Magnon drag induced by magnon-magnon interactions characteristic of noncollinear magnets

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A noncollinear magnet consists of the magnetic moments forming a noncollinear spin structure. Because of this structure, the Hamiltonian of magnons acquires the cubic terms. Although the cubic terms are the magnon-magnon interactions characteristic of noncollinear magnets, their effects on magnon transport have not been clarified yet. Here we show that in a canted antiferromagnet the cubic terms cause a magnon drag that magnons drag magnon spin current and heat current, which can be used to enhance these currents by tuning a magnetic field. For a strong magnetic field, we find that the cubic terms induce low-temperature peaks of a spin-Seebeck coefficient, a magnon conductivity, and a magnon thermal conductivity, and that each value is one order of magnitude larger than the noninteracting value. This enhancement is mainly due to the magnetic field dependence of the coupling constant of the cubic terms through the magnetic-field dependent canting angle. Our magnon drag offers a way for controlling the magnon currents of noncollinear magnets via the many-body effect.

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I. INTRODUCTION

Drag effects are nonequilibrium many-body effects. In contrast to electronic and magnetic properties, transport properties are essentially nonequilibrium because a current makes a system out of equilibrium. Then, transport properties are often described by a theory without interactions, but they are drastically changed by the effects of interactions, many-body effects. One of such examples is the phonon drag [1,2]. The total momentum of electrons or phonons is not conserved with the electron-phonon interaction. As a result, phonons drag an electron charge current in the Seebeck effect [2–4]. This phonon drag sometimes causes a peak of the Seebeck coefficient [1,3,5]. The other drag effects, including the Coulomb drag [6,7], the spin-Coulomb drag [8–10], the spin drag [11–13], and the standard magnon drag [14–17], can be similarly understood. Since the drag effects change transport properties qualitatively, understanding their effects is one of the central issues in condensed-matter physics.

A magnon drag is expected to be realized in noncollinear magnets, but its possibility and effects have not been clarified yet. Magnets are classified into collinear magnets and noncollinear magnets. For collinear magnets the magnetic moments are aligned parallel or antiparallel to each other, whereas for noncollinear magnets those are not. Typical examples of collinear and noncollinear magnets are the Néel state and a canted state, respectively [Figs. 1(a) and 1(b)]. Spintronics or spin-caloritronics phenomena using magnons were initially studied in collinear magnets [18–22], and they have been extended to noncollinear magnets [23–30]. Then, there is another difference between collinear and noncollinear magnets. The dominant interactions between magnons usually come from four-magnon scattering processes [Fig. 1(c)]. Meanwhile, three-magnon scattering processes [Fig. 1(d)], which are described by the cubic terms in the magnon Hamiltonian, appear only for noncollinear magnets [31–33]. By analogy with the phonon drag, the cubic terms may cause a magnon drag that magnons drag a magnon current. This is distinct from the standard magnon drag [14–17] that magnons drag an electron current; the former works for magnetic metals and insulators, whereas the latter works only for magnetic metals. Nevertheless, it is unclear how the cubic terms affect magnon-transport properties of noncollinear magnets.

Here we demonstrate that the magnon drag induced by the cubic terms enhances the magnon spin current and heat current for a noncollinear antiferromagnet. Our noncollinear magnet is a three-dimensional canted antiferromagnet [Fig. 2(a)], such as MnF_2, with a magnetic field along the x axis. We formulate three magnon-transport coefficients using the linear-response theory [4,34–38] in the presence of a temperature gradient or a nonthermal external field along the z axis [Fig. 2(a)]: a spin-Seebeck coefficient S_{m}, a magnon conductivity σ_{m}, and a magnon thermal conductivity κ_{m}. We show that the cubic terms lead to the drag terms of S_{m}, σ_{m}, and κ_{m}, which are proportional to τ^2, the square of the magnon lifetime, whereas the noninteracting ones are proportional to τ. We also show that the drag terms cause low-temperature peaks of S_{m}, σ_{m}, and κ_{m} for a strong magnetic field, at which the cubic terms become large. The S_{m} obtained for a weak magnetic field is consistent with the experiment [24] for MnF_2.

II. MODEL

A. Magnon Hamiltonian of the canted antiferromagnet

Our noncollinear magnet is described by the spin Hamiltonian,

$$H = 2J\sum_{\langle i,j\rangle} \boldsymbol{S}_i \cdot \boldsymbol{S}_j - h\sum_{i=1}^{N/2} S_i^x - h\sum_{j=1}^{N/2} S_j^x. \qquad (1)$$

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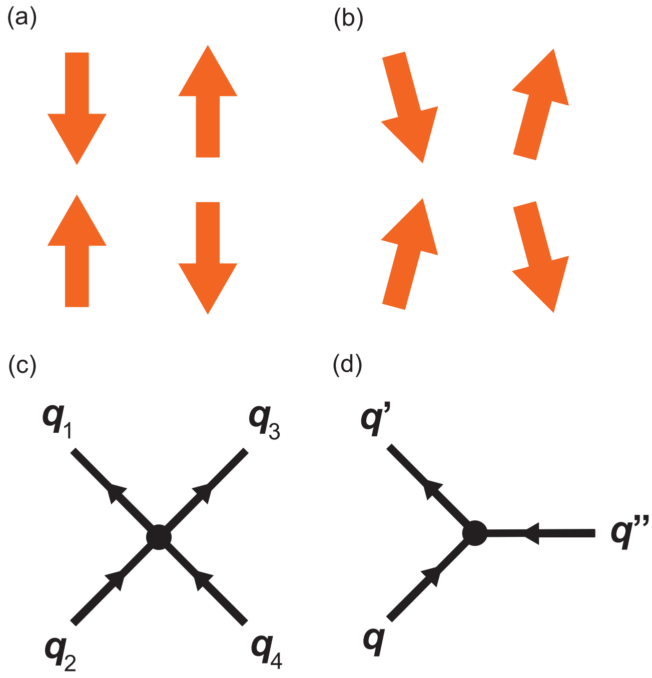

FIG. 1. Spin structures of (a) the Néel state and (b) the canted state. The arrows represent the magnetic moments. In the Néel state, the magnetic moments are aligned antiparallel to each other. Meanwhile, in the canted state, the magnetic moments are canting due to a magnetic field. Examples of (c) the four-magnon and (d) the three-magnon scattering processes. The incoming and outgoing arrows represent the annihilations and creations of a magnon, respectively. The four-magnon scattering processes consist of annihilation and creation processes for four magnons under momentum conservation $\boldsymbol{q}_1 + \boldsymbol{q}_3 = \boldsymbol{q}_2 + \boldsymbol{q}_4$, whereas the three-magnon scattering processes consist of those for three magnons under momentum conservation $\boldsymbol{q} + \boldsymbol{q}'' = \boldsymbol{q}'$. The four-magnon scattering processes are possible for both collinear and noncollinear magnets, whereas the three-magnon scattering processes appear only for noncollinear magnets.

Here the first term is the antiferromagnetic Heisenberg interaction between nearest-neighbor spins, and the others are the couplings with the magnetic field $h = -g\mu_B B$, where g and μ_B are the g factor and Bohr magneton, respectively. We have omitted the dipolar interaction because it may be negligible for MnF_2 (see Appendix A). We consider a three-dimensional case on the body-centered cubic lattice [Fig. 2(a)]; i's and j's in Eq. (1) are site indices for sublattices A and B, respectively. In the range of $0 < h < 4JzS$, where $z = 8$, the canted state for $\boldsymbol{S}_i = {}^t(S\sin\phi \;\; 0 \;\; S\cos\phi)$ and $\boldsymbol{S}_j = {}^t(S\sin\phi \;\; 0 \;\; -S\cos\phi)$ with $\sin\phi = \frac{h}{4JzS}$ is stabilized. For $h = 0$ or $h > 4JzS = h_c$, the stabilized state becomes the Néel or the ferromagnetic state, respectively. (Note that the energy of the canted, the Néel, or the ferromagnetic state divided by $N/2$ is given in the mean-field approximation by $\epsilon_{cAF} = -2JzS^2 - \frac{h^2}{4Jz}$, $\epsilon_{AF} = -2JzS^2$, or $\epsilon_{FM} = 2JzS^2 - 2Sh$, respectively.) Therefore, we choose the magnetic field to be $0 < h < h_c$, in the range of which low-energy excitations can be described by magnons for the canted antiferromagnet. Hereafter we set $k_B = 1$, $\hbar = 1$, and $a = 1$, where a is the lattice constant.

To describe magnon properties, we rewrite Eq. (1) using the Holstein-Primakoff transformation for noncollinear magnets [39–46]. As derived in Appendix B, the magnon Hamiltonian of our canted antiferromagnet is written as

$$H = H_0 + H_{\text{int}}, \tag{2}$$

where the noninteracting part H_0 consists of the quadratic terms,

$$H_0 = \sum_{\boldsymbol{q}} (a_{\boldsymbol{q}}^\dagger \; b_{\boldsymbol{q}}^\dagger \; a_{-\boldsymbol{q}} \; b_{-\boldsymbol{q}}) \begin{pmatrix} A_{\boldsymbol{q}} & B_{\boldsymbol{q}} \\ B_{\boldsymbol{q}} & A_{\boldsymbol{q}} \end{pmatrix} \begin{pmatrix} a_{\boldsymbol{q}} \\ b_{\boldsymbol{q}} \\ a_{-\boldsymbol{q}}^\dagger \\ b_{-\boldsymbol{q}}^\dagger \end{pmatrix}, \tag{3}$$

and the interaction part H_{int} consists of the cubic terms,

$$H_{\text{int}} = \sum_{\boldsymbol{q},\boldsymbol{q}',\boldsymbol{q}''} \delta_{\boldsymbol{q}+\boldsymbol{q}'',\boldsymbol{q}'} J_3(\boldsymbol{q}) (b_{\boldsymbol{q}} a_{\boldsymbol{q}'}^\dagger a_{\boldsymbol{q}''} - a_{\boldsymbol{q}} b_{\boldsymbol{q}'}^\dagger b_{\boldsymbol{q}''}) + (\text{H.c.}). \tag{4}$$

We have omitted the constant terms and quartic terms for simplicity. In Eq. (3), $a_{\boldsymbol{q}}$ and $b_{\boldsymbol{q}}$ are the Fourier coefficients of the magnon operators, the 2×2 matrices $A_{\boldsymbol{q}}$ and $B_{\boldsymbol{q}}$ are given by $(A_{\boldsymbol{q}})_{11} = (A_{\boldsymbol{q}})_{22} = \frac{1}{2}(2Jz\cos 2\phi S + h\sin\phi) = A$, $(A_{\boldsymbol{q}})_{12} = (A_{\boldsymbol{q}})_{21} = -\frac{1}{2}\tilde{J}^{(-)}(\boldsymbol{q})S = A'(\boldsymbol{q})$, $(B_{\boldsymbol{q}})_{12} = (B_{\boldsymbol{q}})_{21} = -\frac{1}{2}\tilde{J}^{(+)}(\boldsymbol{q})S = B'(\boldsymbol{q})$, and $(B_{\boldsymbol{q}})_{11} = (B_{\boldsymbol{q}})_{22} = 0$, $\tilde{J}^{(\mp)}(\boldsymbol{q}) = (\cos 2\phi \mp 1)J(\boldsymbol{q})$, and $J(\boldsymbol{q}) = 8J\cos\frac{q_x}{2}\cos\frac{q_y}{2}\cos\frac{q_z}{2}$. In Eq. (4),

$$J_3(\boldsymbol{q}) = \sqrt{\frac{4S}{N}} \sin 2\phi J(\boldsymbol{q}). \tag{5}$$

Equation (4) is similar to that of the electron-phonon interaction because the former and latter describe the creation and annihilation processes for three magnons and for two electrons and a phonon, respectively.

The coupling constant of the cubic terms depends on the magnetic field though the magnetic field dependence of the canting angle ϕ. Since $\sin 2\phi = \frac{2h\sqrt{(4JzS)^2 - h^2}}{(4JzS)^2}$ in our canted antiferromagnet, $J_3(\boldsymbol{q})$ depends on the magnetic field. Figure 2(b) shows the h/J dependence of $[J_3(\boldsymbol{q})/J(\boldsymbol{q})]^2$ for $S = \frac{5}{2}$ or $\frac{3}{2}$. (Note that $h_c = 4JzS$ for $S = \frac{5}{2}$ or $\frac{3}{2}$ is $80J$ or $48J$, respectively.) We see the coupling constant of the cubic terms for $S = \frac{5}{2}$ or $\frac{3}{2}$ is maximum at $h \sim 57J$ or $34J$, respectively. In addition, the coupling constant for $S = \frac{5}{2}$ at $h = 65J$ is much larger than that at $h = 20J$. This suggests that the effects of the cubic terms are more considerable for strong magnetic fields than those for weak magnetic fields. (In fact, we will show in Sec. III B that the cubic terms cause the huge enhancement of the magnon-transport coefficients at $h = 65J$ compared with that at $h = 20J$.) We emphasize that the magnetic-field dependent coupling constant is characteristic of canted antiferromagnets. (Such a dependence is absent in the case of the phonon drag.)

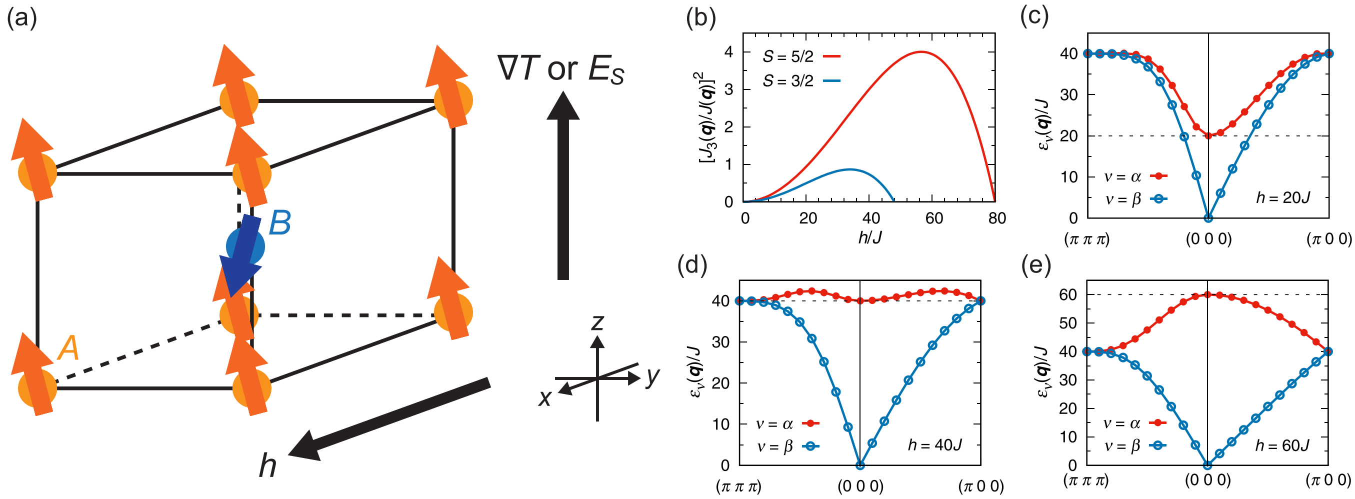

FIG. 2. (a) The spin structure of the canted antiferromagnet on the body-centered cubic lattice with the x, y, and z axes. The case for $S=\frac{5}{2}$ corresponds to MnF_2. The circles on the corners of the cube represent the sites on sublattice A, whereas that on the center represents the site on sublattice B. The arrows in the cube represent the canting spins. The magnetic field $h=-g\mu_B B$ is applied along the x axis, where g is the g factor and μ_B is the Bohr magneton. The temperature gradient ∇T or the nonthermal external field $\boldsymbol{E}_S$ is applied along the z axis; as a result, the magnon spin current and heat current along it are induced. (b) The h/J dependence of $[J_3(\boldsymbol{q})/J(\boldsymbol{q})]^2$ for $S=\frac{5}{2}$ or $\frac{3}{2}$ with $\frac{N}{2}=20^3$ and $J=1$. Here $J_3(\boldsymbol{q})$ is the coupling constant of the cubic terms. The red or blue curve represents that dependence for $S=\frac{5}{2}$ or $\frac{3}{2}$, respectively. The magnon-band dispersion relations along the symmetric lines in the momentum space at (c) $h=20J$, (d) $40J$, and (e) $60J$ for $S=\frac{5}{2}$ with $\frac{N}{2}=20^3$. The blue and red curves represent the energies divided by J for the β-band and α-band magnon [i.e., $\epsilon_\beta(\boldsymbol{q})/J$ and $\epsilon_\alpha(\boldsymbol{q})/J$], respectively. The vertical dashed lines correspond to the values of h.

B. Noninteracting magnon bands

We diagonalize Eq. (3) using the Bogoliubov transformation,

$$\begin{pmatrix} a_{\boldsymbol{q}} \\ b_{\boldsymbol{q}} \\ a^\dagger_{-\boldsymbol{q}} \\ b^\dagger_{-\boldsymbol{q}} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} c_{\boldsymbol{q}} & c'_{\boldsymbol{q}} & s_{\boldsymbol{q}} & s'_{\boldsymbol{q}} \\ c_{\boldsymbol{q}} & -c'_{\boldsymbol{q}} & s_{\boldsymbol{q}} & -s'_{\boldsymbol{q}} \\ s_{\boldsymbol{q}} & s'_{\boldsymbol{q}} & c_{\boldsymbol{q}} & c'_{\boldsymbol{q}} \\ s_{\boldsymbol{q}} & -s'_{\boldsymbol{q}} & c_{\boldsymbol{q}} & c'_{\boldsymbol{q}} \end{pmatrix} \begin{pmatrix} \alpha_{\boldsymbol{q}} \\ \beta_{\boldsymbol{q}} \\ \alpha^\dagger_{-\boldsymbol{q}} \\ \beta^\dagger_{-\boldsymbol{q}} \end{pmatrix}, \tag{6}$$

where $c_{\boldsymbol{q}}=\cosh\theta_{\boldsymbol{q}}$, $s_{\boldsymbol{q}}=\sinh\theta_{\boldsymbol{q}}$, $c'_{\boldsymbol{q}}=\cosh\theta'_{\boldsymbol{q}}$, and $s'_{\boldsymbol{q}}=\sinh\theta'_{\boldsymbol{q}}$. By substituting Eq. (6) into Eq. (3) and setting $\tanh 2\theta_{\boldsymbol{q}}=-\frac{B'(\boldsymbol{q})}{A+A'(\boldsymbol{q})}$ and $\tanh 2\theta'_{\boldsymbol{q}}=\frac{B'(\boldsymbol{q})}{A-A'(\boldsymbol{q})}$, we obtain

$$H_0=\sum_{\boldsymbol{q}}[\epsilon_\alpha(\boldsymbol{q})\alpha^\dagger_{\boldsymbol{q}}\alpha_{\boldsymbol{q}}+\epsilon_\beta(\boldsymbol{q})\beta^\dagger_{\boldsymbol{q}}\beta_{\boldsymbol{q}}], \tag{7}$$

where $\epsilon_\alpha(\boldsymbol{q})=2\sqrt{[A+A'(\boldsymbol{q})]^2-B'(\boldsymbol{q})^2}$ and $\epsilon_\beta(\boldsymbol{q})=2\sqrt{[A-A'(\boldsymbol{q})]^2-B'(\boldsymbol{q})^2}$. (Those choices of the hyperbolic functions are necessary to make the off-diagonal terms zero.) Figures 2(c)–2(e) show the magnon-band dispersion for $S=\frac{5}{2}$ at $h=20J$, $40J$, and $60J$. The band splitting energy at $\boldsymbol{q}=\boldsymbol{0}$ is equal to h and larger than those at the other q's. This property is distinct from the property of a two-sublattice ferrimagnet [21,38,47], in which the band splitting energies at $\boldsymbol{q}=\boldsymbol{0}$ and the others are the same. Moreover, it indicates that even for $T<h$, the upper-branch magnons can contribute to transport properties. (This is true, as shown in Fig. 3.) Note that we do not study the interacting magnon-band dispersion in this paper because the magnon-band energies appearing in the magnon-transport coefficients are the noninteracting ones [see Eqs. (15) and (16)].

