipoles with moment $\pm\hat{\mathbf{a}}$, separated by one lattice site in the $\hat{\mathbf{a}}$ direction. Right: an operator that creates two dipoles with moment $\pm\hat{\mathbf{a}}$, separated by one lattice site in the $\hat{\mathbf{b}} \neq \hat{\mathbf{a}}$ direction.

the total dipole moment, and in particular will only consider states in which it vanishes.[1]

Before starting, we briefly recapitulate the physics of the conventional BHM [26]. The phase diagram of this model is reviewed in the top panel of Fig. 2. When the single-particle hopping $t_{\rm sp}$ vanishes, the system forms a gapped Mott insulating state, with the average boson number n at each site quantized to be an integer determined by the ratio μ/U. Increasing $t_{\rm sp}$ lowers the gap to doped particles via virtual processes in which single particles delocalize around the lattice. When the decrease in energy brought about by these hopping processes brings the gap to zero, the doped particles condense to form a superfluid.

The transition between the Mott insulators and the superfluid takes place along a series of dome-shaped critical lines. The transition generically occurs when particles (or holes, depending on the value of μ) are gradually doped into the parent Mott insulator, and in this case the critical point is described by a dilute Bose gas of particles (or holes), with a dynamical exponent of $z = 2$, and with the average density changing smoothly across the transition. This story is modified at the "tips" of the Mott insulating regions (purple circles in the top panel of Fig. 2). At these multicritical points the energy gaps to doped particles and doped holes simultaneously vanish, thereby producing an effective particle-hole symmetry. In this case the average density is unchanged across the transition, which has $z = 1$ and which is described by the critical point of the $(d + 1)$-dimensional XY model.

With this review out of the way, let us now turn our attention to the DBHM. The $t = t' = 0$ limit of the DBHM Hamiltonian (1) is the same as the $t_{\rm sp} = 0$ of limit of the regular BHM, and consequently in this limit we obtain a series of Mott insulators with fixed integral average particle number per site.

In the opposite limit of large $t/U, t'/U$, the DBHM develops an instability towards states in which the bosons form isolated clumps with large boson number. This is because when acting on a region with local density n, the hopping operators $-b_{i-a}^{\dagger} b_i^2 b_{i+a}^{\dagger}$, $-b_i^{\dagger} b_{i+a} b_{i+a+b}^{\dagger} b_{i+b}$ have eigenvalues

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[1] Focusing on such states lets us preserve the spatial symmetries of the square lattice, and states with zero dipole moment are particularly natural in the context of cold atoms, in which $H_{\rm DBHM}$ is obtained as an effective Hamiltonian describing bosons hopping in a strongly tilted optical lattice.
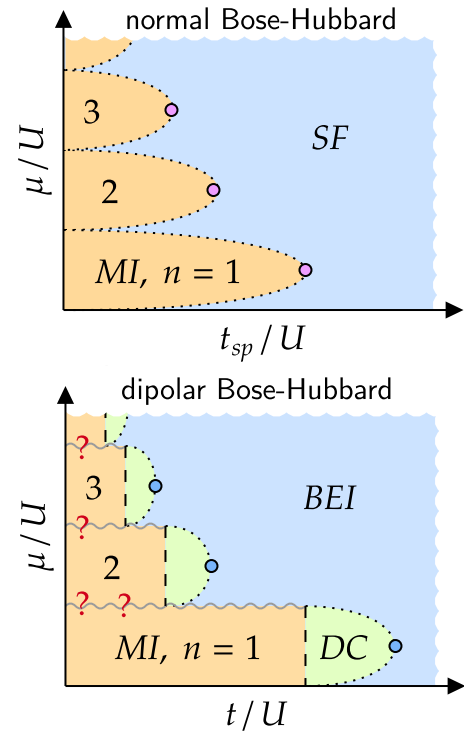

FIG. 2. Mean-field quantum phase diagram of the normal Bose-Hubbard model (top) and the dipolar Bose-Hubbard model (bottom). The orange regions are Mott insulators, with the integer labels denoting the average density of bosons on each site. The green regions are dipole condensates, where dipoles (but not single bosons) have condensed; the average density of these phases is the same as that of their parent Mott insulators. The blue region in the top plot is a conventional superfluid, while the blue region in the bottom plot is a "Bose-Einstein Insulator" (BEI)—a gapless condensate with dynamical exponent $z = 2$ discussed in Sec. III. Dashed lines denote transitions which are either first-order or are described by $(d + 1)$-dimensional XY models, and dotted lines denote transitions described by the d-dimensional dilute Bose gas. The purple circles in the top plot denote multicritical points described by the $(d + 1)$-dimensional XY model, while the blue circles in the bottom point denote the multicritical points discussed in Sec. IV. Red question marks denote various types of crystalline states beyond the present mean-field framework. At large $t/U \gtrsim 1$ and small n, the BEI phase is unstable to the clumped phase (not shown) discussed in the main text.

that scale with n as $-n^2$, which has the same scaling as the Hubbard repulsion $\sim Un^2$. When t, t' are sufficiently larger than U, the energy can thus always be lowered by making the local boson density as large as possible, with the instability arrested only by higher-order interactions like n^4 (note that this instability would not arise in analogous dipole-conserving *spin* models). The physics of this clump formation (which is similar to the fractonic microemulsions of Ref. [28]) will be discussed elsewhere [29], and in the remainder of this paper we will focus on the intermediate $t/U, t'/U$ regime, where as we will see several exotic stable phases of matter exist.

Consider then what happens to the Mott insulators as the boson hopping terms are turned on. The kinematic constraint
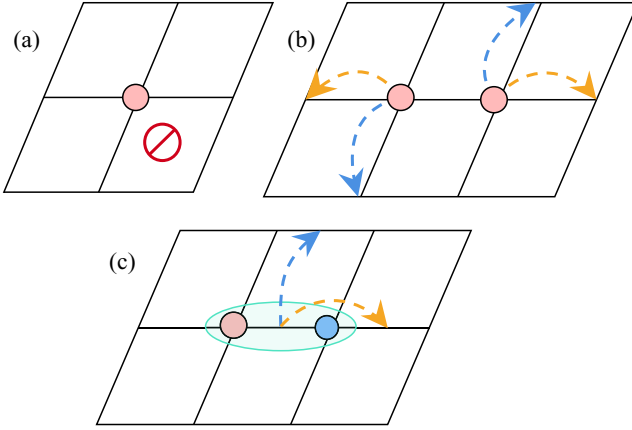
FIG. 3. An illustration of the kinetic constraints imposed by dipole conservation. (a) A single isolated boson (or hole) doped into a given Mott insulator is completely immobile. (b) two bosons located near one another can undergo collective motion which preserves their center of mass, where each boson hops in the $\pm\hat{\mathbf{a}}$ direction. (c) A dipolar bound state of a boson and a hole is free to move in any direction.

imposed by the dipolar conservation law prevents a superfluid from forming in the way that it does in the conventional BHM. Indeed, consider what happens when one dopes particles into a given Mott insulator. Due to dipole conservation, an isolated doped particle is *completely immobile*, and cannot lower its energy through any processes which do not create additional excitations.

Now consider a pair of nearby doped particles. These particles are able to move in a restricted sense, since they can "push" off of one another and move in opposite directions—see panels a and b of Fig. 3 for an illustration. However, once the distance between the particles becomes significantly larger than the range of the hopping terms in the Hamiltonian, their motion will again be frozen out. This means that doped particles will only be able to fully delocalize throughout the lattice when their average density is of order 1.

This suggests that a transition out of the Mott insulator which proceeds directly by condensing single bosons will be first order, as the kinematic constraint means that it is impossible for the bosons to delocalize at arbitrarily small doping levels—the density must therefore jump discontinuously across any direct transition where single bosons condense. In line with these expectations, a naive single-boson mean-field treatment of Eq. (1) [performed by writing $b_i = (b_i - \psi) + \psi$ and working to lowest order in $(b_i - \psi)$] indeed generically yields a first-order transition.

As we discuss in Sec. III, the resulting condensed phase is actually very interesting. Before explaining why, we will take a moment to analyze a more natural two-step condensation process, whereby boson condensation is mediated by the condensation of charge-neutral dipolar "exciton" bound states.

A. Dipole condensation

The most natural way of transitioning out the Mott insulator can be understood by noting that the hopping terms in

(1) are simply conventional kinetic terms for the operators $b_i b_{i+a}^\dagger$, which create dipolar bound states with dipole moment along $\hat{\mathbf{a}}$. This means that while isolated doped particles are localized by the kinematic constraint, dipolar bound states can move freely in all directions (see panel c of Fig. 3). Since dipole motion is the easiest way for the system to lower its kinetic energy, we expect dipole condensation to occur before single-particle condensation does. Note that since the dipoles are charge neutral, the resulting dipole condensate possess the same average density as its parent Mott insulator.[2]

We may analyze this transition within a simple mean-field framework by decoupling the quartic terms in H_{hop}. To do this, we write H_{hop} as

$$H_{\text{hop}} = -\sum_{i,a} \left(d_i^a\right)^\dagger \mathcal{A}_{ij}^a d_j^a, \tag{3}$$

where we have defined the operators

$$d_j^a \equiv b_{j+a}^\dagger b_j, \tag{4}$$

which create boson configurations with dipole moment along the $\hat{\mathbf{a}}$ direction, as well as the matrix

$$[\mathcal{A}^a]_{ij} \equiv \sum_b [t\delta_{a,b} + t'(1 - \delta_{a,b})](\delta_{i,j+b} + \delta_{i,j-b}). \tag{5}$$

Decoupling H_{hop} by introducing a set of dipole fields D_i^a, we obtain an imaginary-time coherent-state path integral with the action

$$S = \int d\tau \left\{ \sum_i b_i^\dagger \partial_\tau b - \sum_{i,a} \left[\left(d_i^a\right)^\dagger D_i^a + \left(D_i^a\right)^\dagger d_i^a \right] \right.$$
$$\left. + \sum_{i,j,a} \left(D_i^a\right)^\dagger [\mathcal{A}^a]_{ij}^{-1} D_j^a + H_{\text{onsite}} \right\}, \tag{6}$$

with the dipole fields satisfying

$$\langle D_i^a \rangle = \sum_j \mathcal{A}_{ij}^a \langle d_j^a \rangle = \sum_b \langle d_{i-b}^a + d_{i+b}^a \rangle [t\delta_{a,b} + t'(1 - \delta_{a,b})]. \tag{7}$$

We now play the usual game of integrating out the b fields to generate an effective continuum action for the D^a dipole variables. On general grounds the most relevant pieces of such an effective action compatible with dipole conservation and spatial symmetries may be written as

$$S_{\text{eff}} = \int d\tau\, d^d x \left(w \sum_a |\partial_\tau D^a|^2 + r \sum_a |D^a|^2 \right.$$
$$\left. + \sum_{a,b} \frac{K_D^{ab}}{2} |\nabla_a D^b|^2 + \sum_{a,b} \frac{u^{ab}}{2} |D^a|^2 |D^b|^2 \right). \tag{8}$$

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[2]Here, as well as in all of what follows, we will restrict our attention to positive values for μ, where the $t = 0$ Mott insulators have a nonzero average density. For $\mu < 0$ by contrast the $t = 0$ state has no particles at all, and it is therefore impossible to create dipoles on top of the ground state. In this case, a direct first-order transition across which single bosons condense seems to be the most natural outcome, but we leave a more detailed investigation to future work.

Here the absence of a linear time derivative term follows under the spatial reflection symmetry which sends D_i^a to $(D_i^a)^\dagger$: A nonzero density of D^a dipoles would break reflection symmetries that send $\hat{\mathbf{a}}$ to $-\hat{\mathbf{a}}$, hence in the presence of such symmetries the dipoles must all be at zero density.

At the mean-field level, the transition occurs when $r = 0$. Expressions for the coefficients in S_{eff} in terms of the microscopic parameters in Eq. (1) can be calculated using standard methods (see Appendix B); in particular, for r one finds

$$r = \frac{1}{2t + 2(d-1)t'} - \frac{2n(n+1)}{U}, \qquad (9)$$

so that e.g. when $t = t'$, the transition occurs at $t_* = U/[4dn(n+1)]$. Note that when n is large, this value allows for a clear parametric separation between the onset of the dipole condensate and the instability caused by the clumping phenomenon mentioned earlier, which occurs when $t \gtrsim U$. The extent of this separation at small n is a question requiring a more detailed numerical treatment [29]; in what follows we will simply assume that a nonzero separation exists (which can always be arranged by working at large n).

The first term in Eq. (9) arises from the zero-momentum contribution to $(D^a)^\dagger [\mathcal{A}^a]^{-1} D^a$, while the second piece arises from performing perturbation theory in the $b_i^\dagger b_{i+a} D_i^a + \text{H.c.}$ term. The U in the denominator of this piece comes from the energy cost of creating a particle-hole pair on top of the Mott insulating ground state. Since this energy is independent of μ to second order in perturbation theory, the shape of the mean-field phase boundary separating the Mott insulator from the dipole-condensed state is *independent* of μ to leading order (see Fig. 2). In particular, at fixed small values of t, t', changing μ will simply induce a direct first-order transition between different Mott insulators, at least at the level of the mean-field analysis considered here. Going beyond the present mean-field approximation, the regions in between distinct Mott insulators will likely host various intermediate density states that spontaneously break lattice symmetries.[3] We leave a detailed investigation of these states to future work.

Now let us now address the nature of the dipole condensate that forms when $r < 0$. Let us write $u^{ab} = u_d \delta^{ab} + u_o(1 - \delta^{ab})$, with $u_d > 0$ assumed to be positive. If $u_o > u_d$, then the system prefers to break the discrete lattice rotational symmetry and condense only a single species of dipole, with the condensed phase possessing a single gapless mode. If $u_o < u_d$, on the other hand, then the system prefers to condense dipoles of all orientations with equal magnitudes. This happens though a continuous transition if $u_o > -u_d/(d-1)$; otherwise, the potential for the dipole fields is unbounded from below to quartic order, and the transition is likely to be rendered first order (which is in fact what happens within a mean-field analysis for the particular Hamiltonian in Eq. (1); see Appendix B for details).

Note that even in the case where all species of dipoles condense, the condensate generically spontaneously breaks both lattice reflections (unless $\text{Im}[\langle D^a \rangle] = 0$ for all a) and lattice rotations (unless $\langle D^a \rangle$ is independent of a). Translation symmetry is unbroken in the condensate however, since the operators which condense are the zero-momentum components of D^a.

B. Single-boson condensation

After dipoles have condensed, there is no longer any kinematic obstruction to condensing single bosons, since the presence of the dipole condensate eases the kinematic constraint—roughly speaking, single bosons are now free to move by absorbing dipoles from the condensate. In the case where all species of dipole condense with equal magnitudes, we may write

$$D^a = \sqrt{\rho_D} e^{i\varphi_D^a}, \qquad (10)$$

with ρ_D a nonzero constant determining the dipole condensate fraction. This substitution yields the effective Hamiltonian

$$H = \sum_{i,a} \sqrt{\rho_D} \Big[\big(d_i^a\big)^\dagger e^{i\varphi_D^a} + d_i^a e^{-i\varphi_D^a} \Big] + H_{\text{onsite}}$$

$$+ \frac{K_L}{2}(\nabla \cdot \boldsymbol{\varphi}_D)^2 + \frac{K_T}{2}(\nabla \times \boldsymbol{\varphi}_D)^2 + \frac{K_A}{2} \sum_a \left(\nabla_a \varphi_D^a\right)^2, \qquad (11)$$

where K_L, K_T, K_A set the stiffness for the phase modes of the dipole condensate, with the anisotropic term K_A allowed by the cubic lattice symmetry. The first term proportional to $\sqrt{\rho_D}$ provides an effective single-boson hopping term (recall $d_i^a = b_{i+a}^\dagger b_i$), and since ρ_D increases as one goes further into the dipole condensed phase, eventually—at least in 2 and 3 dimensions—one triggers a transition at which single bosons condense. In 1d the situation is slightly different, as the effects of vortices in the boson phase need to be taken into account. A detailed analysis of these effects will be given in [29], and in the following we will simply restrict to $d > 1$, where vortices can effectively be ignored for the present purposes.

The location of the phase boundary where single bosons condense can be determined by performing single-particle mean-field theory on Eq. (11) in the standard way [26][4]; this gives rise to the domed parts of the phase diagram in the bottom panel of Fig. 2. We note in passing that this series of transitions—where an intermediate dipolar condensed phase separates the Mott insulating and single boson condensed phases, and provides a way for single bosons to move—is conceptually quite similar to the theory of 2d dislocation-mediated quantum melting put forward in Refs. [22,30–32].

When the interaction matrix u^{ab} is such that only one species D^a of dipole condenses, single-boson hopping is only generated along a single direction, and the condensed phase possesses a quasi-1d character. The consequences that this quasi-1d behavior has for the nature of the condensed phase

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[3]We thank David Huse for this remark.

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[4]In the presence of a dipole condensate, the ground state to perturb about is not given by the usual Mott-insulating ground state, although it does have the same average density. However, in the limit where the condensate fraction of the dipoles is small, treating the ground state as the Mott insulating one will still give accurate results for the phase boundary.

and the character of the condensation transition are discussed in Appendix A. For simplicity, in the rest of the main text we will specialize to the case where all species of dipoles condense.