III. MAGNON-TRANSPORT COEFFICIENTS

A. Magnon-drag terms of S_m, σ_m, and κ_m

The magnon-transport coefficients S_m, σ_m, and κ_m are connected with $\boldsymbol{j}_S$ and $\boldsymbol{j}_Q$, magnon spin and heat current densities:

$$\begin{pmatrix} \boldsymbol{j}_S \\ \boldsymbol{j}_Q \end{pmatrix} = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} \begin{pmatrix} \boldsymbol{E}_S \\ -\frac{\nabla T}{T} \end{pmatrix}, \tag{8}$$

where $L_{11}=\sigma_m$, $L_{12}(=L_{21})=S_m$, $L_{22}=\kappa_m$, $\boldsymbol{E}_S$ is a nonthermal external field, such as a magnetic-field gradient [48], and ∇T is a temperature gradient. (Note that our definition of κ_m is enough to analyze its property at low temperatures at which the magnon picture remains valid [38].) Due to zero magnon chemical potential in equilibrium, $\boldsymbol{j}_Q=\boldsymbol{j}_E$ holds, where $\boldsymbol{j}_E$ is a magnon energy current density. By using the continuity equations, we can express $\boldsymbol{J}_k=N\boldsymbol{j}_k$ $(k=S,E)$ as follows [38,45,49] (see Appendix C):

$$\boldsymbol{J}_k=\sum_{\boldsymbol{q}}\sum_{l,l'=1}^{4}\boldsymbol{v}^k_{ll'}(\boldsymbol{q})x^\dagger_{\boldsymbol{q}l}x_{\boldsymbol{q}l'}, \tag{9}$$

where $x_{\boldsymbol{q}1}=a_{\boldsymbol{q}}$, $x_{\boldsymbol{q}2}=b_{\boldsymbol{q}}$, $x_{\boldsymbol{q}3}=a^\dagger_{-\boldsymbol{q}}$, $x_{\boldsymbol{q}4}=b^\dagger_{-\boldsymbol{q}}$, $\boldsymbol{v}^S_{ll'}(\boldsymbol{q})=\boldsymbol{v}_{ll'}(\boldsymbol{q})=\boldsymbol{v}_{l'l}(\boldsymbol{q})$, and $\boldsymbol{v}^E_{ll'}(\boldsymbol{q})=\boldsymbol{e}_{ll'}(\boldsymbol{q})=\boldsymbol{e}_{l'l}(\boldsymbol{q})$; the finite terms of $\boldsymbol{v}_{ll'}(\boldsymbol{q})$ and $\boldsymbol{e}_{ll'}(\boldsymbol{q})$ are given by $\boldsymbol{v}_{23}(\boldsymbol{q})=-\boldsymbol{v}_{14}(\boldsymbol{q})=\frac{\partial B'(\boldsymbol{q})}{\partial \boldsymbol{q}}$, $\boldsymbol{v}_{12}(\boldsymbol{q})=-\boldsymbol{v}_{34}(\boldsymbol{q})=\frac{\partial A'(\boldsymbol{q})}{\partial \boldsymbol{q}}$, $\boldsymbol{e}_{12}(\boldsymbol{q})=-\boldsymbol{e}_{34}(\boldsymbol{q})=-2A\frac{\partial A'(\boldsymbol{q})}{\partial \boldsymbol{q}}$, and $\boldsymbol{e}_{11}(\boldsymbol{q})=\boldsymbol{e}_{22}(\boldsymbol{q})=-\boldsymbol{e}_{33}(\boldsymbol{q})=-\boldsymbol{e}_{44}(\boldsymbol{q})=2B'(\boldsymbol{q})\frac{\partial B'(\boldsymbol{q})}{\partial \boldsymbol{q}}-2A'(\boldsymbol{q})\frac{\partial A'(\boldsymbol{q})}{\partial \boldsymbol{q}}$. Hereafter we concentrate on the magnon transport with $\boldsymbol{E}_S$ or $(-\nabla T/T)$ applied along the z axis [Fig. 2(a)].

Since the magnon lifetime τ is supposed to be long enough to regard magnons as quasiparticles, we derive L_{12}, L_{11}, and

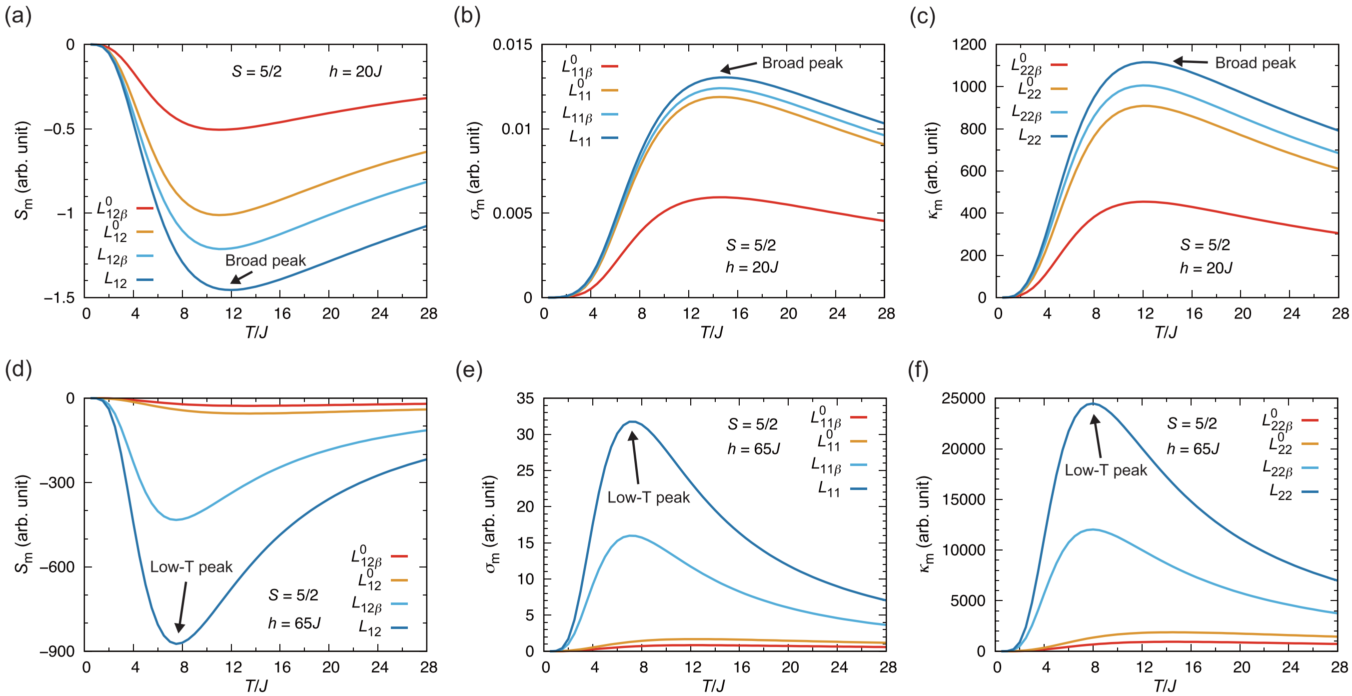

FIG. 3. The temperature dependences of (a) $S_{\rm m}$, (b) $\sigma_{\rm m}$, and (c) $\kappa_{\rm m}$ obtained in the numerical calculations for $S=\frac{5}{2}$ with $\frac{N}{2}=20^3$ and $J=1$ at a weak magnetic field $h=20J$. The red curves represent the T/J dependences of $S_{\rm m}=L^0_{12\beta}$, $\sigma_{\rm m}=L^0_{11\beta}$, and $\kappa_{\rm m}=L^0_{22\beta}$; the yellow curves represent those of $S_{\rm m}=L^0_{12}$, $\sigma_{\rm m}=L^0_{11}$, and $\kappa_{\rm m}=L^0_{22}$; the light blue curves represent those of $S_{\rm m}=L_{12\beta}$, $\sigma_{\rm m}=L_{11\beta}$, and $\kappa_{\rm m}=L_{22\beta}$; and the blue curves represent those of $S_{\rm m}=L_{12}$, $\sigma_{\rm m}=L_{11}$, and $\kappa_{\rm m}=L_{22}$. $L^0_{\mu\eta}=L^0_{\mu\eta\beta}+L^0_{\mu\eta\alpha}$, where $L^0_{\mu\eta\beta}$ and $L^0_{\mu\eta\alpha}$ are the contributions from the lower-branch and higher-branch magnons (i.e., the β-band and α-band magnons), respectively. $L_{\mu\eta\beta}=L^0_{\mu\eta}+L'_{\mu\eta\beta}$, where $L'_{\mu\eta\beta}$ is part of the drag term, the contribution from the term for $(\nu,\nu',\nu'')=(\beta,\beta,\beta)$ in Eq. (16). The temperature dependences of (d) $S_{\rm m}$, (e) $\sigma_{\rm m}$, and (f) $\kappa_{\rm m}$ obtained in the numerical calculations for $S=\frac{5}{2}$ with $\frac{N}{2}=20^3$ and $J=1$ at a strong magnetic field $h=65J$. The same notations as those at $h=20J$ are used.

L_{22} using the linear-response theory [4,10,34–38,50,51] in the limit $\tau\to\infty$. In the linear-response theory, $L_{\mu\eta}$ $(\mu,\eta=1,2)$ is given by

$$L_{\mu\eta}=\lim_{\omega\to 0}\frac{\Phi^{\rm R}_{\mu\eta}(\omega)-\Phi^{\rm R}_{\mu\eta}(0)}{i\omega},\tag{10}$$

where $\Phi^{\rm R}_{\mu\eta}(\omega)=\Phi_{\mu\eta}(i\Omega_n\to\omega+i\delta)$ $(\delta=0+)$, $\Omega_n=2\pi Tn$ $(n>0)$,

$$\Phi_{12}(i\Omega_n)=\int_0^{T^{-1}}d\tau e^{i\Omega_n\tau}\frac{1}{N}\langle T_\tau J^z_S(\tau)J^z_E\rangle,\tag{11}$$

$$\Phi_{11}(i\Omega_n)=\int_0^{T^{-1}}d\tau e^{i\Omega_n\tau}\frac{1}{N}\langle T_\tau J^z_S(\tau)J^z_S\rangle,\tag{12}$$

$$\Phi_{22}(i\Omega_n)=\int_0^{T^{-1}}d\tau e^{i\Omega_n\tau}\frac{1}{N}\langle T_\tau J^z_E(\tau)J^z_E\rangle,\tag{13}$$

and T_τ is the time-ordering operator [51]. Since J^z_S and J^z_E are written as Eq. (9), we can calculate Eqs. (11)–(13) by using a method of Green's functions [4,38,49–51]; in their calculations, we treat $H_{\rm int}$ in the second-order perturbation theory. As derived in Appendix D, $L_{\mu\eta}$ can be written as follows:

$$L_{\mu\eta}=L^0_{\mu\eta}+L'_{\mu\eta},\tag{14}$$

where $L^0_{\mu\eta}$ $(\mu,\eta=1,2)$ is the noninteracting term,

$$L^0_{\mu\eta}=-\frac{2}{N}\sum_{\boldsymbol{q}}\sum_{\nu=\alpha,\beta}j^z_{\mu;\nu\nu}(\boldsymbol{q})j^z_{\eta;\nu\nu}(\boldsymbol{q})\tau\frac{\partial n[\epsilon_\nu(\boldsymbol{q})]}{\partial\epsilon_\nu(\boldsymbol{q})},\tag{15}$$

and $L'_{\mu\eta}$ is the magnon-drag term due to the cubic terms,

$$L'_{\mu\eta}=\frac{\pi}{N^2}\sum_{\boldsymbol{q},\boldsymbol{q}'}\sum_{\nu,\nu',\nu''=\alpha,\beta}j^z_{\mu;\nu\nu}(\boldsymbol{q})j^z_{\eta;\nu'\nu'}(\boldsymbol{q}')\tau^2\frac{\partial n[\epsilon_\nu(\boldsymbol{q})]}{\partial\epsilon_\nu(\boldsymbol{q})}\times S\sin^2 2\phi\sum_{p=1,2,3}F^{(p)}_{\nu\nu'\nu''}(\boldsymbol{q},\boldsymbol{q}').\tag{16}$$

In Eq. (15), $n(x)=1/(e^{x/T}-1)$, $j^z_{1;\nu\nu}(\boldsymbol{q})=v^z_{\nu\nu}(\boldsymbol{q})$, and $j^z_{2;\nu\nu}(\boldsymbol{q})=e^z_{\nu\nu}(\boldsymbol{q})$, where $v^z_{\alpha\alpha}(\boldsymbol{q})=-v^z_{\beta\beta}(\boldsymbol{q})=2v^z_{12}(\boldsymbol{q})$, $e^z_{\alpha\alpha}(\boldsymbol{q})=2[e^z_{12}(\boldsymbol{q})+e^z_{11}(\boldsymbol{q})]$, and $e^z_{\beta\beta}(\boldsymbol{q})=2[-e^z_{12}(\boldsymbol{q})+e^z_{11}(\boldsymbol{q})]$. In Eq. (16),

$$F^{(2)}_{\nu\nu'\nu''}(\boldsymbol{q},\boldsymbol{q}')=\{n[\epsilon_{\nu''}(\boldsymbol{q}-\boldsymbol{q}')]-n[\epsilon_{\nu'}(\boldsymbol{q}')]\}v^{(2)}_{\nu\nu'\nu''}(\boldsymbol{q},\boldsymbol{q}')\times\delta[\epsilon_\nu(\boldsymbol{q})-\epsilon_{\nu'}(\boldsymbol{q}')+\epsilon_{\nu''}(\boldsymbol{q}-\boldsymbol{q}')],\tag{17}$$

$$F^{(3)}_{\nu\nu'\nu''}(\boldsymbol{q},\boldsymbol{q}')=-\{n[\epsilon_{\nu''}(\boldsymbol{q}-\boldsymbol{q}')]-n[\epsilon_{\nu'}(\boldsymbol{q}')]\}v^{(3)}_{\nu\nu'\nu''}(\boldsymbol{q},\boldsymbol{q}')\times\delta[\epsilon_\nu(\boldsymbol{q})+\epsilon_{\nu'}(\boldsymbol{q}')-\epsilon_{\nu''}(\boldsymbol{q}-\boldsymbol{q}')],\tag{18}$$

$$F^{(1)}_{\nu\nu'\nu''}(\boldsymbol{q},\boldsymbol{q}')=\{1+n[\epsilon_{\nu''}(\boldsymbol{q}-\boldsymbol{q}')]+n[\epsilon_{\nu'}(\boldsymbol{q}')]\}v^{(1)}_{\nu\nu'\nu''}(\boldsymbol{q},\boldsymbol{q}')\times\delta[\epsilon_\nu(\boldsymbol{q})-\epsilon_{\nu'}(\boldsymbol{q}')-\epsilon_{\nu''}(\boldsymbol{q}-\boldsymbol{q}')],\tag{19}$$

and the finite components of $v^{(p)}_{\nu\nu'\nu''}(\boldsymbol{q},\boldsymbol{q}')$'s are given by those for $(\nu,\nu',\nu'')=(\beta,\beta,\beta)$, (β,α,α), (α,β,α), and (α,α,β) (for their expressions, see Appendix D). The most important difference between Eqs. (15) and (16) is that $L'_{\mu\eta}\propto\tau^2$, whereas $L^0_{\mu\eta}\propto\tau$. This dependence is different from that of the phonon-drag term of L_{12} [4], which is proportional to $\tau_{\mathrm{el}}\tau_{\mathrm{ph}}$, where τ_{el} and τ_{ph} are the electron and phonon lifetimes, respectively. Then, since Eq. (16) is proportional to the square of the coupling constant of the cubic terms, the magnon-drag term depends on the magnetic field. Because of this property, our magnon drag causes unusual magnetic field dependences of the magnon-transport coefficients (see Sec. III B). Equation (15) is consistent with the expression derived in the Boltzmann theory with the relaxation-time approximation.

B. Magnon-drag induced enhancement and low-temperature peaks of S_{m}, σ_{m}, and κ_{m}

To determine the effects of the cubic terms quantitatively, we evaluate S_{m}, σ_{m}, and κ_{m} numerically. We set $J = 1$ and $S = \frac{5}{2}$. (In the case of $S = \frac{5}{2}$, the magnon picture for the canted antiferromagnet is valid in the range of $0 < h < 80J$.) The transition temperature $T_{\mathrm{c}} = \frac{16}{3}S(S+1)J$ is consistent with the Néel temperature T_{N} of MnF_2 ($S = \frac{5}{2}$) if $J \approx 1.5$ K(≈ 0.13 meV). Note that $h = 20J$ and $65J$ correspond to $|B| \approx 20$ and 65 T, respectively, using $h = -g\mu_{\mathrm{B}}B$, with $g = 2$ and $J = 0.13$ meV. We believe such magnetic fields could be experimentally realized because the magnetic field of the order of $1000T$ is experimentally accessible [52]. We perform the momentum summations by dividing the first Brillouin zone into a N_q-point mesh [38] and setting $N_q = 20^3 (= N/2)$. We consider the temperature range $0 < T \leqslant 28J(\sim 0.6T_{\mathrm{c}})$ because the perturbation theory with magnon-magnon interactions can reproduce the perpendicular susceptibility of MnF_2 up to about $0.6T_{\mathrm{N}}$ [53]. For simplicity, τ is chosen to be $\tau^{-1} = \gamma_0 + \gamma_1 T + \gamma_2 T^2$, where $\gamma_0 = 10^{-2}J$, $\gamma_1 = 10^{-4}$, and $\gamma_2 = 10^{-3}$. We replace the delta functions in Eqs. (17)–(19) by the Lorentzian ones using $\delta(x) \sim \frac{1}{\pi}\frac{3\gamma}{x^2+(3\gamma)^2}$, where $\gamma = 1/2\tau$.

Figures 3(a)–3(c) show the temperature dependences of S_{m}, σ_{m}, and κ_{m} at a weak magnetic field $h = 20J$. The contributions from the upper-branch magnons are non-negligible even at sufficiently low temperatures in the absence of the cubic terms (compare the red and yellow curves of these figures). Even in the presence of the cubic terms, the upper-branch magnons give the non-negligible contributions (compare the light blue and blue curves). Furthermore, the magnon-drag terms enhance S_{m}, σ_{m}, and κ_{m}. For example, the ratios L_{12}/L_{12}^0, L_{11}/L_{11}^0, and L_{22}/L_{22}^0 at $T = 7.5J$ are about 1.4, 1.1, and 1.2, respectively. (As we will show below, these ratios become much larger for $h = 65J$.) The broad peak of S_{m} is consistent with the experimental result of MnF_2 [24] because the voltage observed in the spin-Seebeck effect is proportional to S_{m}.

We turn to the results for a strong magnetic field $h = 65J$. Figure 3(d) shows that the magnon-drag term causes a peak at a low temperature $T = 7.5J \sim 0.16T_{\mathrm{c}}$, at which the ratio L_{12}/L_{12}^0 reaches about 22. This low-temperature peak is similar to that induced by the phonon drag [1,3]. In contrast to the phonon drag, our magnon drag induces a low-temperature peak of σ_{m}, as shown in Fig. 3(e). Thus, our magnon drag could explain a peak observed in σ_{m} [54] if a noncollinear state is stabilized. A similar peak is observed also in κ_{m} [Fig. 3(f)]. The ratios L_{11}/L_{11}^0 and L_{22}/L_{22}^0 at $T = 7.5J$ are about 23 and 20, respectively. These results suggest that our magnon drag can be used to enhance the magnon spin current and heat current by tuning the magnetic field. The contributions from the upper-branch magnons are non-negligible also for $h = 65J$ in the absence and presence of the cubic terms. Note that the larger enhancement of the magnon-transport coefficients for $h = 65J$ than for $h = 20J$ comes mainly from the magnetic field dependence of the coupling constant of the cubic terms.

We emphasize that our magnon drag can induce a similar peak for any transport coefficient described by magnon currents. This is an important difference between our magnon drag and the other drag effects. Therefore, our magnon drag provides a mechanism for a low-temperature peak of a transport coefficient.

IV. DISCUSSION

We discuss the generality of our magnon drag. The mechanism for our magnon drag will work as long as the magnon Hamiltonian contains the cubic terms. This is because the second-order perturbation of the cubic terms leads to the similar magnon-drag term. Thus, the similar enhancement of magnon-transport coefficients may be expected to occur in other noncollinear magnets, such as those with the Dzyaloshinsky-Moriya interaction or the dipolar interaction. We should note that our magnon drag does not necessarily occur in any noncollinear magnets because there is a noncollinear magnet in which the cubic terms are zero [46]. The cubic terms in the magnon Hamiltonian are vital for our magnon drag.

We comment on two ways to reduce the critical magnetic field at which a low-temperature peak appears. One is to make S smaller; in our model for $S = 3/2$, the similar peaks of S_{m}, σ_{m}, and κ_{m} are obtained at $h = 40J$ (see Appendix E). The other is to reduce the dimension; for example, in a two-dimensional canted antiferromagnet, a low-temperature peak could be realized at smaller h's. Thus, the low-temperature peaks due to the magnon drag induced by the cubic terms could be realized in various noncollinear magnets.

Our results suggest a similar drag for phonons or photons. For example, a phonon drag could be realized in the presence of the anharmonicity of lattice forces, which leads to the cubic terms in the phonon Hamiltonian [55]. Our theory is useful to study transport properties for other Bose quasiparticles.

Finally, we discuss the differences between the present magnon drag and another one induced by the quartic terms. The first-order perturbation of the quartic terms causes another magnon drag [38]. In contrast to the present magnon drag, its effect is described by the drag terms proportional to τ. Thus, the effects of the magnon drag induced by the quartic terms are to modify the values of the magnon-transport coefficients. More importantly, it does not cause any peak, and its effects are negligible at low temperatures [38]. Meanwhile, the present magnon drag causes the enhancement of S_{m}, σ_{m}, and κ_{m} even at low temperatures and their low-temperature peaks for the strong magnetic fields. Since many-body effects are usually negligible at low temperatures, the enhancement and low-temperature peaks shown in the present paper may be unusual many-body effects. Note that since the Holstein-Primakoff method is based on the $1/S$ expansion, the effects of the second-order perturbation due to the cubic terms should be compared with those of the first-order perturbation due to the quartic terms. [The second-order terms of the cubic terms and the first-order terms of the quartic terms are both $O(S^0)$.]

V. CONCLUSION

In summary, we showed the magnon drag induced by the cubic terms. Its effects on $S_{\rm m}$, $\sigma_{\rm m}$, and $\kappa_{\rm m}$ are described by the terms proportional to τ^2, whereas the noninteracting terms are proportional to τ. Our magnon drag enhances $S_{\rm m}$, $\sigma_{\rm m}$, and $\kappa_{\rm m}$ even at low temperatures and induces their low-temperature peaks for the strong magnetic field. It provides a mechanism for explaining a peak observed in a transport coefficient. The broad peak of $S_{\rm m}$ for the weak magnetic field agrees with the experimental result of MnF_2 [24]. Our results open a way to control the magnon spin current and heat current of noncollinear magnets by tuning the magnetic field.

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APPENDIX A: ESTIMATE OF THE DIPOLAR INTERACTION ENERGY

We estimate the dipolar interaction energy for MnF_2. According to the argument of Ref [56], the dipolar interaction energy $U_{\rm dip}$ will be estimated from $U_{\rm dip} \approx \frac{(g\mu_{\rm B})^2}{r^3}$, where r is the distance between two magnetic dipoles. This equation can be written as $U_{\rm dip} \approx \frac{e^2}{a_0}\frac{1}{(137)^2}(\frac{a_0}{r})^3 \approx 27.2\frac{1}{(137)^2}(\frac{a_0}{r})^3$ (eV), where $a_0 \approx 0.53$ Å. For MnF_2, the lattice constant along the a or b axis is $a \approx 4.9$ Å, and that along the c axis is $c \approx 3.3$ Å [57]. (This difference in the lattice constant has been neglected in our model for simplicity.) Setting $r = a$ or c in the above relation, we get $U_{\rm dip} \approx 1.4$ or 5.8 μeV, respectively. Since these values are much smaller than the antiferromagnetic Heisenberg interaction, the dipolar interaction may be negligible for MnF_2.

APPENDIX B: HOLSTEIN-PRIMAKOFF TRANSFORMATION FOR A NONCOLLINEAR MAGNET

Before performing the Holstein-Primakoff transformation, we need to rewrite the spin Hamiltonian in terms of rotated spin operators. In general, magnons describe spin fluctuations, the deviations from the ground-state magnetic moments. Since their directions are site dependent in noncollinear magnets, we need to perform a rotation of the spin at each site [42,43,46]. In our case, the ground-state magnetic moments are characterized by $\boldsymbol{S}_i = {}^t(S\sin\phi\ 0\ S\cos\phi)$ and $\boldsymbol{S}_j = {}^t(S\sin\phi\ 0\ \ -S\cos\phi)$ when i and j belong to sublattices A and B, respectively. Thus, we introduce the following rotated spin operators:

$$\boldsymbol{S}'_i = R(-\phi)\boldsymbol{S}_i, \tag{B1}$$

$$\boldsymbol{S}'_j = R(\pi+\phi)\boldsymbol{S}_j, \tag{B2}$$

where the rotation matrix $R(\theta)$ is given by $[R(\theta)]_{xx} = [R(\theta)]_{zz} = \cos\theta$, $[R(\theta)]_{xz} = -[R(\theta)]_{zx} = \sin\theta$, $[R(\theta)]_{yy} = 1$, and $[R(\theta)]_{xy} = [R(\theta)]_{zy} = [R(\theta)]_{yx} = [R(\theta)]_{yz} = 0$. The rotation angles have been chosen in order that $\boldsymbol{S}'_i$ and $\boldsymbol{S}'_j$ satisfy $\boldsymbol{S}'_i = \boldsymbol{S}'_j = {}^t(0\ 0\ S)$ when $\boldsymbol{S}_i = {}^t(S\sin\phi\ 0\ S\cos\phi)$ and $\boldsymbol{S}_j = {}^t(S\sin\phi\ 0\ \ -S\cos\phi)$. Because of this property, we can apply the Holstein-Primakoff transformation similar to that for ferromagnets to the spin Hamiltonian expressed in terms of $\boldsymbol{S}'_i$ and $\boldsymbol{S}'_j$ [42,43,46]. Combining Eqs. (B1) and (B2) with Eq. (1), we obtain

$$\begin{aligned} H =& 2J\sum_{\langle i,j\rangle}\left[-\cos 2\phi\left(S_i'^xS_j'^x + S_i'^zS_j'^z\right) + S_i'^yS_j'^y\right] \\ &+ 2J\sin 2\phi\sum_{\langle i,j\rangle}\left(S_i'^xS_j'^z - S_i'^zS_j'^x\right) \\ &- h\sum_i\left(\cos\phi S_i'^x + \sin\phi S_i'^z\right) \\ &- h\sum_j\left(-\cos\phi S_j'^x + \sin\phi S_j'^z\right). \end{aligned} \tag{B3}$$

We now apply the Holstein-Primakoff transformation,

$$S_i'^z = S - a_i^\dagger a_i, \quad S_i'^+ = \sqrt{2S - a_i^\dagger a_i}a_i, \quad S_i'^- = (S_i'^+)^\dagger, \tag{B4}$$

$$S_j'^z = S - b_j^\dagger b_j, \quad S_j'^+ = \sqrt{2S - b_j^\dagger b_j}b_j, \quad S_j'^- = (S_j'^+)^\dagger, \tag{B5}$$

to Eq. (B3). To consider magnon-magnon interactions, we apply a $1/S$ expansion [46,58] to the above equations of $S_i'^\pm$ and $S_j'^\pm$; the result is

$$S_i'^+ \sim \sqrt{2S}a_i - \frac{1}{2\sqrt{2S}}a_i^\dagger a_i a_i, \tag{B6}$$

$$S_i'^- \sim \sqrt{2S}a_i^\dagger - \frac{1}{2\sqrt{2S}}a_i^\dagger a_i^\dagger a_i, \tag{B7}$$

$$S_j'^+ \sim \sqrt{2S}b_j - \frac{1}{2\sqrt{2S}}b_j^\dagger b_j b_j, \tag{B8}$$

$$S_j'^- \sim \sqrt{2S}b_j^\dagger - \frac{1}{2\sqrt{2S}}b_j^\dagger b_j^\dagger b_j. \tag{B9}$$

Substituting Eqs. (B6)–(B9) and the first equations of Eqs. (B4) and (B5) into Eq. (B3), we obtain Eq. (2) with Eqs. (3) and (4).

APPENDIX C: DERIVATION OF EQ. (9)

We derive Eq. (9) using the continuity equations [49]. This derivation can be performed in a way similar to those for another noncollinear magnet [46] and for a collinear magnet [38].

First, we derive the spin current operator $\boldsymbol{J}_S$, the $k = S$ component of Eq. (9). We suppose that the z components of

$\boldsymbol{S}_i'$ and $\boldsymbol{S}_j'$ satisfy the following continuity equation:

$$\frac{dS_m'^z}{dt} + \nabla \cdot \boldsymbol{j}_m^{(S)} = 0, \tag{C1}$$

where $\boldsymbol{j}_m^{(S)}$ is a spin current operator at site m. Using Eq. (C1), we have

$$\frac{d}{dt}\left(\sum_m \boldsymbol{R}_m S_m'^z\right) = -\sum_m \boldsymbol{R}_m \nabla \cdot \boldsymbol{j}_m^{(S)} = \sum_m \boldsymbol{j}_m^{(S)} = \boldsymbol{J}_l^{(S)}, \tag{C2}$$

where $l = A$ or B for $m \in A$ or B, respectively. (Note that $m \in A$ or B means that site m belongs to sublattice A or B, respectively.) In deriving this equation, we have omitted the surface contributions. Equation (C2) can be rewritten as follows:

$$\boldsymbol{J}_A^{(S)} = i\left[H, \sum_i \boldsymbol{R}_i S_i'^z\right], \tag{C3}$$

$$\boldsymbol{J}_B^{(S)} = i\left[H, \sum_j \boldsymbol{R}_j S_j'^z\right]. \tag{C4}$$

Then, the spin current operator $\boldsymbol{J}_S$ is given by

$$\boldsymbol{J}_S = \boldsymbol{J}_A^{(S)} + \boldsymbol{J}_B^{(S)}. \tag{C5}$$

Since we consider the magnon system described by $H = H_0 + H_{\text{int}}$, we replace the H's, $S_i'^z$, and $S_j'^z$ in Eqs. (C3) and (C4) by the H_0's, $S - a_i^\dagger a_i$, and $S - b_j^\dagger b_j$, respectively. As a result, we have

$$\boldsymbol{J}_A^{(S)} = \sum_i i\boldsymbol{R}_i[a_i^\dagger a_i, H_0] = \sum_i i\boldsymbol{R}_i[a_i^\dagger a_i, H_{AB}], \tag{C6}$$

$$\boldsymbol{J}_B^{(S)} = \sum_j i\boldsymbol{R}_j[b_j^\dagger b_j, H_0] = \sum_j i\boldsymbol{R}_j[b_j^\dagger b_j, H_{AB}], \tag{C7}$$

where

$$H_{AB} = -\tilde{J}^{(+)} S \sum_{\langle i,j\rangle}(a_i b_j + a_i^\dagger b_j^\dagger) - \tilde{J}^{(-)} S \sum_{\langle i,j\rangle}(a_i b_j^\dagger + a_i^\dagger b_j), \tag{C8}$$

and $\tilde{J}^{(\pm)} = (\cos 2\phi \pm 1)J$. After some algebra, Eqs. (C6) and (C7) reduce to

$$\boldsymbol{J}_A^{(S)} = i\sum_{\langle i,j\rangle} \boldsymbol{R}_i S[\tilde{J}^{(+)}(a_i b_j - a_i^\dagger b_j^\dagger) + \tilde{J}^{(-)}(a_i b_j^\dagger - a_i^\dagger b_j)], \tag{C9}$$

$$\boldsymbol{J}_B^{(S)} = i\sum_{\langle i,j\rangle} \boldsymbol{R}_j S[\tilde{J}^{(+)} S(a_i b_j - a_i^\dagger b_j^\dagger) - \tilde{J}^{(-)}(a_i b_j^\dagger - a_i^\dagger b_j)]. \tag{C10}$$

Combining these equations with Eq. (C5), we have

$$\boldsymbol{J}_S = i\sum_{\langle i,j\rangle}(\boldsymbol{R}_i + \boldsymbol{R}_j)\tilde{J}^{(+)} S(a_i b_j - a_i^\dagger b_j^\dagger) + i\sum_{\langle i,j\rangle}(\boldsymbol{R}_i - \boldsymbol{R}_j)\tilde{J}^{(-)} S(a_i b_j^\dagger - a_i^\dagger b_j). \tag{C11}$$

By using the Fourier coefficients of the magnon operators,

$$a_i = \sqrt{\frac{2}{N}}\sum_{\boldsymbol{q}} a_{\boldsymbol{q}} e^{-i\boldsymbol{q}\cdot\boldsymbol{R}_i}, \quad b_j = \sqrt{\frac{2}{N}}\sum_{\boldsymbol{q}} b_{\boldsymbol{q}} e^{-i\boldsymbol{q}\cdot\boldsymbol{R}_j}, \tag{C12}$$

we can express Eq. (C11) as follows:

$$\begin{aligned}\boldsymbol{J}_S &= -\sum_{\boldsymbol{q}} \frac{\partial \tilde{J}^{(+)}(\boldsymbol{q})}{\partial \boldsymbol{q}} S(a_{-\boldsymbol{q}} b_{\boldsymbol{q}} + a_{-\boldsymbol{q}}^\dagger b_{\boldsymbol{q}}^\dagger) \\ &\quad - \sum_{\boldsymbol{q}} \frac{\partial \tilde{J}^{(-)}(\boldsymbol{q})}{\partial \boldsymbol{q}} S(a_{\boldsymbol{q}} b_{\boldsymbol{q}}^\dagger + a_{\boldsymbol{q}}^\dagger b_{\boldsymbol{q}}) \\ &= -\frac{S}{2}\sum_{\boldsymbol{q}}\left[\frac{\partial \tilde{J}^{(+)}(\boldsymbol{q})}{\partial \boldsymbol{q}}(a_{-\boldsymbol{q}} b_{\boldsymbol{q}} + a_{-\boldsymbol{q}}^\dagger b_{\boldsymbol{q}}^\dagger - a_{\boldsymbol{q}} b_{-\boldsymbol{q}} - a_{\boldsymbol{q}}^\dagger b_{-\boldsymbol{q}}^\dagger)\right. \\ &\quad \left. + \frac{\partial \tilde{J}^{(-)}(\boldsymbol{q})}{\partial \boldsymbol{q}}(a_{\boldsymbol{q}} b_{\boldsymbol{q}}^\dagger + a_{\boldsymbol{q}}^\dagger b_{\boldsymbol{q}} - a_{-\boldsymbol{q}} b_{-\boldsymbol{q}}^\dagger - a_{-\boldsymbol{q}}^\dagger b_{-\boldsymbol{q}})\right],\end{aligned} \tag{C13}$$

where $\tilde{J}^{(\pm)}(\boldsymbol{q}) = (\cos 2\phi \pm 1)J(\boldsymbol{q})$. This is equivalent to the $k = S$ component of Eq. (9).