Recall that in the dipole condensed phase, the symmetries of lattice rotations and reflections are generically spontaneously broken, while translation symmetry is preserved. In the single-boson condensed phase, however, we have (assuming that the condensate is not destroyed by fluctuations; the criteria for when this happens will be discussed in the next section)

$$\langle b_i \rangle = |\langle b \rangle| e^{i\alpha + i\boldsymbol{\beta} \cdot \mathbf{i}}, \tag{12}$$

where $\langle e^{i\boldsymbol{\varphi}_D} \rangle \sim e^{i\boldsymbol{\beta}}$ and where $|\langle b_i \rangle|$ is independent of the lattice site i. In particular, as long as $\boldsymbol{\beta}$ is nonzero, translation symmetry is spontaneously broken in the condensate. However, a subgroup mixing global charge conservation and translation symmetries is preserved, as Eq. (12) is left invariant under the transformation

$$b_i \mapsto b_{i+\hat{\mathbf{a}}} e^{-i\boldsymbol{\beta} \cdot \hat{\mathbf{a}}}. \tag{13}$$

Thus, the single-particle condensed phase realizes a type of spiral ordering, with intertwined patterns of phase and translational ordering. Note however that spirals with different pitch are in fact degenerate in energy, as they are related by shifts of $\boldsymbol{\beta}$.

III. PHENOMENOLOGY OF THE SINGLE-PARTICLE CONDENSED PHASE: THE BOSE-EINSTEIN INSULATOR

A. IR field theory and symmetry breaking

In this section, we explore the phenomenology of the phase in which single particles have condensed (still restricting our attention to $d > 1$). In this phase, the resulting IR theory may be written in terms of the phase mode ϕ appearing in $b \sim e^{i\phi}$ as[5]

$$\mathcal{L} = \frac{\kappa}{2}(\partial_\tau \phi)^2 + \frac{w}{2}(\partial_\tau \boldsymbol{\varphi}_D)^2 + \frac{K_\rho}{2}(\nabla \phi - \boldsymbol{\varphi}_D)^2$$
$$+ \frac{K_L}{2}(\nabla \cdot \boldsymbol{\varphi}_D)^2 + \frac{K_T}{2}(\nabla \times \boldsymbol{\varphi}_D)^2 + \frac{K_A}{2}\sum_a \left(\nabla_a \varphi_D^a\right)^2, \tag{14}$$

where K_ρ is a stiffness parameter proportional to $\sqrt{\rho_D}$.

We see from Eq. (14) that the single-particle condensate "Higgses" the dipole Goldstone $\boldsymbol{\varphi}_D$: we may shift $\boldsymbol{\varphi}_D \mapsto \boldsymbol{\varphi}_D + \nabla \phi$, with the term proportional to K_ρ then effectively gapping out the dipolar phase field, allowing us to set $\boldsymbol{\varphi}_D = 0$. That $\boldsymbol{\varphi}_D$ disappears from the IR theory is of course completely physical: in the presence of a single-particle condensate, the phase of $\langle d_i^a \rangle = b_i^\dagger b_{i+a}^\dagger$ is no longer an independent variable, and is determined by the phase of $\langle b_i \rangle$—thus only ϕ (and not $\boldsymbol{\varphi}_D$) should be a low-energy degree of freedom in the IR theory.

With $\boldsymbol{\varphi}_D$ out of the way, we may thus write (dropping $K_T(\nabla \times \nabla \phi)^2/2$, which vanishes away from vortices)

$$\mathcal{L} = \frac{\kappa}{2}(\partial_\tau \phi)^2 + \frac{K_L}{2}(\nabla^2 \phi)^2 + \frac{K_A}{2}\sum_a \left(\nabla_a^2 \phi\right)^2. \tag{15}$$

This is the Lagrangian of an anisotropic quantum Lifshitz model (QLM), which has also appeared in the analysis of the fractonic "superfluids" of Refs. [33,34].[6] In most applications the QLM is realized only at a critical point [36–39], arising when the coefficient of a single gradient term $(\nabla \phi)^2$ is tuned through zero. By contrast, the QLM written down above describes an entire phase of matter, made possible by the dipolar symmetry which forbids the aforementioned single-gradient term.[7]

Let us now examine the pattern of symmetry breaking that occurs in the condensed phase. The equal-time $T = 0$ boson two-point function, which in the IR is determined by the two-point function of $e^{i\phi}$, is

$$\langle e^{i\phi(\mathbf{r})} e^{-i\phi(\mathbf{0})} \rangle = \exp\left[-\int \frac{d\omega \, d^d k}{(2\pi)^{d+1}} \frac{1 - \cos(\mathbf{k} \cdot \mathbf{r})}{\kappa \omega^2 + K_L k^4}\right], \tag{16}$$

where we have momentarily set $K_A = 0$ for simplicity. By looking at the small k behavior of the integral as $r \to \infty$, we see that $e^{i\phi}$ has QLRO in $d = 2$, and LRO in $d = 3$ [11,33].

At finite temperatures, the integral over ω is replaced with a Matsubara sum, and $e^{i\phi}$ is seen to have short-range correlations in *all* $d \leqslant 3$. In $d < 3$, vortices (textures around which ϕ winds by 2π, with a core at which $|\psi| \to 0$) proliferate at any nonzero T, which happens due to the fact that because of the structure of the kinetic term, a single isolated vortex does *not* cost a thermodynamically large amount of gradient energy.[8] Thus, at any $T > 0$ the field ϕ ceases to be well defined and the IR theory is trivial, with all operators exhibiting short-ranged correlation functions. In $d = 3$ by contrast, although $e^{i\phi}$ is disordered at any $T > 0$, vortex lines do not immediately proliferate, since they are extended objects possessing a nonzero core energy. In this case ϕ remains well-defined at small T and the IR theory remains nontrivial, with many operators possessing long-range correlation functions (such as $e^{i\nabla_a \phi}$, $e^{i\partial_\tau \phi}$). At large enough T, the vortices lose their line tension, and the theory passes into the trivial disordered phase by way of an inverted 3d XY transition.

We may also consider correlation functions of dipole operators, which in the IR map to correlators of $e^{i\nabla_a \phi}$. The correlators are calculated as in Eq. (16), except with an extra factor of $1 - \cos(\mathbf{k} \cdot \hat{\mathbf{a}})$ appearing in the integrand. This extra factor means that the integral is never IR-divergent, so that at $T = 0$, $e^{i\nabla_a \phi}$ *always* has long-range order in the boson condensed phase, even in those cases for which $e^{i\phi}$ is disordered.

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[5]The term $K_2(\nabla^2 \phi)^2/2$ is also allowed by symmetry, but as we will see shortly K_2 merely leads to a renormalization of K_L.

[6]The one-dimensional version of this model has also recently studied in Ref. [35].

[7]In the normal QLM, the critical point in, e.g., $d = 2$ has the possibility of being destabilized by marginal terms such as $(\nabla \phi)^4$ [36,40]. In the present setting however such terms are forbidden by dipole conservation, and these issues do not arise.

[8]Thus, in $d = 2$, $T_{\text{BKT}} = 0$. This is also true in the normal QLM [41], but for more subtle reasons.

TABLE I. The symmetry breaking patterns in the dipole condensed phase (top) and single-particle condensed phase (bottom). Here $U(1)_D$ indicates the dipole symmetry, and $U(1)_C$ indicates global charge conservation. A $\sqrt{}$ indicates that the symmetry is preserved while a $\boldsymbol{X}$ indicates that SSB occurs, with $\mathcal{Q}$ standing for a symmetry-preserving phase with QLRO. The black symbols give the symmetry breaking pattern at $T = 0$, with the red symbols corresponding to $T > 0$. $U(1)_C$ is always preserved in the dipole condensed phase, and both symmetries are always preserved in the Mott insulating phase. We write "n/a" in the second table to signify that understanding the 1d theory in this case requires dealing with the effects of vortices [29], which goes beyond the present analysis.

Dip. cond.	$d = 1$	$d = 2$	$d = 3$
$U(1)_D$	$\mathcal{Q}, \sqrt{}$	$\boldsymbol{X}, \mathcal{Q}$	$\boldsymbol{X}, \boldsymbol{X}$
Boson cond.	$d = 1$	$d = 2$	$d = 3$
$U(1)_D$	n/a	$\boldsymbol{X}, \sqrt{}$	$\boldsymbol{X}, \boldsymbol{X}$
$U(1)_C$	n/a	$\mathcal{Q}, \sqrt{}$	$\boldsymbol{X}, \sqrt{}$

The patterns of symmetry breaking that occur throughout the phase diagram are summarized in Table I.

B. Electromagnetic response

Let us now address whether or not the boson condensed phase is as a superfluid. If we were to define a superfluid as a state in which global particle number conservation is either spontaneously broken or has QLRO, then the condensed phase would count as a superfluid for $d = 2, 3$ (provided that $T = 0$). This however is not the correct definition of a superfluid: a more precise definition (albeit one that is frequently used interchangeably with the above statement about symmetry breaking) is a system with a nonzero superfluid weight, viz. a system which exhibits the Meissner effect when coupled to a background electromagnetic field. In this sense, the boson condensed phase is *not* a superfluid, in any dimension. Indeed, the superfluid weight of the boson condensed phase *vanishes*, and it does not display any Meissner effect. Most strikingly, the condensed phase is in fact completely insulating, at any temperature.[9]

To understand these statements, we simply observe that a background field A_μ for the global boson number symmetry couples to the phase action as

$$\mathcal{L} = \frac{\kappa}{2}(\partial_\tau \phi - A_0)^2 + \frac{K_L}{2}(\nabla^2 \phi - \nabla \cdot \mathbf{A})^2$$
$$+ \frac{K_A}{2}\sum_a (\nabla_a^2 \phi - \nabla_a A_a)^2, \quad (17)$$

which can be derived by sending $\partial_\mu \phi \to \partial_\mu \phi - A_\mu$ and $\boldsymbol{\varphi}_D \to \boldsymbol{\varphi}_D - \mathbf{A}$ in Eq. (14) (this procedure also produces the

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[9]That systems with dipole conservation must have zero DC conductivity σ_{DC} (even if the dipole symmetry is spontaneously broken) is essentially due to the fact that dipole conservation prevents motion of the center of mass of the charge carriers. This argument can be made more rigorous by an analysis similar to the one employed in the discussion of Bloch's theorem in Ref. [42].

unimportant term $K_T(\nabla \times \mathbf{A})^2/2$, which simply renormalizes the electric charge).

We see from Eq. (17) that no mass is generated for the vector field $\mathbf{A}$, since only spatial derivatives of $\mathbf{A}$ appear in the above Lagrangian—there thus is no Meissner effect, and both the DC conductivity and superfluid weight vanish, which follows simply from the absence of the $(\nabla \phi)^2$ term in Eq. (15). Note that this occurs despite the fact that particle number conservation is spontaneously broken (at least in $d = 3$). We have thus realized a rather remarkable scenario wherein even though bosons are condensed, the system is insulating, and incapable of transporting charge. Furthermore, as the average density changes continuously in the condensed phase, this system provides an example of a very unusual phase of matter: a translationally invariant compressible insulator! For these reasons, we will refer to the single-particle condensed phase as a *Bose-Einstein insulator* (BEI).

We note as an aside that compressible translation-invariant systems (without a microscopic dipole conservation symmetry) with nonzero resistivity at $T = 0$ were recently studied in Ref. [42], where they were made possible by a phenomenon the authors dubbed "critical drag." As explained in Ref. [42], critical drag is operative in the model (15) at $T = 0$, which implies that σ_{DC} must vanish at zero temperature. The fact that in the present setting we actually have $\sigma_{DC} = 0$ for *all* T is a consequence of the assumed microscopic dipole symmetry.

Furthermore, it is not just the DC conductivity of the BEI that vanishes. Indeed, the electromagnetic response kernel derived from Eq. (17) is

$$-\frac{\delta^2 \ln \mathcal{Z}[\mathbf{A}]}{\delta A_{\omega,\mathbf{q}}^a \delta A_{-\omega,-\mathbf{q}}^b}\Big|_{\mathbf{A}=0}$$
$$= q_a q_b (K_L + K_A \delta_{ab})$$
$$- q_a q_b \frac{(K_L q^2 + K_A q_a^2)(K_L q^2 + K_A q_b^2)}{\kappa\omega^2 + K_L q^4 + K_A \sum_c q_c^4}, \quad (18)$$

whose transverse part vanishes in the zero-momentum limit, in accordance with the vanishing of the superfluid weight. The kernel in Eq. (18) implies that the conductivity $\sigma(\omega, \mathbf{q})$ vanishes at zero momentum for *all* frequencies, viz.

$$\sigma(\omega, \mathbf{0}) = 0. \quad (19)$$

This fact is a simple consequence of dipole symmetry, and holds even if the $U(1)$ particle number symmetry is spontaneously broken. Indeed, if $\rho(x)$ is the charge density and $J^i(x)$ is the charge current, then

$$[H, \int d^d x \, x^i \rho(x)] = -i \int d^d x \, x^i \nabla_j J^j(x) = i \int d^d x \, J^i(x). \quad (20)$$

Since this vanishes by dipole conservation, we have that $J^i = 0$ in any state with uniform current density, thus implying the vanishing of the conductivity in Eq. (19) (note however that the conductivity associated with a rank-2 gauge field that couples linearly to the *dipole* current will be nonzero).

The finite-momentum conductivity by contrast is generically nonvanishing (although the $\omega = 0$ response vanishes at all $\mathbf{q}$ unless $K_A \neq 0$), and even contains a $1/(\omega + i0)$ Drude-type pole. Thus, the BEI can behave like a superconductor at

short distances, but is nevertheless insulating at the longest length scales.

Since the real part of the conductivity extracted from Eq. (18) goes as q^2 at small frequencies, and since the compressibility of the BEI is nonzero, the charge dynamics in the BEI is subdiffusive, with a diffusion "constant" going as q^2. The transition between this subdiffusive behavior and the fully gapped charge response of the Mott insulating and dipole condensed phases could potentially be used as a way to identify the BEI in experiment, with subdiffusion being probed by examining the evolution of the density following a quench, as in Refs. [23,24].

IV. NATURE OF THE PHASE TRANSITIONS

We turn now to analyzing the nature of the phase transitions identified within the above mean-field framework. When $d > 1$ we will continue to assume that all species of dipoles condense, with the case where only a single dipole condenses being treated in Appendix A.

A. Dipole condensation transitions

We first address the simpler case of the transitions that occur when dipoles condense out of a Mott insulator (dashed lines in the bottom panel of Fig. 2). For simplicity we will only consider the case where the dipole hopping is isotropic in space, so that K_D^{ab} in Eq. (8) is independent of a, b.

For $d = 1$, the transition into the dipole condensed phase is simply that of the two-dimensional classical XY model. When $d = 2$, one possible critical point is given by two copies of the three-dimensional classical XY transition (see also Ref. [43]). The most relevant couplings couple the energy operators on each copy, and are (barely) irrelevant [44]: Hence, a transition described by two decoupled three-dimensional classical XY models can occur. In $d = 3$ the quartic couplings between the dipole fields are marginally irrelevant if positive (hence, yielding a stable decoupled fixed point with mean-field exponents), while if they are sufficiently negative they can be made marginally relevant, likely producing an instability toward a first-order transition.

B. Single-particle condensation

More interesting transitions occur when single bosons condense on top of a background dipole condensate (dotted lines in the bottom panel of Fig. 2). The effective field theory describing the transition has the Lagrangian

$$\mathcal{L} = s\psi^\dagger \partial_\tau \psi + p|\partial_\tau \psi|^2 - \mu|\psi|^2 + \frac{u}{2}|\psi|^4$$
$$+ \rho_D|(\nabla - i\boldsymbol{\varphi}_D)\psi|^2 + \mathcal{L}_0[\boldsymbol{\varphi}_D], \quad (21)$$

where ψ is a complex field and $\mathcal{L}_0[\boldsymbol{\varphi}_D]$ contains the Gaussian terms for $\boldsymbol{\varphi}_D$. Note that $\boldsymbol{\varphi}_D$ enters the kinetic term for ψ in the way that an electromagnetic gauge field would, with the structure of the derivative coupling $|(\nabla - i\boldsymbol{\varphi}_D)\psi|^2$ fixed by dipole symmetry. Unlike a gauge field however the kinetic term for $\boldsymbol{\varphi}_D$ is not invariant under shifts of $\boldsymbol{\varphi}_D$ by total derivatives, and there is no corresponding electric potential appearing in the $\psi^\dagger \partial_\tau \psi$ term.

The nature of the critical point where ψ condenses depends on the spatial dimension d, as well as whether or not the transition is generic (occurring at varying density; $s \neq 0$) or multicritical (occurring at fixed density; $s = 0$). Since the analysis in the case of $d = 1$ requires understanding the effects of vortices [29], we will restrict to $d = 2, 3$ in what follows.

1. $s \neq 0$

Consider first the generic transition with $s \neq 0$, where the $p|\partial_\tau \psi|^2$ term is irrelevant and may be dropped. In this case only particles or holes (but not both) are doped into the dipole-condensed phase, and the density changes continuously across the transition. In the absence of the coupling to $\boldsymbol{\varphi}_D$, these transitions would be described by the d-dimensional dilute Bose gas.