Then, we derive the energy current operator $\boldsymbol{J}_E$, the $k = E$ component of Eq. (9). We suppose that the Hamiltonian at site m, h_m, satisfies the following continuity equation:

$$\frac{dh_m}{dt} + \nabla \cdot \boldsymbol{j}_m^{(E)} = 0, \tag{C14}$$

where $\boldsymbol{j}_m^{(E)}$ is an energy current operator at site m. In a way similar to the derivation of $\boldsymbol{J}_S$, we can determine the energy current operator $\boldsymbol{J}_E$ from

$$\boldsymbol{J}_E = i\left[H_0, \sum_n \boldsymbol{R}_n h_n\right] = i\sum_{m,n} \boldsymbol{R}_n[h_m, h_n], \tag{C15}$$

where $\sum_{i=1}^{N/2} h_i + \sum_{j=1}^{N/2} h_j = H_0$, $h_i = h_{iAA} + h_{iAB}$, and $h_j = h_{jBB} + h_{jBA}$. Here h_{iAA}, h_{iAB}, h_{jBB}, and h_{jBA} are given by

$$h_{iAA} = (2Jz\cos 2\phi S + h\sin\phi)a_i^\dagger a_i, \tag{C16}$$

$$h_{iAB} = -\frac{1}{2}S\sum_n[\tilde{J}_{in}^{(+)}(a_i b_n + a_i^\dagger b_n^\dagger) + \tilde{J}_{in}^{(-)}(a_i b_n^\dagger + a_i^\dagger b_n)], \tag{C17}$$

$$h_{jBB} = (2Jz\cos 2\phi S + h\sin\phi)b_j^\dagger b_j, \tag{C18}$$

$$h_{jBA} = -\frac{1}{2}S\sum_m[\tilde{J}_{mj}^{(+)}(a_m b_j + a_m^\dagger b_j^\dagger) + \tilde{J}_{mj}^{(-)}(a_m b_j^\dagger + a_m^\dagger b_j)], \tag{C19}$$

where $\tilde{J}_{ij}^{(\pm)} = (\cos 2\phi \pm 1)J_{ij}$, and $J_{ij} = J$ for nearest-neighbor i and j. Combining these equations with Eq. (C15), we have

$$\begin{aligned}\boldsymbol{J}_E &= i\sum_{m,n}(\boldsymbol{R}_n - \boldsymbol{R}_m)([h_{mAA}, h_{nAB}] + [h_{mAA}, h_{nBA}] \\ &\quad + [h_{mAB}, h_{nBB}] + [h_{mAB}, h_{nBA}] + [h_{mBB}, h_{nBA}]) \\ &\quad + i\sum_{m,n}\boldsymbol{R}_n([h_{mAB}, h_{nAB}] + [h_{mBA}, h_{nBA}]).\end{aligned} \tag{C20}$$

After some calculations, Eq. (C20) reduces to

$$\boldsymbol{J}_E = \sum_{i,j} i(\boldsymbol{R}_j - \boldsymbol{R}_i) S(2Jz\cos 2\phi S + h\sin\phi)\tilde{J}^{(-)}_{ij}(a_i b^\dagger_j - a^\dagger_i b_j)$$
$$+ \sum_{m,n,i}\frac{S^2}{2} i(\boldsymbol{R}_m - \boldsymbol{R}_n)[\tilde{J}^{(+)}_{mi}\tilde{J}^{(+)}_{in} - \tilde{J}^{(-)}_{mi}\tilde{J}^{(-)}_{in}]b^\dagger_m b_n + \sum_{m,n,j}\frac{S^2}{2} i(\boldsymbol{R}_m - \boldsymbol{R}_n)[\tilde{J}^{(+)}_{mj}\tilde{J}^{(+)}_{jn} - \tilde{J}^{(-)}_{mj}\tilde{J}^{(-)}_{jn}]a^\dagger_m a_n. \tag{C21}$$

By using the Fourier coefficients of the magnon operators [Eq. (C12)], Eq. (C21) can be written as follows:

$$\boldsymbol{J}_E = \sum_{\boldsymbol{q}}\left\{(2Jz\cos 2\phi S + h\sin\phi)\frac{\partial \tilde{J}^{(-)}(\boldsymbol{q})}{\partial \boldsymbol{q}} S(a_{\boldsymbol{q}} b^\dagger_{\boldsymbol{q}} + a^\dagger_{\boldsymbol{q}} b_{\boldsymbol{q}}) + S^2\left[\frac{\partial \tilde{J}^{(+)}(\boldsymbol{q})}{\partial \boldsymbol{q}}\tilde{J}^{(+)}(\boldsymbol{q}) - \frac{\partial \tilde{J}^{(-)}(\boldsymbol{q})}{\partial \boldsymbol{q}}\tilde{J}^{(-)}(\boldsymbol{q})\right](a^\dagger_{\boldsymbol{q}} a_{\boldsymbol{q}} + b^\dagger_{\boldsymbol{q}} b_{\boldsymbol{q}})\right\}$$
$$= \sum_{\boldsymbol{q}}\frac{1}{2}(2Jz\cos 2\phi S + h\sin\phi)\frac{\partial \tilde{J}^{(-)}(\boldsymbol{q})}{\partial \boldsymbol{q}} S(a_{\boldsymbol{q}} b^\dagger_{\boldsymbol{q}} + a^\dagger_{\boldsymbol{q}} b_{\boldsymbol{q}} - a_{-\boldsymbol{q}} b^\dagger_{-\boldsymbol{q}} - a^\dagger_{-\boldsymbol{q}} b_{-\boldsymbol{q}})$$
$$+ \sum_{\boldsymbol{q}}\frac{1}{2}S^2\left[\frac{\partial \tilde{J}^{(+)}(\boldsymbol{q})}{\partial \boldsymbol{q}}\tilde{J}^{(+)}(\boldsymbol{q}) - \frac{\partial \tilde{J}^{(-)}(\boldsymbol{q})}{\partial \boldsymbol{q}}\tilde{J}^{(-)}(\boldsymbol{q})\right](a^\dagger_{\boldsymbol{q}} a_{\boldsymbol{q}} + b^\dagger_{\boldsymbol{q}} b_{\boldsymbol{q}} - a_{-\boldsymbol{q}} a^\dagger_{-\boldsymbol{q}} - b_{-\boldsymbol{q}} b^\dagger_{-\boldsymbol{q}}). \tag{C22}$$

This gives the $k = E$ component of Eq. (9).

APPENDIX D: DERIVATIONS OF EQS. (15) AND (16) WITH THE EXPRESSIONS OF $v^{(p)}_{\nu\nu'\nu''}(\boldsymbol{q}, \boldsymbol{q}')$'s APPEARING IN EQS. (17)–(19)

We derive Eqs. (15) and (16), $L^0_{\mu\eta}$ and $L'_{\mu\eta}$ ($\mu, \eta = 1, 2$) in the limit $\tau \to \infty$, and show the expressions of $v^{(p)}_{\nu\nu'\nu''}(\boldsymbol{q}, \boldsymbol{q}')$'s ($p = 1, 2, 3$) appearing in Eqs. (17)–(19). Since we can derive L^0_{11}, L^0_{22}, L'_{11}, and L'_{22} in a way similar to the derivation of L^0_{12} and L'_{12}, we explain the derivations of L^0_{12} and L'_{12} below. Their derivations can be performed in a way similar to those of the spin-Seebeck coefficient of a collinear magnet [38] and of the Seebeck coefficient of a metal [4]. The $v^{(p)}_{\nu\nu'\nu''}(\boldsymbol{q}, \boldsymbol{q}')$'s are given by Eqs. (D64)–(D75) with Eqs. (D76)–(D92).

First, we derive Eq. (15), the expression of $L^0_{\mu\eta}$ in the limit $\tau \to \infty$. After deriving the general expression of L^0_{12} [Eq. (D22)], we derive its expression in the limit $\tau \to \infty$ [Eq. (D28)]. Then, we explain how L^0_{11} and L^0_{22} are obtained from L^0_{12} and show their expressions in the limit $\tau \to \infty$ [Eqs. (D29) and (D30)]. Substituting Eq. (9) into Eq. (11), we have

$$\Phi_{12}(i\Omega_n) = \frac{1}{N}\sum_{\boldsymbol{q},\boldsymbol{q}'}\sum_{l_1,l_2,l_3,l_4=1}^{4} v^z_{l_1 l_2}(\boldsymbol{q}) e^z_{l_3 l_4}(\boldsymbol{q}') G^{\mathrm{II}}_{l_1 l_2 l_3 l_4}(\boldsymbol{q}, \boldsymbol{q}'; i\Omega_n), \tag{D1}$$

where

$$G^{\mathrm{II}}_{l_1 l_2 l_3 l_4}(\boldsymbol{q}, \boldsymbol{q}'; i\Omega_n) = \int_0^{T^{-1}} d\tau e^{i\Omega_n \tau}\langle T_\tau x^\dagger_{\boldsymbol{q} l_1}(\tau) x_{\boldsymbol{q} l_2}(\tau) x^\dagger_{\boldsymbol{q}' l_3} x_{\boldsymbol{q}' l_4}\rangle. \tag{D2}$$

The expectation value in Eq. (D2) can be calculated by using the method of Green's functions [49–51]. Equation (D1) provides a starting point to derive L^0_{12}, and L'_{12}. To derive L^0_{12}, we evaluate Eq. (D2) without the effects of H_{int} using the Wick's theorem; the result is

$$G^{\mathrm{II}(0)}_{l_1 l_2 l_3 l_4}(\boldsymbol{q}, \boldsymbol{q}'; i\Omega_n) = \delta_{\boldsymbol{q},\boldsymbol{q}'} T\sum_m G_{l_4 l_1}(\boldsymbol{q}, i\Omega_m) G_{l_2 l_3}(\boldsymbol{q}, i\Omega_{n+m}), \tag{D3}$$

where $G_{ll'}(\boldsymbol{q}, i\Omega_m)$ is the magnon Green's function in the Matsubara-frequency representation,

$$G_{ll'}(\boldsymbol{q}, i\Omega_m) = \int_0^{T^{-1}} d\tau e^{i\Omega_m \tau} G_{ll'}(\boldsymbol{q}, \tau) = -\int_0^{T^{-1}} d\tau e^{i\Omega_m \tau}\langle T_\tau x_{\boldsymbol{q} l}(\tau) x^\dagger_{\boldsymbol{q} l'}\rangle, \tag{D4}$$

and $\Omega_m = 2\pi T m$. Substituting Eq. (D3) into Eq. (D1), we obtain

$$\Phi^{(0)}_{12}(i\Omega_n) = \frac{1}{N}\sum_{\boldsymbol{q}}\sum_{l_1,l_2,l_3,l_4=1}^{4} v^z_{l_1 l_2}(\boldsymbol{q}) e^z_{l_3 l_4}(\boldsymbol{q}) T\sum_m G_{l_4 l_1}(\boldsymbol{q}, i\Omega_m) G_{l_2 l_3}(\boldsymbol{q}, i\Omega_{n+m}). \tag{D5}$$

By using the Bogoliubov transformation [Eq. (6)],

$$x_{\boldsymbol{q} l} = \sum_{\nu=\alpha_1,\beta_1,\alpha_2,\beta_2} (P_{\boldsymbol{q}})_{l\nu} x'_{\boldsymbol{q}\nu}, \tag{D6}$$

where

$$x'_{\boldsymbol{q}\alpha_1} = \alpha_{\boldsymbol{q}}, \quad x'_{\boldsymbol{q}\beta_1} = \beta_{\boldsymbol{q}}, \quad x'_{\boldsymbol{q}\alpha_2} = \alpha^{\dagger}_{-\boldsymbol{q}}, \quad x'_{\boldsymbol{q}\beta_2} = \beta^{\dagger}_{-\boldsymbol{q}}, \tag{D7}$$

$$(P_{\boldsymbol{q}})_{1\alpha_1} = (P_{\boldsymbol{q}})_{2\alpha_1} = (P_{\boldsymbol{q}})_{3\alpha_2} = (P_{\boldsymbol{q}})_{4\alpha_2} = \frac{1}{\sqrt{2}}\cosh\theta_{\boldsymbol{q}}, \tag{D8}$$

$$(P_{\boldsymbol{q}})_{3\alpha_1} = (P_{\boldsymbol{q}})_{4\alpha_1} = (P_{\boldsymbol{q}})_{1\alpha_2} = (P_{\boldsymbol{q}})_{2\alpha_2} = \frac{1}{\sqrt{2}}\sinh\theta_{\boldsymbol{q}}, \tag{D9}$$

$$(P_{\boldsymbol{q}})_{1\beta_1} = -(P_{\boldsymbol{q}})_{2\beta_1} = (P_{\boldsymbol{q}})_{3\beta_2} = -(P_{\boldsymbol{q}})_{4\beta_2} = \frac{1}{\sqrt{2}}\cosh\theta'_{\boldsymbol{q}}, \tag{D10}$$

$$(P_{\boldsymbol{q}})_{3\beta_1} = -(P_{\boldsymbol{q}})_{4\beta_1} = (P_{\boldsymbol{q}})_{1\beta_2} = -(P_{\boldsymbol{q}})_{2\beta_2} = \frac{1}{\sqrt{2}}\sinh\theta'_{\boldsymbol{q}}, \tag{D11}$$

we can rewrite Eq. (D5) as follows:

$$\Phi^{(0)}_{12}(i\Omega_n) = \frac{1}{N}\sum_{\boldsymbol{q}}\sum_{\nu,\nu'=\alpha_1,\beta_1,\alpha_2,\beta_2} v^z_{\nu'\nu}(\boldsymbol{q})e^z_{\nu\nu'}(\boldsymbol{q})T\sum_m G_{\nu'}(\boldsymbol{q}, i\Omega_m)G_{\nu}(\boldsymbol{q}, i\Omega_{n+m}), \tag{D12}$$

where

$$v^z_{\nu'\nu}(\boldsymbol{q}) = \sum_{l_1,l_2=1}^{4}(P_{\boldsymbol{q}})_{l_1\nu'}(P_{\boldsymbol{q}})_{l_2\nu}v^z_{l_1l_2}(\boldsymbol{q}), \tag{D13}$$

$$e^z_{\nu\nu'}(\boldsymbol{q}) = \sum_{l_3,l_4=1}^{4}(P_{\boldsymbol{q}})_{l_3\nu}(P_{\boldsymbol{q}})_{l_4\nu'}e^z_{l_3l_4}(\boldsymbol{q}), \tag{D14}$$

$$G_{\alpha_1}(\boldsymbol{q}, i\Omega_m) = \frac{1}{i\Omega_m - \epsilon_\alpha(\boldsymbol{q})}, \quad G_{\beta_1}(\boldsymbol{q}, i\Omega_m) = \frac{1}{i\Omega_m - \epsilon_\beta(\boldsymbol{q})}, \tag{D15}$$

$$G_{\alpha_2}(\boldsymbol{q}, i\Omega_m) = -\frac{1}{i\Omega_m + \epsilon_\alpha(\boldsymbol{q})}, \quad G_{\beta_2}(\boldsymbol{q}, i\Omega_m) = -\frac{1}{i\Omega_m + \epsilon_\beta(\boldsymbol{q})}. \tag{D16}$$

Then, to perform the analytic continuation, we replace the Matsubara-frequency summation in Eq. (D12) by the corresponding integral [38,50]; the result is

$$\begin{aligned} T\sum_m & G_{\nu'}(\boldsymbol{q}, i\Omega_m)G_{\nu}(\boldsymbol{q}, i\Omega_{n+m}) \\ &= \int_C \frac{dz}{2\pi i}n(z)G_{\nu'}(\boldsymbol{q}, z)G_{\nu}(\boldsymbol{q}, z + i\Omega_n) + T[G_{\nu'}(\boldsymbol{q}, 0)G_{\nu}(\boldsymbol{q}, i\Omega_n) + G_{\nu'}(\boldsymbol{q}, -i\Omega_n)G_{\nu}(\boldsymbol{q}, 0)] \\ &= \int_{-\infty}^{\infty}\frac{dz}{2\pi i}n(z)\big\{G^{(\mathrm{R})}_{\nu}(\boldsymbol{q}, z + i\Omega_n)\big[G^{(\mathrm{R})}_{\nu'}(\boldsymbol{q}, z) - G^{(\mathrm{A})}_{\nu'}(\boldsymbol{q}, z)\big] + \big[G^{(\mathrm{R})}_{\nu}(\boldsymbol{q}, z) - G^{(\mathrm{A})}_{\nu}(\boldsymbol{q}, z)\big]G^{(\mathrm{A})}_{\nu'}(\boldsymbol{q}, z - i\Omega_n)\big\}, \end{aligned} \tag{D17}$$

where the contour C is shown in Fig. 4(a), $n(z)$ is the Bose distribution function $n(z) = 1/(e^{z/T} - 1)$, $G^{(\mathrm{R})}_{\nu}(\boldsymbol{q}, z)$ and $G^{(\mathrm{A})}_{\nu}(\boldsymbol{q}, z)[= G^{(\mathrm{R})}_{\nu}(\boldsymbol{q}, z)^*]$ are the retarded and advanced magnon Green's functions, respectively,

$$G^{(\mathrm{R})}_{\alpha_1}(\boldsymbol{q}, z) = \frac{1}{z + i\gamma - \epsilon_\alpha(\boldsymbol{q})}, \quad G^{(\mathrm{R})}_{\beta_1}(\boldsymbol{q}, z) = \frac{1}{z + i\gamma - \epsilon_\beta(\boldsymbol{q})}, \tag{D18}$$

$$G^{(\mathrm{R})}_{\alpha_2}(\boldsymbol{q}, z) = -\frac{1}{z + i\gamma + \epsilon_\alpha(\boldsymbol{q})}, \quad G^{(\mathrm{R})}_{\beta_2}(\boldsymbol{q}, z) = -\frac{1}{z + i\gamma + \epsilon_\beta(\boldsymbol{q})}, \tag{D19}$$

and $\gamma(= 1/2\tau)$ is the magnon damping. By combining Eq. (D17) with Eq. (D12) and performing the analytic continuation $i\Omega_n \to \omega + i\delta$ [i.e., $\Phi^{\mathrm{R}(0)}_{12}(\omega) = \Phi^{(0)}_{12}(i\Omega_n \to \omega + i\delta)$], we obtain

$$\begin{aligned} \Phi^{\mathrm{R}(0)}_{12}(\omega) =& \frac{1}{N}\sum_{\boldsymbol{q}}\sum_{\nu,\nu'=\alpha_1,\beta_1,\alpha_2,\beta_2} v^z_{\nu'\nu}(\boldsymbol{q})e^z_{\nu\nu'}(\boldsymbol{q})\int_{-\infty}^{\infty}\frac{dz}{2\pi i}n(z) \\ &\times \big\{G^{(\mathrm{R})}_{\nu}(\boldsymbol{q}, z + \omega)\big[G^{(\mathrm{R})}_{\nu'}(\boldsymbol{q}, z) - G^{(\mathrm{A})}_{\nu'}(\boldsymbol{q}, z)\big] + \big[G^{(\mathrm{R})}_{\nu}(\boldsymbol{q}, z) - G^{(\mathrm{A})}_{\nu}(\boldsymbol{q}, z)\big]G^{(\mathrm{A})}_{\nu'}(\boldsymbol{q}, z - \omega)\big\}. \end{aligned} \tag{D20}$$

After some calculations, Eq. (D20) reduces to

$$\Phi^{\mathrm{R}(0)}_{12}(\omega) \sim \Phi^{\mathrm{R}(0)}_{12}(0) - \frac{\omega}{2N}\sum_{\boldsymbol{q}}\sum_{\nu,\nu'=\alpha_1,\beta_1,\alpha_2,\beta_2} v^z_{\nu'\nu}(\boldsymbol{q})e^z_{\nu\nu'}(\boldsymbol{q})\int_{-\infty}^{\infty}\frac{dz}{2\pi i}\frac{\partial n(z)}{\partial z}\big[-4\mathrm{Im}G^{(\mathrm{R})}_{\nu}(\boldsymbol{q}, z)\mathrm{Im}G^{(\mathrm{R})}_{\nu'}(\boldsymbol{q}, z)\big]. \tag{D21}$$

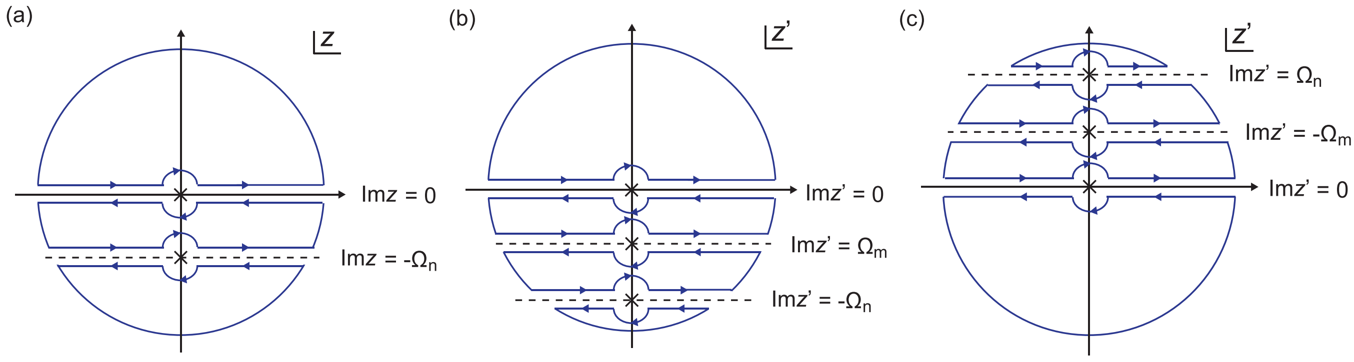

FIG. 4. The contours (a) C, (b) C', and (c) C''. Blue lines and curves correspond to the integral paths. Crosses for C, C', and C'' represent the poles at $\text{Im}z = 0$ and $-\Omega_n$, at $\text{Im}z' = 0$, Ω_m, and $-\Omega_n$, and at $\text{Im}z' = 0$, $-\Omega_m$, and Ω_n, respectively. In these panels, we neglect the horizontal shifts due to the noninteracting energies, such as $\epsilon_\nu(\boldsymbol{q})$, for simplicity because the most important information is about the imaginary parts; in the actual calculations, we consider them correctly. The C is used to derive Eq. (D17); its contributions from the region for $-\Omega_n < \text{Im}z < 0$ are considered to replace the sums over m in Eqs. (D41)–(D43) by the integrals. The C' or C'' is used to replace the sum over m' in Eq. (D41) or (D42) or in Eq. (D43), respectively.

In deriving this equation, we have used $f(z \pm \omega) = f(z) \pm \omega \frac{\partial f(z)}{\partial z} + O(\omega^2)$, $v^z_{\nu'\nu}(\boldsymbol{q}) = v^z_{\nu\nu'}(\boldsymbol{q})$, and $e^z_{\nu\nu'}(\boldsymbol{q}) = e^z_{\nu'\nu}(\boldsymbol{q})$. Combining Eq. (D21) with Eq. (10), we have

$$L^0_{12} = \lim_{\omega\to 0} \frac{\Phi^{\text{R}(0)}_{12}(\omega) - \Phi^{\text{R}(0)}_{12}(0)}{i\omega} = -\frac{1}{N}\sum_{\boldsymbol{q}} \sum_{\nu,\nu'=\alpha_1,\beta_1,\alpha_2,\beta_2} v^z_{\nu'\nu}(\boldsymbol{q}) e^z_{\nu\nu'}(\boldsymbol{q}) \int_{-\infty}^{\infty} \frac{dz}{\pi} \frac{\partial n(z)}{\partial z} \text{Im}G^{(\text{R})}_{\nu}(\boldsymbol{q}, z) \text{Im}G^{(\text{R})}_{\nu'}(\boldsymbol{q}, z). \tag{D22}$$

Then we take the limit $\tau = 1/2\gamma \to \infty$. In this limit, the integral part in Eq. (D22) reduces to

$$I_{\nu\nu'}(\boldsymbol{q}) = \int_{-\infty}^{\infty} \frac{dz}{\pi} \frac{\partial n(z)}{\partial z} \text{Im}G^{(\text{R})}_{\nu}(\boldsymbol{q}, z) \text{Im}G^{(\text{R})}_{\nu'}(\boldsymbol{q}, z) \sim \begin{cases} \frac{1}{2\gamma} \frac{\partial n[\epsilon_\alpha(\boldsymbol{q})]}{\partial \epsilon_\alpha(\boldsymbol{q})} & (\nu = \nu' = \alpha_1, \alpha_2), \\ \frac{1}{2\gamma} \frac{\partial n[\epsilon_\beta(\boldsymbol{q})]}{\partial \epsilon_\beta(\boldsymbol{q})} & (\nu = \nu' = \beta_1, \beta_2), \\ 0 & (\nu \neq \nu'). \end{cases} \tag{D23}$$

These limiting expressions can be obtained by using Eqs. (D18) and (D19) and doing the integral [38]. Combining Eq. (D23) with Eq. (D22), we obtain the expression of L^0_{12} in the limit $\tau = 1/2\gamma \to \infty$,

$$L^0_{12} \sim -\frac{1}{N}\sum_{\boldsymbol{q}} \sum_{\nu=\alpha_1,\beta_1,\alpha_2,\beta_2} v^z_{\nu\nu}(\boldsymbol{q}) e^z_{\nu\nu}(\boldsymbol{q}) \tau \frac{\partial n[\epsilon_\nu(\boldsymbol{q})]}{\partial \epsilon_\nu(\boldsymbol{q})}, \tag{D24}$$

where $\epsilon_{\alpha_1}(\boldsymbol{q}) = \epsilon_{\alpha_2}(\boldsymbol{q}) = \epsilon_\alpha(\boldsymbol{q})$ and $\epsilon_{\beta_1}(\boldsymbol{q}) = \epsilon_{\beta_2}(\boldsymbol{q}) = \epsilon_\beta(\boldsymbol{q})$. In addition, using Eqs. (D13) and (D14) and Eqs. (D8)–(D11), we have

$$v^z_{\alpha_1\alpha_1}(\boldsymbol{q}) = -v^z_{\beta_1\beta_1}(\boldsymbol{q}) = -v^z_{\alpha_2\alpha_2}(\boldsymbol{q}) = v^z_{\beta_2\beta_2}(\boldsymbol{q}) = 2v^z_{12}(\boldsymbol{q}), \tag{D25}$$

$$e^z_{\alpha_1\alpha_1}(\boldsymbol{q}) = -e^z_{\alpha_2\alpha_2}(\boldsymbol{q}) = 2[e^z_{12}(\boldsymbol{q}) + e^z_{11}(\boldsymbol{q})], \tag{D26}$$

$$e^z_{\beta_1\beta_1}(\boldsymbol{q}) = -e^z_{\beta_2\beta_2}(\boldsymbol{q}) = 2[-e^z_{12}(\boldsymbol{q}) + e^z_{11}(\boldsymbol{q})], \tag{D27}$$

where $v^z_{12}(\boldsymbol{q})$, $e^z_{12}(\boldsymbol{q})$, and $e^z_{11}(\boldsymbol{q})$ are defined below Eq. (9). Thus, Eq. (D24) reduces to

$$L^0_{12} \sim -\frac{2}{N}\sum_{\boldsymbol{q}} \sum_{\nu=\alpha,\beta} v^z_{\nu\nu}(\boldsymbol{q}) e^z_{\nu\nu}(\boldsymbol{q}) \tau \frac{\partial n[\epsilon_\nu(\boldsymbol{q})]}{\partial \epsilon_\nu(\boldsymbol{q})}, \tag{D28}$$

where $v^z_{\alpha\alpha}(\boldsymbol{q}) = -v^z_{\beta\beta}(\boldsymbol{q}) = v^z_{\alpha_1\alpha_1}(\boldsymbol{q})$, $e^z_{\alpha\alpha}(\boldsymbol{q}) = e^z_{\alpha_1\alpha_1}(\boldsymbol{q})$, and $e^z_{\beta\beta}(\boldsymbol{q}) = e^z_{\beta_1\beta_1}(\boldsymbol{q})$. Then Eqs. (9) and (11)–(13) show that L^0_{11} and L^0_{22} are obtained by replacing $e^z_{\nu\nu}(\boldsymbol{q})$ in Eq. (D28) by $v^z_{\nu\nu}(\boldsymbol{q})$ and by replacing $v^z_{\nu\nu}(\boldsymbol{q})$ in Eq. (D28) by $e^z_{\nu\nu}(\boldsymbol{q})$, respectively. Therefore, L^0_{11} and L^0_{22} in the limit $\tau \to \infty$ are given by

$$L^0_{11} \sim -\frac{2}{N}\sum_{\boldsymbol{q}} \sum_{\nu=\alpha,\beta} v^z_{\nu\nu}(\boldsymbol{q}) v^z_{\nu\nu}(\boldsymbol{q}) \tau \frac{\partial n[\epsilon_\nu(\boldsymbol{q})]}{\partial \epsilon_\nu(\boldsymbol{q})}, \tag{D29}$$

$$L^0_{22} \sim -\frac{2}{N}\sum_{\boldsymbol{q}} \sum_{\nu=\alpha,\beta} e^z_{\nu\nu}(\boldsymbol{q}) e^z_{\nu\nu}(\boldsymbol{q}) \tau \frac{\partial n[\epsilon_\nu(\boldsymbol{q})]}{\partial \epsilon_\nu(\boldsymbol{q})}. \tag{D30}$$

Equations (D28)–(D30) give Eq. (15).

Next, we derive Eq. (16), the expression of $L'_{\mu\eta}$ in the limit $\tau\to\infty$ [Eqs. (D93), (D97), and (D98)], and show the explicit expressions of $v^{(p)}_{\nu\nu'\nu''}(\boldsymbol{q},\boldsymbol{q}')$'s [Eqs. (D64)–(D75)]. (This derivation can be done in a way similar to that of the phonon-drag term of a metal [4].) Before evaluating Eq. (D2) with the effects of H_{int}, we express H_{int} in terms of the operators $x_{\boldsymbol{q}l}$ and $x^{\dagger}_{\boldsymbol{q}l}$. Since H_{int} is defined as Eq. (4), we have

$$\begin{aligned}H_{\text{int}} =&\frac{1}{2}\sum_{\boldsymbol{q},\boldsymbol{q}',\boldsymbol{q}''}\delta_{\boldsymbol{q}+\boldsymbol{q}'',\boldsymbol{q}'}J_3(\boldsymbol{q})(b_{\boldsymbol{q}}a^{\dagger}_{\boldsymbol{q}'}a_{\boldsymbol{q}''}-a_{\boldsymbol{q}}b^{\dagger}_{\boldsymbol{q}'}b_{\boldsymbol{q}''}+b_{-\boldsymbol{q}}a^{\dagger}_{-\boldsymbol{q}'}a_{-\boldsymbol{q}''}-a_{-\boldsymbol{q}}b^{\dagger}_{-\boldsymbol{q}'}b_{-\boldsymbol{q}''})+(\text{H.c.})\\ =&\frac{1}{2}\sum_{\boldsymbol{q},\boldsymbol{q}',\boldsymbol{q}''}\delta_{\boldsymbol{q}+\boldsymbol{q}'',\boldsymbol{q}'}J_3(\boldsymbol{q})\left[\sum_{l=1}^{2}\text{sgn}(l)(x_{\boldsymbol{q}l}x^{\dagger}_{\boldsymbol{q}'\bar{l}}x_{\boldsymbol{q}''\bar{l}}+x^{\dagger}_{\boldsymbol{q}l}x^{\dagger}_{\boldsymbol{q}''\bar{l}}x_{\boldsymbol{q}'\bar{l}})+\sum_{l=3}^{4}\text{sgn}(l)(x_{\boldsymbol{q}l}x_{\boldsymbol{q}''\bar{l}}x^{\dagger}_{\boldsymbol{q}'\bar{l}}+x^{\dagger}_{\boldsymbol{q}l}x_{\boldsymbol{q}'\bar{l}}x^{\dagger}_{\boldsymbol{q}''\bar{l}})\right],\end{aligned}\tag{D31}$$

where

$$\text{sgn}(l)=\begin{cases}-1 & (l=1,3),\\ 1 & (l=2,4),\end{cases}\qquad \bar{l}=\begin{cases}2 & (l=1),\\ 1 & (l=2),\\ 4 & (l=3),\\ 3 & (l=4).\end{cases}\tag{D32}$$

To derive L'_{12}, we evaluate Eq. (D2) in the second-order perturbation theory [49,51] using the Wick's theorem and Eqs. (D6) and (D31); the result is

$$\begin{aligned}\Delta G^{\text{II}}_{l_1l_2l_3l_4}(\boldsymbol{q},\boldsymbol{q}';i\Omega_n)=&\int_0^{T^{-1}}d\tau e^{i\Omega_n\tau}\int_0^{T^{-1}}d\tau_1\int_0^{T^{-1}}d\tau_2\frac{1}{2}\langle T_\tau x^{\dagger}_{\boldsymbol{q}l_1}(\tau)x_{\boldsymbol{q}l_2}(\tau)x^{\dagger}_{\boldsymbol{q}'l_3}x_{\boldsymbol{q}'l_4}H_{\text{int}}(\tau_1)H_{\text{int}}(\tau_2)\rangle\\ =&\int_0^{T^{-1}}d\tau e^{i\Omega_n\tau}\int_0^{T^{-1}}d\tau_1\int_0^{T^{-1}}d\tau_2\sum_{\nu_1,\nu_2,\nu_3,\nu_4,\nu_5=\alpha_1,\beta_1,\alpha_2,\beta_2}(P_{\boldsymbol{q}})_{l_1\nu_1}(P_{\boldsymbol{q}})_{l_2\nu_2}(P_{\boldsymbol{q}'})_{l_3\nu_3}(P_{\boldsymbol{q}'})_{l_4\nu_4}\\ &\times\sum_{k=a,b,c}\tilde{V}^{(k)}_{\nu_1\nu_2\nu_3\nu_4\nu_5}(\boldsymbol{q},\boldsymbol{q}')f^{(k)}_{\nu_1\nu_2\nu_3\nu_4\nu_5}(\boldsymbol{q},\boldsymbol{q}';\tau,\tau_1,\tau_2),\end{aligned}\tag{D33}$$

where

$$\begin{aligned}\tilde{V}^{(a)}_{\nu_1\nu_2\nu_3\nu_4\nu_5}(\boldsymbol{q},\boldsymbol{q}')=&-\frac{1}{4}\sum_{l,l'=1}^{4}\text{sgn}(l)\text{sgn}(l')\big[J_3(\boldsymbol{q})^2(P_{\boldsymbol{q}})_{l\nu_1}(P_{\boldsymbol{q}})_{l'\nu_2}(P_{\boldsymbol{q}'})_{\bar{l}'\nu_3}(P_{\boldsymbol{q}'})_{\bar{l}\nu_4}(P_{\boldsymbol{q}'-\boldsymbol{q}})_{\bar{l}\nu_5}(P_{\boldsymbol{q}'-\boldsymbol{q}})_{\bar{l}'\nu_5}\\ &+J_3(\boldsymbol{q})J_3(\boldsymbol{q}'-\boldsymbol{q})(P_{\boldsymbol{q}})_{l\nu_1}(P_{\boldsymbol{q}})_{\bar{l}'\nu_2}(P_{\boldsymbol{q}'})_{\bar{l}'\nu_3}(P_{\boldsymbol{q}'})_{\bar{l}\nu_4}(P_{\boldsymbol{q}'-\boldsymbol{q}})_{\bar{l}\nu_5}(P_{\boldsymbol{q}'-\boldsymbol{q}})_{l'\nu_5}\\ &+J_3(\boldsymbol{q}'-\boldsymbol{q})J_3(\boldsymbol{q})(P_{\boldsymbol{q}})_{\bar{l}\nu_1}(P_{\boldsymbol{q}})_{l'\nu_2}(P_{\boldsymbol{q}'})_{\bar{l}'\nu_3}(P_{\boldsymbol{q}'})_{\bar{l}\nu_4}(P_{\boldsymbol{q}'-\boldsymbol{q}})_{l\nu_5}(P_{\boldsymbol{q}'-\boldsymbol{q}})_{\bar{l}'\nu_5}\\ &+J_3(\boldsymbol{q}'-\boldsymbol{q})^2(P_{\boldsymbol{q}})_{\bar{l}\nu_1}(P_{\boldsymbol{q}})_{\bar{l}'\nu_2}(P_{\boldsymbol{q}'})_{\bar{l}'\nu_3}(P_{\boldsymbol{q}'})_{\bar{l}\nu_4}(P_{\boldsymbol{q}'-\boldsymbol{q}})_{l\nu_5}(P_{\boldsymbol{q}'-\boldsymbol{q}})_{l'\nu_5}\big],\end{aligned}\tag{D34}$$

$$\begin{aligned}\tilde{V}^{(b)}_{\nu_1\nu_2\nu_3\nu_4\nu_5}(\boldsymbol{q},\boldsymbol{q}')=&-\frac{1}{4}\sum_{l,l'=1}^{4}\text{sgn}(l)\text{sgn}(l')\big[J_3(\boldsymbol{q}')^2(P_{\boldsymbol{q}})_{\bar{l}'\nu_1}(P_{\boldsymbol{q}})_{\bar{l}\nu_2}(P_{\boldsymbol{q}'})_{l\nu_3}(P_{\boldsymbol{q}'})_{l'\nu_4}(P_{\boldsymbol{q}-\boldsymbol{q}'})_{\bar{l}\nu_5}(P_{\boldsymbol{q}-\boldsymbol{q}'})_{\bar{l}'\nu_5}\\ &+J_3(\boldsymbol{q}')J_3(\boldsymbol{q}-\boldsymbol{q}')(P_{\boldsymbol{q}})_{\bar{l}'\nu_1}(P_{\boldsymbol{q}})_{\bar{l}\nu_2}(P_{\boldsymbol{q}'})_{l\nu_3}(P_{\boldsymbol{q}'})_{\bar{l}'\nu_4}(P_{\boldsymbol{q}-\boldsymbol{q}'})_{\bar{l}\nu_5}(P_{\boldsymbol{q}-\boldsymbol{q}'})_{l'\nu_5}\\ &+J_3(\boldsymbol{q}-\boldsymbol{q}')J_3(\boldsymbol{q}')(P_{\boldsymbol{q}})_{\bar{l}'\nu_1}(P_{\boldsymbol{q}})_{\bar{l}\nu_2}(P_{\boldsymbol{q}'})_{\bar{l}\nu_3}(P_{\boldsymbol{q}'})_{l'\nu_4}(P_{\boldsymbol{q}-\boldsymbol{q}'})_{l\nu_5}(P_{\boldsymbol{q}-\boldsymbol{q}'})_{\bar{l}'\nu_5}\\ &+J_3(\boldsymbol{q}-\boldsymbol{q}')^2(P_{\boldsymbol{q}})_{\bar{l}'\nu_1}(P_{\boldsymbol{q}})_{\bar{l}\nu_2}(P_{\boldsymbol{q}'})_{\bar{l}\nu_3}(P_{\boldsymbol{q}'})_{\bar{l}'\nu_4}(P_{\boldsymbol{q}-\boldsymbol{q}'})_{l\nu_5}(P_{\boldsymbol{q}-\boldsymbol{q}'})_{l'\nu_5}\big],\end{aligned}\tag{D35}$$