Consider first $d = 3$. Under $z = 2$ scaling, the coupling between ψ and $\boldsymbol{\varphi}_D$ is irrelevant; consequently the critical point is simply that of the dilute Bose gas.

In $d = 2$, u is marginal. Taking $K_L = K_T \equiv K$, $K_A = 0$ for simplicity, the flow to leading order in $1/K$ and u is

$$\frac{du}{dt} = -\frac{A}{K^2} - Bu^2, \quad (22)$$

where A and B are positive constants. Since $1/K$ is always marginal (the self energy of $\boldsymbol{\varphi}_D$ is trivial on account of there being no production of virtual ψ particles), the first term means that u is always eventually driven negative, implying that the transition is likely to generically be rendered first-order.

On the condensed side of the transition, the usual mean-field Bogoliubov treatment gives a single mode with dispersion (setting $s = 1$ and taking $K_D^{ab} = K_D$ independent of a, b for simplicity)

$$\omega(\mathbf{k}) = \frac{\omega_B(\mathbf{k})}{\sqrt{1 + 2\rho_D\langle|\psi|^2\rangle/(K_D k^2)}}, \quad (23)$$

where $\omega_B(\mathbf{k}) = \sqrt{k^2\rho_D}\sqrt{k^2\rho_D + 2\mu}$ is the familiar Bogoliubov dispersion of the condensate in the absence of the coupling to $\boldsymbol{\varphi}_D$, and where the square-root factor on the right-hand side of Eq. (23) comes from hybridization with $\boldsymbol{\varphi}_D$. By examining the small k limit of the above expression, we see that the coupling to $\boldsymbol{\varphi}_D$ correctly produces the $z = 2$ dispersion of the BEI, instead of the $z = 1$ of conventional superfluids.

2. $s = 0$

We now examine the nature of the multicritical points where μ is tuned to ensure particle-hole symmetry about the given Mott insulating ground state, so that the condensation transition occurs at fixed density. In the absence of the $\boldsymbol{\varphi}_D$ field, these transitions would be described by the critical point of the $(d + 1)$-dimensional classical XY model.

For $d = 3$, the coupling to $\boldsymbol{\varphi}_D$ renders the transition first order, via essentially the same mechanism as in 3d scalar QED [45–47].

For $d = 2$, the Lagrangian (21) with $s = 0$ is in fact exactly equivalent to the field theory describing the nematic to smectic-A transition, upon identifying imaginary time with the spatial direction normal to the smectic planes and dropping

the presumably unimportant anisotropic stiffness term proportional to K_A [47,48]. This transition has been extensively studied experimentally. When continuous, the exponents are either those of the 3d XY model, or else a slightly anisotropic version thereof [48].[10]

The standard Bogoliubov treatment on the BEI side of the transition yields a massless mode that disperses quadratically at small k as (setting $p = 1$)

$$\omega(\mathbf{k}) = k^2 \sqrt{\frac{K_D}{2\langle|\psi|^2\rangle}}, \tag{24}$$

while at large k the dispersion goes over to the expected $\omega(\mathbf{k}) = \sqrt{\rho_D}k$. Thus, the hybridization with $\boldsymbol{\varphi}_D$ again ensures that the condensed phase correctly has $z = 2$.

V. PARTIAL DIPOLE BREAKING

In this section we consider what happens when only a subset of the components of the total dipole moment are conserved. Such a scenario arises quite naturally in the context of tilted optical lattices, where partial conservation occurs if one or more principal axes of the lattice are orthogonal to the tilt direction (and is in fact the situation realized in the experiment of Ref. [23]). The partial conservation of dipole moment allows for scenarios in which the condensed phases are insulating in some directions and superconducting in others.

A. Two dimensions

In $d = 2$, we consider the Hamiltonian $H = H_{\text{hop}} + H_{\text{onsite}}$, with H_{onsite} the standard onsite part of the BHM [as in Eq. (1)], and with

$$H_{\text{hop}} = -t_{sp}\sum_i d_i^x - t\sum_{i,a}\left(d_i^y\right)^\dagger d_{i+a}^y + \text{H.c}, \tag{25}$$

which conserves only the y-component of the total dipole moment (in the optical lattice context, such a Hamiltonian would arise in a lattice tilted along the y direction). In the following we will sketch the phase diagram of this model as a function of t/t_{sp}, with the chemical potential and interaction strength held fixed.

Consider first the limit where the single-particle hopping vanishes, $t_{sp} = 0$. In this limit we know from previous sections what happens: at small t we have a Mott insulator, at intermediate t a decoupled stack of dipole condensates (with QLRO at $T = 0$) stacked along the x direction, and at large t a decoupled stack of BEIs, with the dipole symmetry on each BEI spontaneously broken. These phases are all stable with respect to turning on a small nonzero t_{sp}: in the Mott insulator and dipole condensate stack the charge gap is nonzero, while in the stack of BEIs, the single-particle hopping generically acts via a perturbation to the Lagrangian of the form

$$\delta\mathcal{L} = t_{sp}\sum_{x,m} g_m \cos(\phi_x - \phi_{x+m}), \tag{26}$$

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[10]Whether or not the observed anisotropy is real or simply an artefact of experimental sensitivity is a longstanding question that we will not attempt to answer, and simply refer the reader to [48] for details.
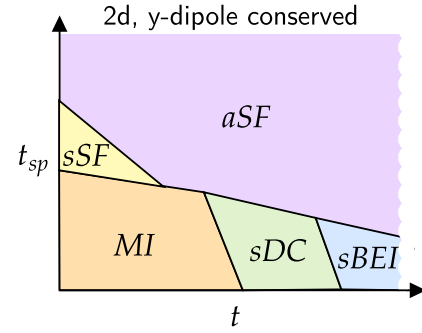

FIG. 4. A rough sketch of a possible phase diagram in the t-t_{sp} plane for the 2d model that only conserves the y component of the total dipole moment (with the chemical potential held fixed at some generic value). "MI" is a Mott insulator, "sDC" a stack of 1d dipole condensates, "sBEI" a stack of 1d BEIs, "sSF" a stack of 1d superfluids, and "aSF" an anisotropic superfluid phase discussed in the main text.

with the x coordinate indexing the BEIs in the stack, and with g_m some function decaying rapidly with $|m|$. Since $\cos(\phi_x - \phi_{x+m})$ has short-range correlations in the BEI phase, the perturbation $\delta\mathcal{L}$ is irrelevant.

Now consider the limit where the dipole hopping vanishes, $t = 0$. At small t_{sp} we of course have a Mott insulator, while at large t_{sp} we have a decoupled stack of superfluids (with QLRO at $T = 0$) stacked along the y direction. The Mott insulating phase is stable with respect to turning on a small nonzero t, while in the stack of superfluids a small t acts as a perturbation like

$$\delta\mathcal{L} = t\sum_{y,m,n} g_{m,n}\cos\left(\Delta_m^y \Delta_n^y \phi_y\right), \tag{27}$$

where $\Delta_m^y \phi_y = \phi_{y+m} - \phi_y$ is the discrete derivative, $g_{m,n}$ decays with $|m|$, $|n|$, and where the subscript on ϕ now indexes the superfluids in the stack. Since $e^{i\phi_y}$ only has QLRO at $T = 0$, the relevance of these perturbations depends on the superfluid density of the superfluids in the stack. At generic values of μ the superfluid density onsets smoothly from zero across the transition out of the Mott insulator, and there is always a regime of t_{sp} for which all of the terms in $\delta\mathcal{L}$ are irrelevant.[11] At large enough t_{sp}, however, Eq. (27) is relevant, so that infinitesimally small t leads to an anisotropic superfluid phase described by the Lagrangian[12]

$$\mathcal{L} = \frac{\kappa}{2}(\partial_\tau\phi)^2 + \frac{K_x}{2}(\partial_x\phi)^2 + \frac{K_y}{2}\left(\partial_y^2\phi\right)^2. \tag{28}$$

It is interesting to note that the physics of this phase is quite similar to that of a quantum smectic whose layers are oriented normal to the $\hat{\mathbf{x}}$ direction [22,32]. The above considerations lead to the schematic phase diagram of Fig. 4.

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[11]If μ is tuned so that the $t = 0$ transition from the Mott insulator into the superfluid stack is of BKT type, then the stiffness jumps across the transition, and the perturbations in Eq. (27) turns out to always be relevant.

[12]The term $(\partial_x\partial_y\phi)^2$ (which is allowed by symmetry) is ignored on the grounds of it being irrelevant under a scaling for which $\tau \sim x \sim y^2$.

By introducing a background electromagnetic field in Eq. (28), we see that the system is superconducting along the x direction and insulating along the y direction. In this phase particle number conservation is spontaneously broken at $T = 0$, as can be demonstrated for example by computing the correlation function

$$\langle e^{i\phi(\mathbf{r})} e^{-i\phi(0)} \rangle = \exp\left[-\int \frac{d\omega\, d^2k}{(2\pi)^3} \frac{1 - \cos(\mathbf{k} \cdot \mathbf{r})}{\kappa\omega^2 + K_x k_x^2 + K_y k_y^4} \right], \tag{29}$$

which asymptotes to a constant as $r \to \infty$. At $T > 0$, however, we see that the above correlator is short-ranged, implying that particle number is not spontaneously broken. y-dipole conservation is spontaneously broken at $T > 0$ however, as can be seen by calculating the $T > 0$ correlator of $e^{i\nabla_y \phi}$ (done by multiplying the integrand in Eq. (29) by $[1 - \cos(k_y a)]^2$, with a the lattice spacing).

B. Three dimensions

In $d = 3$ we may consider two different scenarios.

1. Two components of dipole moment conserved

In the first scenario, two components of the dipole moment are conserved. The appropriate hopping term to study is then

$$H_{\text{hop}} = -t_{\text{sp}} \sum_i d_i^x - t \sum_{i,a} \left(\left(d_i^y \right)^\dagger d_{i+a}^y + \left(d_i^z \right)^\dagger d_{i+a}^z \right) + \text{H.c.} \tag{30}$$

The analysis of the phase diagram at small t is quite similar to that of the two-dimensional model discussed in the previous section. Consider instead the limit where $t_{\text{sp}} = 0$. In this limit we find the usual Mott insulator at small t, an array of two-dimensional dipole condensates stacked along the x direction at intermediate t, and a stack of two-dimensional BEIs at large t, in which charged operators exhibit QLRO at $T = 0$. The Mott insulator and dipole condensate stack are stable with respect to turning on a small t_{sp}, while the stability of the BEI stack depends on the relevance of the terms

$$\delta\mathcal{L} = t_{\text{sp}} \sum_{l,m} g_m \cos(\phi_l - \phi_{l+m}), \tag{31}$$

where l indexes the different layers in the stack. The relevance of the terms in Eq. (31) depends on the stiffness K_D^{ab} appearing in the kinetic term $K_D^{ab}(\nabla_a \nabla_b \phi_l)^2$, with the terms in Eq. (31) being irrelevant at small K_D and relevant at large K_D. These considerations lead to the schematic phase diagram shown in the top panel of Fig. 5.

The anisotropic superfluid phase where both t, t_{sp} are large is described by the Lagrangian

$$\mathcal{L} = \frac{\kappa}{2}(\partial_\tau \phi)^2 + \frac{K_x}{2}(\partial_x \phi)^2 + \frac{K_y}{2}(\partial_y^2 \phi)^2 + \frac{K_z}{2}(\partial_z^2 \phi)^2$$
$$+ \frac{K_{yz}}{2}(\partial_y \partial_z \phi)^2. \tag{32}$$

In this phase the system superconducts along the x direction, but is insulating along y and z. Furthermore, by coupling Eq. (32) to a gauge field one finds that this phase exhibits *no* Meissner effect. This can be understood physically by recalling that the Meissner effect occurs when circulating
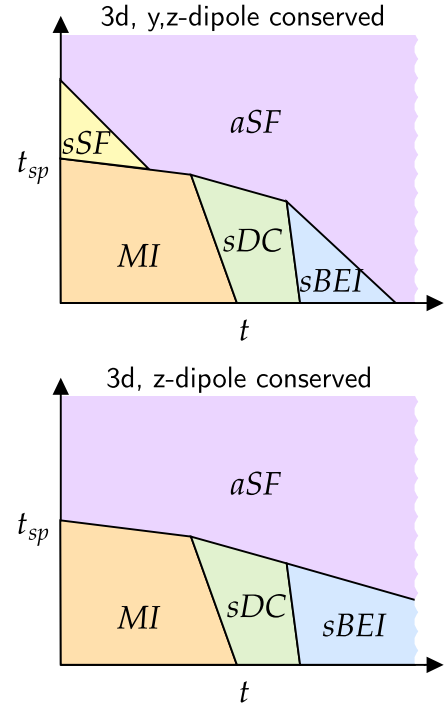

FIG. 5. Schematic phase diagrams for 3d models which conserve both the y, z components of the total dipole moment (top) and just the z component (bottom). The labeling of the various phases is as in Fig. 4.

diamagnetic supercurrents arise to cancel out an applied magnetic field. The existence of circulating supercurrents requires that the system be superconducting in more than one spatial direction, which is not the case in the present model. Finally, computations analogous to that in Eq. (29) show that total charge conservation is spontaneously broken at $T = 0$ and has QLRO at $T > 0$, with dipole conservation being spontaneously broken at all T.

2. One component of dipole moment conserved

We may also consider a scenario in which only one component of the dipole moment is conserved. In this case, we take the hopping term to be

$$H_{\text{hop}} = -t_{\text{sp}} \sum_i \left(d_i^x + d_i^y \right) - t \sum_{i,a} \left(d_i^z \right)^\dagger d_{i+a}^z + \text{H.c.} \tag{33}$$

The small t_{sp} portion of the phase diagram is similar to that of the $d = 2$ anisotropic theory discussed previously. At large t_{sp}, the stack of 2d superfluids that forms at $t = 0$ is unstable to any finite dipole hopping at $T = 0$, leading to the schematic phase diagram in the bottom panel of Fig. 5. The condensed phase at large t, t_{sp} is again an anisotropic superfluid, captured by the Lagrangian

$$\mathcal{L} = \frac{\kappa}{2}(\partial_\tau \phi)^2 + \frac{K_x}{2}(\partial_x \phi)^2 + \frac{K_y}{2}(\partial_y \phi)^2 + \frac{K_z}{2}\left(\partial_z^2 \phi\right)^2. \tag{34}$$

Thus, the system superconducts along the x and y directions, and insulates along the z direction. We also see that the system exhibits a partial Meissner effect: magnetic fields along the z direction are screened, while fields along the x and y directions

are not. As for the pattern of symmetry breaking, one finds that charge conservation is spontaneously broken for all values of T.

VI. DISCUSSION

In this paper we have explored the physics of the dipolar Bose-Hubbard model (DBHM), a simple variant of the conventional Bose-Hubbard model whose dynamics conserves both total charge and total dipole moment. The latter conservation law has a dramatic effect on the physics, significantly changing the phase diagram and producing a highly unusual phase of matter: an insulating Bose condensate, which we dub a "Bose-Einstein insulator" (BEI). We also studied models in which only some components of the total dipole moment are conserved, which were shown to produce phases that superconduct in some directions and insulate in others.

Clearly it is important to understand to what extent dipole moment conservation can be treated as a good symmetry in experimental platforms capable of simulating the DBHM. To date, the most promising experimental platforms appear to be systems of ultracold atoms prepared in tilted optical lattices [13,23,24] (although existing experiments have only studied tilted *Fermi*-Hubbard models). These systems do not exactly conserve dipole moment, and any realistic microscopic Hamiltonian will possess a nonzero single-particle hopping term $-t_{\rm sp} \sum_{i,a} b_i b_{i+a}^\dagger +$ H.c. However, as discussed in Ref. [13], in the presence of a strong tilted potential of strength V, these systems possess dipole-conserving dynamics over a long prethermal timescale t_*, which is exponentially large in $V/t_{\rm sp}$, and which can even become infinite if the strength of the potential is made to scale with the system size. When $t > t_*$ the physics of the DBHM will cross over to that of a strongly tilted Bose-Hubbard model with conventional hopping terms. However, since the ground state of the dipole-conserving model has uniform density, the tilted Bose-Hubbard model describing the dynamics at $t > t_*$ will effectively be initialized at an extremely high energy state, and consequently the detailed behavior at $t > t_*$ is likely to be rather messy. Here we simply content ourselves with the fact that dipole conservation is effectively exact at large tilt potentials and in finite-sized systems, and leave a quantitative analysis of the effects of weak dipole breaking to future work.

It is also possible to consider models where the potential strength V is not the largest energy scale in the problem, but is rather brought down, e.g., to the level of the onsite repulsion U. Previous works analyzing this regime [49,50] have found an interesting array of phases, with dipolar excitations remaining the star of the show. It would be interesting to understand how this physics connects to that of the $V \gg U$ regime considered in this paper.

One of the most interesting aspects of systems whose dynamics conserves dipole moment is the strong sensitivity of the system's dynamics to the choice of initial state, with it often being the case that a large fraction of the Hilbert space is completely inert under time evolution [13]. This however is not likely to be an issue for performing an experimental study of the universal aspects of the equilibrium DBHM phase diagram. This is so because there is always a canonical choice for the initial state prepared in experiments, which belongs to an exponentially large sector of Hilbert space in which the dipolar dynamics act ergodically, in contrast to an inert "shattered" state. For example, in optical lattice realizations, we can imagine first preparing the system in an un-tilted lattice with weak intersite tunneling, placing the system deep in a Mott insulating phase. We can then turn on the tilt potential while remaining in the Mott insulator, and from here one can subsequently increase the tunneling and study the phases that occur at larger hopping strengths. This procedure sidesteps issues of nonstandard thermalization due to Hilbert space shattering.