$$\begin{aligned}\tilde{V}^{(c)}_{\nu_1\nu_2\nu_3\nu_4\nu_5}(\boldsymbol{q},\boldsymbol{q}')=&-\frac{1}{4}\sum_{l,l'=1}^{4}\text{sgn}(l)\text{sgn}(l')\big[J_3(\boldsymbol{q})^2(P_{\boldsymbol{q}})_{l\nu_1}(P_{\boldsymbol{q}})_{l'\nu_2}(P_{\boldsymbol{q}'})_{\bar{l}\nu_3}(P_{\boldsymbol{q}'})_{\bar{l}'\nu_4}(P_{\boldsymbol{q}+\boldsymbol{q}'})_{\bar{l}'\nu_5}(P_{\boldsymbol{q}+\boldsymbol{q}'})_{\bar{l}\nu_5}\\ &+J_3(\boldsymbol{q})J_3(\boldsymbol{q}')(P_{\boldsymbol{q}})_{l\nu_1}(P_{\boldsymbol{q}})_{\bar{l}'\nu_2}(P_{\boldsymbol{q}'})_{\bar{l}\nu_3}(P_{\boldsymbol{q}'})_{l'\nu_4}(P_{\boldsymbol{q}+\boldsymbol{q}'})_{\bar{l}'\nu_5}(P_{\boldsymbol{q}+\boldsymbol{q}'})_{\bar{l}\nu_5}\\ &+J_3(\boldsymbol{q}')J_3(\boldsymbol{q})(P_{\boldsymbol{q}})_{\bar{l}\nu_1}(P_{\boldsymbol{q}})_{l'\nu_2}(P_{\boldsymbol{q}'})_{l\nu_3}(P_{\boldsymbol{q}'})_{\bar{l}'\nu_4}(P_{\boldsymbol{q}+\boldsymbol{q}'})_{\bar{l}'\nu_5}(P_{\boldsymbol{q}+\boldsymbol{q}'})_{\bar{l}\nu_5}\\ &+J_3(\boldsymbol{q}')^2(P_{\boldsymbol{q}})_{\bar{l}\nu_1}(P_{\boldsymbol{q}})_{\bar{l}'\nu_2}(P_{\boldsymbol{q}'})_{l\nu_3}(P_{\boldsymbol{q}'})_{l'\nu_4}(P_{\boldsymbol{q}+\boldsymbol{q}'})_{\bar{l}'\nu_5}(P_{\boldsymbol{q}+\boldsymbol{q}'})_{\bar{l}\nu_5}\big],\end{aligned}\tag{D36}$$

and

$$f^{(a)}_{\nu_1\nu_2\nu_3\nu_4\nu_5}(\boldsymbol{q},\boldsymbol{q}';\tau,\tau_1,\tau_2)=G_{\nu_1}(\boldsymbol{q},\tau_1-\tau)G_{\nu_2}(\boldsymbol{q},\tau-\tau_2)G_{\nu_3}(\boldsymbol{q}',\tau_2)G_{\nu_4}(\boldsymbol{q}',-\tau_1)G_{\nu_5}(\boldsymbol{q}'-\boldsymbol{q},\tau_1-\tau_2),\tag{D37}$$

$$f^{(b)}_{\nu_1\nu_2\nu_3\nu_4\nu_5}(\boldsymbol{q},\boldsymbol{q}';\tau,\tau_1,\tau_2)=G_{\nu_1}(\boldsymbol{q},\tau_2-\tau)G_{\nu_2}(\boldsymbol{q},\tau-\tau_1)G_{\nu_3}(\boldsymbol{q}',\tau_1)G_{\nu_4}(\boldsymbol{q}',-\tau_2)G_{\nu_5}(\boldsymbol{q}-\boldsymbol{q}',\tau_1-\tau_2),\tag{D38}$$

$$f^{(c)}_{\nu_1\nu_2\nu_3\nu_4\nu_5}(\boldsymbol{q},\boldsymbol{q}';\tau,\tau_1,\tau_2)=G_{\nu_1}(\boldsymbol{q},\tau_1-\tau)G_{\nu_2}(\boldsymbol{q},\tau-\tau_2)G_{\nu_3}(\boldsymbol{q}',\tau_1)G_{\nu_4}(\boldsymbol{q}',-\tau_2)G_{\nu_5}(\boldsymbol{q}+\boldsymbol{q}',\tau_2-\tau_1). \tag{D39}$$

By combining Eqs. (D33)–(D39) with Eq. (D1) and doing the integrals about τ, τ_1, and τ_2 in Eq. (D33), we obtain

$$\Delta\Phi_{12}(i\Omega_n)=\frac{1}{N}\sum_{\boldsymbol{q},\boldsymbol{q}'}\sum_{\nu_1,\nu_2,\nu_3,\nu_4,\nu_5=\alpha_1,\beta_1,\alpha_2,\beta_2}v^z_{\nu_1\nu_2}(\boldsymbol{q})e^z_{\nu_3\nu_4}(\boldsymbol{q}')\sum_{k=a,b,c}\tilde{V}^{(k)}_{\nu_1\nu_2\nu_3\nu_4\nu_5}(\boldsymbol{q},\boldsymbol{q}')\tilde{I}^{(k)}_{\nu_1\nu_2\nu_3\nu_4\nu_5}(\boldsymbol{q},\boldsymbol{q}';i\Omega_n), \tag{D40}$$

where

$$\tilde{I}^{(a)}_{\nu_1\nu_2\nu_3\nu_4\nu_5}(\boldsymbol{q},\boldsymbol{q}';i\Omega_n)=T^2\sum_{m,m'}G_{\nu_1}(\boldsymbol{q},i\Omega_m)G_{\nu_2}(\boldsymbol{q},i\Omega_{n+m})G_{\nu_3}(\boldsymbol{q}',i\Omega_{n+m'})G_{\nu_4}(\boldsymbol{q}',i\Omega_{m'})G_{\nu_5}(\boldsymbol{q}'-\boldsymbol{q},i\Omega_{m'-m}), \tag{D41}$$

$$\tilde{I}^{(b)}_{\nu_1\nu_2\nu_3\nu_4\nu_5}(\boldsymbol{q},\boldsymbol{q}';i\Omega_n)=T^2\sum_{m,m'}G_{\nu_1}(\boldsymbol{q},i\Omega_m)G_{\nu_2}(\boldsymbol{q},i\Omega_{n+m})G_{\nu_3}(\boldsymbol{q}',i\Omega_{n+m'})G_{\nu_4}(\boldsymbol{q}',i\Omega_{m'})G_{\nu_5}(\boldsymbol{q}-\boldsymbol{q}',i\Omega_{m-m'}), \tag{D42}$$

$$\tilde{I}^{(c)}_{\nu_1\nu_2\nu_3\nu_4\nu_5}(\boldsymbol{q},\boldsymbol{q}';i\Omega_n)=T^2\sum_{m,m'}G_{\nu_1}(\boldsymbol{q},i\Omega_m)G_{\nu_2}(\boldsymbol{q},i\Omega_{n+m})G_{\nu_3}(\boldsymbol{q}',i\Omega_{m'})G_{\nu_4}(\boldsymbol{q}',i\Omega_{m'-n})G_{\nu_5}(\boldsymbol{q}+\boldsymbol{q}',i\Omega_{m+m'}). \tag{D43}$$

Then, to perform the analytic continuation, we replace the Matsubara-frequency summations in Eqs. (D41)–(D43) by the corresponding integrals in a way similar to that for metals [50]. Namely, since an intraband pair of the retarded and advanced Green's functions, such as $G^{(A)}_\nu(\boldsymbol{q},z)G^{(R)}_\nu(\boldsymbol{q},z)$, gives the leading contribution in the limit $\tau\rightarrow\infty$ [50], we can express Eqs. (D41)–(D43) in this limit as follows:

$$\begin{aligned}
\tilde{I}^{(a)}_{\nu_1\nu_2\nu_3\nu_4\nu_5}(\boldsymbol{q},\boldsymbol{q}';i\Omega_n)\sim{}&\delta_{\nu_1,\nu_2}\delta_{\nu_3,\nu_4}\int_{-\infty}^{\infty}\frac{dz}{2\pi i}n(z)\big[-G^{(A)}_{\nu_1}(\boldsymbol{q},z)G^{(R)}_{\nu_2}(\boldsymbol{q},z+i\Omega_n)\big]\\
&\times\int_{-\infty}^{\infty}\frac{dz'}{2\pi i}n(z')\big\{-G^{(R)}_{\nu_3}(\boldsymbol{q}',z'+i\Omega_n)G^{(A)}_{\nu_4}(\boldsymbol{q}',z')G^{(R)}_{\nu_5}(\boldsymbol{q}'-\boldsymbol{q},z'-z)\\
&+G^{(R)}_{\nu_3}(\boldsymbol{q}',z'+z+i\Omega_n)G^{(A)}_{\nu_4}(\boldsymbol{q}',z'+z)\big[G^{(R)}_{\nu_5}(\boldsymbol{q}'-\boldsymbol{q},z')-G^{(A)}_{\nu_5}(\boldsymbol{q}'-\boldsymbol{q},z')\big]\\
&+G^{(R)}_{\nu_3}(\boldsymbol{q}',z')G^{(A)}_{\nu_4}(\boldsymbol{q}',z'-i\Omega_n)G^{(A)}_{\nu_5}(\boldsymbol{q}'-\boldsymbol{q},z'-z-i\Omega_n)\big\}\\
&+\delta_{\nu_1,\nu_2}\delta_{\nu_3,\nu_4}\int_{-\infty}^{\infty}\frac{dz}{2\pi i}n(z)G^{(A)}_{\nu_1}(\boldsymbol{q},z-i\Omega_n)G^{(R)}_{\nu_2}(\boldsymbol{q},z)\\
&\times\int_{-\infty}^{\infty}\frac{dz'}{2\pi i}n(z')\big\{-G^{(R)}_{\nu_3}(\boldsymbol{q}',z'+i\Omega_n)G^{(A)}_{\nu_4}(\boldsymbol{q}',z')G^{(R)}_{\nu_5}(\boldsymbol{q}'-\boldsymbol{q},z'-z+i\Omega_n)\\
&+G^{(R)}_{\nu_3}(\boldsymbol{q}',z'+z)G^{(A)}_{\nu_4}(\boldsymbol{q}',z'+z-i\Omega_n)\big[G^{(R)}_{\nu_5}(\boldsymbol{q}'-\boldsymbol{q},z')-G^{(A)}_{\nu_5}(\boldsymbol{q}'-\boldsymbol{q},z')\big]\\
&+G^{(R)}_{\nu_3}(\boldsymbol{q}',z')G^{(A)}_{\nu_4}(\boldsymbol{q}',z'-i\Omega_n)G^{(A)}_{\nu_5}(\boldsymbol{q}'-\boldsymbol{q},z'-z)\big\},
\end{aligned} \tag{D44}$$

$$\begin{aligned}
\tilde{I}^{(b)}_{\nu_1\nu_2\nu_3\nu_4\nu_5}(\boldsymbol{q},\boldsymbol{q}';i\Omega_n)\sim{}&\delta_{\nu_1,\nu_2}\delta_{\nu_3,\nu_4}\int_{-\infty}^{\infty}\frac{dz}{2\pi i}n(z)\big[-G^{(A)}_{\nu_1}(\boldsymbol{q},z)G^{(R)}_{\nu_2}(\boldsymbol{q},z+i\Omega_n)\big]\\
&\times\int_{-\infty}^{\infty}\frac{dz'}{2\pi i}n(z')\big\{-G^{(R)}_{\nu_3}(\boldsymbol{q}',z'+i\Omega_n)G^{(A)}_{\nu_4}(\boldsymbol{q}',z')G^{(A)}_{\nu_5}(\boldsymbol{q}-\boldsymbol{q}',z-z')\\
&-G^{(R)}_{\nu_3}(\boldsymbol{q}',z'+z+i\Omega_n)G^{(A)}_{\nu_4}(\boldsymbol{q}',z'+z)\big[G^{(R)}_{\nu_5}(\boldsymbol{q}-\boldsymbol{q}',-z')-G^{(A)}_{\nu_5}(\boldsymbol{q}-\boldsymbol{q}',-z')\big]\\
&+G^{(R)}_{\nu_3}(\boldsymbol{q}',z')G^{(A)}_{\nu_4}(\boldsymbol{q}',z'-i\Omega_n)G^{(R)}_{\nu_5}(\boldsymbol{q}-\boldsymbol{q}',z-z'+i\Omega_n)\big\}\\
&+\delta_{\nu_1,\nu_2}\delta_{\nu_3,\nu_4}\int_{-\infty}^{\infty}\frac{dz}{2\pi i}n(z)G^{(A)}_{\nu_1}(\boldsymbol{q},z-i\Omega_n)G^{(R)}_{\nu_2}(\boldsymbol{q},z)\\
&\times\int_{-\infty}^{\infty}\frac{dz'}{2\pi i}n(z')\big\{-G^{(R)}_{\nu_3}(\boldsymbol{q}',z'+i\Omega_n)G^{(A)}_{\nu_4}(\boldsymbol{q}',z')G^{(A)}_{\nu_5}(\boldsymbol{q}-\boldsymbol{q}',z-z'-i\Omega_n)\\
&-G^{(R)}_{\nu_3}(\boldsymbol{q}',z'+z)G^{(A)}_{\nu_4}(\boldsymbol{q}',z'+z-i\Omega_n)\big[G^{(R)}_{\nu_5}(\boldsymbol{q}-\boldsymbol{q}',-z')-G^{(A)}_{\nu_5}(\boldsymbol{q}-\boldsymbol{q}',-z')\big]\\
&+G^{(R)}_{\nu_3}(\boldsymbol{q}',z')G^{(A)}_{\nu_4}(\boldsymbol{q}',z'-i\Omega_n)G^{(R)}_{\nu_5}(\boldsymbol{q}-\boldsymbol{q}',z-z')\big\},
\end{aligned} \tag{D45}$$

$$\begin{aligned}
\tilde{I}^{(c)}_{\nu_1\nu_2\nu_3\nu_4\nu_5}(\boldsymbol{q},\boldsymbol{q}';i\Omega_n)\sim{}&\delta_{\nu_1,\nu_2}\delta_{\nu_3,\nu_4}\int_{-\infty}^{\infty}\frac{dz}{2\pi i}n(z)\big[-G^{(A)}_{\nu_1}(\boldsymbol{q},z)G^{(R)}_{\nu_2}(\boldsymbol{q},z+i\Omega_n)\big]\\
&\times\int_{-\infty}^{\infty}\frac{dz'}{2\pi i}n(z')\big\{-G^{(R)}_{\nu_3}(\boldsymbol{q}',z'+i\Omega_n)G^{(A)}_{\nu_4}(\boldsymbol{q}',z')G^{(R)}_{\nu_5}(\boldsymbol{q}'+\boldsymbol{q},z'+z+i\Omega_n)\\
&+G^{(R)}_{\nu_3}(\boldsymbol{q}',z'-z)G^{(A)}_{\nu_4}(\boldsymbol{q}',z'-z-i\Omega_n)\big[G^{(R)}_{\nu_5}(\boldsymbol{q}'+\boldsymbol{q},z')-G^{(A)}_{\nu_5}(\boldsymbol{q}'+\boldsymbol{q},z')\big]
\end{aligned}$$

$$+ G_{\nu_3}^{(R)}(\boldsymbol{q}', z')G_{\nu_4}^{(A)}(\boldsymbol{q}', z' - i\Omega_n)G_{\nu_5}^{(A)}(\boldsymbol{q}' + \boldsymbol{q}, z' + z)\big\}$$
$$+ \delta_{\nu_1,\nu_2}\delta_{\nu_3,\nu_4}\int_{-\infty}^{\infty}\frac{dz}{2\pi i}n(z)G_{\nu_1}^{(A)}(\boldsymbol{q}, z - i\Omega_n)G_{\nu_2}^{(R)}(\boldsymbol{q}, z)$$
$$\times\int_{-\infty}^{\infty}\frac{dz'}{2\pi i}n(z')\big\{-G_{\nu_3}^{(R)}(\boldsymbol{q}', z' + i\Omega_n)G_{\nu_4}^{(A)}(\boldsymbol{q}', z')G_{\nu_5}^{(R)}(\boldsymbol{q}' + \boldsymbol{q}, z' + z)$$
$$+ G_{\nu_3}^{(R)}(\boldsymbol{q}', z' - z + i\Omega_n)G_{\nu_4}^{(A)}(\boldsymbol{q}', z' - z)\big[G_{\nu_5}^{(R)}(\boldsymbol{q}' + \boldsymbol{q}, z') - G_{\nu_5}^{(A)}(\boldsymbol{q}' + \boldsymbol{q}, z')\big]$$
$$+ G_{\nu_3}^{(R)}(\boldsymbol{q}', z')G_{\nu_4}^{(A)}(\boldsymbol{q}', z' - i\Omega_n)G_{\nu_5}^{(A)}(\boldsymbol{q}' + \boldsymbol{q}, z' + z - i\Omega_n)\big\}. \quad \text{(D46)}$$

In replacing the sums over m in Eqs. (D41)–(D43) by the contour integrals, we have considered the contributions only from the region for $-\Omega_n < \mathrm{Im}z < 0$ in the contour C shown in Fig. 4(a) because they include the pair of the retarded and advanced Green's functions. Furthermore, in replacing the sums over m' in Eqs. (D41)–(D43) by the integrals, we have used the contours C', C', and C'', respectively; the C' and C'' are shown in Figs. 4(b) and 4(c). We now perform the analytic continuation of Eqs. (D44)–(D46) using the replacement $i\Omega_n \to \omega + i\delta$; the results are

$$\Delta I_{\nu\nu'\nu''}^{R(a)}(\boldsymbol{q}, \boldsymbol{q}'; \omega) = \tilde{I}_{\nu\nu\nu'\nu'\nu''}^{R(a)}(\boldsymbol{q}, \boldsymbol{q}'; \omega) - \tilde{I}_{\nu\nu\nu'\nu'\nu''}^{R(a)}(\boldsymbol{q}, \boldsymbol{q}'; 0)$$
$$\sim i\omega\int_{-\infty}^{\infty}\frac{dz}{2\pi}\frac{\partial n(z)}{\partial z}g_\nu(\boldsymbol{q}, z)\int_{-\infty}^{\infty}\frac{dz'}{\pi}[n(z') - n(z' - z)]g_{\nu'}(\boldsymbol{q}', z')\mathrm{Im}G_{\nu''}^{(R)}(\boldsymbol{q}' - \boldsymbol{q}, z' - z), \quad \text{(D47)}$$
$$\Delta I_{\nu\nu'\nu''}^{R(b)}(\boldsymbol{q}, \boldsymbol{q}'; \omega) = \tilde{I}_{\nu\nu\nu'\nu'\nu''}^{R(b)}(\boldsymbol{q}, \boldsymbol{q}'; \omega) - \tilde{I}_{\nu\nu\nu'\nu'\nu''}^{R(b)}(\boldsymbol{q}, \boldsymbol{q}'; 0)$$
$$\sim i\omega\int_{-\infty}^{\infty}\frac{dz}{2\pi}\frac{\partial n(z)}{\partial z}g_\nu(\boldsymbol{q}, z)\int_{-\infty}^{\infty}\frac{dz'}{\pi}[n(z' - z) - n(z')]g_{\nu'}(\boldsymbol{q}', z')\mathrm{Im}G_{\nu''}^{(R)}(\boldsymbol{q} - \boldsymbol{q}', z - z'), \quad \text{(D48)}$$
$$\Delta I_{\nu\nu'\nu''}^{R(c)}(\boldsymbol{q}, \boldsymbol{q}'; \omega) = \tilde{I}_{\nu\nu\nu'\nu'\nu''}^{R(c)}(\boldsymbol{q}, \boldsymbol{q}'; \omega) - \tilde{I}_{\nu\nu\nu'\nu'\nu''}^{R(c)}(\boldsymbol{q}, \boldsymbol{q}'; 0)$$
$$\sim i\omega\int_{-\infty}^{\infty}\frac{dz}{2\pi}\frac{\partial n(z)}{\partial z}g_\nu(\boldsymbol{q}, z)\int_{-\infty}^{\infty}\frac{dz'}{\pi}[n(z') - n(z' + z)]g_{\nu'}(\boldsymbol{q}', z')\mathrm{Im}G_{\nu''}^{(R)}(\boldsymbol{q}' + \boldsymbol{q}, z' + z), \quad \text{(D49)}$$