APPENDIX A: SINGLE DIPOLE CONDENSATE IN $d > 1$

In this Appendix we consider what happens in $d > 1$ dimensions when interactions favor a scenario in which only a single species of dipole moment condenses (which we take without loss of generality to be d^x).

The phase where d^x dipoles have condensed but individual bosons are gapped is described simply by a single compact scalar φ, the phase mode of the dipole condensate. More interesting is the regime in which individual bosons are condensed. Because individual bosons in the dipole condensed phase only possess an effective single-particle hopping along the $\hat{\mathbf{x}}$ direction, we may analyze this regime by way of a quasi-one-dimensional description in terms of a coupled array of Luttinger liquids, in a manner quite similar to the subdimensional critical points considered in Ref. [51].

We will index the Luttinger liquids by a $(d-1)$-dimensional vector $\boldsymbol{\lambda}$, which runs over the sites of a $(d-1)$-dimensional square lattice. Writing the phase field for the Luttinger liquid at site $\boldsymbol{\lambda}$ as $\phi_{\boldsymbol{\lambda}}$, we then define fields Fourier transformed in the $d-1$ directions normal to $\hat{\mathbf{x}}$ as $\phi_{\mathbf{p}}(x, \tau) = \sum_{\boldsymbol{\lambda}} e^{-i\boldsymbol{\lambda}\cdot\mathbf{p}}\phi_{\boldsymbol{\lambda}}(x, \tau)$. The most general action we may write down for the $\phi_{\mathbf{p}}, \varphi_{\mathbf{p}}$ fields is

$$S = \int d\tau \, dx \left[\frac{1}{2} \int \frac{d^{d-1}p}{(2\pi)^{d-1}} \left(\mathcal{J}_{\mathbf{p}}|\partial_\tau \phi_{\mathbf{p}}|^2 + \mathcal{K}_{\mathbf{p}}|\partial_x \phi_{\mathbf{p}} - \varphi_{\mathbf{p}}|^2 + \mathcal{M}_{\mathbf{p}}|\partial_x \phi_{\mathbf{p}}|^2 \right) + \sum_{\{\mathcal{N}_{\mathbf{p}}\}} \cos\left(\int \frac{d^{d-1}p}{(2\pi)^{d-1}} \mathcal{N}_{\mathbf{p}} \phi_{\mathbf{p}} \right) \right] + S_{0,\varphi}. \quad \text{(A1)}$$

where φ is the phase of the condensed dipole field, $S_{0,\varphi}$ is the free spin-wave action for φ, and where the integrals over all components of **p** run from $-\pi/a$ to π/a, with a the lattice spacing.

The functions $\mathcal{J}, \mathcal{K}, \mathcal{M}, \mathcal{N}$ appearing in S are required to respect the symmetries of the square lattice, and to be compatible with charge and dipole conservation. Dipole conservation imposes that $\mathcal{M}_\mathbf{0} = 0$, but hopping in directions normal to $\hat{\mathbf{x}}$ nevertheless allows for the single derivative terms $|\partial_x \phi_\mathbf{p}|^2$ to be present for all $\mathbf{p} \neq \mathbf{0}$. Conservation of charge, dipole moment, and compactness of ϕ requires that $\mathcal{N}_\mathbf{p}$ be the Fourier transform of an integer-valued function such that $\mathcal{N}_\mathbf{0} = 0$ and $(\frac{\partial \mathcal{N}}{\partial p^a})|_{\mathbf{p}=\mathbf{0}} = 0$. In real space, the simplest terms appearing in the $\mathcal{M}_\mathbf{p}$ term are $(\partial_x \Delta_b \phi)^2$ with Δ_b the discrete lattice derivative along $\hat{\mathbf{b}} \neq \hat{\mathbf{x}}$, while the simplest terms appearing in the cosine are $\Delta_a \Delta_b \phi$, with $\hat{\mathbf{a}}, \hat{\mathbf{b}} \neq \hat{\mathbf{x}}$.[13]

As in the analysis of case where all of the d^a condense, the dipole phase field φ can effectively be dropped, since after shifting $\varphi_\mathbf{p}$ by $\partial_x \phi_\mathbf{p}$ we generate a mass term for $\varphi_\mathbf{p}$, along with unimportant terms that either are irrelevant, or can be absorbed by a redefinition of $\mathcal{M}_\mathbf{p}$. Performing this shift, we then write the free term for $\phi_\mathbf{p}$ as

$$S_{0,\phi} = \int \frac{d^{d-1}p}{(2\pi)^{d-1}} \frac{R_\mathbf{p}^2}{4\pi} \int d\tau\, dx \left(\frac{1}{v_\mathbf{p}} |\partial_\tau \phi_\mathbf{p}|^2 + v_\mathbf{p} |\partial_x \phi_\mathbf{p}|^2 \right), \tag{A2}$$

where we have defined

$$R_\mathbf{p}^2 \equiv 2\pi \sqrt{\mathcal{J}_\mathbf{p} \mathcal{M}_\mathbf{p}}, \qquad v_\mathbf{p} \equiv \sqrt{\mathcal{M}_\mathbf{p}/\mathcal{J}_\mathbf{p}}. \tag{A3}$$

The IR theory described by the above free action contains fields which disperse in a quasi-1d fashion. This quasi-1d behavior will persist so long as cosines containing discrete derivatives along directions normal to $\hat{\mathbf{x}}$ are irrelevant. The simplest of these cosines are $\cos(\Delta_a^2 \phi)$ with $\hat{\mathbf{a}} \neq \hat{\mathbf{x}}$ in $d = 2, 3$ and $\cos(\Delta_y \Delta_z \phi)$ in $d = 3$, which have scaling dimensions determined by

$$\Delta_{\cos(\Delta_a \Delta_b \phi)} = 8 \int \frac{d^{d-1}p}{(2\pi)^{d-1}} \frac{\sin^2(p^a/2) \sin^2(p^b/2)}{R_\mathbf{p}^2}. \tag{A4}$$

Note that while dipole conservation imposes $\mathcal{M}_\mathbf{0} = 0 \Rightarrow R_\mathbf{0}^2 = 0$, the small-$p$ behavior of the numerator means that the integral is still finite. However, the integral diverges in the absense of the sines in the numerator, meaning that the scaling dimension of, e.g., $\cos(\phi)$ is infinite (as is the case for all operators which do not conserve dipole moment).

$R_\mathbf{p}^2$ generically increases as we proceed deeper into the single-boson condensed phase, hence the above terms will eventually become relevant (which happens when their scaling dimensions drops below 2, for the same reasons as explained in Ref. [52]). When this happens we may replace the discrete derivatives with continuum ones and Taylor expand the cosines, so that the quasi-1d theory (with $z = 1$) crosses over to the $z = 2$ QLM theory of the BEI discussed in the main text.

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[13]If either of $\hat{\mathbf{a}}, \hat{\mathbf{b}}$ are equal to $\hat{\mathbf{x}}$, then they can be replaced with $a\partial_x$, and the cosine can then be Taylor expanded—the resulting term then simply makes a contribution to the $\mathcal{M}_\mathbf{p}$ term.

The nature of the transitions where single bosons condense depend as usual on whether or not the transition occurs at fixed density. For the special case where the transition occurs at fixed density, the transition can presumably be identified by determining when the smallest-dimension cosine involving $\theta_\mathbf{p}$, the field dual to $\phi_\mathbf{p}$, becomes irrelevant. Since the bosons are at fixed integer filling, the simplest such translation-invariant cosine is simply $\cos(\theta)$, which has scaling dimension

$$\Delta_{\cos(\theta)} = \frac{1}{2} \int \frac{d^{d-1}p}{(2\pi)^{d-1}} R_\mathbf{p}^2. \tag{A5}$$

Due to the quasi-1d nature of the problem it seems reasonable to expect that in this case, the transition where $\cos(\theta)$ becomes irrelevant is of BKT character.