where we have introduced $g_\nu(\boldsymbol{q}, z) = G_\nu^{(A)}(\boldsymbol{q}, z)G_\nu^{(R)}(\boldsymbol{q}, z)$, used $n(z) - n(z + \omega) \sim -\omega\frac{\partial n(z)}{\partial z}$, and neglected the $O(\omega^2)$ terms. Combining Eqs. (D47)–(D49) with Eq. (D40) and $\Delta\Phi_{12}^{R}(\omega) = \Delta\Phi_{12}(i\Omega_n \to \omega + i\delta)$, we obtain

$$L'_{12} = \lim_{\omega\to 0}\frac{\Delta\Phi_{12}^{R}(\omega) - \Delta\Phi_{12}^{R}(0)}{i\omega} = \frac{1}{N}\sum_{\boldsymbol{q},\boldsymbol{q}'}\sum_{\nu,\nu',\nu''=\alpha_1,\beta_1,\alpha_2,\beta_2}v_{\nu\nu}^{z}(\boldsymbol{q})e_{\nu'\nu'}^{z}(\boldsymbol{q}')\sum_{k=a,b,c}V_{\nu\nu'\nu''}^{(k)}(\boldsymbol{q}, \boldsymbol{q}')I_{\nu\nu'\nu''}^{(k)}(\boldsymbol{q}, \boldsymbol{q}'), \quad \text{(D50)}$$

where

$$V_{\nu\nu'\nu''}^{(k)}(\boldsymbol{q}, \boldsymbol{q}') = \tilde{V}_{\nu\nu\nu'\nu'\nu''}^{(k)}(\boldsymbol{q}, \boldsymbol{q}'), \quad \text{(D51)}$$

$$I_{\nu\nu'\nu''}^{(a)}(\boldsymbol{q}, \boldsymbol{q}') = \int_{-\infty}^{\infty}\frac{dz}{2\pi}\frac{\partial n(z)}{\partial z}g_\nu(\boldsymbol{q}, z)\int_{-\infty}^{\infty}\frac{dz'}{\pi}[n(z') - n(z' - z)]g_{\nu'}(\boldsymbol{q}', z')\mathrm{Im}G_{\nu''}^{(R)}(\boldsymbol{q}' - \boldsymbol{q}, z' - z), \quad \text{(D52)}$$

$$I_{\nu\nu'\nu''}^{(b)}(\boldsymbol{q}, \boldsymbol{q}') = \int_{-\infty}^{\infty}\frac{dz}{2\pi}\frac{\partial n(z)}{\partial z}g_\nu(\boldsymbol{q}, z)\int_{-\infty}^{\infty}\frac{dz'}{\pi}[n(z' - z) - n(z')]g_{\nu'}(\boldsymbol{q}', z')\mathrm{Im}G_{\nu''}^{(R)}(\boldsymbol{q} - \boldsymbol{q}', z - z'), \quad \text{(D53)}$$

$$I_{\nu\nu'\nu''}^{(c)}(\boldsymbol{q}, \boldsymbol{q}') = \int_{-\infty}^{\infty}\frac{dz}{2\pi}\frac{\partial n(z)}{\partial z}g_\nu(\boldsymbol{q}, z)\int_{-\infty}^{\infty}\frac{dz'}{\pi}[n(z') - n(z' + z)]g_{\nu'}(\boldsymbol{q}', z')\mathrm{Im}G_{\nu''}^{(R)}(\boldsymbol{q}' + \boldsymbol{q}, z' + z). \quad \text{(D54)}$$

Note that $\tilde{V}_{\nu\nu\nu'\nu'\nu''}^{(k)}(\boldsymbol{q}, \boldsymbol{q}')$'s have been given by Eqs. (D34)–(D36). In the limit $\tau = 1/2\gamma \to \infty$, we can easily do the integrals in Eqs. (D52)–(D54) by using the approximate relations,

$$g_\nu(\boldsymbol{q}, z) = G_\nu^{(A)}(\boldsymbol{q}, z)G_\nu^{(R)}(\boldsymbol{q}, z) = \frac{1}{[z + (-1)^\nu\epsilon_\nu(\boldsymbol{q})]^2 + \gamma^2} \sim \frac{\pi}{\gamma}\delta[z + (-1)^\nu\epsilon_\nu(\boldsymbol{q})], \quad \text{(D55)}$$
$$\mathrm{Im}G_\nu^{(R)}(\boldsymbol{q}, z) = (-1)^\nu\frac{\gamma}{[z + (-1)^\nu\epsilon_\nu(\boldsymbol{q})]^2 + \gamma^2} \sim (-1)^\nu\pi\delta[z + (-1)^\nu\epsilon_\nu(\boldsymbol{q})], \quad \text{(D56)}$$

where $(-1)^\nu = -1$ for $\nu = \alpha_1, \beta_1$ and 1 for $\nu = \alpha_2, \beta_2$. Combining these equations with Eqs. (D52)–(D54), we obtain

$$I_{\nu\nu'\nu''}^{(a)}(\boldsymbol{q}, \boldsymbol{q}') \sim \frac{\pi}{2\gamma^2}\frac{\partial n[\epsilon_\nu(\boldsymbol{q})]}{\partial\epsilon_\nu(\boldsymbol{q})}\{n[(-1)^{\nu'+1}\epsilon_{\nu'}(\boldsymbol{q}')] - n[(-1)^{\nu''+1}\epsilon_{\nu''}(\boldsymbol{q}' - \boldsymbol{q})]\}(-1)^{\nu''}$$
$$\times\,\delta[(-1)^\nu\epsilon_\nu(\boldsymbol{q}) - (-1)^{\nu'}\epsilon_{\nu'}(\boldsymbol{q}') + (-1)^{\nu''}\epsilon_{\nu''}(\boldsymbol{q}' - \boldsymbol{q})], \quad \text{(D57)}$$

$$I^{(b)}_{\nu\nu'\nu''}(\boldsymbol{q},\boldsymbol{q}') \sim \frac{\pi}{2\gamma^2}\frac{\partial n[\epsilon_\nu(\boldsymbol{q})]}{\partial\epsilon_\nu(\boldsymbol{q})}\{n[(-1)^{\nu''}\epsilon_{\nu''}(\boldsymbol{q}-\boldsymbol{q}')]-n[(-1)^{\nu'+1}\epsilon_{\nu'}(\boldsymbol{q}')]\}(-1)^{\nu''} \times\delta[(-1)^{\nu}\epsilon_\nu(\boldsymbol{q})-(-1)^{\nu'}\epsilon_{\nu'}(\boldsymbol{q}')-(-1)^{\nu''}\epsilon_{\nu''}(\boldsymbol{q}-\boldsymbol{q}')], \quad \text{(D58)}$$

$$I^{(c)}_{\nu\nu'\nu''}(\boldsymbol{q},\boldsymbol{q}') \sim \frac{\pi}{2\gamma^2}\frac{\partial n[\epsilon_\nu(\boldsymbol{q})]}{\partial\epsilon_\nu(\boldsymbol{q})}\{n[(-1)^{\nu'+1}\epsilon_{\nu'}(\boldsymbol{q}')]-n[(-1)^{\nu''+1}\epsilon_{\nu''}(\boldsymbol{q}'+\boldsymbol{q})]\}(-1)^{\nu''} \times\delta[(-1)^{\nu}\epsilon_\nu(\boldsymbol{q})+(-1)^{\nu'}\epsilon_{\nu'}(\boldsymbol{q}')-(-1)^{\nu''}\epsilon_{\nu''}(\boldsymbol{q}'+\boldsymbol{q})], \quad \text{(D59)}$$

where the delta functions represent the energy conservation relations in the scattering processes due to the second-order H_{int}. These equations can be obtained also by using Eqs. (D18) and (D19) and the relation $\frac{3}{x^2+(3\gamma)^2} \sim \frac{\pi}{\gamma}\delta(x)$, instead of Eqs. (D55) and (D56), and doing the integrals in Eqs. (D52)–(D54). This is the reason why we have used that relation about the Lorentzian function in the numerical evaluations of S_{m}, σ_{m}, and κ_{m}. Then, performing some calculations using Eqs. (D51), (D34)–(D36), and (D8)–(D11), we find that the finite terms of $V^{(p)}_{\nu\nu'\nu''}(\boldsymbol{q},\boldsymbol{q}')$'s ($p=1,2,3$) are given by those for $(\nu,\nu',\nu'')=(\beta,\beta,\beta)$, (β,α,α), (α,β,α), and (α,α,β), which are expressed as follows:

$$V^{(1)}_{\nu\nu'\nu''}(\boldsymbol{q},\boldsymbol{q}') = V^{(a)}_{\nu_1\nu_1'\nu_2''}(\boldsymbol{q},\boldsymbol{q}')+V^{(a)}_{\nu_2\nu_2'\nu_1''}(\boldsymbol{q},\boldsymbol{q}')+V^{(b)}_{\nu_1\nu_1'\nu_1''}(\boldsymbol{q},\boldsymbol{q}')+V^{(b)}_{\nu_2\nu_2'\nu_2''}(\boldsymbol{q},\boldsymbol{q}')+V^{(c)}_{\nu_1\nu_2'\nu_1''}(\boldsymbol{q},-\boldsymbol{q}')+V^{(c)}_{\nu_2\nu_1'\nu_2''}(\boldsymbol{q},-\boldsymbol{q}'), \quad \text{(D60)}$$

$$V^{(2)}_{\nu\nu'\nu''}(\boldsymbol{q},\boldsymbol{q}') = V^{(a)}_{\nu_1\nu_1'\nu_1''}(\boldsymbol{q},\boldsymbol{q}')+V^{(a)}_{\nu_2\nu_2'\nu_2''}(\boldsymbol{q},\boldsymbol{q}')+V^{(b)}_{\nu_1\nu_1'\nu_2''}(\boldsymbol{q},\boldsymbol{q}')+V^{(b)}_{\nu_2\nu_2'\nu_1''}(\boldsymbol{q},\boldsymbol{q}')+V^{(c)}_{\nu_2\nu_1'\nu_1''}(\boldsymbol{q},-\boldsymbol{q}')+V^{(c)}_{\nu_1\nu_2'\nu_2''}(\boldsymbol{q},-\boldsymbol{q}'), \quad \text{(D61)}$$

$$V^{(3)}_{\nu\nu'\nu''}(\boldsymbol{q},\boldsymbol{q}') = V^{(a)}_{\nu_2\nu_1'\nu_1''}(\boldsymbol{q},\boldsymbol{q}')+V^{(a)}_{\nu_1\nu_2'\nu_2''}(\boldsymbol{q},\boldsymbol{q}')+V^{(b)}_{\nu_1\nu_2'\nu_1''}(\boldsymbol{q},\boldsymbol{q}')+V^{(b)}_{\nu_2\nu_1'\nu_2''}(\boldsymbol{q},\boldsymbol{q}')+V^{(c)}_{\nu_1\nu_1'\nu_1''}(\boldsymbol{q},-\boldsymbol{q}')+V^{(c)}_{\nu_2\nu_2'\nu_2''}(\boldsymbol{q},-\boldsymbol{q}'). \quad \text{(D62)}$$

[Note that if $(\nu,\nu',\nu'')=(\beta,\alpha,\alpha)$, we have $(\nu_1,\nu_1',\nu_2'')=(\beta_1,\alpha_1,\alpha_2)$, $(\nu_2,\nu_2',\nu_1'')=(\beta_2,\alpha_2,\alpha_1)$, etc.] Since $V^{(k)}_{\nu\nu'\nu''}(\boldsymbol{q},\boldsymbol{q}')$'s ($k=a,b,c$) include the square of the coupling constant of H_{int} [see Eqs. (D34)–(D36) with Eq. (D51)] and $J_3(\boldsymbol{q})=\sqrt{\frac{4S}{N}}\sin 2\phi J(\boldsymbol{q})$, we can write the finite terms of $V^{(p)}_{\nu\nu'\nu''}(\boldsymbol{q},\boldsymbol{q}')$'s ($p=1,2,3$) as follows:

$$V^{(p)}_{\nu\nu'\nu''}(\boldsymbol{q},\boldsymbol{q}') = v^{(p)}_{\nu\nu'\nu''}(\boldsymbol{q},\boldsymbol{q}')\frac{S}{2N}\sin^2 2\phi, \quad \text{(D63)}$$

where

$$v^{(1)}_{\beta\beta\beta}(\boldsymbol{q},\boldsymbol{q}') = +v_{a0}(\boldsymbol{q},\boldsymbol{q}')C'_{\boldsymbol{q}}-v_{b0}(\boldsymbol{q},\boldsymbol{q}')C'_{\boldsymbol{q}'}-v_{c0}(\boldsymbol{q},\boldsymbol{q}')C'_{\boldsymbol{q}-\boldsymbol{q}'}-v_{d0}(\boldsymbol{q},\boldsymbol{q}')(C'_{\boldsymbol{q}}C'_{\boldsymbol{q}'}C'_{\boldsymbol{q}-\boldsymbol{q}'}+S'_{\boldsymbol{q}}S'_{\boldsymbol{q}'}S'_{\boldsymbol{q}-\boldsymbol{q}'}), \quad \text{(D64)}$$

$$v^{(2)}_{\beta\beta\beta}(\boldsymbol{q},\boldsymbol{q}') = -v_{a0}(\boldsymbol{q},\boldsymbol{q}')C'_{\boldsymbol{q}}+v_{b0}(\boldsymbol{q},\boldsymbol{q}')C'_{\boldsymbol{q}'}-v_{c0}(\boldsymbol{q},\boldsymbol{q}')C'_{\boldsymbol{q}-\boldsymbol{q}'}-v_{d0}(\boldsymbol{q},\boldsymbol{q}')(C'_{\boldsymbol{q}}C'_{\boldsymbol{q}'}C'_{\boldsymbol{q}-\boldsymbol{q}'}+S'_{\boldsymbol{q}}S'_{\boldsymbol{q}'}S'_{\boldsymbol{q}-\boldsymbol{q}'}), \quad \text{(D65)}$$

$$v^{(3)}_{\beta\beta\beta}(\boldsymbol{q},\boldsymbol{q}') = -v_{a0}(\boldsymbol{q},\boldsymbol{q}')C'_{\boldsymbol{q}}-v_{b0}(\boldsymbol{q},\boldsymbol{q}')C'_{\boldsymbol{q}'}+v_{c0}(\boldsymbol{q},\boldsymbol{q}')C'_{\boldsymbol{q}-\boldsymbol{q}'}-v_{d0}(\boldsymbol{q},\boldsymbol{q}')(C'_{\boldsymbol{q}}C'_{\boldsymbol{q}'}C'_{\boldsymbol{q}-\boldsymbol{q}'}+S'_{\boldsymbol{q}}S'_{\boldsymbol{q}'}S'_{\boldsymbol{q}-\boldsymbol{q}'}), \quad \text{(D66)}$$

$$v^{(1)}_{\beta\alpha\alpha}(\boldsymbol{q},\boldsymbol{q}') = +v_{a1}(\boldsymbol{q},\boldsymbol{q}')C'_{\boldsymbol{q}}-v_{b1}(\boldsymbol{q},\boldsymbol{q}')C_{\boldsymbol{q}'}-v_{c1}(\boldsymbol{q},\boldsymbol{q}')C_{\boldsymbol{q}-\boldsymbol{q}'}-v_{d1}(\boldsymbol{q},\boldsymbol{q}')(C'_{\boldsymbol{q}}C_{\boldsymbol{q}'}C_{\boldsymbol{q}-\boldsymbol{q}'}+S'_{\boldsymbol{q}}S_{\boldsymbol{q}'}S_{\boldsymbol{q}-\boldsymbol{q}'}), \quad \text{(D67)}$$

$$v^{(2)}_{\beta\alpha\alpha}(\boldsymbol{q},\boldsymbol{q}') = -v_{a1}(\boldsymbol{q},\boldsymbol{q}')C'_{\boldsymbol{q}}+v_{b1}(\boldsymbol{q},\boldsymbol{q}')C_{\boldsymbol{q}'}-v_{c1}(\boldsymbol{q},\boldsymbol{q}')C_{\boldsymbol{q}-\boldsymbol{q}'}-v_{d1}(\boldsymbol{q},\boldsymbol{q}')(C'_{\boldsymbol{q}}C_{\boldsymbol{q}'}C_{\boldsymbol{q}-\boldsymbol{q}'}+S'_{\boldsymbol{q}}S_{\boldsymbol{q}'}S_{\boldsymbol{q}-\boldsymbol{q}'}), \quad \text{(D68)}$$

$$v^{(3)}_{\beta\alpha\alpha}(\boldsymbol{q},\boldsymbol{q}') = -v_{a1}(\boldsymbol{q},\boldsymbol{q}')C'_{\boldsymbol{q}}-v_{b1}(\boldsymbol{q},\boldsymbol{q}')C_{\boldsymbol{q}'}+v_{c1}(\boldsymbol{q},\boldsymbol{q}')C_{\boldsymbol{q}-\boldsymbol{q}'}-v_{d1}(\boldsymbol{q},\boldsymbol{q}')(C'_{\boldsymbol{q}}C_{\boldsymbol{q}'}C_{\boldsymbol{q}-\boldsymbol{q}'}+S'_{\boldsymbol{q}}S_{\boldsymbol{q}'}S_{\boldsymbol{q}-\boldsymbol{q}'}), \quad \text{(D69)}$$

$$v^{(1)}_{\alpha\beta\alpha}(\boldsymbol{q},\boldsymbol{q}') = +v_{a2}(\boldsymbol{q},\boldsymbol{q}')C_{\boldsymbol{q}}-v_{b2}(\boldsymbol{q},\boldsymbol{q}')C'_{\boldsymbol{q}'}-v_{c2}(\boldsymbol{q},\boldsymbol{q}')C_{\boldsymbol{q}-\boldsymbol{q}'}-v_{d2}(\boldsymbol{q},\boldsymbol{q}')(C_{\boldsymbol{q}}C'_{\boldsymbol{q}'}C_{\boldsymbol{q}-\boldsymbol{q}'}+S_{\boldsymbol{q}}S'_{\boldsymbol{q}'}S_{\boldsymbol{q}-\boldsymbol{q}'}), \quad \text{(D70)}$$

$$v^{(2)}_{\alpha\beta\alpha}(\boldsymbol{q},\boldsymbol{q}') = -v_{a2}(\boldsymbol{q},\boldsymbol{q}')C_{\boldsymbol{q}}+v_{b2}(\boldsymbol{q},\boldsymbol{q}')C'_{\boldsymbol{q}'}-v_{c2}(\boldsymbol{q},\boldsymbol{q}')C_{\boldsymbol{q}-\boldsymbol{q}'}-v_{d2}(\boldsymbol{q},\boldsymbol{q}')(C_{\boldsymbol{q}}C'_{\boldsymbol{q}'}C_{\boldsymbol{q}-\boldsymbol{q}'}+S_{\boldsymbol{q}}S'_{\boldsymbol{q}'}S_{\boldsymbol{q}-\boldsymbol{q}'}), \quad \text{(D71)}$$

$$v^{(3)}_{\alpha\beta\alpha}(\boldsymbol{q},\boldsymbol{q}') = -v_{a2}(\boldsymbol{q},\boldsymbol{q}')C_{\boldsymbol{q}}-v_{b2}(\boldsymbol{q},\boldsymbol{q}')C'_{\boldsymbol{q}'}+v_{c2}(\boldsymbol{q},\boldsymbol{q}')C_{\boldsymbol{q}-\boldsymbol{q}'}-v_{d2}(\boldsymbol{q},\boldsymbol{q}')(C_{\boldsymbol{q}}C'_{\boldsymbol{q}'}C_{\boldsymbol{q}-\boldsymbol{q}'}+S_{\boldsymbol{q}}S'_{\boldsymbol{q}'}S_{\boldsymbol{q}-\boldsymbol{q}'}), \quad \text{(D72)}$$

$$v^{(1)}_{\alpha\alpha\beta}(\boldsymbol{q},\boldsymbol{q}') = +v_{a3}(\boldsymbol{q},\boldsymbol{q}')C_{\boldsymbol{q}}-v_{b3}(\boldsymbol{q},\boldsymbol{q}')C_{\boldsymbol{q}'}-v_{c3}(\boldsymbol{q},\boldsymbol{q}')C'_{\boldsymbol{q}-\boldsymbol{q}'}-v_{d3}(\boldsymbol{q},\boldsymbol{q}')(C_{\boldsymbol{q}}C_{\boldsymbol{q}'}C'_{\boldsymbol{q}-\boldsymbol{q}'}+S_{\boldsymbol{q}}S_{\boldsymbol{q}'}S'_{\boldsymbol{q}-\boldsymbol{q}'}), \quad \text{(D73)}$$

$$v^{(2)}_{\alpha\alpha\beta}(\boldsymbol{q},\boldsymbol{q}') = -v_{a3}(\boldsymbol{q},\boldsymbol{q}')C_{\boldsymbol{q}}+v_{b3}(\boldsymbol{q},\boldsymbol{q}')C_{\boldsymbol{q}'}-v_{c3}(\boldsymbol{q},\boldsymbol{q}')C'_{\boldsymbol{q}-\boldsymbol{q}'}-v_{d3}(\boldsymbol{q},\boldsymbol{q}')(C_{\boldsymbol{q}}C_{\boldsymbol{q}'}C'_{\boldsymbol{q}-\boldsymbol{q}'}+S_{\boldsymbol{q}}S_{\boldsymbol{q}'}S'_{\boldsymbol{q}-\boldsymbol{q}'}), \quad \text{(D74)}$$

$$v^{(3)}_{\alpha\alpha\beta}(\boldsymbol{q},\boldsymbol{q}') = -v_{a3}(\boldsymbol{q},\boldsymbol{q}')C_{\boldsymbol{q}}-v_{b3}(\boldsymbol{q},\boldsymbol{q}')C_{\boldsymbol{q}'}+v_{c3}(\boldsymbol{q},\boldsymbol{q}')C'_{\boldsymbol{q}-\boldsymbol{q}'}-v_{d3}(\boldsymbol{q},\boldsymbol{q}')(C_{\boldsymbol{q}}C_{\boldsymbol{q}'}C'_{\boldsymbol{q}-\boldsymbol{q}'}+S_{\boldsymbol{q}}S_{\boldsymbol{q}'}S'_{\boldsymbol{q}-\boldsymbol{q}'}), \quad \text{(D75)}$$

and

$$C'_{\boldsymbol{q}}=\cosh 2\theta'_{\boldsymbol{q}},\quad S'_{\boldsymbol{q}}=\sinh 2\theta'_{\boldsymbol{q}},\quad C_{\boldsymbol{q}}=\cosh 2\theta_{\boldsymbol{q}},\quad S_{\boldsymbol{q}}=\sinh 2\theta_{\boldsymbol{q}}, \quad \text{(D76)}$$

$$v_{a0}(\boldsymbol{q},\boldsymbol{q}') = J(\boldsymbol{q})[J(\boldsymbol{q})+J(\boldsymbol{q}')]+J(\boldsymbol{q}-\boldsymbol{q}')[J(\boldsymbol{q})-J(\boldsymbol{q}')], \quad \text{(D77)}$$

$$v_{b0}(\boldsymbol{q},\boldsymbol{q}') = J(\boldsymbol{q}')[J(\boldsymbol{q}')+J(\boldsymbol{q})]-J(\boldsymbol{q}-\boldsymbol{q}')[J(\boldsymbol{q})-J(\boldsymbol{q}')], \quad \text{(D78)}$$

$$v_{c0}(\boldsymbol{q},\boldsymbol{q}')=J(\boldsymbol{q}-\boldsymbol{q}')[J(\boldsymbol{q})+J(\boldsymbol{q}')+J(\boldsymbol{q}-\boldsymbol{q}')]-J(\boldsymbol{q})J(\boldsymbol{q}'), \tag{D79}$$

$$v_{d0}(\boldsymbol{q},\boldsymbol{q}')=[J(\boldsymbol{q})+J(\boldsymbol{q}')]^2-J(\boldsymbol{q})J(\boldsymbol{q}')+J(\boldsymbol{q}-\boldsymbol{q}')[J(\boldsymbol{q})+J(\boldsymbol{q}')+J(\boldsymbol{q}-\boldsymbol{q}')], \tag{D80}$$

$$v_{a1}(\boldsymbol{q},\boldsymbol{q}')=J(\boldsymbol{q})[J(\boldsymbol{q})-J(\boldsymbol{q}')]-J(\boldsymbol{q}-\boldsymbol{q}')[J(\boldsymbol{q})+J(\boldsymbol{q}')], \tag{D81}$$

$$v_{b1}(\boldsymbol{q},\boldsymbol{q}')=J(\boldsymbol{q}')[J(\boldsymbol{q}')-J(\boldsymbol{q})]+J(\boldsymbol{q}-\boldsymbol{q}')[J(\boldsymbol{q})+J(\boldsymbol{q}')], \tag{D82}$$

$$v_{c1}(\boldsymbol{q},\boldsymbol{q}')=J(\boldsymbol{q}-\boldsymbol{q}')[J(\boldsymbol{q}-\boldsymbol{q}')-J(\boldsymbol{q})+J(\boldsymbol{q}')]+J(\boldsymbol{q})J(\boldsymbol{q}'), \tag{D83}$$

$$v_{d1}(\boldsymbol{q},\boldsymbol{q}')=[J(\boldsymbol{q})-J(\boldsymbol{q}')]^2+J(\boldsymbol{q})J(\boldsymbol{q}')+J(\boldsymbol{q}-\boldsymbol{q}')[J(\boldsymbol{q}-\boldsymbol{q}')-J(\boldsymbol{q})+J(\boldsymbol{q}')], \tag{D84}$$

$$v_{a2}(\boldsymbol{q},\boldsymbol{q}')=J(\boldsymbol{q})[J(\boldsymbol{q})-J(\boldsymbol{q}')]+J(\boldsymbol{q}-\boldsymbol{q}')[J(\boldsymbol{q})+J(\boldsymbol{q}')], \tag{D85}$$

$$v_{b2}(\boldsymbol{q},\boldsymbol{q}')=J(\boldsymbol{q}')[J(\boldsymbol{q}')-J(\boldsymbol{q})]-J(\boldsymbol{q}-\boldsymbol{q}')[J(\boldsymbol{q})+J(\boldsymbol{q}')], \tag{D86}$$

$$v_{c2}(\boldsymbol{q},\boldsymbol{q}')=J(\boldsymbol{q}-\boldsymbol{q}')[J(\boldsymbol{q})-J(\boldsymbol{q}')+J(\boldsymbol{q}-\boldsymbol{q}')]+J(\boldsymbol{q})J(\boldsymbol{q}'), \tag{D87}$$

$$v_{d2}(\boldsymbol{q},\boldsymbol{q}')=[J(\boldsymbol{q})-J(\boldsymbol{q}')]^2+J(\boldsymbol{q})J(\boldsymbol{q}')+J(\boldsymbol{q}-\boldsymbol{q}')[J(\boldsymbol{q})-J(\boldsymbol{q}')+J(\boldsymbol{q}-\boldsymbol{q}')], \tag{D88}$$

$$v_{a3}(\boldsymbol{q},\boldsymbol{q}')=J(\boldsymbol{q})[J(\boldsymbol{q})+J(\boldsymbol{q}')]-J(\boldsymbol{q}-\boldsymbol{q}')[J(\boldsymbol{q})-J(\boldsymbol{q}')], \tag{D89}$$

$$v_{b3}(\boldsymbol{q},\boldsymbol{q}')=J(\boldsymbol{q}')[J(\boldsymbol{q}')+J(\boldsymbol{q})]+J(\boldsymbol{q}-\boldsymbol{q}')[J(\boldsymbol{q})-J(\boldsymbol{q}')], \tag{D90}$$

$$v_{c3}(\boldsymbol{q},\boldsymbol{q}')=J(\boldsymbol{q}-\boldsymbol{q}')[J(\boldsymbol{q}-\boldsymbol{q}')-J(\boldsymbol{q})-J(\boldsymbol{q}')]-J(\boldsymbol{q})J(\boldsymbol{q}'), \tag{D91}$$

$$v_{d3}(\boldsymbol{q},\boldsymbol{q}')=[J(\boldsymbol{q})+J(\boldsymbol{q}')]^2-J(\boldsymbol{q})J(\boldsymbol{q}')-J(\boldsymbol{q}-\boldsymbol{q}')[J(\boldsymbol{q})+J(\boldsymbol{q}')-J(\boldsymbol{q}-\boldsymbol{q}')]. \tag{D92}$$

[Note that the hyperbolic functions Eq. (D76) satisfy $\tanh 2\theta_{\boldsymbol{q}}=-\frac{B'(\boldsymbol{q})}{A+A'(\boldsymbol{q})}$ and $\tanh 2\theta'_{\boldsymbol{q}}=\frac{B'(\boldsymbol{q})}{A-A'(\boldsymbol{q})}$, as described in Sec. II B.] Equations (D64)–(D75) with Eqs. (D76)–(D92) give the expressions of the $v^{(p)}_{\nu\nu'\nu''}(\boldsymbol{q},\boldsymbol{q}')$'s ($p=1,2,3$) appearing in Eqs. (17)–(19). By combining Eqs. (D63)–(D92), (D57)–(D59), and (D25)–(D27) with Eq. (D50), we can express L'_{12} in the limit $\tau\to\infty$ as follows:

$$L'_{12}=\frac{\pi}{N^2}\sum_{\boldsymbol{q},\boldsymbol{q}'}\sum_{\nu,\nu',\nu''=\alpha,\beta}v^z_{\nu\nu}(\boldsymbol{q})e^z_{\nu'\nu'}(\boldsymbol{q}')\tau^2\frac{\partial n[\epsilon_\nu(\boldsymbol{q})]}{\partial\epsilon_\nu(\boldsymbol{q})}S\sin^2 2\phi\sum_{p=1,2,3}F^{(p)}_{\nu\nu'\nu''}(\boldsymbol{q},\boldsymbol{q}'), \tag{D93}$$

where

$$F^{(1)}_{\nu\nu'\nu''}(\boldsymbol{q},\boldsymbol{q}')=v^{(1)}_{\nu\nu'\nu''}(\boldsymbol{q},\boldsymbol{q}')\{1+n[\epsilon_{\nu''}(\boldsymbol{q}-\boldsymbol{q}')]+n[\epsilon_{\nu'}(\boldsymbol{q}')]\}\delta[\epsilon_\nu(\boldsymbol{q})-\epsilon_{\nu'}(\boldsymbol{q}')-\epsilon_{\nu''}(\boldsymbol{q}-\boldsymbol{q}')], \tag{D94}$$

$$F^{(2)}_{\nu\nu'\nu''}(\boldsymbol{q},\boldsymbol{q}')=v^{(2)}_{\nu\nu'\nu''}(\boldsymbol{q},\boldsymbol{q}')\{n[\epsilon_{\nu''}(\boldsymbol{q}-\boldsymbol{q}')]-n[\epsilon_{\nu'}(\boldsymbol{q}')]\}\delta[\epsilon_\nu(\boldsymbol{q})-\epsilon_{\nu'}(\boldsymbol{q}')+\epsilon_{\nu''}(\boldsymbol{q}-\boldsymbol{q}')], \tag{D95}$$

$$F^{(3)}_{\nu\nu'\nu''}(\boldsymbol{q},\boldsymbol{q}')=-v^{(3)}_{\nu\nu'\nu''}(\boldsymbol{q},\boldsymbol{q}')\{n[\epsilon_{\nu''}(\boldsymbol{q}-\boldsymbol{q}')]-n[\epsilon_{\nu'}(\boldsymbol{q}')]\}\delta[\epsilon_\nu(\boldsymbol{q})+\epsilon_{\nu'}(\boldsymbol{q}')-\epsilon_{\nu''}(\boldsymbol{q}-\boldsymbol{q}')]. \tag{D96}$$

In deriving them, we have used the identity $n(-x)=-1-n(x)$. Then, since Eqs. (9) and (11)–(13) show that L'_{11} and L'_{22} are obtained by replacing $e^z_{\nu'\nu'}(\boldsymbol{q}')$ in Eq. (D93) by $v^z_{\nu'\nu'}(\boldsymbol{q}')$ and by replacing $v^z_{\nu\nu}(\boldsymbol{q})$ in Eq. (D93) by $e^z_{\nu\nu}(\boldsymbol{q})$, respectively, we can express L'_{11} and L'_{22} in the limit $\tau\to\infty$ as follows:

$$L'_{11}=\frac{\pi}{N^2}\sum_{\boldsymbol{q},\boldsymbol{q}'}\sum_{\nu,\nu',\nu''=\alpha,\beta}v^z_{\nu\nu}(\boldsymbol{q})v^z_{\nu'\nu'}(\boldsymbol{q}')\tau^2\frac{\partial n[\epsilon_\nu(\boldsymbol{q})]}{\partial\epsilon_\nu(\boldsymbol{q})}S\sin^2 2\phi\sum_{p=1,2,3}F^{(p)}_{\nu\nu'\nu''}(\boldsymbol{q},\boldsymbol{q}'), \tag{D97}$$

$$L'_{22}=\frac{\pi}{N^2}\sum_{\boldsymbol{q},\boldsymbol{q}'}\sum_{\nu,\nu',\nu''=\alpha,\beta}e^z_{\nu\nu}(\boldsymbol{q})e^z_{\nu'\nu'}(\boldsymbol{q}')\tau^2\frac{\partial n[\epsilon_\nu(\boldsymbol{q})]}{\partial\epsilon_\nu(\boldsymbol{q})}S\sin^2 2\phi\sum_{p=1,2,3}F^{(p)}_{\nu\nu'\nu''}(\boldsymbol{q},\boldsymbol{q}'). \tag{D98}$$

Equations (D93), (D97), and (D98) yield Eq. (16).

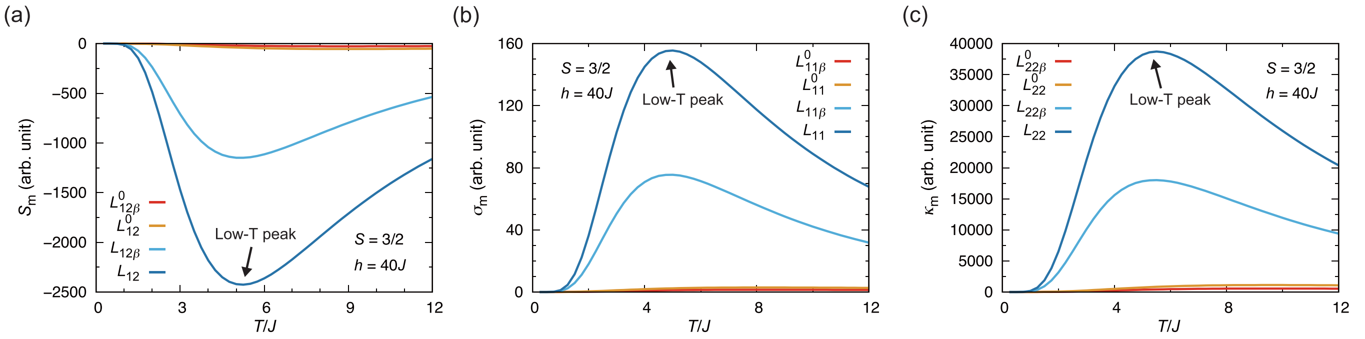

FIG. 5. The temperature dependences of (a) S_m, (b) σ_m, and (c) κ_m obtained in the numerical calculations for $S = \frac{3}{2}$ with $\frac{N}{2} = 20^3$ and $J = 1$ at $h = 40J$. The red, yellow, light blue, and blue curves represent the T/J dependences of $S_m = L^0_{12\beta}$, $\sigma_m = L^0_{11\beta}$, and $\kappa_m = L^0_{22\beta}$, those of $S_m = L^0_{12}$, $\sigma_m = L^0_{11}$, and $\kappa_m = L^0_{22}$, those of $S_m = L_{12\beta}$, $\sigma_m = L_{11\beta}$, and $\kappa_m = L_{22\beta}$, and those of $S_m = L_{12}$, $\sigma_m = L_{11}$, and $\kappa_m = L_{22}$, respectively. $L^0_{\mu\eta\beta}$ is part of the noninteracting term, the contribution from the lower-branch magnons (i.e., the β-band magnons); $L^0_{\mu\eta}$ and $L'_{\mu\eta}(= L_{\mu\eta} - L^0_{\mu\eta})$ are the noninteracting and drag terms, respectively. $L_{\mu\eta\beta} = L^0_{\mu\eta} + L'_{\mu\eta\beta}$, where $L'_{\mu\eta\beta}$ is part of the drag term, the contribution from the term for $(\nu, \nu', \nu'') = (\beta, \beta, \beta)$ in Eq. (16).

APPENDIX E: ADDITIONAL NUMERICAL RESULTS OF S_m, σ_m, AND κ_m

We present the additional results of the numerically evaluated S_m, σ_m, and κ_m for $S = \frac{3}{2}$ with $\frac{N}{2} = 20^3$ and $J = 1$. (In the case of $S = \frac{3}{2}$, the magnon picture for the canted antiferromagnet is valid in the range of $0 < h < 48J$.) Since the transition temperature for $S = \frac{3}{2}$ becomes $T_c = 20J$, we choose the temperature range to be $0 < T \leqslant 12J(= 0.6T_c)$. Figures 5(a)–5(c) show the temperature dependences of S_m, σ_m, and κ_m for $S = \frac{3}{2}$ at $h = 40J$. For $S = \frac{3}{2}$, the low-temperature peaks are observed at the h lower than $65J$. Then, the ratios L_{12}/L^0_{12}, L_{11}/L^0_{11}, and L_{22}/L^0_{22} at $T = 5J(= 0.25T_c)$ reach about 60, 66, and 52, respectively. The larger enhancement for $S = \frac{3}{2}$ than that for $S = \frac{5}{2}$ comes from the property that the smaller the S is, the more considerable the effects of magnon-magnon interactions become. This general property is due to the difference between the S dependences of H_0 and H_{int}.

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