For the generic case of variable density, the transitions are described by a coupled array of dilute Bose gasses, with Lagrangian

$$\mathcal{L} = \sum_{\boldsymbol{\lambda}} \left(\psi_{\boldsymbol{\lambda}}^\dagger \partial_\tau \psi_{\boldsymbol{\lambda}} + \frac{1}{2m} |(i\nabla_x - \varphi)\psi_{\boldsymbol{\lambda}}|^2 - \mu |\psi_{\boldsymbol{\lambda}}|^2 \right) + \sum_{\boldsymbol{\lambda},\boldsymbol{\lambda}'} \mathcal{V}_{\boldsymbol{\lambda},\boldsymbol{\lambda}'} |\psi_{\boldsymbol{\lambda}}|^2 |\psi_{\boldsymbol{\lambda}'}|^2$$

$$+ \sum_{\boldsymbol{\lambda}} \left(\sum_{a \neq x} \mathcal{P}_{\boldsymbol{\lambda},a} \psi_{\boldsymbol{\lambda}-\hat{\mathbf{a}}}^\dagger \psi_{\boldsymbol{\lambda}}^2 \psi_{\boldsymbol{\lambda}+\hat{\mathbf{a}}}^\dagger + \sum_{b \neq a \neq x} \mathcal{Q}_{\boldsymbol{\lambda},a,b} \psi_{\boldsymbol{\lambda}}^\dagger \psi_{\boldsymbol{\lambda}+\hat{\mathbf{a}}} \psi_{\boldsymbol{\lambda}+\hat{\mathbf{a}}+\hat{\mathbf{b}}}^\dagger \psi_{\boldsymbol{\lambda}+\hat{\mathbf{b}}} + \text{H.c.} \right) + \mathcal{L}_{0,\varphi}, \tag{A6}$$

where again $\boldsymbol{\lambda}$ indexes coordinates transverse to $\hat{\mathbf{x}}$. The coupling to φ appears to complicate the analysis of the fixed point slightly, and we defer a detailed RG analysis to future work.

APPENDIX B: EFFECTIVE DIPOLE ACTION

In this Appendix we derive an effective action for the dipole fields D^a which is valid in the Mott insulating phases, and which allows us to map out at a mean-field level the transitions from the Mott insulators into the dipole condensed phases.

We begin by writing the hopping term in the microscopic DBHM Hamiltonian (1) as

$$H_{\text{hop}} = -\sum_{i,j,a} b_i^\dagger b_{i+a} [\mathcal{A}^a]_{ij} b_{j+a}^\dagger b_j, \tag{B1}$$

where the matrix $\mathcal{A}$ is defined as

$$[\mathcal{A}^a]_{ij} = \sum_b [t \delta_{a,b} + t'(1 - \delta_{a,b})](\delta_{i,j+b} + \delta_{i,j-b}). \tag{B2}$$

As in the main text, we may then decouple the hopping term in terms of dipole fields D_i^a as

$$H_{\text{hop}} = -\sum_{i,a} \left[b_i^\dagger b_{i+a} D_i^a + (D_i^a)^\dagger b_{i+a}^\dagger b_i \right]$$

$$+ \sum_{i,j,a} (D_i^a)^\dagger [\mathcal{A}^a]_{ij}^{-1} D_j^a. \tag{B3}$$

We are now interested in integrating out the boson fields b_i to produce an effective IR action for the dipole fields. On general grounds we may write the dipole action as

$$S = \int d\tau \left(\frac{w}{2} \sum_a |\partial_\tau D^a|^2 + \frac{1}{2} \sum_{a,b} K_D^{ab} |\nabla_a D^b|^2 \right.$$

$$\left. + r \sum_a |D^a|^2 + \frac{1}{2} \sum_{a,b} u_{ab} |D^a|^2 |D^b|^2 \right) + \cdots$$

$$\equiv S_2 + S_4 + \cdots, \tag{B4}$$

where S_n denotes the terms containing n powers of the dipole fields. These are given explicitly by

$$S_2 = -C_2 + \int d\tau \sum_{i,j,a} (D_i^a)^\dagger [\mathcal{A}^a]_{ij}^{-1} D_j^a,$$

$$S_4 = -C_4 + \frac{1}{2} C_2^2, \tag{B5}$$

where

$$C_n \equiv \frac{1}{n!} \int \prod_{i=1}^n d\tau_i \left\langle \mathcal{T} \left[\prod_{j=1}^n H_{Db}(\tau_j) \right] \right\rangle, \tag{B6}$$

with $H_{Db} \equiv -\sum_{i,a} b_i^\dagger b_{i+a} D_i^a + \text{H.c.}$, and where the expectation value above is taken with respect to the ground state of the site-diagonal Mott insulating Hamiltonian $H_{\text{onsite}} = \sum_i [-\mu n_i + U n_i(n_i-1)/2]$. In what follows we will assume μ is chosen so that the ground state of H_{onsite} is a Mott insulator with $n > 0$ bosons per site.

We first calculate C_2 as

$$C_2 = \int \frac{d\omega}{2\pi} \sum_{i,a} \left| D_i^a(\omega) \right|^2 \int d\tau\, e^{i\omega\tau} \langle T[(b_i^\dagger b_{i+a})(\tau)(b_i b_{i+a}^\dagger)(0)] \rangle$$

$$= \int \frac{d\omega}{2\pi} \sum_{i,a} \left| D_i^a(\omega) \right|^2 \int d\tau\, e^{i\omega\tau} \sum_l \left(\Theta(\tau) e^{-\tau(E_l-E_0)} |\langle 0|b_i^\dagger b_{i+a}|l\rangle|^2 + \Theta(-\tau) e^{\tau(E_l-E_0)} |\langle 0|b_{i+a}^\dagger b_i|l\rangle|^2 \right)$$

$$= \int \frac{d\omega}{2\pi} \sum_{i,a} \left| D_i^a(\omega) \right|^2 \sum_l |\langle 0|b_i^\dagger b_{i+1}|l\rangle|^2 \left(\frac{1}{i\omega + E_l - E_0} + \frac{1}{-i\omega + E_l - E_0} \right), \tag{B7}$$

where E_0 is the ground-state energy of H_{onsite} and l runs over all of H_{onsite}'s eigenstates. Since the energy of a particle hole excitation above the Mott insulator is always U regardless of n or μ, we may expand in small $\omega \ll U$ and write

$$C_2 = \frac{2n(n+1)}{U} \int \frac{d\omega}{2\pi} \sum_{i,a} |D_i^a(\omega)|^2 \left(1 - \frac{\omega^2}{U} \right). \tag{B8}$$

This determines the coefficient w of the time derivative term appearing in Eq. (B4) as

$$w = \frac{4n(n+1)}{U^2}. \tag{B9}$$

Note that as claimed, no linear time derivative term of the form $(D^a)^\dagger \partial_\tau D^a$ appears, due to the particle-hole symmetry present in the expression for C_2.

To derive r and K_D^{ab}, we use the fact that

$$\sum_{j,m} [\mathcal{A}^a]_{j,m}^{-1} e^{i\mathbf{j}\cdot\mathbf{q} + i\mathbf{m}\cdot\mathbf{p}} = \frac{\delta_{\mathbf{q},-\mathbf{p}}}{2\sum_b [t\delta_{a,b} + t'(1-\delta_{a,b})] \cos(p_b)} \tag{B10}$$

to obtain

$$r = \frac{1}{2[dt + (d-1)t']} - \frac{2n(n+1)}{U} \tag{B11}$$

and

$$K_D^{ab} = \frac{1}{dt + (d-1)t'} \left[t\delta^{ab} + \frac{t'}{2}(1-\delta^{ab}) \right]. \tag{B12}$$

Now for C_4. For the purposes of determining the most relevant terms in S_4, we may select out the part of C_4 which is local in time. This is the part that provides the fourth-order correction to the ground-state energy of H_{onsite} when perturbing in powers of H_{Db}, and a straightforward calculation gives

$$S_4 = \int d\tau \left(\sum_{lmn\neq 0} \frac{\mathcal{O}_{0l}\mathcal{O}_{lm}\mathcal{O}_{mn}\mathcal{O}_{n0}}{E_{0l}E_{0m}E_{0n}} - \sum_{lm\neq 0} \frac{|\mathcal{O}_{0l}|^2|\mathcal{O}_{0m}|^2}{E_{0l}^2 E_{0m}} \right), \tag{B13}$$

where $E_{0p} \equiv E_0 - E_p$ and where we have defined the operator

$$\mathcal{O} \equiv \sum_{i,a} (b_i^\dagger b_{i+a})(0) D^a(\tau) + \text{H.c.} \tag{B14}$$

The evaluation of S_4 by way of Eq. (B13), hence the determination of u_{ab}, is straightforward but tedious, and here we only quote the result. Dropping derivatives of D^a and writing $u_{ab} = \delta_{ab} u_d + (1 - \delta_{ab}) u_o$, we find

$$u_d = \frac{n(n+1)[4 + n(n+1)]}{3U^3},$$

$$u_o = -\frac{4n(n+1)[10 + 19n(n+1)]}{3U^3}. \tag{B15}$$

Note that $u_o < 0$, so that in mean field the system favors condensation of all components of D^a, and that $|u_o| > u_d$, so that the potential as derived in mean field is unbounded from below, most likely leading to a first-order transition (this conclusion is of course nonuniversal, however).

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